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Bidirectional Assisted Cloning via Four-Qubit Symmetric State

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Abstract

In this work, we propose a scheme of bidirectional assisted cloning via four-qubit symmetric state or the Dicke-like state as channel. The proposed scheme comprises of two stages. The first stage of the protocol consists of transferring the two unknown arbitrary single-qubit states from sender Alice to receiver Bob and vice-versa. This is known as the bidirectional teleportation scheme. In the second stage, the perfect clones of the single-qubit states and their orthogonal complements have been constructed at the sender's and the receiver's end, respectively, with the assistance of the state preparer called Victor. As the real quantum systems are more prone to noise, this leads the pure entangled quantum channel to turn into a non-maximally entangled one. Therefore, to fix the noise, the suitable ancilla quantum systems have been introduced in the non-maximally entangled channel in an adequate manner and from the perspective of projective measurement (PM) and positive operator-value measurement (POVM), the analysis of the proposed protocol has been done in the noisy environment.

Keywords

Teleportation, Dicke-Like States, Cloning, PM, POVM

1. Introduction

Quantum communication comes up with its astonishing communication schemes like the quantum teleportation (QT) proposed in the year (1993) by Bennett *et al.* [1], quantum key distribution (QKD) by Bennett and Brassard in (1984) [2], quantum cryptography (QC) using non-orthogonal states by Bennett in (1992) and entanglement based QC by Ekert in (1991) [3] [4] and in (2000) quantum remote state preparation (QRSP) by Bennett *et al.* [5]. In recent times, they have

become highly focused research areas and are gaining popularity among researchers due to their astonishing security and efficiency in surpassing their classical counterparts. The area of quantum teleportation, in the current era has witnessed its large scale advent in terms of its various types of variants like: controlled quantum teleportation (CQT) by Karlsson *et al.* in (1998), bidirectional quantum teleportation (BQT) by Huelga *et al.* in (2001), bidirectional quantum controlled teleportation (BCQT) by Zha *et al.* in (2013), cyclic quantum teleportation (CYQT) by Chen *et al.* in (2017), back in 2018, the cyclic controlled quantum teleportation by Sang *et al.*, bidirectional cyclic quantum teleportation (BCYQT) by Zhou *et al.* in (2019), bidirectional cyclic controlled quantum teleportation (BCYCQT) by Sun and Zhang in (2021) etc. [6]-[12]. The QT protocols have not only been theoretically developed, but have undergone sophisticated experimental realization also. A recent review work which explores the overall advancements in the QT protocols in theory and experiment was done by Pirondola *et al.* back in (2015) and more recently by Hu *et al.* in (2023) [13] [14].

In addition, the QT has extended its dimensions from being a mode of transferring quantum information from one place to another, along with a mode which can transport energy from one place to another via leveraging the ground-state entanglement in the many-body quantum systems. The protocol is known as quantum energy teleportation (QET). The pioneer work in this direction was carried out by M. Hotta back in 2009 [15] [16], and its experimental demonstration on the superconducting hardware was done by K. Ikeda in (2023) [17].

The various types of quantum communication schemes discussed above, are driven by two bizarre quantum properties like quantum superposition and quantum entanglement [18]. Quantum entanglement is the strongest existing non-classical correlation in contrast to any classical statistical correlation. Also, it acts like a resource, which means that it can be spent like bucks in order to accomplish these quantum communication tasks [19]. Thus, in order to execute any kind of quantum information processing task like the quantum teleportation protocol (QT) and its various types of variants, the highly entangled multi-partite quantum states are required.

In the present work, we have considered a four-qubit entangled state known as the dihedral symmetric state, also known as the Dicke-like state, as a communication channel for the proposed assisted cloning scheme, which we are going to describe briefly in the subsequent paragraphs. The Dicke-state, symmetric states or Dicke-like states and their properties have been discussed briefly in [20]-[22].

Furthermore, the four-qubit symmetric state can be represented as

$$\left|\mathcal{D}_4^2\right\rangle = \frac{1}{2}\left[\left|1100\right\rangle + \left|0110\right\rangle + \left|1001\right\rangle + \left|0011\right\rangle\right]_{1234} \tag{1}$$

These states are distributed between Alice and Bob in such a way that the qubits 1 and 4 are possessed by Alice and the qubits 2 and 3 are possessed by Bob. By leveraging the above quantum channel $\left|\mathcal{D}_4^2\right\rangle$, we implement our bidirectional assisted cloning protocol in two stages: the first stage teleports two unknown quan-

tum states bidirectionally to the legitimate users Alice and Bob. In the second stage, with the assistance of a state preparer Victor, the perfect cloning of the quantum states or messages of Alice and Bob is reconstructed at Alice and Bob's end.

In the traditional quantum state teleportation protocols, sometimes the legitimate users also want to have the copies of their messages which they transfer to each other. The QT protocols obey no-cloning theorem put forward by Wooterz and Zurek [23] so it does not permit to have the copy of the states transferred during the process of QT. Let alone the no-cloning theorem, it is the Bell-state measurement which also destroys the message or the unknown quantum state at the sender's end. This assumption also rules out the possibility of copying the quantum states. But in case of photons, it is a valid statement because a photon gets absorbed during the process of measurement [24]. While for other particles like electrons, it may not be true. On the contrary, quantum mechanics never claims that the measurements or more specifically the Bell-state measurement destroy the measuring particles [25].

Although we cannot exactly clone a completely unknown quantum state by any quantum process according to the no-cloning theorem, nevertheless, the possibility of producing approximate copies of a quantum state has been considered by Bužek and Hillery, Gisin and Massar [26] [27] and Bruß *et al.* [28] constructed an optimal state dependent clone that runs in any two non orthogonal states. In (1998), Duan and Guo [29] first proposed a probabilistic cloning machine using a general unitary reduction operation with a postelection of the measurement results. Murao *et al.* in (1999), proposed a quantum telecloning process that combines quantum teleportation with optimal quantum cloning from one input to M outputs [30].

In 2000, A. K. Pati [31] put forward a cloning scheme to produce perfect copy of an arbitrary unknown quantum state and its orthogonal-complement with the minimal assistance of the state preparer called Victor. The scheme includes two steps: the first step requires the usual QT protocol, while in the second step, the preparer conducts single particle measurements and communicates some classical information to different parties, achieving the perfect cloning of the unknown state and its orthogonal complement in a probabilistic manner. Since its inception, the assisted cloning has evolved to its different variants [32]-[35]. More recently (2024-25), a group of researchers have proposed the various types of cloning protocols via different entangled channels namely: bidirectional assisted cloning, asymmetric bidirectional controlled assisted cloning, cyclic cloning, controlled cyclic assisted cloning [36]-[38].

