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Additive and Porous Media Physics in Fractional Spaces

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Abstract

This paper is the first paper in a series of papers devoted to developing continuum thermodynamics for additive and porous media. The porous material volume V in the physical coordinate space (x-frame) is mapped into a curvilinear coordinate space x^{β} with axes $x_1^{\beta_1}$, $x_2^{\beta_2}$ and $x_3^{\beta_3}$; $\beta_i \in (0,1]$, into a volume V_β such that volume V_β contains no pore space and is homogenized by dimensional regularization. Hence, in volume V_β we have isotropic homogeneous solid matter of the same physical properties as the solid matter in the porous volume V but without the pore space. The curvilinear coordinate space with axes $x_i^{\beta_i}$ (no sum) is called fractional curvilinear coordinate space. Fractional indices β_i are measures of porosity in three mutual directions. It is shown that when $\beta_i = 1$, volume V has no pore space; in this case, $V_\beta = V$. Progressively decreasing values of β_i define progressively increasing volume of pore space. Upon deformation $V \to \bar{V}$ and $V_\beta \to \bar{V}_\beta$, the solid matter in V_β and $\bar{V}_\beta$ remains isotropic and homogeneous. Thus, the conservation and balance laws and the constitutive theories for porous media in $V \to \bar{V}$ must be derived using volumes $V_\beta \to \bar{V}_\beta$ in which conservation and balance laws and the methodology of deriving constitutive theories for classical continuum mechanics hold, as the volumes V_β and $\bar{V}_\beta$ contain homogeneous and isotropic solid matter. There is another aspect of the deformation physics that exists in x- and $\bar{x}$-frames that must be preserved by the mapping of volume V in x-frame into the volume V_β in x^{β}-frame. Since volumes V_β and $\bar{V}_\beta$ are isotropic and homogeneous, the Cauchy stress tensors ${}^{\beta}\boldsymbol{\sigma}^{(0)}$ and ${}^{\beta}\bar{\boldsymbol{\sigma}}^{(0)}$ in x^{β}- and $\bar{x}^{\beta}$-frames are symmetric. When the Cauchy stress tensor ${}^{\beta}\boldsymbol{\sigma}^{(0)}$ is mapped from x^{β}-frame to $\boldsymbol{\sigma}^{(0)}$ in x-frame, we find that $\boldsymbol{\sigma}^{(0)}$ is not symmetric. The symmetric com-

ponents of $\boldsymbol{\sigma}^{(0)}$ are self-equilibrating, but the antisymmetric components generate moments in the volume $\bar{V}$ which must have a balancing mechanism for equilibrium in the volume V. Resistance provided by the anisotropy of porous media to the rotation field in volume V (or $\bar{V}$) generates additional moments that balance with the moments generated by the antisymmetric components of the Cauchy stress tensor. This physics is identical to micropolar physics in microcontinuum theories. Thus, when mapping $V \to V_\beta$, the pore space is eliminated, matter is homogenized by dimensional regularization; additionally, the micropolar physics in x- and $\bar{x}$-frames must also be mapped into volume V_β in fractional space x^β. Thus, we must consider solid matter in volume V_β to be micropolar, isotropic, and homogeneous. This is accomplished by assuming rigid microconstituents in volume V_β that can only experience rigid body rotations. A classical rotation field in volume V_β or $\bar{V}_\beta$ that is a free field in the absence of microconstituents becomes rigid body rotations of the microconstituents when they are present in the volume V_β. Interaction of the rigid body rotations of the microconstituents and the classical rotation field in volume V_β is assumed to generate the same moment field as exists in volume V (or $\bar{V}$) in x- or $\bar{x}$-frames. Henceforth we always refer to solid matter in volume V_β as micropolar solid matter. Thus, the Cauchy stress tensors ${}^\beta\boldsymbol{\sigma}^{(0)}$ and ${}^\beta\bar{\boldsymbol{\sigma}}^{(0)}$ in volumes V_β and $\bar{V}_\beta$ are no longer symmetric. Conservation and balance laws and the constitutive theories are derived in fractional spaces x^β and $\bar{x}^\beta$ using volumes V_β and $\bar{V}_\beta$ containing compressible micropolar solid matter, assuming finite deformation, finite strain physics (nonlinear) and assuming the material to be thermoelastic and thermoviscoelastic with dissipation, with and without rheology. This paper is the first paper in a series of papers to follow. In this paper, we present introductory material, new concepts, definitions, various measures, and kinematics that are essential for deriving conservation and balance laws in x^β- and $\bar{x}^\beta$-frames presented in the following paper. Mappings of points, lengths, areas, and volumes are presented between x- and x^β-frames. Using $V_\beta \to \bar{V}_\beta$ as undeformed and deformed configurations, details of the deformation gradient tensor, inverse deformation gradient, differential calculus, material derivative, covariant Lagrangian measures of Cauchy and Green's strain tensors, contravariant and covariant Cauchy stress and moment tensors, first and second Piola-Kirchhoff stress and moment tensors derived using contravariant Cauchy stress and moment tensors are presented. Mapping of stress tensors and moment tensors between x- and x^β-frames is also presented. This material is the essential basic material in deriving conservation and the balance laws in x^β- and $\bar{x}^\beta$-frames to be presented in a follow-up paper. In this paper, when needed and when possible, we compare our approach with the published works.

