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Ranking Trapezoidal Fuzzy Numbers Using Some Total Orders

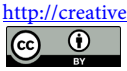
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Abstract

Fuzzy number ranking is a fundamental issue in fuzzy decision-making and fuzzy optimization. Due to information overlaps among fuzzy numbers, traditional methods are hard to obtain complete order relations and may lead to indifferent situations where magnitudes cannot be judged. Starting from several total orders on bounded closed intervals, this paper investigates the construction of order relations for fuzzy numbers. Several partial orders of fuzzy numbers are constructed based on properties of interval total orders, and their basic properties are analyzed. Furthermore, aiming at trapezoidal fuzzy numbers, several total order relations are established to resolve the incomparability of partial orders. These total orders can distinguish the priority of fuzzy numbers between them. Finally, the proposed total order relations are applied to numerical examples of fuzzy number ranking to verify feasibility and effectiveness. The results provide theoretical support for practical problems such as multi-attribute fuzzy decision-making and fuzzy games.

Keywords

Bounded Closed Intervals, Trapezoidal Fuzzy Numbers, Component Wise Order, Total Order

1. Introduction

Ranking fuzzy numbers constitutes a well studied research topic due to its significant applications in approximate reasoning [1], decision-making [2], forecasting [3], and other fuzzy systems. As a fundamental problem within fuzzy decision-making, it has drawn considerable research interest. Numerous ranking methodologies have been proposed, and early comprehensive reviews are available in [4]-[6]. Representative works include the relative-preference-relation-based approach

for triangular and trapezoidal fuzzy numbers developed by Wang [7], as well as the possibility-theory-driven method reported by Gu and Xuan [8].

As summarized by Ma and Yang [9], existing fuzzy number ranking techniques can be grouped into three classes: based on defuzzification methods, based on ranking-index methods, and pairwise-comparison methods. Although based on defuzzification methods are computationally convenient, heuristic manipulation may lead to the loss of inherent fuzzy information [10]. In comparison, based on ranking-index and pairwise-comparison schemes preserve full fuzzy information yet suffer from higher computational complexity. Along this line, Yuan [11] advocated constructing fuzzy ranking via fuzzy preference relations. Wang [7] further pointed out that such preference relations are capable of quantifying preference degrees with membership functions.

Preference assessment for fuzzy numbers is closely linked to the ranking of interval-valued quantities, which has also received substantial research attention. Interval numbers serve as a powerful tool for modelling expert judgements [12], and interval-valued weights are widely adopted in various decision-making scenarios [13]. Moreover, interval ordering occupies a critical position in multiple-attribute decision-making [14].

This paper focuses on constructing order relations for fuzzy numbers. Inspired by the properties of interval total orders, several partial orders for fuzzy numbers are established, and their fundamental characteristics are investigated. To address the incomparability limitation inherent to partial orders, we further develop a set of total order relations specifically for trapezoidal fuzzy numbers. These total orders enable clear discrimination of priority levels among distinct fuzzy numbers. Finally, numerical examples concerning fuzzy number ranking are presented to validate the feasibility and effectiveness of the proposed total order relations. The findings offer theoretical underpinnings for real-world applications such as multi-attribute fuzzy decision-making and fuzzy games.

2. Preliminaries

Proposed by Zadeh [15], a fuzzy set defined on a nonempty set X is a mapping $A: X \to [0,1]$. For any element $x \in X$, the value $A(x)$ is termed the membership degree of X, which characterizes the degree to which x belongs to the fuzzy set A. As a special and widely investigated class of fuzzy sets on the real number field $\mathbb{R}$, fuzzy numbers extend the conventional definition of real numbers. Different from precise single-valued real numbers, fuzzy numbers permit a set of plausible values to describe uncertain quantities. The fundamental concept of fuzzy numbers was first proposed and systematically investigated by Dubois and Prade [16].

Definition 1. ([16]) *A fuzzy set A on $\mathbb{R}$ is called a fuzzy number, if it satisfies the following conditions*:

1) A *is normal, i.e., there exists* $x_0 \in \mathbb{R}$ *such that* $A(x_0) = 1$;

2) A *is fuzzy convex, i.e.,* $A(\lambda x + (1-\lambda) y) \geq \min\{A(x), A(y)\}$ *for all*

$x, y \in \mathbb{R}$ *and* $\lambda \in [0,1]$;

3) A *is upper semi-continuous on* $\mathbb{R}$, *i.e.,* $\forall \varepsilon > 0$, $\exists \delta > 0$ *with* $A(x) - A(x_0) < \varepsilon$ *whenever* $|x - x_0| < \delta$; *and*

4) *the closure of* $\operatorname{supp}(A)$, *i.e.,* $cl\{x \in \mathbb{R} : A(x) > 0\}$ *is bounded. Where the symbol* cl *is a topological closure operator with respect to the usual topology on* $\mathbb{R}$.

