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Existence of Positive Solutions for a Class of Nonlocal Elliptic Differential Equations

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Abstract

This paper investigates the existence and multiplicity of positive solutions for the Kirchhoff-type elliptic equation

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\Delta u = \lambda f(x,u), & x\in\Omega, \\ u(x)=0, & x\in\partial\Omega, \end{cases}$$

where $\lambda \geq 0$ is a parameter, $\Omega \subset \mathbb{R}^N (N\geq 1)$ is a bounded smooth domain, and $M:[0,+\infty)\to\mathbb{R}$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ are smooth functions satisfying $f(x,0)>0$ for all $x\in\Omega$. By employing the bifurcation method, the existence of positive solutions is successfully proved under appropriate hypotheses regarding M and f. F Furthermore, when the nonlinearity f admits two zeros on the intery $[0,+\infty)$, further multiplicity results for the positive solutions are derived.

Keywords

Positive Solutions, Bifurcation Methods, Eigenvalues, Kirchhoff-Type Elliptic Equation

1. Introduction

This paper investigates the existence and multiplicity of positive solutions for the Kirchhoff-type elliptic equation

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\Delta u = \lambda f(x,u), & x\in\Omega, \\ u(x)=0, & x\in\partial\Omega \end{cases} \tag{1}$$

where $\lambda \geq 0$ is a parameter, $\Omega \subset \mathbb{R}^N (N\geq 1)$ is a bounded smooth domain, $M:[0,+\infty)\to\mathbb{R}$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ are smooth functions. The Kirchhoff-type equation is a class of non-classical elliptic partial differential equations. It was

proposed by Kirchhoff [1] to describing the equilibrium state of strings or membrane vibrations, and it has a wide range of applications in theoretical physics, biochemistry, engineering mathematics, and other disciplines. When the Kirchhoff-type equation does not contain a nonlocal term, that is, when $M \equiv 1$ in Equation (1), there have been some results on the existence of positive solutions for this class of equations; see references [2]-[5]. For example, Carmona and Martnez [2] studied the semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda f(u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded smooth open set, and f is a nonnegative continuous real-valued function with multiple zeros. Based on topological degree theory and the Rabinowitz global bifurcation theorem, they obtained.

Theorem A ([2] Theorem 1.1) Let f be a nonnegative continuous function, and let $0 < \alpha < \beta$ be two adjacent positive zeros of f, that is, $f(\alpha) = f(\beta) = 0$, and $f(s) > 0$ for any $s \in (\alpha, \beta)$. Assume that for $\sigma = \alpha$ and $\sigma = \beta$, the following condition holds: there exist $M > 0$, $\epsilon > 0$ such that the function $f(s) + Ms$ is increasing on the interval $(\sigma - \epsilon, \sigma)$. Define

$$S_{\alpha\beta} := \left\{ (\lambda, u) \in [0, +\infty) \times L^{\infty}(\Omega) : (2) \text{ has a solution } u \text{ satisfying } \alpha < \|u\|_{L^{\infty}(\Omega)} < \beta \right\}.$$

Then there exists an unbounded connected component $\Sigma \subset S_{\alpha,\beta}$, and the following conclusions hold:

1) $\lambda_{\alpha\beta} = \inf\{\lambda : (\lambda, u) \in \Sigma\}$;

2) $\lambda_{\alpha\beta} > 0$, and for any $\lambda \geq \lambda_{\alpha\beta}$, problem (2) has at least one solution u, satisfying $\alpha < \|u\|_{L^{\infty}(\Omega)} < \beta$;

3) for any $\lambda > \lambda_{\alpha\beta}$, there exist at least two distinct solutions $u_1 \neq u_2$ in Σ, such that $(\lambda, u_1), (\lambda, u_2) \in \Sigma$, and $\alpha < \|u_i\|_{L^{\infty}(\Omega)} < \beta$ $(i = 1, 2)$.