Keywords

Additive and Porous Media, Pore Space, Fractional Spaces, Anisotropic, Finite Deformation, Finite Strain, Nonlinear Deformation, Moment Tensors, Stress Tensors, Micropolar, Homogenization, Dimensional Regularization

1. Introduction and Scope of Work

1.1. Introduction

Additive and porous media contain solid matter with interspersed pores. The varying sizes of the pores and their varying density of distribution in the solid matter make such materials highly anisotropic. If the pore size is much smaller than the macro continuum scale, then the pores may be ignored. On the other hand, if the pore sizes approach the macro continuum scale, then in the development of the theory that describes deformation physics of such material, the varying pore sizes, their density and distribution in the solid matter play an important role.

There are many published works that describe how these complex pore spaces within the solid matter can be regularized in three mutually consistent directions so that the material may now be considered as having regularized but different anisotropies in these directions. This problem has been found in the published literature to be more tractable, but the approaches used vary drastically.

Another simple proposition is that we assume in each of three mutually perpendicular (or any other) directions there is a surrogate configuration of pores uniformly distributed in the volume and each pore is of the same size, and they collectively produce the same deformation physics and/or flow physics as the original configuration pore space does. With this assumption, the original porous material is characterized by one in which a characteristic pore length describes the level of anisotropy in each of the three directions. This idealized porous solid matter forms the basis of the foundational concepts (in reference [1]) and derivations presented here in this paper that will eventually help in establishing continuum thermodynamics for additive and porous media.

Since additively manufactured materials are inherently porous, in this paper it suffices to refer to both as porous media. In the following, we present a review of the published works, concentrating just on the fundamental concepts used rather than intricate details of the derivations. The motivation for doing so is to clearly show that our views and concepts presented in this paper on the approach towards establishing a continuum thermodynamic framework for porous media are different than those used currently in the published works, and the principles and the concepts considered in our approach are strictly supported by classical continuum thermodynamics, thus ensuring that the resulting new continuum thermodynamic framework is thermodynamically consistent.

At present, classical thermodynamics is the only valid thermodynamics we know for continuous matter. Thus, in deriving a new thermodynamic framework

for newer physics (like porous media), we may require modifications of the existing conservation and/or balance laws and may also require the introduction of new conservation and/or balance laws. We must make sure both of these are supported by classical thermodynamics; if not, then the resulting new thermodynamic framework is not thermodynamically valid based on classical thermodynamics.

Perhaps Mandelbrot [2] and Schroeder [3] were the first to introduce the concept of mass fractional dimension. Tarasov [4] explains how this concept is more practical and can be used to measure fractional dimension as opposed to fractal dimension associated with limits.

In a porous medium, properties like mass obey a power law relation. If $M(R)$ is the mass of the porous media and R is the box size and D_m is the fractional dimension, then

$$M(R) = kR^{D_m} \tag{1}$$

Taking the natural logarithm of (1), we get

$$\ln M(R) = D_m \ln R + \ln k \tag{2}$$

When measured data from the different-sized boxes R are plotted using $\ln M(R)$ versus $\ln(R)$, the slope of this line gives the value of D_m. Tarasov [4] showed that relation (1) can be derived using a fractional integral. Derivation of the fractional equation of continuity is also presented in [4]. In another paper [5], Tarasov presents elasticity of fractal material using a continuum model with non-integer dimensional space. This work also intrinsically uses the fractional integral definition of mass. In reference [6] the author considers fractional generalization of phase volume, volume element, and Poisson brackets. Tarasov [7] considers dynamical fractal solids using Euler's equations and integrals of fractional order. In reference [8] Tarasov uses fractional integrals for hydrodynamic equations of fractal media. Equations describing conservation of mass, balance of linear momentum, and the first law of thermodynamics are derived. The book by Tarasov [9], Fractional Dynamics, has a treatment of dynamics of fractal media using fractional calculus. All these works by Tarasov are based on fractional integral formulation. Works based on fractional integrals can also be found in references [10]-[12]. Nonlocal continuum models and fractional calculus have also been used in references [13] [14] for continuum mechanics. This approach, based on fractional integrals, is widely used and is the subject matter of many other papers.