Throughout this paper, $\mathcal{F}(\mathbb{R})$ denotes the collection of all fuzzy numbers. For any fuzzy number $A \in \mathcal{F}(\mathbb{R})$ and each $\alpha \in (0,1]$, the α-level set of A is defined as $A_\alpha = \{x \in X : A(x) \geq \alpha\}$. The kernel of A, denoted by $\ker(A)$, is given by $\ker(A) = \{x \in X : A(x) = 1\}$, which is equivalent to the 1-level set A_1. In addition, the 0-level set A_0 corresponds to the support $\operatorname{supp}(A)$ of the fuzzy number A. It is well known that a fuzzy number can be uniquely determined by its family of α-level sets. Accordingly, for any two fuzzy numbers $A, B \in \mathcal{F}(\mathbb{R})$, $A = B$ holds if and only if $A_\alpha = B_\alpha$ for all $\alpha \in [0,1]$.

Let $\mathbb{IR}$ represent the set of all nonempty, closed, and bounded intervals, which is formally defined as:

$$\mathbb{IR} = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}.$$

Each interval $[a,b] \in \mathbb{IR}$ can be correspondingly regarded as a specific fuzzy number via its characteristic function. Specifically, the fuzzy valued extension $\overline{[a,b]} : \mathbb{R} \to [0,1]$ of the interval $[a,b]$ is defined as:

$$\overline{[a,b]}(x) = \begin{cases} 1, & x \in [a,b], \\ 0, & x \notin [a,b]. \end{cases}$$

Due to its rectangular membership distribution, $\overline{[a,b]}$ is termed a rectangular fuzzy number. In the degenerate case where $a = b$, the single-point interval $[a,a]$ degenerates to a standard real number. For simplicity, we adopt the notation $\overline{a}$ instead of $\overline{[a,a]}$ or $\overline{[a]}$ for such degenerate intervals. In this framework, closed real intervals can be regarded as a special class of fuzzy numbers, such that $\mathbb{IR} \subseteq \mathcal{F}(\mathbb{R})$. This further implies the embedding relationship $\mathbb{R} \subseteq \mathcal{F}(\mathbb{R})$, demonstrating that fuzzy numbers serve as a generalized form of real numbers and real intervals. According to Definition 1, the α-level set A_α of any fuzzy number A is a nonempty closed interval for all $\alpha \in [0,1]$. This interval is uniformly written as $[A_\alpha^-, A_\alpha^+]$, where the finite real numbers A_α^- and A_α^+ satisfy $A_\alpha^- \leq A_\alpha^+$, and are defined as the left and right endpoints of A_α, respectively. Xiao and Wang [17] systematically summarized the fundamental properties of the α-level sets ($\alpha \in [0,1]$) of fuzzy numbers and established an equivalent characterization principle for fuzzy numbers based on these interval families, which is presented in the following proposition.

Proposition 1 ([17]) *A fuzzy set* A *on* $\mathbb{R}$ *is a fuzzy number if and only if* A_α *is a non-empty and closed interval for every* $\alpha \in [0,1]$.

A trapezoidal fuzzy number is defined as a quadruple $A = (a_1, a_2, a_3, a_4)$, where a_1, a_2, a_3 and a_4 are all real numbers satisfying $a_1 \leq a_2 \leq a_3 \leq a_4$, and its mem-

bership function is given by

$$A(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1}, & \text{if } a_1\le x<a_2,\\ 1, & \text{if } a_2\le x\le a_3,\\ \dfrac{a_4-x}{a_4-a_3}, & \text{if } a_3<x\le a_4,\\ 0, & \text{if } x<a_1 \text{ or } x>a_4.\end{cases}$$

Its α -level set takes the form

$$A_\alpha=\left[a_1(1-\alpha)+\alpha a_2, a_4(1-\alpha)+\alpha a_3\right].$$

In the special case where $a_1=a_2$ and $a_3=a_4$, A will degenerate as a closed interval and is denoted by $A=(a_1,a_1,a_3,a_3)$ or $A=[a_1,a_3]$. Throughout this paper, $\mathrm{TrFN}(\mathbb{R})$ denotes the collection of all trapezoidal fuzzy numbers.

3. Some Total Orders of Trapezoidal Fuzzy Numbers

Let X be a nonempty set. A binary relation $\le$ defined on X is termed a partial order if it satisfies reflexivity, antisymmetry and transitivity. X is referred to as an ordered set (or partially ordered set). Note that an order $\le$ on X gives rise to a relation $<$ of strict inequality: $x<y$ on X if and only if $x\le y$ and $x\ne y$. Furthermore, an order $\le$ in X is called total or linear if, for any $x,y\in X$, $x\le y$ or $y\le x$.

