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Existence and Multiplicity of Solutions for Double Phase Problems with Indefinite Weight in $\mathbb{R}^N$

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Abstract

In this paper, we study a class of double phase problems with indefinite weight: $-\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u+|\nabla u|^{q-2}\nabla u\right)+V(x)|u|^{q-2}u=K(x)f(u)$ in $\mathbb{R}^N$, where $N\geq 3$, $1<q<N$, $q<p<q^*=\frac{Nq}{N-q}$, $a\in C^{0,\alpha}\left(\mathbb{R}^N,[0,\infty)\right)$, $a\not\equiv 0$, $\alpha\in(0,1]$, V,K are nonnegative continuous functions and f is a continuous function with a quasicritical growth. Using variational methods, we prove the existence of nontrivial solutions via the mountain pass theorem. In addition, employing the symmetric mountain pass theorem, we prove the existence of infinitely many solutions when the nonlinearity f is odd, thereby enriching the multiplicity results for double phase problems in $\mathbb{R}^N$.

Keywords

Double Phase Problem, Indefinite Weight, Variational Methods, Critical Point Theory

1. Introduction

In this paper, we study the following double phase problem with indefinite weight:

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u+|\nabla u|^{q-2}\nabla u\right)+V(x)|u|^{q-2}u=K(x)f(u) & \text{in } \mathbb{R}^N, \\ u(x)>0 & \text{in } \mathbb{R}^N, \\ u\in D^{1,\xi}\left(\mathbb{R}^N\right), \end{cases} \quad (1.1)$$

where $N\geq 3$, $1<q<N$, $q<p<q^*=\frac{Nq}{N-q}$, $a\in C^{0,\alpha}\left(\mathbb{R}^N,[0,\infty)\right)$, $a\not\equiv 0$,

$\alpha \in (0,1]$, $V, K : \mathbb{R}^N \to \mathbb{R}$ are nonnegative continuous functions, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with quasicritical growth.

In recent years, quasilinear elliptic equations with nonstandard growth conditions have been widely applied in fields such as nonlinear elasticity, non-Newtonian fluids, strongly anisotropic materials, and phase transition theory. The study of such problems has long been an important direction in the fields of nonlinear partial differential equations and variational analysis, see [1]-[9]. Among various problems with nonstandard growth, the double phase operator initially introduced and systematically investigated by Zhikov has attracted considerable attention, as it can characterize composite materials with both p-growth and q-growth behavior in [10]-[13]. In 1986, he introduced the energy functional

$$u \mapsto \int \left(\frac{a(x)}{p} |\nabla u|^p + \frac{1}{q} |\nabla u|^q \right) \mathrm{d}x, \tag{1.2}$$

which serves as a prototype for functionals whose ellipticity behavior changes according to the vanishing set of the coefficient $a(x)$, we refer the reader to [14] [15]. This model describes strongly anisotropic materials, where the modulation coefficient $a(x)$ determines a composite material with two distinct power hardening exponents p and q. In this context, Zhikov discovered a gap phenomenon with further mathematical applications in homogenization theory.

Li *et al.* [16] investigated ground states for nonlinear Schrödinger equations in $\mathbb{R}^N$:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

weakened the standard Ambrosetti-Rabinowitz super-linear condition and relaxed the superlinear conditions of nonlinearities to guarantee ground states via Nehari manifold arguments. Later, Alves and Souto [17] further studied Schrödinger equations with weighted nonlinearity $K(x) f(u)$ and potentials vanishing at infinity in $\mathbb{R}^N$:

$$\begin{cases} -\Delta u + V(x)u = K(x) f(u) & \text{in } \mathbb{R}^N, \\ u(x) > 0 & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$, the potentials V and K are nonnegative continuous functions, and the nonlinearity f satisfies the quasicritical growth condition. They proved the existence of positive ground state solutions via Hardy-type inequalities and variational arguments. However, the above works only deal with linear Laplace operator and nonnegative weight functions, and ignored the nonstandard growth induced by double phase diffusion.

Ge *et al.* [18] investigated double phase equations on bounded domains:

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2} \nabla u + a(x) |\nabla u|^{q-2} \nabla u \right) = \lambda f(x,u) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $N \geq 2$, $1 < p < q < N$ and $\lambda > 0$, they derived the existence of infinitely many solutions via the fountain theorem, dual fountain theorem and Krasnoselskii's genus theory. Furthermore, researchers have investigated double phase elliptic problems with indefinite weights in $\mathbb{R}^N$, and establish the existence and multiplicity of solutions.

Liu and Dai [19] studied multiplicity results for double phase problems in $\mathbb{R}^N$:

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) + |u|^{p-2}u + a(x)|u|^{q-2}u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,\mathcal{H}}\left(\mathbb{R}^N\right), \end{cases}$$

where $1 < p < q < N$, $\frac{q}{p} \leq 1 + \frac{\alpha}{N}$, $0 \leq a \in C^{0,\alpha}\left(\mathbb{R}^N\right)$, $\alpha \in (0,1]$. After weakening the Ambrosetti-Rabinowitz condition, they obtained positive and negative ground state solutions as well as one sign-changing solution with a single sign transition. Meanwhile, they established the existence of infinitely many solutions for even functionals.

Motivated by the above analysis, we discuss the existence and multiplicity results for the problem (1.1). Our primary difficulty of the problem (1.1) is the loss of compactness in $\mathbb{R}^N$. To overcome this, we establish the compactness by applying the Alves's some proof and complete relevant verification with Hardy-type inequalities together with variational methods. Furthermore, we establish the existence of infinitely many solutions via the symmetric mountain pass theorem.

Let us now state the precise hypotheses to problem (1.1) and the main results. Throughout of this paper, we suppose that the potential $(V,K) \in \mathcal{K}$ satisfy the following conditions:

(I) $V(x), K(x) > 0 \quad \forall x \in \mathbb{R}^N$ and $K \in L^{\infty}\left(\mathbb{R}^N\right)$.

(II) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \leq R$, for all n and some $R > 0$, then

$$\lim_{r \to +\infty} \int_{A_n \cap B_r^c(0)} K(x)\mathrm{d}x = 0, \text{ uniformly in } n \in \mathbb{N}. \tag{K_1}$$

(III) One of the below conditions occurs:

$$\frac{K}{V} \in L^{\infty}\left(\mathbb{R}^N\right) \tag{K_2}$$

or there is $s \in \left(q, q^*\right)$ such that

$$\frac{K(x)}{\left[V(x)\right]^{\frac{q^*-s}{q^*-q}}} \to 0 \text{ as } |x| \to +\infty. \tag{K_3}$$

Related to the function f, we assume the following conditions:

(f_1)

$$\limsup_{t \to 0} \frac{|f(t)|}{t^{q-1}} = 0 \quad \text{if } (K_2) \text{ holds,}$$

or

$$\limsup_{t \to 0^+} \frac{|f(t)|}{t^{p-1}} < +\infty \quad \text{if } (K_3) \text{ holds.}$$

(f_2) f has a quasicritical growth, that is

$$\limsup_{t\to+\infty}\frac{f(t)}{t^{q^*-1}}=0.$$

(f_3) the cap $t\mapsto\dfrac{f(t)}{t^{p-1}}$ is a non-decreasing function in $(0,+\infty)$ and its primitive F is super-p-power at infinity, that is

$$\limsup_{t\to+\infty}\frac{F(t)}{t^{p}}=+\infty.$$

Now, we present the main results of this paper.