The bidirectional assisted cloning protocol proposed in [36] utilizes a robust four-qubit entangled cluster channel in order to clone the two single-qubit states. Whereas, our proposed bidirectional assisted cloning protocol uses a four-qubit maximally entangled channel which has already been described in the preceding paragraphs. It is a dihedral symmetric state not as robust as cluster state but this entangled channel still deserves a cloning analysis although this quantum channel

may be unitarily equivalent to product of two Bell-pairs. Also, the aforementioned A. K. Pati's assisted cloning scheme has been well established by using a Bell-pair and a four-qubit GHZ entangled channel. A new variant of cyclic cloning via hyper-entangled state has been proposed by J. Shi in (2024) [39]. The controlled assisted cloning protocol has been developed by Lu *et al.* in unidirectional mode and bidirectional mode for m-qudit state via $2m+1$ qudit entangled channel [40].

The above mentioned cloning protocols have been realized via different multipartite entangled channel. Here, in this work, we are proposing an assisted cloning protocol executed in two stages. The first stage of the assisted cloning protocol consists of teleporting two unknown quantum states from Alice to Bob and vice-versa respectively. In the second stage, the assisted cloning is achieved with the help of a state preparer Victor. The detailed discussion will be done in the following sections. This manuscript has been organized in four sections. Section 1 is about introduction, Section 2 discusses the bidirectional teleportation as well as the bidirectional assisted cloning of arbitrary two unknown single-qubit states. In Section 3, the effect of noise is considered on the quantum channel and it is fixed with the introduction of ancilla quantum systems and then performing projective and POVM measurements on the ancilla systems to extract pure quantum channel. the Section 4 is about the conclusion of the proposed work in this manuscript.

2. Assisted Cloning for Arbitrary Unknown Single-Qubit States via Maximally Entangled Symmetric Four-Qubit State

The assisted cloning scheme is executed in two stages: in the first stage, the bidirectional teleportation of two unknown single-qubits is carried out by using the maximally-entangled channel in (1), and in the second stage, the bidirectional assisted cloning is executed with the help of a state preparer, Victor.

2.1. Stage 1: Bidirectional Teleportation

Considering the three legitimate participants Alice, Bob and Victor. The state preparer Victor prepares two pure single-qubit states as

$$\left|\phi\right\rangle_A = a_1\left|0\right\rangle + a_2\left|1\right\rangle,\ \left|\psi\right\rangle_B = b_1\left|0\right\rangle + b_2\left|1\right\rangle \tag{2}$$

where a_1 and b_1 are real numbers and a_2 and b_2 are complex numbers and satisfy the normalization conditions $|a_1|^2 + |a_2|^2 = 1$ and $|b_1|^2 + |b_2|^2 = 1$ and dispatches to Alice and Bob. Victor being the state preparer, has the complete knowledge of these coefficients but Alice and Bob are ignorant about these coefficients. The four-qubit symmetric state shared by Alice and Bob is given by

$$\left|\mathcal{D}_4^2\right\rangle = \frac{1}{2}\left[\left|1100\right\rangle + \left|0110\right\rangle + \left|1001\right\rangle + \left|0011\right\rangle\right]_{1234} \tag{3}$$

where qubits 1 and 4 are shared by the Alice and qubits 2 and 3 by the Bob. The initial joint state of the composite system of qubits A, B, 1, 2, 3 and 4 can be cast into the form

$$\left|\Omega\right\rangle_{AB1234} = \left|\phi\right\rangle_A \otimes \left|\psi\right\rangle_B \otimes \left|\mathcal{D}_4^2\right\rangle \tag{4}$$

In the proposed assisted cloning scheme, the first step consists of teleporting the unknown single-qubit $\left|\phi\right\rangle_A$ at the Alice's end to Bob and the unknown single-qubit state $\left|\psi\right\rangle_B$ at the Bob's end to Alice. The task of bidirectional teleportation can be accomplished using the Bell basis measurement on the qubits $\left(\left|\phi\right\rangle_A, 1\right)$ by Alice and similar Bell basis measurement on the qubits $\left(\left|\psi\right\rangle_B, 2\right)$ by Bob.

Using the Bell-states $\left|\mathcal{B}^{(l,k)}\right\rangle = \frac{1}{\sqrt{2}}\left[(-1)^l \left|l,k\right\rangle + \left|1\oplus l, 1\oplus k\right\rangle\right]$ where $\oplus$ denotes the addition modulo 2 and $l,k \in [0,1]$. The state $\left|\Omega\right\rangle_{AB1234}$ can be written in the following form

$$\left|\Omega\right\rangle_{AB1234} = \frac{1}{4}\sum_{l,k,m,n=0,1}^{1} \left|\mathcal{B}^{(l,k)}\right\rangle_{A,1} \left|\mathcal{B}^{(m,n)}\right\rangle_{B,2} \left[\sigma^{(s,t)}\left|\phi_3\right\rangle\right]\left[\sigma^{(i,j)}\left|\psi\right\rangle_4\right] \tag{5}$$

where $\sigma^{(s,t)} = \left|0\right\rangle\left\langle s\oplus t\right| + (-1)^k \left|1\right\rangle\left\langle 1+k\oplus l\right|$, $s,t \in [0,1]$ and $\sigma^{(i,j)} = \left|0\right\rangle\left\langle i\oplus j\right| + (-1)^i \left|1\right\rangle\left\langle 1+i\oplus j\right|$, $i,j \in [0,1]$ are the Pauli's operators corresponding to the Bell-state measurements on the qubits $(A,1)$ and $(B,2)$ which are to be applied on the qubits 3 and 4 in order to recover the states $\left|\phi_A\right\rangle$ and $\left|\psi_B\right\rangle$. Therefore, defining the Pauli operators for $(s,t,i,j) \in [0,1]$ as $\sigma^{(0,0)} = I$, $\sigma^{(0,1)} = \sigma_x$, $\sigma^{(1,0)} = i\sigma_y$ and $\sigma^{(1,1)} = \sigma_z$.

In the view of equation (5), it can be observed that the Bell-basis measurements performed by Alice and Bob on the qubits $(A,1)$ and $(B,2)$ respectively, has left the qubits 3 and 4 to their unitary equivalent of the states $\left|\phi\right\rangle_A$ and $\left|\psi\right\rangle_B$ respectively. In order to avoid the lengthy description of the sixteen types of Bell-state measurement outcomes and corresponding states of the qubits 1 and 3 at the Alice's and Bob's end respectively, we have skipped it.

In general, when such type of Bell-state measurement $\left|\mathcal{B}^{(l,k)}\right\rangle_{A,1}$ and $\left|\mathcal{B}^{(m,n)}\right\rangle_{B,2}$ will be performed by Alice and Bob, then the composite state $\left|\Omega\right\rangle_{AB1234}$ collapses to the following form

$$\left|\Omega\right\rangle_{3,4} = \frac{1}{4}\left[\sigma^{(s,t)}\left|\phi_3\right\rangle\right]\left[\sigma^{(i,j)}\left|\psi\right\rangle_4\right] \tag{6}$$

Thus, in the bidirectional QT scheme, to reconstruct the original state $\left|\phi\right\rangle_A$ teleported from the Alice's end to the Bob's end and the state $\left|\psi\right\rangle_B$ which has been teleported from Bob's end to Alice's end, both Alice and Bob have to apply the unitary operations $\sigma^{(s,t)}$ and $\sigma^{(i,j)}$ on the qubits $\left|\phi_3\right\rangle$ and $\left|\psi_4\right\rangle$ respectively. As evident from the expression (6), both these output states can be retrieved at their respective ends with a probability of $\frac{1}{16}$.