3.1. Some Total Orders of Closed and Bounded Intervals

For two intervals $\left[a^L,a^U\right],\left[b^L,b^U\right]\in\mathbb{IR}$, if we compare their left and right endpoints, then the following component-wise order (also named LU order in [18]) is defined.

Definition 2. $\left[a^L,a^U\right]\le_{LU}\left[b^L,b^U\right]$ *if and only if* $a^L\le b^L$ *and* $a^U\le b^U$, $\left[a^L,a^U\right]<_{LU}\left[b^L,b^U\right]$ *if and only if* $\left[a^L,a^U\right]\le_{LU}\left[b^L,b^U\right]$ *and* $\left[a^L,a^U\right]\ne\left[b^L,b^U\right]$.

On the set $\mathbb{IR}$, the component-wise order is a frequently used partial order. Some pairs of intervals remain incomparable, which complicates the identification of a unique optimal interval solution in practical problems. To mitigate this difficulty, some total orders were introduced for the set of closed subintervals of $[0,1]$ that refines the component-wise order. Some important classes of admissible orders in $\left(\mathcal{I}(\mathbb{R}),\le_{LU}\right)$ are defined as following.

1) Lexicographical orders.

$$[a,b]\le_{Lex1}^{\mathcal{I}(\mathbb{R})}[c,d]:=a<c,\text{ or }\begin{cases}a=c\\ b\le d\end{cases}.\tag{1}$$

$$[a,b]\le_{Lex2}^{\mathcal{I}(\mathbb{R})}[c,d]:=b<d,\text{ or }\begin{cases}b=d\\ a\le c\end{cases}.\tag{2}$$

2) Xu-Yager order with preference for higher uncertainty. If the decision maker

prefers the alternative with the larger expected value, and when expectations coincide, the one with greater uncertainty, the $XY1$ order [19] is

$$[a,b] \leq_{XY1}^{\mathcal{I}(\mathbb{R})} [c,d] := a+b<c+d, \text{ or } \begin{cases} a+b=c+d \\ b-a \leq d-c \end{cases}. \tag{3}$$

3) Xu-Yager order with preference for lower uncertainty. Conversely, if the decision maker still favors the larger expected value but when expectations coincide, prefers lower uncertainty, the order is

$$[a,b] \leq_{XY2}^{\mathcal{I}(\mathbb{R})} [c,d] := a+b<c+d, \text{ or } \begin{cases} a+b=c+d \\ b-a \geq d-c \end{cases}. \tag{4}$$

In **Figure 1(a)**, the collection of bounded closed intervals, composed of both the orange and blue ones, represents the set of intervals that are smaller than $[c,d]$ according to the order $\leq_{Lex1}^{\mathcal{I}(\mathbb{R})}$. Specifically, these intervals satisfy the ranking criteria defined by $\leq_{Lex1}^{\mathcal{I}(\mathbb{R})}$ relative to $[c,d]$. In **Figure 1(b)**, the set of orange and blue intervals corresponds to the collection of those bounded closed intervals that are ranked lower than $[c,d]$ under the order $\leq_{Lex2}^{\mathcal{I}(\mathbb{R})}$. This order $\leq_{Lex2}^{\mathcal{I}(\mathbb{R})}$ determines the relative size of these intervals with respect to $[c,d]$. For **Figure 1(c)**, the set of orange and blue bounded closed intervals precisely constitutes the set of all intervals that are smaller than $[c,d]$ in terms of the order $\leq_{XY2}^{\mathcal{I}(\mathbb{R})}$. That is, based on the rules of the $\leq_{XY2}^{\mathcal{I}(\mathbb{R})}$ order, these intervals are considered to be of a smaller magnitude compared to $[c,d]$.