Theorem 1.1 *Suppose that* f *satisfies* (f_1)-(f_3) *and* $(V,K)\in\mathcal{K}$, *then problem* (*1.1*) *possesses a positive ground state solution.*

Theorem 1.2 *Suppose that* f *satisfies* (f_1)-(f_3) *and* $(V,K)\in\mathcal{K}$, *and that* $f(x,-t)=-f(x,t)$, *then problem* (*1.1*) *admits infinitely many solutions with negative energy values.*

This paper is organized as follows. Section 2 lists necessary preliminary results. Section 3 is devoted to prove the energy functional of problem (1.1) satisfying the Cerami condition, further obtain the existence of a positive solution. In Section 4, we prove the existence of infinite many solutions by the symmetric mountain pass theorem.

2. Preliminary Results

This section first introduces some notations, and proves that the space X is a separable and uniformly convex Banach space. Similar to reference [20], the space X is given by

$$X=\left\{u\in D^{1,\xi}\left(\mathbb{R}^N\right):\int_{\mathbb{R}^N}V(x)|u|^q\,\mathrm{d}x<+\infty\right\},$$

endowed with the norm

$$\|u\|_X^q=\int_{\mathbb{R}^N}\left(a(x)|\nabla u|^p+|\nabla u|^q+V(x)|u|^q\right)\mathrm{d}x,$$

where

$$D^{1,\xi}\left(\mathbb{R}^N\right)=\left\{u\in L^1_{loc}\left(\mathbb{R}^N\right):\int_{\mathbb{R}^N}\left(a(x)|\nabla u|^p+|\nabla u|^q\right)\mathrm{d}x<+\infty\right\},$$

the space $D^{1,\xi}\left(\mathbb{R}^N\right)$ is defined as the completion of $C_0^\infty\left(\mathbb{R}^N\right)$ under the norm $\|\nabla u\|_\xi$, see Proposition 2.18(ii) of [21]. Furthermore, we will prove that X is compactly embedded into the weighted Lebesgue space

$$L^s_K\left(\mathbb{R}^N\right)=\left\{u:\mathbb{R}^N\to\mathbb{R}\ u\text{ is measurable and }\int_{\mathbb{R}^N}K(x)|u|^s\,\mathrm{d}x<\infty\right\}.$$

Next we present the energy functional associated with problem (1.1) is given by

$$J(u)=\frac{1}{p}\int_{\mathbb{R}^N}a(x)|\nabla u|^p\,\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\,\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}V(x)|u|^q\,\mathrm{d}x-\int_{\mathbb{R}^N}K(x)F(u)\mathrm{d}x,$$

for all $u \in X$. Thanks to the regularity assumptions imposed on f, we obtain $J \in C^1(X, \mathbb{R})$, and its Gateaux derivative is

$$J'(u)v = \int_{\mathbb{R}^N} \left(a(x)|\nabla u|^{p-2} \nabla u \nabla v + |\nabla u|^{q-2} \nabla u \nabla v + V(x)|u|^{q-2} uv \right) \mathrm{d}x$$
$$- \int_{\mathbb{R}^N} K(x) f(u) v \mathrm{d}x, \text{ for all } u, v \in X.$$

It is easy to check that critical points of J are weak solutions of problem (1.1). Once that we intend to find positive solution, we will assume that

$$f(t) = 0 \quad \forall t \in (-\infty, 0].$$

Accordingly, there is a sequence $\{u_n\} \subset X$ such that

$$J(u_n) \to c \text{ and } \left(1 + \|u_n\|\right) \|J'(u_n)\|_{X^*} \to 0, \tag{2.1}$$

where c is the mountain pass level given by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

and construct the set

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

The above sequence is the so-called Cerami sequence for J at level c, arguing as in Willem [22].

Now, we are ready to show our main results of continuous embedding theorem, see the reference [23] for more details.

Lemma 2.1 *Let* $\xi : \mathbb{R}^N \times [0, \infty) \to [0, \infty)$, *and* $\rho_\xi(u) = \int_{\mathbb{R}^N} \left(a(x)|u|^p + |u|^q \right) \mathrm{d}x$. *Then, the embedding* $L^\xi(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ *is continuous.*

As a consequence of Lemma 2.1 and Sobolev embedding in $W^{1,q}(\mathbb{R}^N)$ and $D^{1,q}(\mathbb{R}^N)$ for $1 < q < N$, we have the following embedding result.

Lemma 2.2 *Let* $\xi : \mathbb{R}^N \times [0, \infty) \to [0, \infty)$, $\rho_\xi(u) = \int_{\mathbb{R}^N} \left(a(x)|u|^p + |u|^q \right) \mathrm{d}x$. *Then, the embedding* $W^{1,\xi}(\mathbb{R}^N) \hookrightarrow W^{1,q}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ *is continuous, for any* $q \le r \le q^*$. *Moreover, the embedding* $D^{1,\xi}(\mathbb{R}^N) \hookrightarrow D^{1,q}(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$ *is continuous.*

Next, we present some properties of space X.

Lemma 2.3 *The space* $X = \left(X, \|\cdot\|_X\right)$ *is a separable and uniformly convex Banach space.*

Proof. We first verify that X is complete. Let $\{u_n\}$ be a Cauchy sequence in X, then for any $\varepsilon > 0$, there exists $N > 0$ such that if $n, m \ge N$, we have

$$\|u_n - u_m\|_X < \varepsilon. \tag{2.2}$$

By the definition of X, we know that

$$\int_{\mathbb{R}^N} a(x)|\nabla u_n - \nabla u_m|^p \mathrm{d}x \le \|u_n - u_m\|_X^q < \varepsilon^q,$$

$$\int_{\mathbb{R}^N} |\nabla u_n - \nabla u_m|^q \mathrm{d}x \le \|u_n - u_m\|_X^q < \varepsilon^q,$$

By the completeness of $D^{1,\xi}(\mathbb{R}^N)$, there exists $u \in D^{1,\xi}(\mathbb{R}^N)$ such that

$u_n \to u$ in $D^{1,\xi}\left(\mathbb{R}^N\right)$, we can extract a subsequence $\left\{u_{n_k}\right\}$ such that $u_{n_k} \to u$ a.e. in $\mathbb{R}^N$ as $k \to \infty$. Therefore, by the Fatou's Lemma and (2.2), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)|u|^q \,\mathrm{d}x &\le \liminf_{k\to\infty} \int_{\mathbb{R}^N} V(x)\left|u_{n_k}\right|^q \mathrm{d}x \\ &\le \liminf_{k\to\infty} \left(\left\|u_{n_k}-u_N\right\|_X + \left\|u_N\right\|_X\right)^q \\ &\le \left(\varepsilon + \left\|u_N\right\|_X\right)^q < \infty,\end{aligned}$$

which implies that $u \in X$. Let $n \ge N$, then we get

$$\left\|u_n - u\right\|_X \le \liminf_{k\to\infty} \left\|u_n - u_{n_k}\right\|_X \le \varepsilon,$$

that is $u_n \to u$ in X. Hence, the space X is complete.