More specifically, assuming that the outcomes of the Bell-state measurements performed by the Alice and Bob respectively, are $\left|\mathcal{B}^{(0,1)}\right\rangle_{A,1}$ and $\left|\mathcal{B}^{(0,0)}\right\rangle_{B,2}$, the

state of the qubits 3 and 4 take the form

$$|\Omega\rangle'_{3,4} = {}_{A,1}\langle \mathcal{B}^{(0,1)}|_{B,2}\langle \mathcal{B}^{(0,0)}| \otimes |\Omega\rangle_{AB1234} \tag{7}$$

or

$$|\Omega_1\rangle'_{3,4} = \frac{1}{4}\left[\sigma^{(0,0)}|\phi_3\rangle\right]\left[\sigma^{(1,0)}|\psi\rangle_4\right] \tag{8}$$

where $|\phi\rangle_3 = \left(a_1|0\rangle + a_2|1\rangle\right)_3$ and $|\psi\rangle_4 = \left(b_1|1\rangle - b_2|0\rangle\right)_4$ and upon performing these operations $\sigma^{(0,0)} = I$ and $\sigma^{(1,0)} = i\sigma_y$ on the states $|\phi\rangle_3$ and $|\psi\rangle_4$, the original messages on $|\phi\rangle_A$ and on $|\psi\rangle_4$ can be retrieved at the respective communication ends with a probability of $\frac{1}{16}$. The quantum circuit for the proposed BQT protocol is given in the Figure 1. The four-qubit quantum channel $|\mathcal{D}_4^2\rangle$ has been constructed by applying the two Hadamard gates, one X gate and five $CNOT$ gates on appropriate qubits as shown in the Figure 1 and a brief discussion is provided in the **Appendix**.

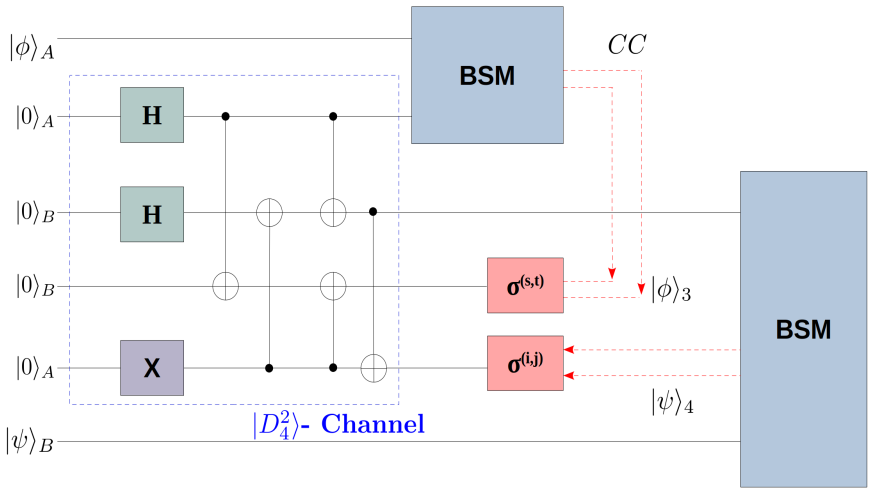


Figure 1. The diagram represents the quantum circuit for bidirectional teleportation scheme. The quantum gates inside the blue dotted box construct the four-qubit entangled channel while the subscripts A and B on the qubit lines denote the qubits held by Alice and Bob respectively. The red dotted lines represent the classical-bits communicated by Alice to Bob and vice-versa.

The following Section 2.2, discusses the bidirectional assisted cloning scheme.

2.2. Stage 2: The Assisted Cloning Scheme

In assisted cloning, the state preparer Victor plays a key role in creating the copies of unknown states $|\phi\rangle_A$ and $|\psi\rangle_B$ along with their orthogonal complements $|\phi_\perp\rangle_A$ and $|\psi_\perp\rangle_B$ at the communicating ends of the Alice and Bob respectively. This is due to the fact that Victor being the state preparer has the complete knowledge of the coefficients a_1, a_2 and b_1, b_2 of the states which Alice and Bob

wants to communicate to each other via quantum teleportation. In the usual quantum teleportation protocol, the Bell-state measurement is usually believed to destroy the qubit pairs $(A,1)$ and $(B,2)$ (the pairs considered in our case) [24]. Hence, the question of cloning the states, doesn't get raised. Also, the no-cloning theorem as discussed earlier in the manuscript, does not allow Alice and Bob to have the copy of their messages. But later on it was figured out that the no-cloning theorem has nothing to do with the measurement postulate of the quantum mechanics (QM) [31]. QM never says that the particles have to be destroyed in measurement process. Therefore, in the Bell-state measurement, the particles at the Alice's end and at the Bob's end remain intact. In a more general way, the measured particles are mathematically still represented in the post-measurement state *i.e.* a maximally entangled Bell-state. Whether they physically survive depends on the implementation of the measurement. If the detectors absorb them, they cannot subsequently be reused. If a suitable nondestructive Bell-state measurement is implemented, the particles may remain available [41].

In fact, the only particles which get destroyed in the measurement processes are photons. Now, even for photons Quantum Non-demolition Measurements (QND) have been carried out which do not destroy the quantum state of the photons [41]. Thus, the qubit pairs $(A,1)$ and $(B,2)$ do not get destroyed in the measurement process. So moving on to the cloning part of the proposed protocol, the projectors used by Alice and Bob for Bell-state measurements $\left|\mathcal{B}^{(0,1)}\right\rangle_{A,1}\left\langle\mathcal{B}^{(0,1)}\right|$ and $\left|\mathcal{B}^{(0,0)}\right\rangle_{B,2}\left\langle\mathcal{B}^{(0,0)}\right|$ respectively, collapses the state of the particles $A,1,B,2$ to the following product state $\left|\mathcal{B}^{(0,1)}\right\rangle_{A,1}\otimes\left|\mathcal{B}^{(0,0)}\right\rangle_{B,2}$.