The second approach that is commonly found is based on the method of dimensional regularization using fractional integrals and then converting the model to the x-frame (Cartesian). Ostoja-Starzewski *et al.* [15] present what they refer to as homogenization of fractal media via dimensional regularization. The authors point out drawbacks of the approaches based on fractional-order integrals used by Tarasov. Their starting point is the following representation of mass

$$m(L_1, L_2, L_3) \simeq L_1^{\alpha_1} L_2^{\alpha_2} L_3^{\alpha_3} \tag{3}$$

followed by a product measure. The authors [15] show that this approach and Tarasov's work result in the same homogenization of the porous media. This approach eliminates fractional integrals with gamma function used in Tarasov's approach. The authors introduce coefficients to scale length, followed by scaling of areas and volumes. The paper presents many details using this approach for deriving conservation and balance laws, treatment of elastic solids under finite strain, and many other topics. The approach described in reference [15] is subsequently used by Ostoja-Starzewski [16] for thermomechanics of fractal media, and by Ostoja-Starzewski *et al.* [17] for beams and fracture mechanics.

Li *et al.* present fractal solids, product measures and continuum mechanics in Chapter 33 of reference [18]. Ostoja-Starzewski in reference [19] presents continuum of fractal and porous media: integral relations and extremum principles and variational principles for elastic and inelastic media with fractal geometries; in reference [20] Li *et al.* discuss fractal solids, product measures and fractional wave equations; Ostoja-Starzewski in reference [21] discusses turbulence in porous media, and in reference [22], Joumaa *et al.* present wave propagation in isotropic fractal media; Balankin [23] [24] also considers fractal media, mechanics of continuum with fractal matrix. Nonsymmetry of the Cauchy stress tensor in the x-frame has been pointed out and used by Ostoja-Starzewski [25] to point out the need for micropolar physics in mechanics of porous media. We do not find any organized thermodynamic approach for addressing porous media physics in the published works.

1.2. Scope of Work

Additively manufactured and porous media are highly anisotropic due to varying pore sizes and their random distribution in the volume of matter. In additive materials, pore sizes and their distribution are not as random and can be controlled by the manufacturing methods employed. In the present work, the additively manufactured materials are treated in the same manner as porous media in general.

Consider a volume V of the porous media in the x-frame, original undeformed state of porous media. Let x^β-frame with axes $x_1^{\beta_1}$, $x_2^{\beta_2}$, $x_3^{\beta_3}$ be a curvilinear coordinate frame in which $(\beta_1,\beta_2,\beta_3)\in(0,1]$, *i.e.* β_i are fractions; a value of one is admissible for each of the three. We refer to x^β-space as fractional curvilinear coordinate space. Fractional indices β_i are chosen such that when volume V from x-frame is mapped into volume V_β in the x^β-frame, the pore space is completely eliminated, and the matter is homogenized through dimensional regularization. Thus, the volume V_β only has solid matter, homogeneous and isotropic. We use mapping $T:\boldsymbol{x}^\beta=\boldsymbol{x}^\beta(\boldsymbol{x})$ and its inverse $T^{-1}:\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{x}^\beta)$ that are completely general, thus permitting description of highly complex porous media for $V\to V_\beta$; that is, there are no restrictions in the mapping that restrict the presence of porosity in volume V in the x-frame. We show that the relationship between volumes V and V_β due to a general map-

ping T reduces to relationship (1) for a specific choice of mapping T, confirming that the general form of the mapping T used here is a more general case of homogenization by dimensional regularization.

When disturbance is applied to volume V, volume V deforms into $\bar{V}$, *i.e.* $V \to \bar{V}$, likewise volume V_β deforms into $\bar{V}_\beta$, *i.e.* $V_\beta \to \bar{V}_\beta$. In volumes V and $\bar{V}$, the matter is porous, whereas V_β and $\bar{V}_\beta$ contain undeformed and deformed isotropic, homogeneous solid matter without any pore space. Thus, the conservation and balance laws of classical thermodynamics can be applied to volumes $\bar{V}_\beta$ and V_β to derive conservation and balance laws in the $\bar{x}^\beta$-frame (Eulerian description) and x^β-frame (Lagrangian description). The same holds for constitutive theories.