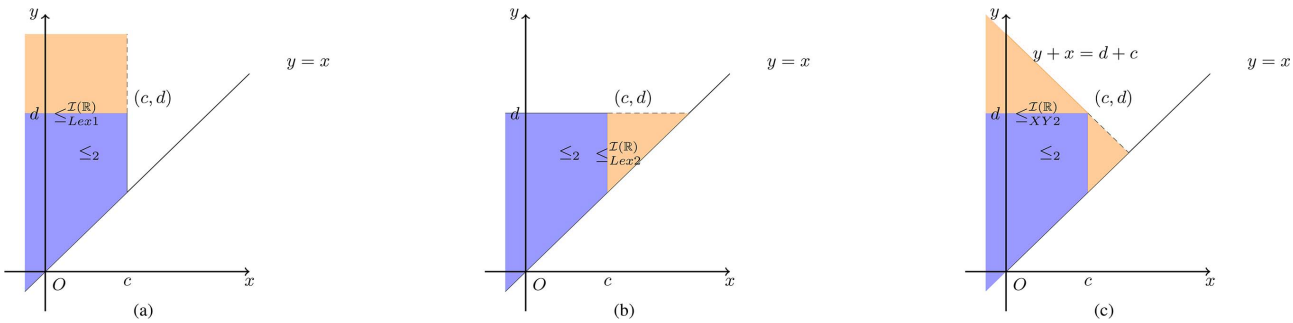


Figure 1. Blue region is the set of intervals smaller than $[c,d]$ under the order $\leq_2$; Orange and blue regions in (a), (b) and (c) are the set of intervals smaller than $[c,d]$ under the order $\leq_{Lex1}^{\mathcal{I}(\mathbb{R})}$, $\leq_{Lex2}^{\mathcal{I}(\mathbb{R})}$ and $\leq_{XY2}^{\mathcal{I}(\mathbb{R})}$ respectively.

3.2. Some Orders of Fuzzy Numbers and Trapezoidal Fuzzy Numbers

For two fuzzy numbers $A, B \in \mathcal{F}(\mathbb{R})$, Klir and Yuan [20] introduced the KY order:

$$A \leq_{KY} B := A \wedge B = A.$$

which can equivalently be expressed via the component-wise order $\leq_{LU}$ on α-level sets:

$$A \leq_{KY} B \Leftrightarrow A_\alpha \leq_{LU} B_\alpha \quad \forall \alpha \in [0,1]. \tag{5}$$

For two TrFNs $A=(a_1,a_2,a_3,a_4), B=(b_1,b_2,b_3,b_4) \in \text{TrFN}(\mathbb{R})$, from Defini-

tions 1, 2 and Equation (5), we have

$$A \leq_{KY} B \Leftrightarrow a_i \leq b_i, i = 1,2,3,4. \tag{6}$$

Figure 2(a) shows a case where $A \leq_{KY} B$, the meaning expressed by the KY order is that B is greater than A, **Figure 2(b)** shows a case where $A \not\leq_{KY} B$.

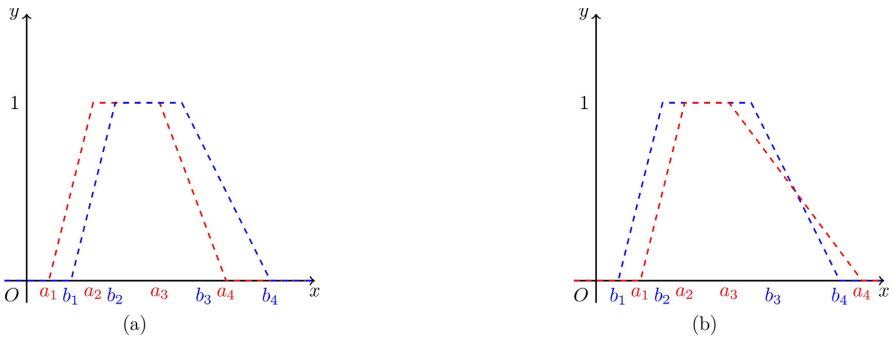


Figure 2. The red graph and the blue graph represent membership functions of trapezoidal fuzzy numbers A and B, respectively. (a) $A \leq_{KY} B$; (b) A, B are not comparable in $\leq_{KY}$ and $\leq_{Lex2}$, $B \leq_{Lex1} A$.

As previously stated, establishing a total order for fuzzy numbers is quite intricate. In the subsequent part of this section, we will construct total orders specifically for $\mathrm{TrFN}(\mathbb{R})$. In Equations (1-4), we know $\leq_{Lex1}^{\mathcal{I}(\mathbb{R})}$, $\leq_{Lex2}^{\mathcal{I}(\mathbb{R})}$, $\leq_{XY1}^{\mathbb{IR}}$ and $\leq_{XY2}^{\mathbb{IR}}$ are total orders in partially ordered sets $(\mathbb{IR}, \leq_{LU})$. We can generalize these four orders from $\mathbb{IR}$ to the set of fuzzy numbers as follows.