Now, Alice transmits her qubit A to the state preparer Victor while keeping the qubit 1 in her possession and similarly Bob also transmits his particle B to Victor and retaining the particle 2 in his possession. After receiving the qubits from Alice and Bob, Victor performs a two-qubit projective measurement on the particles A and B in the following basis:

$$\begin{pmatrix}\left|\chi_1\right\rangle_{AB}\\ \left|\chi_2\right\rangle_{AB}\\ \left|\chi_3\right\rangle_{AB}\\ \left|\chi_4\right\rangle_{AB}\end{pmatrix}=\begin{pmatrix}a_1b_1 & a_1b_2 & a_2b_1 & a_2b_2\\ a_1b_2^* & -a_1b_2 & a_2b_2^* & -a_2b_1\\ a_2^*b_1 & a_2^*b_2 & -a_1b_1 & -a_1b_2\\ a_1^*b_2^* & -a_2^*b_1 & -a_1b_2^* & a_1b_1\end{pmatrix}\begin{pmatrix}\left|00\right\rangle_{AB}\\ \left|01\right\rangle_{AB}\\ \left|10\right\rangle_{AB}\\ \left|11\right\rangle_{AB}\end{pmatrix}$$

In terms of the above measurement basis, the state $\left|\zeta\right\rangle_{A1B2}=\left|\mathcal{B}^{(0,1)}\right\rangle_{A,1}\otimes\left|\mathcal{B}^{(0,0)}\right\rangle_{B,2}$ can be written as

$$\begin{aligned}\left|\zeta\right\rangle_{A1B2}=\frac{1}{2}\Big[&\left|\chi_1\right\rangle_{AB}\left(a_1\left|0\right\rangle+a_2^*\left|1\right\rangle\right)_1\left(b_1\left|1\right\rangle-b_2^*\left|0\right\rangle\right)_2\\ &+\left|\chi_2\right\rangle_{AB}\left(a_1\left|0\right\rangle+a_2^*\left|1\right\rangle\right)_1\left(b_1\left|0\right\rangle+b_2\left|1\right\rangle\right)_2\\ &+\left|\chi_3\right\rangle_{AB}\left(a_2\left|0\right\rangle-a_1\left|1\right\rangle\right)_1\left(b_1\left|1\right\rangle-b_2^*\left|0\right\rangle\right)_2\\ &+\left|\chi_4\right\rangle_{AB}\left(a_2\left|0\right\rangle-a_1\left|1\right\rangle\right)_1\left(b_2\left|1\right\rangle+b_1\left|0\right\rangle\right)_2\Big]\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\Big[|\chi_1\rangle_{AB} \sigma^{(1,0)} \left(a_1|1\rangle - a_2^*|0\rangle\right)_1 \left(b_1|1\rangle - b_2^*|0\rangle\right)_2 \\
&+ |\chi_2\rangle_{AB} \sigma^{(1,0)} \left(a_1|1\rangle - a_2^*|0\rangle\right)_1 \left(b_1|0\rangle + b_2|1\rangle\right)_2 \\
&+ |\chi_3\rangle_{AB} \sigma^{(1,0)} \left(a_1|0\rangle + a_2|1\rangle\right)_1 \left(b_1|1\rangle - b_2^*|0\rangle\right)_2 \\
&+ |\chi_4\rangle_{AB} \sigma^{(1,0)} \left(a_1|0\rangle + a_2|1\rangle\right)_1 \sigma^{(0,1)} \left(b_1|0\rangle + b_2|1\rangle\right)_2 \Big]
\end{aligned} \tag{9}
$$

Table 1. The expression (9) has been cast into the tabular form for the Alice and Bob's operations on the qubits in their possession so as to get the perfect copies of the states $|\phi\rangle_A$ and $|\psi\rangle_B$.

Victor's outcome	Alice's State	Alice's Operation Required	Bob's State	Bob's Operation Required
$\|\chi_1\rangle_{AB}$	$\sigma^{(1,0)}\left(a_1\|1\rangle - a_2^*\|0\rangle\right)_1$	$\sigma^{(1,0)}$	$\left(b_1\|1\rangle - b_2^*\|0\rangle\right)_2$	I
$\|\chi_2\rangle_{AB}$	$\sigma^{(1,0)}\left(a_1\|1\rangle - a_2^*\|0\rangle\right)_1$	$\sigma^{(1,0)}$	$\left(b_1\|0\rangle + b_2^*\|1\rangle\right)_2$	I
$\|\chi_3\rangle_{AB}$	$\sigma^{(1,0)}\left(a_1\|0\rangle + a_2\|1\rangle\right)_1$	$\sigma^{(1,0)}$	$\left(b_1\|1\rangle - b_2^*\|0\rangle\right)_2$	I
$\|\chi_4\rangle_{AB}$	$\sigma^{(1,0)}\left(a_1\|0\rangle + a_2\|1\rangle\right)_1$	$\sigma^{(1,0)}$	$\left(b_1\|0\rangle + b_2\|1\rangle\right)_2$	I

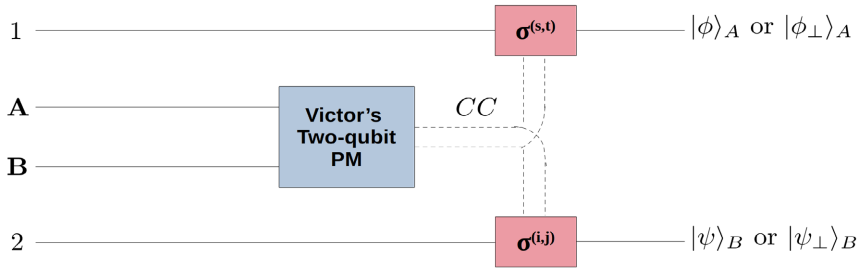


Figure 2. This figure shows the bidirectional assisted cloning circuit. Here, the dotted lines represents the classical bits communicated by Victor to Alice and Bob in order to copy the states: $|\phi\rangle_A$ and $|\psi\rangle_B$.

In the view of the expression (9) and (2), it can be observed that the states $|\phi_\perp\rangle = \left(a_1|1\rangle - a_2^*|0\rangle\right)_1$ and $|\psi_\perp\rangle_2 = \left(b_1|1\rangle - b_2^*|0\rangle\right)$ are the orthogonal-complement of the states $|\phi_A\rangle$ and $|\psi_B\rangle$ Table 1 respectively. When Victor performs a two-qubit projective measurement on the qubits A, B, he obtains four possible outcomes $|\chi\rangle_1, |\chi\rangle_2, |\chi\rangle_3, |\chi\rangle_4$ with a probability of $\frac{1}{4}$. After performing the measurement Victor sends the classical results to Alice and Bob respectively Figure 2.

Suppose the Victor's measurement result is $|\chi_1\rangle_{AB}$, then he communicates the result classically to the Alice who then, in accordance with expression (9) applies a unitary operation $\sigma^{(1,0)}$ on her qubit 1 to retrieve an orthogonal complement copy of the state $|\phi\rangle_A$. Similarly, Bob after receiving the classical information from Victor, he can retrieve the original copy of the orthogonal complement of the state $|\psi\rangle_B$ at his end by performing identity operation or doing nothing on his qubit 2. The complete evolution of the process is given by

$$\begin{aligned}&\left|\chi_1\right\rangle_{AB}\left\langle\chi_1\right|_{A,1}\left\langle\mathcal{B}^{(0,1)}\right|_{B,2}\left\langle\mathcal{B}^{(0,0)}\right|\left.\Omega\right\rangle_{AB1234}\\&=-\frac{1}{8}\left|\chi_1\right\rangle_{AB}\otimes\left(\sigma^{(1,0)}\left|\phi_\perp\right\rangle_1\right)\otimes\left(\sigma^{(1,0)}\left|\psi_\perp\right\rangle_2\right)\otimes\sigma^{(0,0)}\left|\phi\right\rangle_3\otimes\sigma^{(1,0)}\left|\psi\right\rangle_4\end{aligned}\tag{10}$$

the states of the qubits 3 and 4 corresponding to Bell-state measurements have already been discussed in (8).