Unlike semilinear elliptic equations, a prominent feature of Equation (1) is that it contains a nonlocal term $\int_{\Omega} |\nabla u|^2 \, dx$, which makes Equation (1) not a pointwise identity, and its solutions cannot be verified pointwise, thus bringing essential difficulties to the study of the problem. In recent years, many scholars have adopted different methods to study this class of equations and have obtained a series of results; see references [6]-[12]. Recently, Ambrosetti and Arcoya [6] used the Rabinowitz global bifurcation theorem to study the existence of positive solutions for problem (1), and assumed the following:

(A1) $f(x, 0) = 0$, $\forall x \in \Omega$, $a(x) := D_u f(x, 0) > 0$;

(A2) $f(x, u) = b(x)u + g(x, u)$, where $b(x) > 0$, $\lim_{u \to \infty} \frac{g(x, u)}{u} = 0$;

(A3) there exists $\alpha > 0$ such that $f(x, u) \geq \alpha u$, $\forall (x, u) \in \Omega \times \mathbb{R}$;

(A4) there exists $m_0 > 0$ such that $M(t) \geq m_0$, $\forall t \geq 0$;

(A5) there exists $M_{\infty} > 0$ such that $\lim_{t \to +\infty} M(t) = M_{\infty}$.

Denote

$$\Lambda_0 = M(0)\lambda_1[a],\ \Lambda_\infty = M_\infty\lambda_1[b].$$

where $\lambda_1[a]$ is the principal eigenvalue of

$$\begin{cases} -\Delta u = \lambda a(x)u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \end{cases}$$

and $a(x) \in C(\bar{\Omega})$ with $a(x) > 0$. The main results obtained in this paper are as follows:

Theorem B ([6] Theorem 2.6) Assume that (A1)-(A5) hold. Then there exists a connected branch Γ of positive solutions of problem (1) bifurcating from $(\Lambda_0, 0)$, satisfying the following properties:

1) Γ connects to the point (Λ_∞, ∞);

2) if $\lim_{t\to\infty} M(t) = \infty$, then $\mathrm{Proj}_{\mathbb{R}}\Gamma = [l, +\infty)$, where $l \le \Lambda_1^0$.

It should be pointed out that the results in reference [6] depend on condition (A1), and their branch of positive solutions emanates from $(\Lambda_0, 0)$. However, this paper allows $f(x,0) > 0$, in which case $u \equiv 0$ is no longer a solution of the original equation, and the original bifurcation framework is no longer applicable; furthermore, when the nonlinear term f has two zeros on $[0, +\infty)$, this paper will continue to explore the multiplicity of positive solutions of problem (1). This makes the model possess a broader physical background. Therefore, this paper investigates the existence and multiplicity of positive solutions of problem (1) under the condition $f(x,0) > 0$.

The general assumptions of this paper are as follows:

(A6) $f(x,0) > 0$, $\forall x \in \Omega$;

(A7) there exist $0 < s' < s''$ such that when $u \in [s', s'']$, $f(x,u) \le 0$, and when $u \in (0, s') \cup (s'', +\infty)$, $f(x,u) > 0$, $\forall x \in \Omega$.

Let u, v belong to the Sobolev space $X := H_0^1(\Omega)$, then the inner product and norm on it are respectively

$$(u,v) = \int_\Omega \nabla u \nabla v \mathrm{d}x,\ \|u\| = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Theorem 1 Assume that (A2)-(A4) and (A6) hold. Then problem (1) has two unbounded connected continua $\Sigma_0 \subset R^+ \times X$ and $\Sigma_\infty \subset R^+ \times X$, satisfying:

1) Assume that (A5) holds. Then Σ_0 connects to the point (Λ_∞, ∞), where $\Lambda_\infty := M_\infty\lambda_1[b]$. That is, $\Sigma_0 = \Sigma_\infty$;

2) if $\lim_{t\to\infty} M(t) = \infty$, then $\mathrm{Proj}_{\mathbb{R}}\Sigma_0 = \mathrm{Proj}_{\mathbb{R}}\Sigma_\infty = (0, +\infty)$, where $\mathrm{Proj}_{\mathbb{R}}\Sigma_0$ and $\mathrm{Proj}_{\mathbb{R}}\Sigma_\infty$ denote the projections of Σ_0 and Σ_∞ in the R direction, respectively.

Theorem 2 Assume that (A2), (A4)-(A7) hold. Then problem (1) has no positive solution u satisfying $\max_{x\in\bar{\Omega}} |u(x)| \in [s', s'']$. In this case, the connected continua Σ_0 and Σ_∞ are disjoint.