Next, we prove that X is uniformly convex. Fix $\varepsilon > 0$, let $u, v \in X$ with $\|u\|_X = \|v\|_X = 1$ and $\|u-v\|_X \ge \varepsilon$. For $q \ge 2$, by the following inequality (see [24])

$$\left|\frac{a+b}{2}\right|^q + \left|\frac{a-b}{2}\right|^q \le \frac{|a|^q + |b|^q}{2} \quad \text{for } a, b \in \mathbb{R},$$

we have

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|_X^q + \left\|\frac{u-v}{2}\right\|_X^q &= \left[\int_{\mathbb{R}^N} a(x)\left|\nabla\left(\frac{u+v}{2}\right)\right|^p \mathrm{d}x + \int_{\mathbb{R}^N} a(x)\left|\nabla\left(\frac{u-v}{2}\right)\right|^p \mathrm{d}x\right]^{\frac{q}{p}} \\ &\quad + \left[\int_{\mathbb{R}^N}\left|\nabla\left(\frac{u+v}{2}\right)\right|^q \mathrm{d}x + \int_{\mathbb{R}^N}\left|\nabla\left(\frac{u-v}{2}\right)\right|^q \mathrm{d}x\right] \\ &\quad + \int_{\mathbb{R}^N} V(x)\left(\left|\frac{u+v}{2}\right|^q + \left|\frac{u-v}{2}\right|^q\right)\mathrm{d}x \\ &\le \frac{1}{2}\left(\int_{\mathbb{R}^N} a(x)|\nabla u|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} a(x)|\nabla v|^p \,\mathrm{d}x\right)^{\frac{q}{p}} \\ &\quad + \frac{1}{2}\left(\int_{\mathbb{R}^N}|\nabla u|^q \,\mathrm{d}x + \int_{\mathbb{R}^N}|\nabla v|^q \,\mathrm{d}x\right) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left(|u|^q + |v|^q\right)\mathrm{d}x \\ &= \frac{1}{2}\left(\|u\|_X^q + \|v\|_X^q\right) = 1.\end{aligned} \tag{2.3}$$

Then we have

$$\left\|\frac{u+v}{2}\right\|_X^q \le 1 - \left(\frac{\varepsilon}{2}\right)^q.$$

Taking $\delta = 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^q\right]^{1/q} > 0$, we obtain that

$$\left\|\frac{u+v}{2}\right\|_X \le 1 - \delta.$$

For $1<q<2$, let $q'=\frac{q}{q-1}$. We can write

$$\begin{aligned}\|u\|_X^{q'} &= \left(\int_{\mathbb{R}^N} a(x)|\nabla u|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|u|^q \,\mathrm{d}x\right)^{\frac{1}{q-1}} \\ &\le 2^{\frac{1}{q-1}}\left(\left(\int_{\mathbb{R}^N} a(x)|\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{q-1}} + \left(\int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x\right)^{\frac{1}{q-1}} + \left(\int_{\mathbb{R}^N} V(x)|u|^q \,\mathrm{d}x\right)^{\frac{1}{q-1}}\right).\end{aligned} \tag{2.4}$$

In view of (2.4) and the reverse Minkowski inequality (see [24] Theorem 2.13), we deduce

$$\begin{aligned}&2^{\frac{1}{1-q}}\left(\left\|\frac{u+v}{2}\right\|_X^{q'} + \left\|\frac{u-v}{2}\right\|_X^{q'}\right) \\ &\le \left(\int_{\mathbb{R}^N} a(x)\left|\frac{\nabla u+\nabla v}{2}\right|^p \mathrm{d}x\right)^{\frac{1}{q-1}} + \left(\int_{\mathbb{R}^N} a(x)\left|\frac{\nabla u-\nabla v}{2}\right|^p \mathrm{d}x\right)^{\frac{1}{q-1}} \\ &\quad + \left(\int_{\mathbb{R}^N} \left|\frac{\nabla u+\nabla v}{2}\right|^q \mathrm{d}x\right)^{\frac{1}{q-1}} + \left(\int_{\mathbb{R}^N} \left|\frac{\nabla u-\nabla v}{2}\right|^q \mathrm{d}x\right)^{\frac{1}{q-1}} \\ &\quad + \left(\int_{\mathbb{R}^N} V(x)\left|\frac{u+v}{2}\right|^q \mathrm{d}x\right)^{\frac{1}{q-1}} + \left(\int_{\mathbb{R}^N} V(x)\left|\frac{u-v}{2}\right|^q \mathrm{d}x\right)^{\frac{1}{q-1}} \\ &\le \left(\int_{\mathbb{R}^N} a(x)\left(\left|\frac{\nabla u+\nabla v}{2}\right|^{\frac{p}{q-1}} + \left|\frac{\nabla u-\nabla v}{2}\right|^{\frac{p}{q-1}}\right)^{q-1} \mathrm{d}x\right)^{\frac{1}{q-1}} \\ &\quad + \left(\int_{\mathbb{R}^N} \left(\left|\frac{\nabla u+\nabla v}{2}\right|^{q'} + \left|\frac{\nabla u-\nabla v}{2}\right|^{q'}\right)^{q-1} \mathrm{d}x\right)^{\frac{1}{q-1}} \\ &\quad + \left(\int_{\mathbb{R}^N} V(x)\left(\left|\frac{u+v}{2}\right|^{q'} + \left|\frac{u-v}{2}\right|^{q'}\right)^{q-1} \mathrm{d}x\right)^{\frac{1}{q-1}}.\end{aligned} \tag{2.5}$$

By the following inequality (see [24])

$$\left|\frac{a+b}{2}\right|^{q'} + \left|\frac{a-b}{2}\right|^{q'} \le \left(\frac{|a|^q+|b|^q}{2}\right)^{\frac{1}{q-1}} \quad \text{for } a,b\in\mathbb{R},$$

it follows from (2.5) that

$$\begin{aligned}2^{\frac{1}{1-q}}\left(\left\|\frac{u+v}{2}\right\|_X^{q'} + \left\|\frac{u-v}{2}\right\|_X^{q'}\right) &\le 2^{\frac{1}{1-q}}\left(\left(\int_{\mathbb{R}^N} a(x)|\nabla u|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} a(x)|\nabla v|^p \,\mathrm{d}x\right)^{\frac{1}{q-1}}\right. \\ &\quad + \left(\int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v|^q \,\mathrm{d}x\right)^{\frac{1}{q-1}} \\ &\quad \left. + \left(\int_{\mathbb{R}^N} V(x)|u|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|v|^q \,\mathrm{d}x\right)^{\frac{1}{q-1}}\right) \\ &\le 2^{\frac{1}{1-q}}\left(\|u\|_X^q + \|v\|_X^q\right)^{\frac{1}{q-1}} = 2^{\frac{1}{1-q}}\cdot 2^{\frac{1}{q-1}} = 1.\end{aligned}$$

Then we have

$$\left\|\frac{u+v}{2}\right\|_X^{q'} \le 1-\left(\frac{\varepsilon}{2}\right)^{q'}.$$

Taking $\delta = 1-\left[1-\left(\frac{\varepsilon}{2}\right)^{q'}\right]^{1/q'}$, we can get

$$\left\|\frac{u+v}{2}\right\|_X \le 1-\delta.$$

Hence, the space X is uniformly convex.

Finally, we show that X is separable in four steps.