1) Compare the left endpoints of each α-level set. If the left endpoints are equal, compare their right endpoints. The lexicographical orders are defined as follows: $\forall \alpha \in [0,1]$,

$$A \leq_{Lex1} B \Leftrightarrow A_\alpha^- < B_\alpha^-, \text{ or } \begin{cases} A_\alpha^- = B_\alpha^- \\ A_\alpha^+ \leq B_\alpha^+ \end{cases}, \ \forall \alpha \in [0,1]. \tag{7}$$

$$A \leq_{Lex2} B \Leftrightarrow A_\alpha^+ < B_\alpha^+, \text{ or } \begin{cases} A_\alpha^+ = B_\alpha^+ \\ A_\alpha^- \leq B_\alpha^- \end{cases}. \tag{8}$$

2) If experts prioritize alternatives with a higher expected value for each α-level set, and in case of equal expected values, prefer those with greater uncertainty, the order $\leq_{XY1}$ is defined as: $\forall \alpha \in [0,1]$,

$$A \leq_{XY1} B \Leftrightarrow A_\alpha^- + A_\alpha^+ < B_\alpha^- + B_\alpha^+, \text{ or } \begin{cases} A_\alpha^- + A_\alpha^+ = B_\alpha^- + B_\alpha^+ \\ A_\alpha^+ - A_\alpha^- \leq B_\alpha^+ - B_\alpha^- \end{cases}. \tag{9}$$

3) If experts prioritize alternatives with a higher expected value for each α-level set, and in case of equal expected values, prefer those with lower uncertainty, the order $\leq_{XY2}$ is defined as: $\forall \alpha \in [0,1]$,

$$A \leq_{XY2} B \Leftrightarrow A_\alpha^- + A_\alpha^+ < B_\alpha^- + B_\alpha^+, \text{ or } \begin{cases} A_\alpha^- + A_\alpha^+ = B_\alpha^- + B_\alpha^+ \\ A_\alpha^+ - A_\alpha^- \geq B_\alpha^+ - B_\alpha^- \end{cases}. \tag{10}$$

Theorem 1. $\le_{Lex1}$, $\le_{Lex2}$, $\le_{XY1}$ *and* $\le_{XY2}$ *are partial orders on fuzzy numbers, moreover* $A \le_{Lex1} B$, $A \le_{Lex2} B$, $A \le_{XY1} B$ *and* $A \le_{XY2} B$ *whenever* $A \le_{KY} B$.

Proof. We shall only prove that $\le_{Lex1}$ is a partial order and $A \le_{Lex1} B$ whenever $A \le_{KY} B$, with the proofs for the remaining three following analogously. Let $A = \left[A_\alpha^-, A_\alpha^+\right]$, $B = \left[B_\alpha^-, B_\alpha^+\right]$, $C = \left[C_\alpha^-, C_\alpha^+\right]$ be fuzzy numbers.

1) $A \le_{Lex1} A$ (reflexivity) is trivial.

2) Suppose $A \le_{Lex1} B$ and $B \le_{Lex1} A$, then $\forall \alpha \in [0,1]$, 1) $A_\alpha^- < B_\alpha^-$, or $\begin{cases} A_\alpha^- = B_\alpha^- \\ A_\alpha^+ \le B_\alpha^+ \end{cases}$ and 2) $B_\alpha^- < A_\alpha^-$, or $\begin{cases} B_\alpha^- = A_\alpha^- \\ B_\alpha^+ \le A_\alpha^+ \end{cases}$ hold. Thus, $\begin{cases} A_\alpha^- = B_\alpha^- \\ A_\alpha^+ = B_\alpha^+ \end{cases}$. Therefore, $A = B$ (antisymmetry) holds.

3) Suppose $A \le_{Lex1} B$ and $B \le_{Lex1} C$, then $\forall \alpha \in [0,1]$, a) $A_\alpha^- < B_\alpha^-$, or $\begin{cases} A_\alpha^- = B_\alpha^- \\ A_\alpha^+ \le B_\alpha^+ \end{cases}$ and b) $B_\alpha^- < C_\alpha^-$, or $\begin{cases} B_\alpha^- = C_\alpha^- \\ B_\alpha^+ \le C_\alpha^+ \end{cases}$ hold. Thus, $A_\alpha^- < C_\alpha^-$, or $\begin{cases} A_\alpha^- = C_\alpha^- \\ A_\alpha^+ \le C_\alpha^+ \end{cases}$. Therefore, $A \le_{Lex1} C$ (transitivity) holds.

Suppose $A \le_{KY} B$, then $\forall \alpha \in [0,1]$, $A_\alpha \le_{LU} B_\alpha$. Therefore, $\begin{cases} A_\alpha^- \le B_\alpha^- \\ A_\alpha^+ \le B_\alpha^+ \end{cases}$. Thus, $A_\alpha^- < B_\alpha^-$, or $\begin{cases} A_\alpha^- = B_\alpha^- \\ A_\alpha^+ \le B_\alpha^+ \end{cases}$ holds, *i.e.*, $A \le_{Lex1} B$. □

The above theorem proves that Equations. (7-10) define a partial order, yet they do not constitute a total order. Figure 2(b) shows a case where A, B are not comparable in $\le_{KY}$ and $\le_{Lex2}$, $B \le_{Lex1} A$.