Corollary 1 Assume that (A2)-(A4) and (A6) hold. Then

1) Assume that (A5) holds. If $\lambda \in (0, \Lambda_\infty)$, then problem (1) has one positive

solution.

2) If $\lim_{t\to\infty} M(t)=\infty$, then when $\lambda>0$, problem (1) has one positive solution.

Corollary 2 Assume that (A2), (A4)-(A7) hold. If $\lambda>\Lambda_\infty$, then problem (1) has at least two positive solutions u_1,u_2, where u_1 satisfies $\max_{x\in\bar{\Omega}}|u_1(x)|<s'$, and u_2 satisfies $\max_{x\in\bar{\Omega}}|u_2(x)|>s''$.

The paper is organized as follows. Section 2 is devoted to the preliminary knowledge. More precisely, we introduce the solution operator T associated with the Kirchhoff-type elliptic equation, prove that T is compact, and then establish the existence of an unbounded connected continuum Σ_0 connecting $(0,0)$ to infinity as well as an unbounded connected continuum Σ_∞ emanating from (Λ_∞,∞). The proofs of our main results are provided in Section 3. We first obtain the existence of positive solutions by using the global bifurcation method, and then, under the assumption that f possesses two zeros on $[0,+\infty)$, we prove the multiplicity of positive solutions and show that the two branches Σ_0 and Σ_∞ are disjoint.

2. Preliminary Knowledge

Let F denote the Nemitski operator associated with f, and define the linear operator $L:L^2(\Omega)\to X$ by

$$Lv:=(-\Delta)^{-1}v.$$

Since $M(\|u\|^2)>0$, we define the operator $T:\mathbb{R}\times X\to X$ by

$$T(\lambda,u)=\frac{\lambda LF(u)}{M(\|u\|^2)},\ u\in X.$$

Clearly, a solution of $u=T(\lambda,u)$ is a solution of (1).

Lemma 1 Assume that (A2), (A4) and (A5) hold, and that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathodory function satisfying the growth condition

$$|f(x,u)|\le C\left(1+|u|^{p-1}\right),\ p\in(2,2^*),\ 2^*=\frac{2N}{N-2}\ (N\ge 3),\ p<\infty\ (N=1,2).$$

Then $T:\mathbb{R}\times X\to X$ is a compact operator.

Proof. According Theorem 1.2.1 in [13] and (A2), the Nemitski operator F is continuous from X to $L^2(\Omega)$. To see this, note that the convergence $u_n\to u$ in X implies strong convergence in $L^p(\Omega)$ and pointwise convergence almost everywhere. Combining this with the growth condition and Lebesgue's dominated convergence theorem yields $F(u_n)\to F(u)$ in $L^2(\Omega)$. Since the inverse Laplacian $L=(-\Delta)^{-1}:L^2(\Omega)\to X$ is a compact mapping from $L^2(\Omega)$ to X, the $LF:X\to X$ is consequently compact.

Now choose an arbitrary bounded sequence $\{u_k\}\subset X$, In view of (A4) and (A5), after passing to a subsequence, we may assume $\|u_k\|\to r$, or at least that $\{\|u_k\|\}$ is bounded. The continuity of M then gives

$$M\left(\|u_k\|^2\right) \to M\left(r^2\right) =: C_0 > 0,$$

where C_0 is a constant. Hence

$$T\left(\lambda, u_k\right) = \frac{\lambda LF\left(u_k\right)}{M\left(\|u_k\|^2\right)}$$

admits a convergent subsequence in X, this proves that T is compact. □

Lemma 2 Assume that (A2), (A4) and (A5) hold. Then problem (1) possesses an un-bounded connected continuum $\Sigma_0 \subset \mathbb{R}^+ \times X$ connecting $(0,0)$ to infinity.

Proof. Since the operator $T: \mathbb{R} \times X \to X$ is continuous and compact, and $T(0,u) \equiv 0$, By Theorem 3.2 in [14] that the equation $u = T(\lambda, u)$ has an un-bounded connected continuum Σ_0 contained in $\mathbb{R}^+ \times X$ and containing $(0,0)$, that is,

$$(0,0) \in \Sigma_0,\ \Sigma_0 \subset \mathbb{R}^+ \times X,$$

and Σ_0 is unbounded in $\mathbb{R}^+ \times X$, *i.e.*, it connects $(0,0)$ to infinity.