Step 1: We establish the basic density property. The space $C_0^\infty\left(\mathbb{R}^N\right)$ is dense in X, *i.e.*, every element in X can be approximated by mollifiers:

$$\overline{C_0^\infty\left(\mathbb{R}^N\right)}^{\|\cdot\|_X} = X.$$

That is, for any $u \in X$, there exists a sequence $\{\varphi_n\} \subset C_0^\infty\left(\mathbb{R}^N\right)$ such that

$$\|\varphi_n - u\|_X \to 0.$$

Step 2: Proof of separability. Consider the map $T: X \to L_a^p\left(\mathbb{R}^N\right) \times L^q\left(\mathbb{R}^N\right) \times L_V^q\left(\mathbb{R}^N\right)$, defined by

$$T(u) = \left(a^{1/p}\nabla u, \nabla u, V^{1/q}u\right),$$

where $L_a^p\left(\mathbb{R}^N\right) = \left\{v : \int_{\mathbb{R}^N} a(x)|v|^p \,\mathrm{d}x < \infty\right\}$, and $L_V^q\left(\mathbb{R}^N\right) = \left\{v : \int_{\mathbb{R}^N} V(x)|v|^q \,\mathrm{d}x < \infty\right\}$. Then T is an isometric embedding of X into the product space $L_a^p\left(\mathbb{R}^N\right) \times L^q\left(\mathbb{R}^N\right) \times L_V^q\left(\mathbb{R}^N\right)$. Since $1 < p, q < \infty$, each of the weighted Lebesgue spaces L_a^p, L^q, and L_V^q is separable. Therefore their product is separable. Any subspace of a separable metric space is separable, and hence X is separable. The uniform convexity follows from the uniform convexity of L_a^p, L^q, and L_V^q together with the fact that the norm $\|\cdot\|_X$ is equivalent to the product norm. This completes the proof.

3. Existence of Nontrivial Solutions

Proposition 3.1 (*Hardy-type inequality*) *Assume that* $(V,K) \in \mathcal{K}$ *hold. Then,* X *is compactly embedded in* $L_K^m\left(\mathbb{R}^N\right)$ *for all* $m \in \left(q, q^*\right)$ *if* (K_2) *holds. If* (K_3) *holds,* X *is compactly embedded in* $L_K^s\left(\mathbb{R}^N\right)$.

Proof. The proof will be made into two parts, we first derive the conclusion under condition (K_2), then handle the case for (K_3).

Suppose (K_2) holds. Fix $m \in \left(q, q^*\right)$, for any $\varepsilon > 0$, there exist $0 < t_0 < t_1$ and $C > 0$ such that

$$K(x)|t|^m \le \varepsilon C\left(V(x)|t|^q + |t|^{q^*}\right) + CK(x)\chi_{[t_0,t_1]}(|t|)|t|^{q^*} \quad \forall t \in \mathbb{R}. \tag{3.1}$$

Hence,

$$\int_{B_r^c(0)} K(x)|u|^m \,\mathrm{d}x \le \varepsilon C Q(u) + C|t_1|^{q^*} \int_{A \cap B_r^c(0)} K(x)\,\mathrm{d}x \quad \forall u \in X \tag{3.2}$$

where

$$Q(u)=\int_{\mathbb{R}^N} V(x)|u|^q \,\mathrm{d}x+\int_{\mathbb{R}^N}|u|^{q^*}\,\mathrm{d}x,$$

and

$$A=\left\{x\in\mathbb{R}^N : t_0\le|u(x)|\le t_1\right\}.$$

If $\{v_n\}$ is a sequence such that $v_n \rightharpoonup v$ in X, then there exists a constant $M_1>0$ satisfying

$$\int_{\mathbb{R}^N}\left(|\nabla v_n|^q+V(x)|v_n|^q\right)\mathrm{d}x\le M_1 \text{ and } \int_{\mathbb{R}^N}|v_n|^{q^*}\,\mathrm{d}x\le M_1 \ \ \forall n\in\mathbb{N},$$

this implies that the sequence $\{Q(v_n)\}$ is bounded. On the other hand, let

$$A_n=\left\{x\in\mathbb{R}^N : t_0\le|v_n(x)|\le t_1\right\},$$

the last inequality implies that

$$t_0^{q^*}|A_n|\le\int_{A_n}|v_n|^{q^*}\,\mathrm{d}x\le M_1 \ \ \forall n\in\mathbb{N},$$

showing that $\sup_{n\in\mathbb{N}}|A_n|<+\infty$. Therefore, by virtue of (K_1), there exists $r>0$ such that

$$\int_{A_n\cap B_r^c(0)}K(x)\,\mathrm{d}x<\frac{\varepsilon}{t_1^{q^*}} \text{ for all } n\in\mathbb{N}. \tag{3.3}$$

Now, (3.2) and (3.3) lead to

$$\int_{B_r^c(0)}K(x)|v_n|^m\,\mathrm{d}x\le\varepsilon CM_1+t_1^{q^*}\int_{A_n\cap B_r^c(0)}K(x)\,\mathrm{d}x<(CM_1+1)\varepsilon \ \ \forall n\in\mathbb{N}. \tag{3.4}$$

Since $m\in(q,q^*)$ and K is a continuous function, Sobolev embeddings imply that

$$\lim_{n\to+\infty}\int_{B_r(0)}K(x)|v_n|^m\,\mathrm{d}x=\int_{B_r(0)}K(x)|v|^m\,\mathrm{d}x. \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}K(x)|v_n|^m\,\mathrm{d}x=\int_{\mathbb{R}^N}K(x)|v|^m\,\mathrm{d}x, \tag{3.6}$$

which gives

$$v_n\to v \text{ in } L_K^m(\mathbb{R}^N) \ \ \forall m\in(q,q^*).$$

We now assume that (K_3) holds. First of all, it is important to observe that for any fixed $x\in\mathbb{R}^N$, the function

$$h(t)=V(x)t^{q-s}+t^{q^*-s}, \ \ \forall t>0,$$

attains its minimum value $C_pV(x)^{\frac{q^*-s}{q^*-q}}$, where

$$C_p=\left(\frac{q^*-q}{q^*-s}\right)\left(\frac{s-q}{q^*-q}\right)^{\frac{q-s}{q^*-q}}.$$

Hence,

$$C_p V(x)^{\frac{q^*-s}{q^*-q}} \le V(x)t^{q-s} + t^{q^*-s}, \ \forall x \in \mathbb{R}^N \text{ and } t > 0.$$

Combining the last inequality with (K_3), for any $\varepsilon \in (0, C_p)$, there exists $r > 0$ sufficiently large, such that

$$K(x)|t|^s \le \varepsilon\left(V(x)|t|^q + |t|^{q^*}\right) \ \forall t \in \mathbb{R} \text{ and } |x| \ge r, \tag{3.7}$$

which implies

$$\int_{B_r^c(0)} K(x)|u|^s \,\mathrm{d}x \le \varepsilon \int_{B_r^c(0)} \left(V(x)|u|^q + |u|^{q^*}\right)\mathrm{d}x \ \forall u \in X.$$

If $\{v_n\}$ is a sequence such that $v_n \rightharpoonup v$ in X, there is $M_1 > 0$ such that

$$\int_{\mathbb{R}^N} V(x)|v_n|^q \,\mathrm{d}x \le M_1 \text{ and } \int_{\mathbb{R}^N} |v_n|^{q^*} \,\mathrm{d}x \le M_1 \ \forall n \in \mathbb{N},$$

consequently,

$$\int_{B_r^c(0)} K(x)|v_n|^s \,\mathrm{d}x \le 2\varepsilon M_1 \ \forall n \in \mathbb{N}. \tag{3.8}$$

Since $p \in (q, q^*)$ and K is a continuous function, the Sobolev embeddings yield

$$\lim_{n\to+\infty} \int_{B_r(0)} K(x)|v_n|^s \,\mathrm{d}x = \int_{B_r(0)} K(x)|v|^s \,\mathrm{d}x. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)|v_n|^s \,\mathrm{d}x = \int_{\mathbb{R}^N} K(x)|v|^s \,\mathrm{d}x,$$

which implies

$$v_n \to v \text{ in } L_K^s(\mathbb{R}^N).$$

Thus, we completes the proof of the proposition.

The following lemma provides a key step for establishing the boundedness of the Cerami sequence given in (2.1).