If the outcome of the Victor's two-qubit projective measurement is $\left|\chi_2\right\rangle_{AB}$ as the (9) implies. In that case, the two-classical bits communicated by Victor to Alice and Bob, assist to retrieve the orthogonal complement of $\left|\phi_A\right\rangle$ at the Alice's place and Bob's retrieves the original copy of $\left|\psi_B\right\rangle$ at his site. The whole operation can be represented as

$$\begin{aligned}&\left|\chi_2\right\rangle_{AB}\left\langle\chi_2\right|_{A,1}\left\langle\mathcal{B}^{(0,1)}\right|_{B,2}\left\langle\mathcal{B}^{(0,0)}\right|\left.\Omega\right\rangle_{AB1234}\\&=-\frac{1}{8}\left|\chi_1\right\rangle_{AB}\otimes\left(\sigma^{(1,0)}\left|\phi_\perp\right\rangle_1\right)\otimes\left(I\left|\psi\right\rangle_2\right)\otimes\sigma^{(0,0)}\left|\phi\right\rangle_3\otimes\sigma^{(1,0)}\left|\psi\right\rangle_4\end{aligned}\tag{11}$$

If the outcome of the Victor's two-qubit projective measurement is $\left|\chi_3\right\rangle_{AB}$ then from (9), it can be seen that the copies of $\left|\phi\right\rangle_A$ and $\left|\psi\right\rangle_B$ received by Alice and Bob are $\sigma^{(1,0)}\left|\phi\right\rangle_A$ *i.e.* Alice can copy her qubit by just applying $\sigma^{(1,0)}$ on her qubit and Bob directly gets the orthogonal complement of $\left|\psi\right\rangle_B$ which is $\left|\psi_\perp\right\rangle_B$. Therefore, this process can be shown as

$$\begin{aligned}&\left|\chi_3\right\rangle_{AB}\left\langle\chi_3\right|_{A,1}\left\langle\mathcal{B}^{(0,1)}\right|_{B,2}\left\langle\mathcal{B}^{(0,0)}\right|\left.\Omega\right\rangle_{AB1234}\\&=-\frac{1}{8}\left|\chi_1\right\rangle_{AB}\otimes\left(\sigma^{(1,0)}\left|\phi_\perp\right\rangle_1\right)\otimes\left(\sigma^{(1,0)}\left|\psi_\perp\right\rangle_2\right)\otimes\sigma^{(0,0)}\left|\phi\right\rangle_3\otimes\sigma^{(1,0)}\left|\psi\right\rangle_4\end{aligned}\tag{12}$$

If the outcome of the Victor's two-qubit projective measurement is $\left|\chi_4\right\rangle_{AB}$ then as implied by (9), the Alice applies $\sigma^{(1,0)}$ on her qubit 1 to receive the copy of her state and for the Bob the state has been directly copied. The expression (13) shows the complete evolution process.

$$\begin{aligned}&\left|\chi_4\right\rangle_{AB}\left\langle\chi_4\right|_{A,1}\left\langle\mathcal{B}^{(0,1)}\right|_{B,2}\left\langle\mathcal{B}^{(0,0)}\right|\left.\Omega\right\rangle_{AB1234}\\&=-\frac{1}{8}\left|\chi_1\right\rangle_{AB}\otimes\left(\sigma^{(1,0)}\left|\phi_\perp\right\rangle_1\right)\otimes\left(\sigma^{(1,0)}\left|\psi_\perp\right\rangle_2\right)\otimes\sigma^{(0,0)}\left|\phi\right\rangle_3\otimes\sigma^{(1,0)}\left|\psi\right\rangle_4\end{aligned}\tag{13}$$

In summary, if Victor's two-qubit projective measurement outcome is $\left|\chi_1\right\rangle_{AB}$, then both Alice and Bob can get the perfect copies of unitary equivalent to orthogonal complement of their teleported unknown single-qubit states. If the Victor's measurement produces $\left|\chi_2\right\rangle_{AB}$, in that case, the Alice receives the copy of unitary equivalent to orthogonal complement of her state and Bob directly receives the original copy of his state. In case, the Victor's outcome is $\left|\chi_3\right\rangle_{AB}$, Alice receives the original copy and Bob gets the orthogonal complement of the copy. For $\left|\chi_4\right\rangle_{AB}$ outcome, Alice and Bob can get the perfect copies of the states. Furthermore, the exact copies of the bidirectionally teleported unknown single-qubit states are retrieved with a probability of 0.5 and their orthogonal complements are copied with probability 0.5 as evident from the expression (9).

3. Assisted Cloning of Arbitrary Unknown Single-Qubit States via Non-Maximally Entangled Four-Qubit Symmetric State

In quantum mechanics, projective measurements (PM) and positive operator valued measures (POVM) are two important measurements. Here, in this section we are replacing the maximally-entangled four-qubit symmetric channel with a four-qubit symmetric non-maximally entangled channel as

$$\left|\mathcal{D}_4^2\right\rangle' = \left[\alpha\left|1100\right\rangle + \beta\left|0110\right\rangle + \gamma\left|1001\right\rangle + \delta\left|0011\right\rangle\right]_{1234} \tag{14}$$

where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$, the normalization condition.

3.1. Assisted Cloning Using Non-Maximally Entangled Four-Qubit State via PM

Let alice and Bob share the non-maximally entangled channel as

$$\left|\mathcal{D}_4^2\right\rangle' = \left[\alpha\left|1100\right\rangle + \beta\left|0110\right\rangle + \gamma\left|1001\right\rangle + \delta\left|0011\right\rangle\right]_{1234} \tag{15}$$

Now, choosing $|\alpha| = \min\{|\alpha|, |\beta|, |\gamma|, |\delta|\}$.

As any BQT protocol requires maximally-entangled channel for its successful execution, therefore, Bob introduces an ancillary qubit F in the ground state $|0\rangle_F$ and applies unitary transformation on the qubit group $(2,3,F)$ under the orthogonal basis $\{|000\rangle_{23F}, |001\rangle_{2,3,F}, |010\rangle_{2,3,F}, |100\rangle_{2,3,F}, |011\rangle_{2,3,F}, |110\rangle_{2,3,F}, |101\rangle_{2,3,F}, |111\rangle_{2,3,F}\}$ and the unitary transformation U is defined as

$$U = \operatorname{diag}(E_3, E_2, I, E_1) \tag{16}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} \frac{\alpha}{\delta} & \sqrt{1-\frac{\alpha^2}{\delta^2}} \\ \sqrt{1-\frac{\alpha^2}{\delta^2}} & -\frac{\alpha}{\delta} \end{pmatrix}$, $E_2 = \begin{pmatrix} \frac{\alpha}{\gamma} & \sqrt{1-\frac{\alpha^2}{\gamma^2}} \\ \sqrt{1-\frac{\alpha^2}{\gamma^2}} & -\frac{\alpha}{\gamma} \end{pmatrix}$ and $E_1 = \begin{pmatrix} \frac{\alpha}{\beta} & \sqrt{1-\frac{\alpha^2}{\beta^2}} \\ \sqrt{1-\frac{\alpha^2}{\gamma^2}} & -\frac{\alpha}{\beta} \end{pmatrix}$