When the symmetric Cauchy stress tensor ${}^\beta\boldsymbol{\sigma}^{(0)}$ in volume V_β in x^β-frame is transformed into Cauchy stress tensor $\boldsymbol{\sigma}^{(0)}$ in the undeformed volume V in x-frame we find that $\boldsymbol{\sigma}^{(0)}$ is not symmetric. The symmetric part of the Cauchy stress tensor $\boldsymbol{\sigma}^{(0)}$ is self-equilibrating, while the antisymmetric components of $\boldsymbol{\sigma}^{(0)}$ generate moments within the deformed volume in the x-frame that must have a mechanism of equilibrating with moments due to some other mechanism so that the deforming volume in the x-frame is in thermodynamic equilibrium. The anisotropy of the porous matter provides resistance to the rotation field in the deforming volume $\bar{V}$, and a consequent moment field is generated in the deformed volume. This moment field is balanced by the one created by the antisymmetric components of $\boldsymbol{\sigma}^{(0)}$, hence preserving thermodynamic equilibrium. The physics of the moment field due to anisotropy of the porous media is exactly parallel to the micropolar physics in microcontinuum theories in which rigid body rotations of the rigid microconstituents interacting with the classical rotation field generate a moment field within the deforming volume. When volume V is mapped into V_β, this physics in volume V in x-frame must also be mapped into the volume V_β in the x^β-frame. This can be accomplished by assuming that the solid matter in volume V_β is micropolar, isotropic, and homogeneous containing rigid microconstituents that only experience rigid body rotations. The classical rotation field in V_β is a free field in the absence of microconstituents. The presence of microconstituents interferes with the free field; as a consequence, the classical rotation field becomes the rigid rotations of the microconstituents [1]. Micropolar physics due to the moment field in volume V_β generated due to microconstituents and the classical rotation field is the map of the micropolar physics in the porous volume in the x-frame. Henceforth we refer to the solid matter in volume V_β as micropolar medium; therefore, the Cauchy stress tensor ${}^\beta\boldsymbol{\sigma}^{(0)}$ in volume V_β is not symmetric any more.

For the mathematical models in x^β- and $\bar{x}^\beta$-frames to be useful in actual applications, some quantities need to be mapped in $\bar{x}$- or x-frames, as our experiments, measurements, and numerical simulations are all in $\bar{x}$- and x-frames. First, we note that the laws of thermodynamics are not applicable to volumes V and $\bar{V}$ in x- and $\bar{x}$-frames. But, there are a few facts that help us in arriving

at a workable correspondence between the x^β - and x -frames.

The volume of a material point in V_β in the x^β -frame has a map to the corresponding volume V in the x -frame. Both have a center of mass c_β and c in x^β - and x -frames. When the matter is disturbed, both c_β and c have translations (displacements). Between $\bar{x}^\beta$ - and x^β -frames, the definition of the deformation gradient tensor $\boldsymbol{J}_\beta$ exists, but the deformation gradient tensor $\boldsymbol{J}$ between $\bar{x}$ - and x -frames does not exist (as it requires continuum mechanics); however, the gradients of u_i with respect to x_j do exist at the center c in volume V in the x -frame, as these are simply rates of change in the spatial directions (simple calculus). Thus, components of ${}^d\boldsymbol{J}$ exist, but ${}^d\boldsymbol{J}$ does not. Also, time derivatives of u_i exist at point c, as these are simply rates of change with respect to time. In the strain measures in x^β -frame, we need components of ${}^d\boldsymbol{J}$ when transforming them to x -frame, we take the liberty to use ${}^d\boldsymbol{J}$ for these components for convenience of notation, but keeping in mind that the use of ${}^d\boldsymbol{J}$ only implies that we know its components realizing full well that it cannot be defined using deformation gradient tensor $\boldsymbol{J}$ between $\bar{x}$ - and x -frames as the laws of classical thermodynamics do not hold in $\bar{x}$ - and x -frames.

The concepts described above toward continuum thermodynamics for porous media are organized in a series of four additional papers. In this paper we present the mapping of the x -frame $\to x^\beta$ -frame and its inverse. We consider the mapping of points, lengths, areas, and volumes between the two frames. This is followed by the kinematics of motion in x^β -frame and likewise in the $\bar{x}^\beta$ -frame; displacement; deformation gradient tensor; inverse deformation gradient tensor; mapping of points, lengths, areas and volumes between x^β - and $\bar{x}^\beta$ -frames; covariant and contravariant basis in $\bar{x}^\beta$ -frame; some useful differential relations in x^β - and $\bar{x}^\beta$ -frames and their map in x -frame; gradient of a scalar; divergence of a tensor of rank one; gradient of a tensor of rank one; divergence of a tensor of rank two; Laplacian of a scalar field and material derivative. This is followed by the derivation of nonlinear strain measures, Cauchy strain tensor ${}^\beta\boldsymbol{C}_{[0]}$ and Green's strain tensor ${}^\beta\boldsymbol{\varepsilon}_{[0]}$ (covariant measures) in the Lagrangian description. Derivation of Eulerian strain measure such as Cauchy ${}^\beta\bar{\boldsymbol{C}}^{[0]}$ and Almansi strain ${}^\beta\bar{\boldsymbol{\varepsilon}}^{[0]}$ follows a similar procedure. Derivation of the contravariant Cauchy stress tensor ${}^\beta\boldsymbol{\sigma}^{(0)}$ in x^β -frame and the moment tensor ${}^\beta\boldsymbol{m}^{(0)}$ is presented followed by first and second Piola-Kirchhoff stress and moment tensors in x^β -frame based on contravariant Cauchy stress and moment tensors ${}^\beta\boldsymbol{\sigma}^{(0)}$ and ${}^\beta\boldsymbol{m}^{(0)}$. This is basic material needed for deriving conservation and the balance laws in Eulerian and Lagrangian descriptions in $\bar{x}^\beta$ - and x^β -frames. These will be presented in forthcoming follow-up papers.