If $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ are trapezoidal fuzzy numbers, we can define $\trianglelefteq_{Lex1}$, $\trianglelefteq_{Lex2}$, $\trianglelefteq_{XY1}$ and $\trianglelefteq_{XY2}$ as follows.

$$A \trianglelefteq_{Lex1} B \Leftrightarrow a_2 < b_2, \text{or} \begin{cases} a_2 = b_2 \\ a_1 < b_1 \end{cases}, \text{or} \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 < b_3 \end{cases}, \text{or} \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \\ a_4 \le b_4 \end{cases}. \tag{11}$$

$$A \trianglelefteq_{Lex2} B \Leftrightarrow a_3 < b_3, \text{or} \begin{cases} a_3 = b_3 \\ a_4 < b_4 \end{cases}, \text{or} \begin{cases} a_3 = b_3 \\ a_4 = b_4 \\ a_2 < b_2 \end{cases}, \text{or} \begin{cases} a_3 = b_3 \\ a_4 = b_4 \\ a_2 = b_2 \\ a_1 \le b_1 \end{cases}. \tag{12}$$

$$A \trianglelefteq_{XY1} B \Leftrightarrow a_2 + a_3 < b_2 + b_3, \text{or} \begin{cases} a_2 + a_3 = b_2 + b_3 \\ a_1 + a_4 < b_1 + b_4 \end{cases}, \text{or}$$
$$\begin{cases} a_2 + a_3 = b_2 + b_3 \\ a_1 + a_4 = b_1 + b_4 \\ a_3 - a_2 < b_3 - b_2 \end{cases}, \text{or} \begin{cases} a_2 + a_3 = b_2 + b_3 \\ a_1 + a_4 = b_1 + b_4 \\ a_3 - a_2 = b_3 - b_2 \\ a_4 - a_1 \le b_4 - b_1 \end{cases}. \tag{13}$$

$$A \trianglelefteq_{XY2} B \Leftrightarrow a_2 + a_3 < b_2 + b_3, \text{ or } \begin{cases} a_2 + a_3 = b_2 + b_3 \\ a_1 + a_4 > b_1 + b_4 \end{cases}, \text{ or}$$

$$\begin{cases} a_2 + a_3 = b_2 + b_3 \\ a_1 + a_4 = b_1 + b_4 \\ a_3 - a_2 > b_3 - b_2 \end{cases}, \text{ or } \begin{cases} a_2 + a_3 = b_2 + b_3 \\ a_1 + a_4 = b_1 + b_4 \\ a_3 - a_2 = b_3 - b_2 \\ a_4 - a_1 \geq b_4 - b_1 \end{cases}. \tag{14}$$

Theorem 2. $\trianglelefteq_{Lex1}$, $\trianglelefteq_{Lex2}$, $\trianglelefteq_{XY1}$ *and* $\trianglelefteq_{XY2}$ *are total orders trapezoidal fuzzy numbers, moreover* $A \trianglelefteq_{Lex1} B$ *whenever* $A \leq_{Lex1} B$, $A \trianglelefteq_{Lex2} B$ *whenever* $A \leq_{Lex2} B$, $A \trianglelefteq_{XY1} B$ *whenever* $A \leq_{XY1} B$ *and* $A \trianglelefteq_{XY2} B$ *whenever* $A \leq_{XY2} B$ *in* $\mathrm{TrFN}(\mathbb{R})$.

Proof. We shall only prove that $\trianglelefteq_{Lex1}$ is a total order and $A \trianglelefteq_{Lex1} B$ whenever $A \leq_{Lex1} B$, with the proofs for the remaining three following analogously. Let $A = (a_1, a_2, a_3, a_4)$, $B = (b_1, b_2, b_3, b_4)$ and $C = (c_1, c_2, c_3, c_4)$ be trapezoidal fuzzy numbers.

1) $A \trianglelefteq_{Lex1} A$ (reflexivity) is trivial.

2) Suppose $A \trianglelefteq_{Lex1} B$ and $B \trianglelefteq_{Lex1} A$, then from Equation (11), we have $a_i \leq b_i$ and $b_i \leq a_i$, $i = 1,2,3,4$. Thus, $A = B$ (antisymmetry) holds.