The composite system of non-maximally entangled channel and ancilla state can be written as

$$\left|\mathcal{D}_4^2\right\rangle' |0\rangle_F = \left[\alpha\left|1100\right\rangle + \beta\left|0110\right\rangle + \gamma\left|1001\right\rangle + \delta\left|0011\right\rangle\right]_{1234} |0\rangle_F \tag{17}$$

after the application of U on (11), it takes the form as

$$\begin{aligned} \left|\mathcal{Q}_1\right\rangle &= U\left|\mathcal{D}_4^2\right\rangle' |0\rangle_F \\ &= \alpha\left(\left|1100\right\rangle + \left|0110\right\rangle + \left|0011\right\rangle + \left|1001\right\rangle\right)_{1234} |0\rangle_F \\ &+ \left(\beta\sqrt{1-\frac{\alpha^2}{\beta^2}}\left|0110\right\rangle + \gamma\sqrt{1-\frac{\alpha^2}{\gamma^2}}\left|0011\right\rangle + \delta\sqrt{1-\frac{\alpha^2}{\delta^2}}\left|1001\right\rangle\right)_{1234} |1\rangle_F \end{aligned} \tag{18}$$

Now, Bob performs a measurement of his ancilla qubit F in the computational basis states $\{|0\rangle,|1\rangle\}$. As is evident from (18) that if the outcome of Bob's measurement is $|0\rangle_F$, then we get the pure channel $|\mathcal{D}_4^2\rangle_{1234}$ with probability $4\alpha^2$. If the outcome is $|1\rangle_F$, then obtaining $|\mathcal{D}_4^2\rangle_{1234}$ is a failure. Thus, by the introduction of ancilla qubit in the $|0\rangle$ state, applying an appropriate unitary operator U on the qubit group $(2,3,F)$ and performing a single-quibt projective measurement in $\{|0\rangle,|1\rangle\}$ basis on the ancilla-qubit, the required channel state $|\mathcal{D}_4^2\rangle$ can be recovered. After the recovery of the quantum channel $|\mathcal{D}_4^2\rangle$, the cloning scheme described in the Stage. 2 of the section 2.2 can be implemented to create the copies of the unknown single-qubit states. The probability of obtaining the copy of original state is given by $4\alpha^2\times 0.5 = 2\alpha^2$ and copy of the orthogonal complement of the original state is $2\alpha^2$.

3.2. Assisted Cloning Using Non-Maximally Entangled Four-Qubit State via POVM

As considered in the previous section, the non-maximally entangled state shared by Alice and Bob

$$|\mathcal{D}_4^2\rangle' = \left[\alpha|1100\rangle+\beta|0110\rangle+\gamma|1001\rangle+\delta|0011\rangle\right]_{1234} \tag{19}$$

To extract the pure maximally-entangled channel out of non-maximally one, Bob introduces two ancillary qubits initialized in the state $|00\rangle_{mn}$ and the state becomes

$$|\mathcal{D}_4^2\rangle'|00\rangle_{mn} = \left[\alpha|1100\rangle+\beta|0110\rangle+\gamma|1001\rangle+\delta|0011\rangle\right]_{1234}|00\rangle_{mn} \tag{20}$$

also, he implements two-CNOT operations on the qubits $(2,m)$ and $(3,n)$ where the control qubits are $(2,3)$ and target qubits are (m,n) respectively, and as a result of that the final state is given by

$$\begin{aligned}|\mathcal{Q}\rangle_2 &= \mathrm{CNOT}_{(2,m)}\mathrm{CNOT}_{(3,n)}|\mathcal{D}_4^2\rangle'|00\rangle_{mn}\\ &= \left(\alpha|110010\rangle+\beta|011011\rangle+\gamma|001101\rangle+\delta|100100\rangle\right)_{1234mn}\\ &= \frac{1}{2}\left(|D_1\rangle|E_1\rangle+|D_2\rangle|E_2\rangle+|D_3\rangle|E_3\rangle+|D_4\rangle|E_4\rangle\right)\end{aligned} \tag{21}$$

where

$$|D_1\rangle = \frac{1}{2}\left(|1100\rangle+|0110\rangle+|0011\rangle+|1001\rangle\right)_{12345}$$

$$|E_1\rangle = \left(\alpha|10\rangle+\beta|11\rangle+\gamma|01\rangle+\delta|00\rangle\right)_{mn}$$

$$|D_2\rangle = \frac{1}{2}\left(|1100\rangle-|0110\rangle+|0011\rangle-|1001\rangle\right)_{1234}$$

$$|E_2\rangle = \left(\alpha|10\rangle-\beta|11\rangle+\gamma|01\rangle-\delta|00\rangle\right)_{mn}$$

$$|D_3\rangle = \frac{1}{2}\left(|1100\rangle-|0110\rangle-|0011\rangle+|1001\rangle\right)_{1234}$$

$$|E_3\rangle = (\alpha|10\rangle - \beta|11\rangle - \gamma|01\rangle + \delta|00\rangle)_{mn}$$

$$|D_4\rangle = \frac{1}{2}(|1100\rangle + |0110\rangle - |0011\rangle - |1001\rangle)_{1234}$$

$$|E_4\rangle = (\alpha|10\rangle + \beta|11\rangle - \gamma|01\rangle - \delta|00\rangle)_{mn} \tag{22}$$

It is evident from the above state $|\mathcal{Q}\rangle_2$, Bob can identify the pure channel state $|\mathcal{D}_4^2\rangle$ by performing the positive operator valued measurements (POVM) on ancillary qubits (m,n) to distinguish the non-orthogonal states: $|E_1\rangle$, $|E_2\rangle$, $|E_3\rangle$ and $|E_4\rangle$. The POVM operators are defined as

$$W_k = \frac{1}{\mu}|U_k\rangle\langle U_k| \ (k \in \{1,2,3,4\}), \ |W_5\rangle = I - \frac{1}{\mu}\sum_{k=1}^{4}|U_k\rangle\langle U_k| \tag{23}$$

where I denotes the identity operator, and

$$\begin{aligned}
|U_1\rangle &= \frac{1}{\sqrt{\lambda}}\left(\frac{1}{\alpha}|10\rangle + \frac{1}{\beta}|11\rangle + \frac{1}{\gamma}|01\rangle + \frac{1}{\delta}|00\rangle\right)\\
|U_2\rangle &= \frac{1}{\sqrt{\lambda}}\left(\frac{1}{\alpha}|10\rangle - \frac{1}{\beta}|11\rangle + \frac{1}{\gamma}|01\rangle - \frac{1}{\delta}|00\rangle\right)\\
|U_3\rangle &= \frac{1}{\sqrt{\lambda}}\left(\frac{1}{\alpha}|10\rangle - \frac{1}{\beta}|11\rangle - \frac{1}{\gamma}|01\rangle + \frac{1}{\delta}|00\rangle\right)\\
|U_4\rangle &= \frac{1}{\sqrt{\lambda}}\left(\frac{1}{\alpha}|10\rangle + \frac{1}{\beta}|11\rangle - \frac{1}{\gamma}|01\rangle - \frac{1}{\delta}|00\rangle\right)
\end{aligned} \tag{24}$$