2. General Considerations

We consider regularized porous media by using surrogate pore lengths in three mutually perpendicular or any other convenient directions, thus defining three characteristic pore lengths in three directions. Following Tarasov, let us consider

the mass of solid matter in a volume that has anisotropies in three mutual directions. If our objective is to derive conservation and the balance laws for a porous medium that is anisotropic, then we must show how to map this matter into a homogeneous isotropic medium for which we can use classical thermodynamics. In simplistic terms, there are two basic approaches possible: in the first approach, keep the volume the same as V in x-space but change the density such that the solid matter fills the volume and the pore space is eliminated. In the second approach, we keep the density the same as the density of the solid matter but change the volume. In both cases, the end result is that the pores are eliminated. Additionally, we must ensure that in both approaches the matter is homogenized, hence remains isotropic and homogeneous. The first approach requires density change while the volume remains unchanged. In the second approach, preserving the density of solid matter in porous media requires that the volume must change. This little illustration shows that the problem is not mass or density; it is the pore space that must be eliminated by transforming the matter from x-space to another space while preserving isotropic, homogeneous nature of matter through intrinsic homogenization in the mapping. If we are studying transient heat conduction in a porous medium, then in the case of the first approach, keeping the volume the same we must now determine surrogate specific heat and thermal conductivity. Use of a fractional index for mass is obviously not appropriate in this case. This suggests that the second approach, in which the solid material properties are preserved and the volume is changed to achieve pore space free isotropic homogeneous solid matter through homogenization by dimensional regularization, is preferable.

Based on surrogate porosity lengths in three mutual directions, we have three fractional indices that must define the new volume in which the original properties of the solid matter are preserved without pore space.

We have pointed out in the Abstract and the Introduction that the porous medium in x- and $\bar{x}$-frames has micropolar physics which must also be mapped in the x^β-frame in volume V_β. This is done by assuming that the solid matter in volume V_β is micropolar, *i.e.* it is isotropic and homogeneous but contains rigid microconstituents that can only experience rigid body rotations. Interaction of the rigid body rotations of the microconstituents and the classical rotation field (a free field in the absence of microconstituents) provides a map of the micropolar physics in x-, $\bar{x}$-frames in volumes V and $\bar{V}$ into the volumes V_β and $\bar{V}_\beta$ in x^β- and $\bar{x}^\beta$-frames. Thus, the volume V_β certainly contains isotropic and homogeneous solid matter without void space, but this solid matter is micropolar. Thus the continuum thermodynamics of porous media requires derivation of conservation and balance laws and the constitutive theories for micropolar solid media in curvilinear fractional spaces x^β and $\bar{x}^\beta$. In the remainder of this paper, it is always understood that we are considering micropolar theory in curvilinear fractional spaces x^β and $\bar{x}^\beta$.

3. Fractional Spaces

Let x-space $Ox_1x_2x_3$ be a Cartesian coordinate space. Let β_1, β_2 and β_3 be

fractional indices describing porosity in x_i -directions. We consider a nonorthogonal curvilinear coordinate space with axes $x_1^{\beta_1}$, $x_2^{\beta_2}$ and $x_3^{\beta_3}$ and β_1, β_2 and $\beta_3 \in (0,1]$. This curvilinear coordinate space $x_i^{\beta_i}$ -frame (no sum) or x^β -frame will be referred to as curvilinear fractional space or fractional space. The surrogate porosities define β_1, β_2, β_3, and the volume V transformed from the x -frame to V_β in the x^β -frame eliminates the pore space and preserves the original properties of the solid matter in the porous media. Consideration of micropolar physics in x^β -space provides a map of micropolar physics present in volumes V and $\bar{V}$ in x - and $\bar{x}$ -spaces. Conservation and the balance laws of classical continuum thermodynamics can be applied to the micropolar volume V_β in the x^β -frame, as the matter is isotropic and homogeneous in volume V_β, to derive details of the mathematical model in the x^β -frame suitable for porous media in the x -frame.