3) Suppose $A \trianglelefteq_{Lex1} B$ and $B \trianglelefteq_{Lex1} C$, then from Equation (11), we have $a_i \leq b_i$ and $b_i \leq c_i$, $i = 1,2,3,4$. Thus, $a_i \leq c_i$. a) If $a_2 < b_2$ or $b_2 < c_2$, then $a_2 < c_2$. Thus, $B \trianglelefteq_{Lex1} C$. b) If $\begin{cases} a_2 = b_2 \\ a_1 < b_1 \end{cases}$, and $\begin{cases} b_2 = c_2 \\ b_1 < c_1 \end{cases}$, then $\begin{cases} a_2 = c_2 \\ a_1 < c_1 \end{cases}$. Thus, $B \trianglelefteq_{Lex1} C$. c) If $\begin{cases} a_2 = b_2 \\ a_1 < b_1 \end{cases}$, and $\begin{cases} b_1 = c_1 \\ b_2 = c_2 \\ b_3 < c_3 \end{cases}$, then $\begin{cases} a_2 = c_2 \\ a_1 < c_1 \end{cases}$. Thus, $B \trianglelefteq_{Lex1} C$. (d) If $\begin{cases} a_2 = b_2 \\ a_1 < b_1 \end{cases}$, and $\begin{cases} b_1 = c_1 \\ b_2 = c_2 \\ b_3 = c_3 \\ b_4 \leq c_4 \end{cases}$, then $\begin{cases} a_2 = c_2 \\ a_1 < c_1 \end{cases}$. Thus, $B \trianglelefteq_{Lex1} C$. By a similar argument, the conclusion is valid for the other six cases. Therefore, $A \trianglelefteq_{Lex1} C$ (transitivity) holds.

Suppose $A \leq_{Lex1} B$, then $\forall \alpha \in [0,1]$, $A_\alpha^- < B_\alpha^-$, or $\begin{cases} A_\alpha^- = B_\alpha^- \\ A_\alpha^+ \leq B_\alpha^+ \end{cases}$. Therefore, $\begin{cases} a_1 < b_1 \\ a_2 < b_2 \end{cases}$, or $\begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 \leq b_3 \\ a_4 \leq b_4 \end{cases}$. Thus, $a_2 < b_2$, or $\begin{cases} a_2 = b_2 \\ a_1 < b_1 \end{cases}$, or $\begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 < b_3 \end{cases}$, or $\begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \\ a_4 \leq b_4 \end{cases}$ holds, *i.e.*, $A \trianglelefteq_{Lex1} B$. □

4. Application to Rank Trapezoidal Fuzzy Numbers

In this section, we apply the main results derived in the aforementioned sections to the ranking of trapezoidal fuzzy numbers. A total order imposes clear, well-

defined sequential relationships on the elements of a set. In a totally ordered set, any two elements can be compared in terms of their sequence with absolute clarity, *i.e.*, leaving no room for ambiguity or uncertainty regarding their order. Consequently, the findings of this paper can be applied to the ranking of trapezoidal fuzzy numbers.

The four trapezoidal fuzzy numbers featured in the subsequent example are drawn from [21]. This example is primarily intended to demonstrate that these total orders can be used for ranking trapezoidal fuzzy numbers. Furthermore, this approach is able to resolve issues that remain unaddressed by the magnitude-based methods proposed in [22] [23]. Notably, when evaluated from different perspectives, the total orders in the set of trapezoidal fuzzy numbers lead to distinct conclusions regarding the ranking of these four trapezoidal fuzzy numbers.

Example 1. Consider the four trapezoidal fuzzy numbers $A=(0.3,0.5,0.6,0.8)$, $B=(0.3,0.45,0.65,0.8)$, $C=(0.4,0.5,0.6,0.7)$ and $D=(0.4,0.45,0.65,0.7)$, which are graphically represented in **Figure 3**, where A is green, and B is red, and C is blue, and D is yellow.

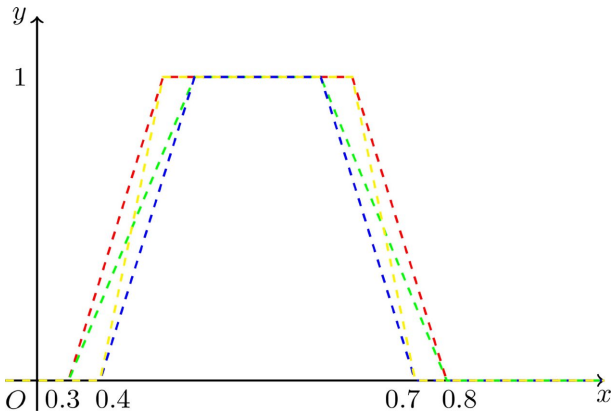


Figure 3. Membership functions of the trapezoidal fuzzy numbers A, B, C and D.