From the normalization condition, $\lambda = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$. The parameter μ is determined by α, β, γ and δ such that W_5 is a semi-positive operator. The operators U_k can be cast into the form

$$W_1 = \frac{1}{\mu}\begin{pmatrix} \frac{1}{\delta^2} & \frac{1}{\delta\gamma} & \frac{1}{\delta\alpha} & \frac{1}{\delta\beta} \\ \frac{1}{\gamma\delta} & \frac{1}{\gamma^2} & \frac{1}{\gamma\alpha} & \frac{1}{\gamma\beta} \\ \frac{1}{\alpha\delta} & \frac{1}{\alpha\gamma} & \frac{1}{\alpha^2} & \frac{1}{\alpha\beta} \\ \frac{1}{\beta\delta} & \frac{1}{\beta\gamma} & \frac{1}{\beta\alpha} & \frac{1}{\beta^2} \end{pmatrix} \tag{25}$$

$$W_2 = \frac{1}{\mu}\begin{pmatrix} \frac{1}{\delta^2} & -\frac{1}{\delta\gamma} & -\frac{1}{\delta\alpha} & \frac{1}{\delta\beta} \\ -\frac{1}{\gamma\delta} & \frac{1}{\gamma^2} & \frac{1}{\gamma\alpha} & -\frac{1}{\gamma\beta} \\ -\frac{1}{\alpha\delta} & \frac{1}{\alpha\gamma} & \frac{1}{\alpha^2} & -\frac{1}{\alpha\beta} \\ \frac{1}{\beta\delta} & -\frac{1}{\beta\gamma} & -\frac{1}{\beta\alpha} & \frac{1}{\beta^2} \end{pmatrix} \tag{26}$$

$$W_3 = \frac{1}{\mu}\begin{pmatrix} \frac{1}{\delta^2} & -\frac{1}{\delta\gamma} & \frac{1}{\delta\alpha} & -\frac{1}{\delta\beta} \\ -\frac{1}{\gamma\delta} & \frac{1}{\gamma^2} & -\frac{1}{\gamma\alpha} & \frac{1}{\gamma\beta} \\ \frac{1}{\alpha\delta} & -\frac{1}{\alpha\gamma} & \frac{1}{\alpha^2} & -\frac{1}{\alpha\beta} \\ -\frac{1}{\beta\delta} & \frac{1}{\beta\gamma} & -\frac{1}{\beta\alpha} & \frac{1}{\beta^2} \end{pmatrix} \tag{27}$$

$$W_4 = \frac{1}{\mu}\begin{pmatrix} \frac{1}{\delta^2} & -\frac{1}{\delta\gamma} & -\frac{1}{\delta\alpha} & \frac{1}{\delta\beta} \\ \frac{1}{\gamma\delta} & \frac{1}{\gamma^2} & -\frac{1}{\gamma\alpha} & \frac{1}{\gamma\beta} \\ -\frac{1}{\alpha\delta} & \frac{1}{\alpha\gamma} & \frac{1}{\alpha^2} & \frac{1}{\alpha\beta} \\ -\frac{1}{\beta\delta} & \frac{1}{\beta\gamma} & \frac{1}{\beta\alpha} & \frac{1}{\beta^2} \end{pmatrix} \tag{28}$$

and

$$W_5 = \frac{1}{\mu}\begin{pmatrix} 1-\frac{4}{\lambda\mu\delta^2} & 0 & 0 & 0 \\ 0 & 1-\frac{4}{\lambda\mu\gamma^2} & 0 & 0 \\ 0 & 0 & 1-\frac{4}{\lambda\mu\alpha^2} & 0 \\ 0 & 0 & 0 & 1-\frac{4}{\lambda\mu\beta^2} \end{pmatrix} \tag{29}$$

The semi-positive definiteness of W_5 is ensured by calculating the condition on μ as $\mu \geq \max\left\{\frac{4}{\lambda\alpha^2}, \frac{4}{\lambda\beta^2}, \frac{4}{\lambda\gamma^2}, \frac{4}{\lambda\delta^2}\right\}$. After executing the POVM, Bob can obtain $W_k\ (k=1,2,3,4)$ with probability

$$p(W_k) = \langle Q | W_k | Q \rangle = \frac{4}{\lambda\mu} \tag{30}$$

and

$$p(W_5) = 1 - \frac{16}{\lambda\mu} \tag{31}$$

Now, it is evident from (30), the POVM performed by the Bob on ancilla qubits mn, decides the state $|E_k\rangle_{mn}$ $k \in (1,2,3,4)$ with the success probability $\frac{4}{\lambda\mu}$. For W_5, the POVM fails to identify the state $|E_k\rangle_{mn}$ with a probability, $1-\frac{16}{\lambda\mu}$. When the state $|E_k\rangle_{mn}$ $k \in (1,2,3,4)$ gets identified, then as implied by (21), Bob can apply unitary operation on the qubits owned by him corresponding to each of the $|E_k\rangle_{mn}$ obtained to recover the noise free channel state $|\mathcal{D}_4^2\rangle$.

4. Conclusions

The current work proposes an assisted cloning protocol via four-qubit entangled state. The protocol has been executed in two stages. The first stage focuses on the simultaneous exchange of the two unknown single-qubit quantum states between Alice and Bob *i.e.* bidirectional teleportation using the four-qubit symmetric maximally entangled state represented as $\left|\mathcal{D}_4^2\right\rangle$. The entangled quantum channel $\left|\mathcal{D}_4^2\right\rangle$ in consideration here, is a subset of multipartite highly entangled states known as the Dicke-states [21]. Also, the quantum circuit for the four-qubit entangled channel $\left|\mathcal{D}_4^2\right\rangle$ can be constructed by using two Hadamard gates, one NOT gate and five CNOT gates and the complete detail is in the **Appendix**.

In the second stage as described in the Sec.2, the two perfect copies of the unknown arbitrary single-qubit quantum states at the Alice's and Bob's end respectively, have been reconstructed with the assistance of the state preparer known as the Victor [31]. The Victor's role is to performs a two-qubit projective measurement on the qubits received from Alice and Bob after the completion of the bidirectional teleportation protocol. Then, by communicating the outputs *i.e.* two classical-bits to Alice and Bob respectively, can make the perfect copies of their states.

As we aware of the fact that the noise is a unavoidable aspect of the quantum systems, and consequently, the pure maximally-entangled state gets turned into a non-maximally one. As we know that the quantum information cannot be perfectly transferred from one place to another via noisy entangled channel. Therefore, the bidirectional assisted cloning scheme can be carried out probabilistically as outlined in Sections 3.1 and 3.2. The maximally entangled quantum channel can then be separated probabilistically by introducing the necessary ancillary quantum systems into the channel and performing projective measurements (PM) and positive operator valued measurements (POVMs) on the ancillary quantum systems.