4. Mapping of Points, Lengths, Areas and Volumes in x - and x^β -Spaces

Let the mapping T be such that

$$T: x_i^{\beta_i} = x_i^{\beta_i}\left(x_1, x_2, x_3\right) \ (\text{no sum}) \tag{4}$$

then

$$T^{-1}: x_i = x_i\left(x_1^{\beta_1}, x_2^{\beta_2}, x_3^{\beta_3}\right) \tag{5}$$

The mapping T maps x_i from the x -frame to the corresponding points in the x^β -frame. T^{-1} is the inverse of the mapping T that maps points from x^β -frame to x -frame. This is a mapping of points between the cartesian and curvilinear fractional frames.

4.1. Mapping of Lengths

Consider material points $P\left(x_i\right)$ and $Q\left(x_i + \mathrm{d}x_i\right)$ in the x -frame. These map into material points $P^\beta\left(x_i^\beta\right)$ and $Q^\beta\left(x_i^\beta + \mathrm{d}x_i^\beta\right)$ in x^β -frame. Lengths $\boldsymbol{PQ}$ and $\boldsymbol{P}^\beta\boldsymbol{Q}^\beta$ are not the same, and their directions are not the same either. Using the mapping given by T in (4) we can write

$$\boldsymbol{P}^\beta\boldsymbol{Q}^\beta = \left(x_i^\beta\left(x_1 + \mathrm{d}x_1, x_2 + \mathrm{d}x_2, x_3 + \mathrm{d}x_3\right) - x_i^\beta\left(x_1, x_2, x_3\right)\right)\boldsymbol{e}_i^\beta \tag{6}$$

Using the Taylor series expansion of the first term on the right-hand side of (6) and retaining only up to linear terms in $\mathrm{d}x_i$, we obtain

$$\boldsymbol{P}^\beta\boldsymbol{Q}^\beta = \left(x_i^\beta\left(x_1, x_2, x_3\right) + \frac{\partial x_i^\beta}{\partial x_j}\mathrm{d}x_j + \mathcal{O}\left(\left\|\mathbf{d}\boldsymbol{x}\right\|\right)^2 - x_i^\beta\left(x_1, x_2, x_3\right)\right)\boldsymbol{e}_i^\beta = \boldsymbol{e}_i^\beta \mathrm{d}x_i^\beta \tag{7}$$

Neglecting $\mathcal{O}\left(\left\|\mathrm{d}\boldsymbol{x}\right\|\right)^2$ term and equating coefficient of $\boldsymbol{e}_i^\beta$ in both sides of (7) we obtain

$$\mathrm{d}x_i^\beta = \frac{\partial x_i^\beta}{\partial x_j}\mathrm{d}x_j \tag{8}$$

or

$$\left\{\mathrm{d}x^{\beta}\right\}=\left[J_{x\beta}\right]\left\{\mathrm{d}x\right\} \tag{9}$$

Equation (9) maps elemental lengths $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$ from x-frame to elemental lengths $\mathrm{d}x_1^{\beta_1}$, $\mathrm{d}x_2^{\beta_2}$, $\mathrm{d}x_3^{\beta_3}$ in x^{β}-frame. Likewise, if we consider

$$\boldsymbol{PQ}=\left(x_i\left(x_1^{\beta_1}+\mathrm{d}x_1^{\beta_1},x_2^{\beta_2}+\mathrm{d}x_2^{\beta_2},x_3^{\beta_3}+\mathrm{d}x_3^{\beta_3}\right)-x_i\left(x_1^{\beta_1},x_2^{\beta_2},x_3^{\beta_3}\right)\right)\boldsymbol{e}_i \tag{10}$$

Then, following the same procedure, we can obtain

$$\mathrm{d}x_i=\frac{\partial x_i}{\partial x_j^{\beta}}\mathrm{d}x_j^{\beta} \tag{11}$$

or

$$\left\{\mathrm{d}x\right\}=\left[J_{\beta x}\right]\left\{\mathrm{d}x^{\beta}\right\} \tag{12}$$

Equation (12) maps $\left\{\mathrm{d}x^{\beta}\right\}$ from x^{β}-frame to $\left\{\mathrm{d}x\right\}$ in x-frame. From (9) and (12), we note that

$$\left[J_{x\beta}\right]=\left[J_{\beta x}\right]^{-1}\ \text{and}\ \left[J_{\beta x}\right]=\left[J_{x\beta}\right]^{-1} \tag{13}$$