Lemma 3.2 *Assume that* f *satisfies* (f_1)-(f_3) *and* $(V, K) \in \mathcal{K}$. *Let* $\{v_n\}$ *be a sequence such that* $v_n \rightharpoonup v$ *in* X. *Then*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)F(v_n)\mathrm{d}x = \int_{\mathbb{R}^N} K(x)F(v)\mathrm{d}x,$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)f(v_n)v_n \mathrm{d}x = \int_{\mathbb{R}^N} K(x)f(v)v\mathrm{d}x.$$

Proof. We start the argument under condition (K_2). Given fixed $m \in (q, q^*)$ and $\varepsilon > 0$, there is $C > 0$, the assumptions (f_1)-(f_3) yield that

$$|K(x)F(t)| \le \varepsilon C\left(V(x)|t|^q + |t|^{q^*}\right) + K(x)|t|^m \text{ for all } t \in \mathbb{R}. \tag{3.10}$$

By Proposition 3.1,

$$\int_{\mathbb{R}^N} K(x)|v_n|^m \,\mathrm{d}x \to \int_{\mathbb{R}^N} K(x)|v|^m \,\mathrm{d}x,$$

then there is $r > 0$ such that

$$\int_{B_r^c(0)} K(x)|v_n|^m \,\mathrm{d}x < \varepsilon \quad \forall n \in \mathbb{N}. \tag{3.11}$$

Since $\{v_n\}$ is bounded in X, there is $M_1 > 0$ such that

$$\int_{\mathbb{R}^N} V(x)|v_n|^q \,\mathrm{d}x \le M_1 \text{ and } \int_{\mathbb{R}^N} |v_n|^{q^*} \,\mathrm{d}x \le M_1.$$

Combining the last inequalities with (3.10) and (3.11),

$$\left|\int_{B_r^c(0)} K(x)F(v_n)\,\mathrm{d}x\right| < (2CM_1+1)\varepsilon \quad \forall n \in \mathbb{N}. \tag{3.12}$$

Now, if (K_3) holds, we employ the repeating the same arguments explored in the proof of Proposition 3.1. For sufficiently small $\varepsilon > 0$, taking $r > 0$ large enough such that

$$K(x) \le \varepsilon\left(V(x)|t|^{q-s} + |t|^{q^*-s}\right) \text{ for all } t \in \mathbb{R} \text{ and } |x| \ge r.$$

Hence,

$$K(x)|F(t)| \le \varepsilon\left(V(x)|F(t)||t|^{q-s} + |F(t)||t|^{q^*-s}\right) \quad \forall t \in \mathbb{R} \text{ and } |x| \ge r.$$

From (f_1) and (f_2), there are $C, t_0, t_1 > 0$ verifying

$$K(x)|F(t)| \le \varepsilon C\left(V(x)|t|^q + |t|^{q^*}\right) \quad \forall t \in I \text{ and } |x| \ge r,$$

where $I = \{t \in \mathbb{R} : |t| < t_0 \text{ or } |t| > t_1\}$. Thus, for any $u \in X$, we have the following estimate

$$\int_{B_r^c(0)} K(x)F(u)\,\mathrm{d}x \le \varepsilon C Q(u) + C\int_{A\cap B_r^c(0)} K(x)\,\mathrm{d}x$$

with

$$Q(u) = \int_{\mathbb{R}^N} V(x)|u|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^{q^*} \,\mathrm{d}x,$$

and

$$A = \{x \in \mathbb{R}^N : t_0 \le |u(x)| \le t_1\}.$$

Once that $\{v_n\}$ is bounded in X, there is $M_1 > 0$ such that

$$\int_{\mathbb{R}^N} V(x)|v_n|^q \,\mathrm{d}x \le M_1 \text{ and } \int_{\mathbb{R}^N} |v_n|^{q^*} \,\mathrm{d}x \le M_1.$$

Thereby,

$$\int_{B_r^c(0)} K(x)F(v_n)\,\mathrm{d}x \le 2M_1\varepsilon + C\int_{A_n\cap B_r^c(0)} K(x)\,\mathrm{d}x,$$

where

$$A_n = \{x \in \mathbb{R}^N : t_0 \le |v_n(x)| \le t_1\}.$$

Repeating the same arguments used in the proof of Proposition 3.1, it follows that

$$\int_{A_n\cap B_r^c(0)} K(x)\,\mathrm{d}x \to 0 \text{ as } |x| \to +\infty,$$

and so, for n large enough

$$\left|\int_{B_r^c(0)} K(x)F(v_n)\mathrm{d}x\right| \le (2M_1+1)\varepsilon.$$

To complete the proof, we need to verify that

$$\lim_{n\to+\infty}\int_{B_r(0)} K(x)F(v_n)\mathrm{d}x = \int_{B_r(0)} K(x)F(v)\mathrm{d}x.$$

However, this limit result can be derived via using the compactness lemma of Strauss ([25], Theorem A.I). Specifically, $B_r(0)$ is a bounded domain, the sequence $|v_n|_{L^{q^*}(B_r(0))}$ is bounded. Assumption (f_2) together with almost everywhere convergence imply the required limit convergence.

Next, we prove that the Cerami sequence sequence associated with (2.1) is bounded in X.

Lemma 3.3 *The Cerami sequence* $\{u_n\}\subset X$ *defined by* (2.1) *is bounded.*

Proof. Let $t_n\in[0,1]$ satisfy $J(t_nu_n)=\max_{t\ge0}J(tu_n)$. We first show that $J(t_nu_n)$ is bounded from above. In fact, if either $t_n=0$ or $t_n=1$, we are done. Thus, we can assume $t_n\in(0,1)$, and so, $J'(t_nu_n)t_nu_n=0$. Then,

$$\begin{aligned}
pJ(t_nu_n) &= pJ(t_nu_n)-J'(t_nu_n)t_nu_n\\
&=\left(\frac{p}{q}-1\right)t_n^q\int_{\mathbb{R}^N}|\nabla u_n|^q\,\mathrm{d}x+\left(\frac{p}{q}-1\right)t_n^q\int_{\mathbb{R}^N}V(x)|u_n|^q\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}K(x)\left(f(t_nu_n)t_nu_n-pF(t_nu_n)\right)\mathrm{d}x\\
&=\left(\frac{p}{q}-1\right)t_n^q\left(\int_{\mathbb{R}^N}|\nabla u_n|^q\,\mathrm{d}x+\int_{\mathbb{R}^N}V(x)|u_n|^q\,\mathrm{d}x\right)+\int_{\mathbb{R}^N}K(x)H(t_nu_n)\mathrm{d}x\\
&\le\left(\frac{p}{q}-1\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^q\,\mathrm{d}x+\int_{\mathbb{R}^N}V(x)|u_n|^q\,\mathrm{d}x\right)+\int_{\mathbb{R}^N}K(x)H(u_n)\mathrm{d}x,
\end{aligned}$$

where

$$H(t)=tf(t)-pF(t)\quad\forall t\in\mathbb{R}.$$

We may assume $u_n\ge0$ without loss of generality, the condition (f_3) implies that H is non-decreasing for each x. Hence,

$$\begin{aligned}
pJ(t_nu_n) &\le\left(\frac{p}{q}-1\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^q\,\mathrm{d}x+\int_{\mathbb{R}^N}V(x)|u_n|^q\,\mathrm{d}x\right)+\int_{\mathbb{R}^N}K(x)H(u_n)\mathrm{d}x\\
&= pJ(u_n)-J'(u_n)u_n\\
&= pJ(u_n)+o_n(1).
\end{aligned}$$

Since $J(u_n)$ converges, we conclude that $J(t_nu_n)$ is bounded from above.