In summary, we have realized an assisted cloning protocol using a special class of entangled states, the symmetric or Dicke-like states as mentioned in Sec.1. As is well known that the entanglement is an enigmatic phenomenon in quantum mechanics. The entangled states come in variety of classes of entangled multipartite states like graph states, cluster states [42] [43], W-states and GHZ-sates [19], Dicke-states [21], AME states [44] [45], PME-states [46]. All these sets of highly entangled states have their own special properties and applications in different branches of QIP, and our considered quantum channel also falls in this category. So to realize any kind of quantum information processing protocols using above mentioned states has its own importance. Thus, our work not only realize an assisted cloning protocol via symmetric states but also opens up new doors to explore the new types of QIP protocols using the above mentioned states which may be proposed in near future.

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Author Contributions

Dev Dutt and Jagat Ram performed the theoretical calculations and analysis. Mukul Kumar prepared the figures and graphical illustrations. All authors—Dev Dutt, Jagat Ram, Mukul Kumar, Sunil Kumar, Raj Kumar, and Shashi K. Dhiman—contributed to the writing, reviewing, and editing of the manuscript. All authors read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

The detailed description of $|\Omega\rangle_{AB1234}$ in (5) can be as follows:

$$\begin{aligned}|\Omega\rangle_{AB1234} = \frac{1}{4}\Big[&|\phi^+\rangle|\phi^+\rangle(a_1b_1|00\rangle + a_1b_2|01\rangle + a_2b_1|10\rangle + a_2b_2|11\rangle)\\ &+|\phi^+\rangle|\phi^-\rangle(a_1b_1|00\rangle - a_1b_2|01\rangle + a_2b_1|10\rangle - a_2b_2|11\rangle)\\ &+|\phi^-\rangle|\phi^+\rangle(a_1b_1|00\rangle + a_1b_2|01\rangle - a_2b_1|10\rangle - a_2b_2|11\rangle)\\ &+|\phi^-\rangle|\phi^-\rangle(a_1b_1|00\rangle - a_1b_2|01\rangle - a_2b_1|10\rangle + a_2b_2|11\rangle)\\ &+|\phi^+\rangle|\psi^+\rangle(a_1b_1|01\rangle + a_1b_2|00\rangle + a_2b_1|11\rangle + a_2b_2|10\rangle)\\ &+|\phi^+\rangle|\psi^-\rangle(a_1b_1|01\rangle - a_1b_2|00\rangle + a_2b_1|11\rangle - a_2b_2|10\rangle)\\ &+|\phi^-\rangle|\psi^+\rangle(a_1b_1|01\rangle + a_1b_2|00\rangle - a_2b_1|11\rangle - a_2b_2|10\rangle)\\ &+|\phi^-\rangle|\psi^-\rangle(a_1b_1|01\rangle - a_1b_2|00\rangle - a_2b_1|11\rangle + a_2b_2|10\rangle)\\ &+|\psi^+\rangle|\phi^+\rangle(a_1b_1|10\rangle + a_1b_2|11\rangle + a_2b_1|00\rangle + a_2b_2|01\rangle)\\ &+|\psi^+\rangle|\phi^-\rangle(a_1b_1|10\rangle - a_1b_2|11\rangle + a_2b_1|00\rangle - a_2b_2|01\rangle)\\ &+|\psi^-\rangle|\phi^+\rangle(a_1b_1|10\rangle + a_1b_2|11\rangle - a_2b_1|00\rangle - a_2b_2|01\rangle)\\ &+|\psi^-\rangle|\phi^-\rangle(a_1b_1|10\rangle - a_1b_2|11\rangle - a_2b_1|00\rangle + a_2b_2|01\rangle)\\ &+|\psi^+\rangle|\psi^+\rangle(a_1b_1|11\rangle + a_1b_2|10\rangle + a_2b_1|01\rangle + a_2b_2|00\rangle)\\ &+|\psi^+\rangle|\psi^-\rangle(a_1b_1|11\rangle - a_1b_2|10\rangle + a_2b_1|01\rangle - a_2b_2|00\rangle)\\ &+|\psi^-\rangle|\psi^+\rangle(a_1b_1|11\rangle + a_1b_2|10\rangle - a_2b_1|01\rangle - a_2b_2|00\rangle)\\ &+|\psi^-\rangle|\psi^-\rangle(a_1b_1|11\rangle - a_1b_2|10\rangle - a_2b_1|01\rangle + a_2b_2|00\rangle)\Big]\end{aligned} \tag{32}$$

The construction of the quantum channel $|\mathcal{D}_4^2\rangle$ has been done in the following way: We start with four qubits initialized in the $|0\rangle$ state.

$$|Q_1\rangle_{1234} = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4$$

Then, we applied two Hadamard gates, $H = \frac{1}{\sqrt{2}}(|+\rangle\langle 0| + |-\rangle\langle 1|)$ on qubit 1 and 2, which transforms the state $|Q_1\rangle$ to $|Q_2\rangle$ as

$$|Q_2\rangle_{1234} = (|0000\rangle + |0100\rangle + |1000\rangle + |1100\rangle)_{1234}$$

On qubit 4 of the state $|Q_2\rangle$, the NOT gate denoted as, $X = |1\rangle\langle 0| + |0\rangle\langle 1|$ is applied, we obtained the state $|Q\rangle_{1234}$:

$$|Q_3\rangle_{1234} = (|0001\rangle + |0101\rangle + |1001\rangle + |1101\rangle)_{1234}$$

Finally, a series of CNOT (controlled-not) operations have been applied on the qubits of the state $|Q_3\rangle_{1234}$ in the following sequence:

$$\mathrm{CNOT}_{1-3}|Q_3\rangle_{1234} = (|0001\rangle + |0101\rangle + |1011\rangle + |1111\rangle)_{1234} = |Q_4\rangle_{1234}$$

$$\mathrm{CNOT}_{4-2}|Q_4\rangle_{1234} = (|0101\rangle + |0001\rangle + |1111\rangle + |1011\rangle)_{1234} = |Q_5\rangle_{1234}$$

$$\mathrm{CNOT}_{1-2}\left|Q_5\right\rangle_{1234}=\left(\left|0101\right\rangle+\left|0001\right\rangle+\left|1011\right\rangle+\left|1111\right\rangle\right)_{1234}=\left|Q_6\right\rangle_{1234}$$

$$\mathrm{CNOT}_{4-3}\left|Q_6\right\rangle_{1234}=\left(\left|0111\right\rangle+\left|0011\right\rangle+\left|1001\right\rangle+\left|1101\right\rangle\right)_{1234}=\left|Q_7\right\rangle_{1234}$$

$$\mathrm{CNOT}_{2-4}\left|Q_7\right\rangle_{1234}=\left(\left|0110\right\rangle+\left|0011\right\rangle+\left|1001\right\rangle+\left|1100\right\rangle\right)_{1234}=\left|\mathcal{D}_4^2\right\rangle_{1234}$$

which is the required entangled channel for the proposed protocol.