Also

$$\left[J_{x\beta}\right]=\left[\left\{{}^{x\beta}\tilde{g}_1\right\},\left\{{}^{x\beta}\tilde{g}_2\right\},\left\{{}^{x\beta}\tilde{g}_3\right\}\right]$$

$$\left[J_{\beta x}\right]=\begin{bmatrix}\left\{{}^{\beta x}\tilde{g}^1\right\}^{\mathrm{T}}\\ \left\{{}^{\beta x}\tilde{g}^2\right\}^{\mathrm{T}}\\ \left\{{}^{\beta x}\tilde{g}^3\right\}^{\mathrm{T}}\end{bmatrix} \tag{14}$$

$\left\{{}^{x\beta}\tilde{g}_i\right\}$ and $\left\{{}^{\beta x}\tilde{g}^i\right\}$ are covariant and contravariant vectors. $\left\{{}^{x\beta}\tilde{g}_i\right\}$ are tangent to $x_1^{\beta_1}$, $x_2^{\beta_2}$ and $x_3^{\beta_3}$ at a material point. $\left\{{}^{x\beta}\tilde{g}_i\right\}$ and $\left\{{}^{\beta x}\tilde{g}^i\right\}$ form nonorthogonal reciprocal basis in x^{β}-frame.

4.2. Mapping of Areas

If $\boldsymbol{PQ}$, $\boldsymbol{PR}$ and $\boldsymbol{PS}$ are originating from a material point P, these are mapped into $\boldsymbol{P}^{\beta}\boldsymbol{Q}^{\beta}$, $\boldsymbol{P}^{\beta}\boldsymbol{R}^{\beta}$ and $\boldsymbol{P}^{\beta}\boldsymbol{S}^{\beta}$ in x^{β}-frame. Area formed by $\boldsymbol{P}^{\beta}\boldsymbol{Q}^{\beta}$, $\boldsymbol{P}^{\beta}\boldsymbol{R}^{\beta}$ is $\mathrm{d}\boldsymbol{A}^{\beta}$. Likewise the area formed by $\boldsymbol{PQ}$, $\boldsymbol{PR}$ is $\mathrm{d}\boldsymbol{A}$.

$$\mathrm{d}\boldsymbol{A}^{\beta}=\boldsymbol{P}^{\beta}\boldsymbol{Q}^{\beta}\times\boldsymbol{P}^{\beta}\boldsymbol{R}^{\beta}=\boldsymbol{e}_l^{\beta}\left(P^{\beta}Q^{\beta}\right)_l\times\boldsymbol{e}_m^{\beta}\left(P^{\beta}R^{\beta}\right)_m \tag{15}$$

or

$$\mathrm{d}\boldsymbol{A}^{\beta}=\boldsymbol{e}_n^{\beta}\epsilon_{lmn}\left(P^{\beta}Q^{\beta}\right)_l\left(P^{\beta}R^{\beta}\right)_m \tag{16}$$

$$\mathrm{d}A_n^{\beta}=\epsilon_{lmn}\left(P^{\beta}Q^{\beta}\right)_l\left(P^{\beta}R^{\beta}\right)_m \tag{17}$$

Similarly

$$\mathrm{d}A_k=\epsilon_{ijk}\left(PQ\right)_i\left(PR\right)_j \tag{18}$$

using

$$\left(P^{\beta}Q^{\beta}\right)_l = \left(J_{x\beta}\right)_{lp}\left(PQ\right)_p$$
$$\left(P^{\beta}R^{\beta}\right)_m = \left(J_{x\beta}\right)_{mq}\left(PR\right)_q \tag{19}$$

in (15) and following details in reference [26] we can derive the following

$$\left\{\mathrm{d}A^{\beta}\right\} = \det\left[J_{x\beta}\right]\left[\left[J_{x\beta}\right]^{-1}\right]^{\mathrm{T}}\left\{\mathrm{d}A\right\} \tag{20}$$

4.3. Mapping of Volumes

$$\mathrm{d}V_{\beta} = \left(\boldsymbol{P}^{\beta}\boldsymbol{Q}^{\beta}\times\boldsymbol{P}^{\beta}\boldsymbol{R}^{\beta}\right)\cdot\boldsymbol{P}^{\beta}\boldsymbol{S}^{\beta} = \epsilon_{lmn}\left(P^{\beta}Q^{\beta}\right)_l\left(P^{\beta}R^{\beta}\right)_m\left(P^{\beta}S^{\beta}\right)_n \tag{21}$$

$$\mathrm{d}V = \left(\boldsymbol{PQ}\times\boldsymbol{PR}\right)\cdot\boldsymbol{PS} = \epsilon_{ijk}\left(PQ\right)_i\left(PR\right)_j\left(PS\right)_k \tag{22}$$

using

$$\left(P^{\beta}Q^{\beta}\right)_l = \left(J_{x\beta}\right)_{lp}\left(PQ\right)_p$$
$$\left(P^{\beta}R^{\beta}\right)_m = \left(J_{x\beta}\right)_{mq}\left(PR\right)_q$$
$$\left(P^{\beta}S^{\beta}\right)_n = \left(J_{x\beta}\right)_{nr}\left(PS\right)_r \tag{23}$$