Now, we will prove that $\{u_n\}$ is bounded. For this purpose, suppose by contradiction that $\|u_n\|_X\to\infty$ and set $v_n=\dfrac{u_n}{\|u_n\|_X}$. Since $\|v_n\|=1$, we note that $\{v_n\}$ is bounded. Thus, after passing to a subsequence, we may assume that $v_n\rightharpoonup v$ in X, as $n\to+\infty$.

If $v\ne0$, the condition (f_3) implies that for each $\tau>0$, there is $\xi>0$ such that

$$\frac{F(t)}{|t|^p} \geq \tau \quad \text{for } |t| \geq \xi.$$

Thus, we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|_X^p} = \lim_{n\to\infty} \frac{J(u_n)}{\|u_n\|_X^p} \\
&= \lim_{n\to\infty} \frac{1}{\|u_n\|_X^p} \left(\frac{1}{p} \int_{\mathbb{R}^N} a(x) |\nabla u_n|^p \,\mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u_n|^q \,\mathrm{d}x \right. \\
&\quad \left. + \frac{1}{q} \int_{\mathbb{R}^N} V(x) |u_n|^q \,\mathrm{d}x - \int_{\mathbb{R}^N} K(x) F(u_n) \,\mathrm{d}x \right) \\
&\leq \lim_{n\to\infty} \frac{1}{\|u_n\|_X^p} \left(\frac{1}{q} \|u_n\|_X^q - \int_{\mathbb{R}^N} K(x) F(u_n) \,\mathrm{d}x \right) \\
&= \lim_{n\to\infty} \frac{1}{q \|u_n\|_X^{p-q}} - \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{K(x) F(u_n)}{\|u_n\|_X^p} \,\mathrm{d}x \\
&\leq -\lim_{n\to\infty} \int_{\Omega \cap \{|u_n| \geq \xi\}} \frac{K(x) F(u_n)}{|u_n|^p} |v_n|^p \,\mathrm{d}x \\
&\leq -\tau \lim_{n\to\infty} \int_{\Omega \cap \{|u_n| \geq \xi\}} K(x) |v_n|^p \,\mathrm{d}x,
\end{aligned}$$

where $\Omega = \{x \in \mathbb{R}^N : w(x) \neq 0\}$. By Fatou's Lemma

$$0 \leq -\tau \int_\Omega K(x) v^p \mathrm{d}x \quad \forall \tau > 0,$$

which is a contradiction.

If $v = 0$. For each $B > 0$, we have $\frac{B}{\|u_n\|_X} \in [0,1]$ for n large enough. Hence

$$\begin{aligned}
J(t_n u_n) &\geq J\left(\frac{B}{\|u_n\|_X} u_n \right) = J(B v_n) \\
&= \frac{B^p}{p} \int_{\mathbb{R}^N} a(x) |\nabla v_n|^p \,\mathrm{d}x + \frac{B^q}{q} \int_{\mathbb{R}^N} |\nabla v_n|^q \,\mathrm{d}x \\
&\quad + \frac{B^q}{q} \int_{\mathbb{R}^N} V(x) |v_n|^q \,\mathrm{d}x - \int_{\mathbb{R}^N} K(x) F(B v_n) \,\mathrm{d}x \\
&\geq \frac{B^p}{p} \|v_n\|_X^q - \int_{\mathbb{R}^N} K(x) F(B v_n) \,\mathrm{d}x \\
&= \frac{B^p}{p} - \int_{\mathbb{R}^N} K(x) F(B v_n) \,\mathrm{d}x.
\end{aligned}$$

By Lemma 3.2

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(x) F(B w_n) \,\mathrm{d}x = 0, \tag{3.13}$$

from where it follows that

$$\liminf_{n\to\infty} J(t_n u_n) \geq \frac{B^q}{p} \quad \text{for every } B > 0,$$

which is a contradiction with the fact that $J(t_n u_n)$ is bounded from above.

Next, we will prove that the functional J satisfies the Cerami condition at the level c.

Lemma 3.4 *Let hypotheses* (f_1)-(f_3) *be satisfied. Then the functional* J *fulfills the Cerami condition.*

Proof. By Lemma 3.3, it follows that $\{u_n\}$ is bounded, up to some subsequence, we can assume that there is $u \in X$ such that

$$u_n \rightharpoonup u \text{ in } X.$$

Since $J'(u_n)u_n = o_n(1)$, we can deduce

$$\begin{aligned}&\int_{\mathbb{R}^N} a(x)|\nabla u_n|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u_n|^q \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|u_n|^q \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} K(x) f(u_n) u_n \mathrm{d}x,\end{aligned}$$

that is

$$\lim_{n\to+\infty} \|u_n\|_X^q = \lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x) f(u_n) u_n \mathrm{d}x, \tag{3.14}$$

By Lemma 3.2,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x) f(u_n) u_n \mathrm{d}x = \int_{\mathbb{R}^N} K(x) f(u) u \mathrm{d}x,$$

then,

$$\lim_{n\to+\infty} \|u_n\|_X^q = \int_{\mathbb{R}^N} K(x) f(u) u \mathrm{d}x.$$

Since $J'(u_n)u = o_n(1)$,

$$\|u\|_X^q = \int_{\mathbb{R}^N} K(x) f(u) u \mathrm{d}x. \tag{3.15}$$

Therefore, combining (3.14) and (3.15), we obtain

$$\lim_{n\to+\infty} \|u_n\|_X^q = \|u\|_X^q,$$

which implies that

$$u_n \to u \text{ in } X.$$

Consequently

$$J(u) = c \text{ and } J'(u) = 0,$$

Thus, the proof is completed.

To verify that J has the geometric structure required by the mountain pass theorem, we need to introduce the following results.

Lemma 3.5 *Let* f *satisfies* (f_1)-(f_3) *and* $(V,K) \in \mathcal{K}$. *The following properties are fulfilled for the functional* J:

- there exists $\alpha, \rho > 0$ such that $J(u) \geq \alpha$ for all $u \in X$ such that $\|u\|_X = \rho$.
- there exists $e \in X$ satisfying $J(e) < 0$ and $\|e\|_X > \rho$.

Proof. (i) If (K_2) holds, then $K(x) \leq CV(x)$, by (f_1), we have $|F(t)| \leq \frac{\varepsilon}{q}|t|^q$.

Then, for $0 < \varepsilon < \frac{q}{Cp}$, we can deduce

$$\begin{aligned}J(u)&=\frac{1}{p}\int_{\mathbb{R}^N}a(x)|\nabla u|^p\,\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\,\mathrm{d}x\\&\quad+\frac{1}{q}\int_{\mathbb{R}^N}V(x)|u|^q\,\mathrm{d}x-\int_{\mathbb{R}^N}K(x)F(u)\mathrm{d}x\\&\geq\frac{1}{p}\|u\|_X^q-\int_{\mathbb{R}^N}K(x)F(u)\mathrm{d}x\\&\geq\frac{1}{p}\|u\|_X^q-\frac{\varepsilon}{q}C\int_{\mathbb{R}^N}V(x)|u|^q\,\mathrm{d}x\\&\geq\frac{1}{p}\|u\|_X^q-\frac{\varepsilon}{q}C\|u\|_X^q\\&=\left(\frac{1}{p}-\frac{\varepsilon}{q}C\right)\|u\|_X^q>0,\end{aligned}$$

that is

$$J(u)\geq\left(\frac{1}{p}-\frac{\varepsilon}{q}C\right)\rho^q=:\alpha>0=J(0).$$