On the other hand, from $f(x,0) > 0$ we have

$$F(0) \neq 0,\ T(\lambda, 0) = \frac{\lambda LF(0)}{M(0)} \neq 0\ (\lambda \neq 0).$$

Hence $(\lambda, 0)$ is a solution of problem (1) only when $\lambda = 0$, while $(0,u)$ is a solution of problem (1) only when $u = 0$. By (A6) and the continuity of the positive solution branch, it follows that those sufficiently small solutions bifurcating from $(0,0)$ satisfy $u > 0$. Hence all solutions in $\Sigma_0 \setminus \{(0,0)\}$ are positive. Therefore, problem (1) possesses an unbounded connected continuum $\Sigma_0 \subset \mathbb{R}^+ \times X$, which connects $(0,0)$ to infinity. □

Lemma 3 Assume that (A2), (A4) and (A5) hold. Then problem (1) possesses an unbounded connected continuum Σ_∞ emanating from (Λ_∞, ∞), and (Λ_∞, ∞) is the unique bifurcation point of problem (1) at infinity.

Proof. Let

$$z = \frac{u}{\|u\|^2}\ (u \neq 0).$$

Then problem (1) is transformed into

$$\begin{cases} -M\left(\dfrac{1}{\|z\|^2}\right)\Delta z = \lambda \|z\|^2 f\left(x, \dfrac{z}{\|z\|^2}\right), & x \in \Omega, \\ z(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

The linearization of problem (3) at $z = 0$ is

$$\begin{cases} -M_\infty \Delta v = \lambda b(x) v, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

Since the bifurcation of (1) at infinity corresponds to the bifurcation of (3) at the trivial solution, by the Rabinowitz global bifurcation theorem ([14], Theorem 2.3) there exists an unbounded connected continuum Σ_∞ of positive solutions of

problem (3) bifurcating from (Λ_∞, ∞).

Since the principal eigenvalue of Equation (4) is $M_\infty \lambda_1[b]$, and the corresponding eigenfunction does not change sign, (Λ_∞, ∞) is the unique bifurcation point of problem (1) at infinity. □

Lemma 4 Assume that (A3)-(A5) hold. Then there exists $\lambda_* > 0$ such that $u = T(\lambda, u)$ has no positive solution (λ, u) whenever $\lambda > \lambda_*$.

Proof. Suppose, on the contrary, that there exists a positive solution (λ, u) of $u = T(\lambda, u)$. By integration by parts, we obtain

$$-M\left(\|u\|^2\right)\int_\Omega \Delta u\phi = \lambda\int_\Omega f(x,u)\phi = \lambda_1 M\left(\|u\|^2\right)\int_\Omega u\phi,$$

where $\lambda_1 = \lambda_1[1]$ is the principal eigenvalue of

$$\begin{cases} -\Delta v = \lambda v, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases}$$

and ϕ is the corresponding eigenfunction. By (A4)-(A5), the function M is bounded above. Set

$$\bar{M} = \sup_{t\geq 0} M(t).$$

Then

$$\lambda\int_\Omega f(x,u)\phi \leq \lambda_1 \bar{M}\int_\Omega u\phi.$$

From (A3) it follows that

$$\lambda\alpha\int_\Omega u\phi \leq \lambda\int_\Omega f(x,u)\phi \leq \lambda_1\bar{M}\int_\Omega u\phi,$$

that is,

$$\lambda \leq \frac{\lambda_1\bar{M}}{\alpha}.$$

Taking $\lambda_* = \frac{\lambda_1\bar{M}}{\alpha}$, we conclude that $u = T(\lambda, u)$ has no positive solution (λ, u) when $\lambda > \lambda_*$. □

3. Proofs of the Main Results

Proof of Theorem 1

1) By Lemma 2, problem (1) has an unbounded connected continuum $\Sigma_0 \subset R^+ \times X$ connecting $(0,0)$ to infinity. Then by Lemma 3, problem (1) has an unbounded connected continuum Σ_∞ emanating from (Λ_∞, ∞). Furthermore, (Λ_∞, ∞) is proven to be the sole bifurcation point of positive solutions of problem (1) at infinity. On the other hand, by Lemma 4, both Σ_0 and Σ_∞ are bounded in the λ direction, so the unbounded connected continuum Σ_0 must intersect (Λ_∞, ∞), that is, $\Sigma_0 = \Sigma_\infty$.