If we aim to ascertain which of these four trapezoidal fuzzy numbers is larger with respect to the magnitude relationship among the numbers, we can utilize the KY order for the comparison. Nevertheless, the ultimate result of this comparison shows that, within the framework of the KY order, any two of these four trapezoidal fuzzy numbers are incomparable. Further, the following results can be obtained.

1) $B\le_{Lex1}A\le_{Lex1}C$ and $B\le_{Lex1}D\le_{Lex1}C$, but A and D are incomparable in the order $\le_{Lex1}$. And $B\trianglelefteq_{Lex1}D\trianglelefteq_{Lex1}A\trianglelefteq_{Lex1}C$.

2) $C\le_{XY1}A\le_{XY1}B$ and $C\le_{XY1}D\le_{XY1}B$, but A and D are incomparable in the order $\le_{XY1}$. And $C\trianglelefteq_{XY1}A\trianglelefteq_{XY1}D\trianglelefteq_{XY1}B$.

Example 2. This example taken from ([9] [10]). The collection $F=\{A_1,A_2,\cdots,A_{10}\}$ of fuzzy sets with triangular membership functions. The parameter values of $A_1,A_2,\cdots,A_{10}$ are shown in **Table 1**.

Table 1. Trapezoidal fuzzy numbers in Example 2.

A_1	(33, 65, 68, 74)	A_6	(65, 68, 72, 83)
A_2	(55, 58, 62, 73)	A_7	(22, 80, 83, 86)
A_3	(67, 76, 125, 130)	A_8	(28, 31, 34, 61)
A_4	(38, 67, 83, 107)	A_9	(2, 5, 20, 66)
A_5	(72, 85, 88, 91)	A_{10}	(11, 70, 73, 76)

1) Based on a parametric relation pair, the method proposed by Dombi and Jonas [10] enables the ranking of a collection of trapezoidal fuzzy numbers. Applying this method, the resulting ranking is determined as:
$A_9 \prec A_8 \prec A_1 =_{>}^{<} A_2 =_{>}^{<} A_{10} \prec A_6 =_{>}^{<} A_7 \prec A_4 \prec A_5 \prec A_3$. Here, the notation $A_1 =_{>}^{<} A_2$ indicates that A_1 and A_2 are non-comparable.

2) Ma et al. [9] put forward a novel approach for ranking fuzzy numbers, which is built on additively consistent fuzzy preference relations. When this method is adopted, the specific ranking of the fuzzy numbers is obtained as:
$A_8 \prec A_9 \prec A_2 \prec A_1 \prec A_{10} \prec A_6 \prec A_5 \prec A_7 \prec A_4 \prec A_3$.

3) To judge the ordering relationship among these ten trapezoidal fuzzy numbers according to their magnitude relations, the KY order should be adopted for comparison. As mentioned above, since the KY order is a partial order, incomparable cases may exist. Therefore, the total order defined by Equations (11-14) can be used for comparison. If Equation (13) is employed, the resulting ranking is determined as:
$A_9 \trianglelefteq_{XY1} A_8 \trianglelefteq_{XY1} A_2 \trianglelefteq_{XY1} A_1 \trianglelefteq_{XY1} a_6 \trianglelefteq_{XY1} A_{10} \trianglelefteq_{XY1} A_4 \trianglelefteq_{XY1} A_7 \trianglelefteq_{XY1} A_5 \trianglelefteq_{XY1} A_3$.

5. Conclusion

In this paper, multiple partial order relations for fuzzy numbers are constructed based on the characteristics of interval total orders. Aiming at the incomparability defect of traditional partial order relations, a corresponding total order relation system for trapezoidal fuzzy numbers is further established in this study. This system effectively solves the problem of priority indistinguishability between fuzzy numbers and realizes the ordered discrimination of different fuzzy numbers. Numerical examples of fuzzy number ranking verify the feasibility and practical effectiveness of the proposed total order relations. The research results can provide important theoretical basis and technical support for the practical applications of fuzzy systems such as multi-attribute fuzzy decision-making and fuzzy games.

Future research can examine the performance of the proposed ranking orders under standard arithmetic operations (addition and scalar multiplication) to validate the robustness of the ranking function. It is also valuable to explore practical implications of inconsistent ranking outcomes and develop guidance for interpreting gaps between the proposed approach and benchmark methods in real decision scenarios. Moreover, the current framework is limited to a single geometric fuzzy set type; extending it to widely adopted continuous fuzzy representations

like bell curves and Gaussian distributions merits further investigation. Additionally, future theoretical work may adapt the strict hierarchical parameter-checking scheme to nonlinear and more complex fuzzy models.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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