If (K_3) holds, then $K(x)\leq\varepsilon\left[V(x)\right]^{\frac{q^*-s}{q^*-q}}$. In view of ($f_1$), we obtain $|F(t)|\leq C_1|t|^p$. Therefore, for $0<\varepsilon<\frac{1}{pC_1}$, combined with Lemma 2.2 and (3.7), we have

$$\begin{aligned}J(u)&=\frac{1}{p}\int_{\mathbb{R}^N}a(x)|\nabla u|^p\,\mathrm{d}x+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\,\mathrm{d}x\\&\quad+\frac{1}{q}\int_{\mathbb{R}^N}V(x)|u|^q\,\mathrm{d}x-\int_{\mathbb{R}^N}K(x)F(u)\mathrm{d}x\\&\geq\frac{1}{p}\|u\|_X^q-C_1\int_{B_r(0)}K(x)|u|^p\,\mathrm{d}x-C_1\int_{B_r^c(0)}K(x)|u|^p\,\mathrm{d}x\\&\geq\frac{1}{p}\|u\|_X^q-\varepsilon C_1\|u\|_X^p-\varepsilon C_1\int_{B_r^c(0)}V(x)^{\frac{q^*-s}{q^*-q}}|u|^p\,\mathrm{d}x\\&\geq\frac{1}{p}\|u\|_X^q-\varepsilon C_1\|u\|_X^p-\varepsilon C_1\int_{B_r^c(0)}\left(V(x)|u|^q+|u|^{q^*}\right)\mathrm{d}x\\&\geq\frac{1}{p}\|u\|_X^q-\varepsilon C_1\|u\|_X^p-\varepsilon C_1\int_{\mathbb{R}^N}V(x)|u|^q\,\mathrm{d}x-\varepsilon C_1\int_{\mathbb{R}^N}|u|^{q^*}\,\mathrm{d}x\\&\geq\frac{1}{p}\|u\|_X^q-\varepsilon C_1\|u\|_X^p-\varepsilon C_1\|u\|_X^q-\varepsilon C_1\int_{\mathbb{R}^N}|u|^{q^*}\,\mathrm{d}x\\&=\left(\frac{1}{p}-\varepsilon C_1\right)\|u\|_X^q-\varepsilon C_1\|u\|_X^p-\varepsilon C_1\|u\|_X^{q^*},\end{aligned}$$

for $q<p<q^*$ and $\|u\|$ sufficiently small with $\|u\|<1$, then

$$J(u)\geq\left(\frac{1}{p}-\varepsilon C_1\right)\rho^q-\varepsilon C_1\rho^p-\varepsilon C_1\rho^{q^*}=:\alpha>0=J(0).$$

Thus there exists $\rho,\alpha>0$ such that $J(u)\geq\alpha>0$ for all $u\in X$ such that $\|u\|_X=\rho$.

(ii) It follows from (f_3) that, for any $M>0$, such that $F(x,tu)\geq Mt^p|u|^p$ for $t>0$. Then

$$\begin{aligned} J(tu) &= \frac{t^p}{p}\int_{\mathbb{R}^N} a(x)|\nabla u|^p \,\mathrm{d}x + \frac{t^q}{q}\int_{\mathbb{R}^N} |\nabla u|^q \,\mathrm{d}x \\ &\quad + \frac{t^q}{q}\int_{\mathbb{R}^N} V(x)|u|^q \,\mathrm{d}x - \int_{\mathbb{R}^N} K(x)F(tu)\,\mathrm{d}x \\ &\le \frac{t^p}{q}\|u\|_X^q - \int_{\mathbb{R}^N} K(x)F(tu)\,\mathrm{d}x \\ &\le \frac{t^p}{q}\|u\|_X^q - Mt^p\int_{\mathbb{R}^N} K(x)|u|^p \,\mathrm{d}x, \end{aligned}$$

since $q < p$, when $M > 0$ sufficiently large and $t \to +\infty$, we deduce $J(tu) \to -\infty$. Let $e = tu$, $\|e\| > \rho$, and $J(e) < 0$. Hence, there exists $e \in X$ such that $J(e) < 0$ and $\|e\|_X > \rho$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.3, any Cerami sequences of the functional J is bounded, by Lemma 3.4, J satisfies the Cerami condition, and by Lemma 3.5, J satisfies the mountain pass geometric structure. Since $J(u) \ge 0 = J(0)$ implies $u \ne 0$ and $u \ge 0$. Then the strong maximum principle for the double phase operator yields $u > 0$ in $\mathbb{R}^N$.

4. Existence of Infinitely Many Solutions

In this section, we aim to prove the existence of infinitely many solutions of problem (1.1). More precisely, by using the symmetric mountain pass theorem (see [26]), we obtain an unbounded sequence of critical values of the energy functional J. If, in addition, the corresponding critical values are negative, then the associated solutions have negative energy. This completes the proof of Theorem 1.2.

First, by the assumption $f(x,-t) = -f(x,t)$, which implies that the associated energy functional J is even, so that $J(-u) = J(u)$ for every $u \in X$. Furthermore, $J(0) = 0$ and $J \in C^1(X, \mathbb{R})$.

To prove Theorem 1.2, we will use the following symmetric mountain pass theorem of Rabinowitz [26], which we explicitly state below for reference.

Proposition 4.1 (*symmetric mountain pass theorem* [26]) *Let X be an infinite-dimensional real Banach space, $J \in C^1(X, \mathbb{R})$ is even, satisfying $(C)_c$ condition, and $J(0) = 0$. Suppose:*

1. There exist constants $\rho > 0$, $\alpha > 0$ such that $J(u) \ge \alpha$ for all $u \in X$ with $\|u\| = \rho$;

2. For every finite-dimensional subspace $\tilde{X} \subset X$, there exists $R = R(\tilde{X}) > 0$ such that $J(u) \le 0$ for all $u \in \tilde{X}$ with $\|u\| \ge R$.

Then I possesses an unbounded sequence of critical values.

First, we state the following lemma, which will be needed in the sequel.

Lemma 4.2 *Suppose that (f_1)-(f_3) and $(V,K) \in \mathcal{K}$ are satisfied. Then for any finite dimensional subspace $\tilde{X} \subset X$, we have*

$$J(u) \to -\infty \quad \text{as} \quad \|u\| \to \infty \quad \text{with} \quad u \in \tilde{X}.$$

Proof. Suppose that for some sequence $\{u_n\} \subset \tilde{X}$ with $\|u_n\|_X \to +\infty$, then exists $M > 0$ such that

$$J(u_n) \geq -M \quad \text{for all } n \in \mathbb{N} \tag{4.1}$$

Write $v_n = \dfrac{u_n}{\|u_n\|_X}$, then it is obvious that $\|v_n\| = 1$, we note that $\{v_n\}$ is bounded. Since $\tilde{X}$ is finite dimensional, up to a subsequence, we may assume that

$$\begin{cases} v_n \to v & \text{in } X, \\ v_n \to v & \text{a.e. on } \mathbb{R}^N, \end{cases} \tag{4.2}$$

which implies that $\|v\| = 1$ and so $v \neq 0$. It follows that there is $\varepsilon_0 > 0$, such that

$$\operatorname{meas}\left\{x \in \mathbb{R}^N : |v(x)| \geq \varepsilon_0\right\} \geq \varepsilon_0.$$