2) By Lemma 2, problem (1) has an unbounded connected branch Σ_0. However, if $\lim_{t\to+\infty} M(t) = \infty$, then no bifurcation occurs at infinity. For this reason, we will prove that the solutions u of problem (1) have a priori bounds.

Let $\lambda \geq 0$, and suppose $u \in X$ is a positive solution of problem (1) corre-

sponding to λ. Taking u as a test function in problem (1), we obtain

$$M\left(\|u\|^2\right)\|u\|^2 = \lambda\int_\Omega f(x,u)u,$$

From condition (A2), it follows that there exists a constant $K>0$ such that for any $u\geq 0$, $f(x,u)\leq Ku$. Hence

$$M\left(\|u\|^2\right)\|u\|^2 \leq \lambda K\int_\Omega u^2 \leq \frac{\lambda K}{\lambda_1[a]}\|u\|^2,$$

that is,

$$M\left(\|u\|^2\right)\leq \frac{\lambda K}{\lambda_1[a]}.$$

Moreover, since $\lim_{t\to+\infty} M(t)=\infty$, there exists a constant $C_\lambda>0$ such that $\|u\|^2\leq C_\lambda$. Therefore $\|u\|^2$ has a priori bounds, and since the connected branch Σ_0 is unbounded, we have $\mathrm{Proj}_{\mathbb{R}}\Sigma_0=(0,+\infty)$. □

Proof of Theorem 2

Assume that $u\in X$ and $u(x_0)=\max_{x\in\bar{\Omega}}|u(x)|=s_0\in[s',s'']$. Then

$$u(x_0)=T(\lambda,u(x_0)).$$

On the one hand, since $u(x_0)=\max_{x\in\bar{\Omega}}|u(x)|=s_0\in[s',s'']$, we have

$$\Delta u(x_0)=-\frac{\lambda f(x_0,u(x_0))}{M\left(\|u\|^2\right)}\leq 0.$$

On the other hand, by condition (A4), we have $M\left(\|u\|^2\right)\geq m_0>0$, so

$$-M\left(\|u\|^2\right)\Delta u(x_0)=\lambda f(x_0,u(x_0))=\lambda f(x_0,s_0).$$

Combining the assumption $s_0\in[s',s'']$ with condition (A7), we get $f(x_0,s_0)\leq 0$. Therefore

$$\Delta u(x_0)=-\frac{\lambda f(x_0,s_0)}{M(s_0)}\geq 0.$$

Hence $\Delta u(x_0)=0$. Applying maximum principle ([15], Theorem 2.2), if the function u is not identically constant, it cannot achieve a nonnegative maximum within the interior of Ω_+. This forces $u\equiv 0$, which stands in direct contradiction to the initial condition, which contradicts $u(x_0)=s_0>0$. Therefore, problem (1) has no positive solution satisfying $\max_{x\in\bar{\Omega}}|u(x)|\in[s',s'']$.

Next, we prove that the connected branches Σ_0 and Σ_∞ are disjoint. By Lemma 2, problem (1) has a connected branch $\Sigma_0\subset R^+\times X$ emanating from $(0,0)$ and going to infinity, and it has been proved that the solutions on this branch do not satisfy $\max_{x\in\bar{\Omega}}|u(x)|\in[s',s'']$. By the connectedness of solutions, all positive solutions on Σ_0 satisfy

$$\max_{x\in\bar{\Omega}} u(x) < s'.$$

On the other hand, by Lemma 3, problem (1) has an unbounded connected branch Σ_∞ emanating from (Λ_∞, ∞), and (Λ_∞, ∞) is the unique bifurcation point of positive solutions of problem (1) at infinity. From $\|u\| \to +\infty$, together with the connectedness of solutions and the fact that solutions cannot lie in the interval $[s', s'']$, all positive solutions on Σ_∞ satisfy

$$\max_{x\in\bar{\Omega}} u(x) > s''.$$

Since $0 < s' < s''$, in summary, we finally obtain that Σ_0 and Σ_∞ are disjoint. □

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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