Set $A = \left\{x \in \mathbb{R}^N : |v(x)| \geq \varepsilon_0\right\}$. For a.e. $x \in A$, from (4.2) we can easily see that $\lim_{n\to\infty} |u_n(x)| = \infty$. For $0 \leq a < b$, let

$$\Omega_n(a,b) = \left\{x \in \mathbb{R}^N : a \leq |u_n(x)| < b\right\}.$$

A direct argument shows that $A \subset \Omega_n(r_0, \infty)$ for $n \in \mathbb{N}$ large enough. By (f_3), we have

$$\int_A \frac{F(x,u_n(x))}{\|u_n\|_X^p} = \int_A \frac{F(x,u_n(x))}{|u_n(x)|^p} |v_n(x)|^p \, \mathrm{d}x \to +\infty \quad \text{as } n \to \infty. \tag{4.3}$$

If (K_2) holds, by (f_1) and (f_2), for $q < p < q^*$, there exists $\varepsilon > 0$ small enough, and exists $C_\varepsilon > 0$, such that

$$|F(x,t)| \leq \frac{\varepsilon}{q}|t|^q + C_\varepsilon |t|^{q^*}. \tag{4.4}$$

It then follows from (4.2)-(4.4), and by the assumption (I) and Fatou's Lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{-M}{\|u_n\|_X^p} \leq \limsup_{n\to\infty} \frac{J(u_n)}{\|u_n\|_X^p} \\
&= \limsup_{n\to\infty} \left(\frac{1}{p} \int_{\mathbb{R}^N} \frac{a(x)|\nabla u_n|^p}{\|u_n\|_X^p} \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} \frac{|\nabla u_n|^q}{\|u_n\|_X^p} \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} \frac{V(x)|u_n|^q}{\|u_n\|_X^p} \mathrm{d}x \right) \\
&\quad - \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{K(x)F(u_n)}{\|u_n\|_X^p} \mathrm{d}x \\
&\leq \frac{1}{q\|u_n\|_X^{p-q}} - \limsup_{n\to\infty} \int_{\Omega_n(0,r_0)} \frac{K(x)F(u_n)}{|u_n|^p} |v_n|^p \, \mathrm{d}x \\
&\quad - \limsup_{n\to\infty} \int_{\Omega_n(r_0,\infty)} \frac{K(x)F(u_n)}{|u_n|^p} |v_n|^p \, \mathrm{d}x \\
&\leq C\left(\frac{\varepsilon}{q} + C_\varepsilon r_0^{q^*-q} \right) - \liminf_{n\to\infty} \int_A \frac{K(x)F(u_n)}{|u_n|^p} |v_n|^p \, \mathrm{d}x \\
&\leq C\left(\frac{\varepsilon}{q} + C_\varepsilon r_0^{q^*-q} \right) - \int_A \liminf_{n\to\infty} \frac{K(x)F(u_n)}{|u_n|^p} |v_n|^p \, \mathrm{d}x \\
&= -\infty.
\end{aligned}$$

If (K_3) holds, by (f_1) and (f_2), for $q < p < q^*$, there exists $C_1, C_\varepsilon > 0$, such that

$$\left|F(x,t)\right| \le C_1|t|^p + C_\varepsilon|t|^{q^*}. \tag{4.5}$$

By (4.2), (4.3), (4.5) and the assumption (I), combining (f_1) - (f_2) with Fatou's Lemma, we obtain

$$\begin{aligned}
0 &= \lim_{n\to\infty}\frac{-M}{\|u_n\|^p} \le \limsup_{n\to\infty}\frac{J(u_n)}{\|u_n\|^p}\\
&\le \frac{1}{q\|u_n\|_X^{p-q}} - \limsup_{n\to\infty}\int_{\Omega_n(0,r_0)}\frac{K(x)F(u_n)}{|u_n|^p}|v_n|^p\,\mathrm{d}x\\
&\quad - \limsup_{n\to\infty}\int_{\Omega_n(r_0,\infty)}\frac{K(x)F(u_n)}{|u_n|^p}|v_n|^p\,\mathrm{d}x\\
&\le C\left(C_1 + C_\varepsilon r_0^{q^*-p}\right) - \liminf_{n\to\infty}\int_A\frac{K(x)F(u_n)}{|u_n|^p}|v_n|^p\,\mathrm{d}x\\
&\le C\left(C_1 + C_\varepsilon r_0^{q^*-p}\right) - \int_A\liminf_{n\to\infty}\frac{K(x)F(u_n)}{|u_n|^p}|v_n|^p\,\mathrm{d}x\\
&= -\infty,
\end{aligned}$$

which implies that whether (K_2) or (K_3) holds, we all have a contradiction for ε sufficiently small. The proof of Lemma 4.2 is complete.

In the following, we will prove that the functional J has the geometric property, which is needed by Proposition 4.1.

Lemma 4.3 *Suppose that* (f_1)-(f_3) *and* $(V,K)\in\mathcal{K}$ *are satisfied. Then the following results are satisfied for the functional* J*:*

1. There exist constants $\rho,\alpha > 0$ such that $J(u)\ge\alpha$ for all $u\in X$ with $\|u\| = \rho$;

2. For every finite-dimensional subspace $\tilde{X}\subset X$, there exists $R = R(\tilde{X}) > 0$ such that $J(u)\le 0$ for all $u\in\tilde{X}$ with $\|u\|\ge R$.

Proof. (i) From the prove of Lemma 3.5 in Section 3, we know that J satisfies the geometric condition (i) of the mountain pass theorem.

(ii) For $u\in X$ with $\|u\| > 1$ sufficiently large. By (f_3), we obtain that there exists $T > 0$ such that $F(t)\ge M|t|^p$, for $|t| > T$ and $q < p < q^*$. Then we have

$$\begin{aligned}
J(u) &= \frac{1}{p}\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\,\mathrm{d}x\\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N} V(x)|u|^q\,\mathrm{d}x - \int_{\mathbb{R}^N} K(x)F(u)\,\mathrm{d}x\\
&\le \frac{1}{q}\|u\|_X^q - \int_{\mathbb{R}^N} K(x)F(u)\,\mathrm{d}x\\
&\le \frac{1}{q}\|u\|_X^q - M\int_{\mathbb{R}^N} K(x)|u|^p\,\mathrm{d}x
\end{aligned}$$

since $q < p$, when $M > 0$ sufficiently large, we deduce $J(u)\le 0$. Thus for every finite-dimensional subspace $\tilde{X}\subset X$, there exists $R = R(\tilde{X}) > 0$ such that $J(u)\le 0$ for all $u\in\tilde{X}$ with $\|u\|\ge R$.

Therefore, all the assumptions of Proposition 4.1 are satisfied, so that problem (1.1) admits infinitely many solutions with negative energy in X. This completes the proof.

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We know that J satisfies the Cerami condition from Lemma 3.4. Thus, it follows from Lemma 4.3 that all conditions of Proposition 4.1 are satisfied. Namely, problem (1.1) possesses an unbounded sequence of critical values and hence infinitely many nontrivial solutions in X. If the additional negativity condition on the critical values is imposed, these solutions have negative energy.

5. Conclusion

In this paper, we have studied a class of double phase elliptic problems with indefinite weight in $\mathbb{R}^N$. By combining weighted compact embedding results with variational methods, we first established the boundedness of Cerami sequences and verified the Cerami condition for the associated energy functional. Using the mountain pass theorem, we obtained the existence of at least one nontrivial solution. Furthermore, when the nonlinearity f is odd, we applied the symmetric mountain pass theorem and proved the existence of an unbounded sequence of critical values, which yields infinitely many nontrivial solutions. The main difficulty caused by the lack of compactness in $\mathbb{R}^N$ was overcome by Hardy-type inequalities and weighted embedding theorems. The results obtained here enrich the multiplicity theory for double phase problems with indefinite weights. It would be interesting to further study the regularity and asymptotic behavior of these solutions, as well as the case of variable exponents and more general double phase operators.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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