Using (23) in (21) and then using (22) (see [26]) we can obtain

$$\mathrm{d}V_{\beta} = \det\left[J_{x\beta}\right]\mathrm{d}V \tag{24}$$

This is a fundamental relationship that shows $\mathrm{d}V$ in x-space is reduced to volume $\mathrm{d}V_{\beta}$ in x^{β}-space such that the pore space is completely eliminated in the volume $\mathrm{d}V_{\beta}$. That is β_1, β_2, β_3 in $\left[J_{x\beta}\right]$ are such that $\mathrm{d}V_{\beta}$ has no pore space.

Remarks

1) Mapping T in (4) and its inverse in (5) is the most general case for which $\left[J_{x\beta}\right]$ in (24) is a fully populated matrix. The explicit form of the mapping is necessary to further study (24).

2) β_1, β_2, β_3 are fractions associated with the porosities that have been used to define the curvilinear fractional space x^{β}.

3) We consider a simpler case of mapping T in order to demonstrate that (24) indeed describes the porosity physics correctly. Let

$$T:\begin{cases} x_1 = x_1\left(x_1^{\beta_1}\right) \\ x_2 = x_2\left(x_2^{\beta_2}\right) \\ x_3 = x_3\left(x_3^{\beta_3}\right) \end{cases} \tag{25}$$

$$T^{-1}:\begin{cases} x_1^{\beta_1} = x_1^{\beta_1}\left(x_1\right) \\ x_2^{\beta_2} = x_2^{\beta_2}\left(x_2\right) \\ x_3^{\beta_3} = x_3^{\beta_3}\left(x_3\right) \end{cases} \tag{26}$$

Mappings (25) and (26) imply that porosity in one direction does not influence the porosity in the other two directions (could be orthogonal). That is, porosity in each direction is isotropic and is defined by β_i only. Using mapping T in (25), $\left[J_{x\beta}\right]$ becomes a diagonal matrix

$$\left[J_{x\beta}\right]=\begin{bmatrix}\dfrac{\partial x_1^{\beta_1}}{\partial x_1} & 0 & 0\\ 0 & \dfrac{\partial x_2^{\beta_2}}{\partial x_2} & 0\\ 0 & 0 & \dfrac{\partial x_3^{\beta_3}}{\partial x_3}\end{bmatrix}=\begin{bmatrix}\beta_1 x_1^{\beta_1-1} & 0 & 0\\ 0 & \beta_2 x_2^{\beta_2-1} & 0\\ 0 & 0 & \beta_3 x_3^{\beta_3-1}\end{bmatrix} \tag{27}$$

$$\mathrm{d}V_\beta=\left(\beta_1\beta_2\beta_3 x_1^{\beta_1-1}x_2^{\beta_2-1}x_3^{\beta_3-1}\right)\mathrm{d}V \tag{28}$$

Based on the isotropic surrogate pore space assumption in each of the three directions, (28) is a fundamental relationship. Consider $x_1=x_2=x_3=l$ (a cube with edge length of l), then (28) reduces to

$$\mathrm{d}V_\beta=\left(\beta_1\beta_2\beta_3 l^{\beta_1+\beta_2+\beta_3-3}\right)\mathrm{d}V \tag{29}$$

a) $\beta_1=\beta_2=\beta_3=1$ implies that $\mathrm{d}V_\beta=\mathrm{d}V$, *i.e.* $\mathrm{d}V$ has no pore space, *i.e.*, 1 is the right limit point of the interval for each β_i, and each β_i must be greater than zero. This establishes that

$$\beta_i\in(0,1] \tag{30}$$

b) Progressively decreasing values of β_i from one imply progressively increasing pore space.

c) If we choose

$$\beta_1=\beta_2=\beta_3=\beta \tag{31}$$

implying completely isotropic pore space in all three directions in the entire volume of matter, then (29) reduces to

$$\mathrm{d}V_\beta=\beta^3 l^{3\beta-3}\mathrm{d}V \tag{32}$$

We note:

1) when $\beta=1$, the volume $\mathrm{d}V$ has no pore space, hence in this case (32) yields $\mathrm{d}V_\beta=\mathrm{d}V$.

2) when $0<\beta<1$, $\mathrm{d}V_\beta<\mathrm{d}V$ and a progressively decreasing value of β defines porous media with progressively increasing pore space.

Thus, with the same assumptions as in references [4]-[9] [15], we are able to show that our approach presented here yields the same results as in references [4]-[9] [15], confirming that for this special case of mapping T, our approach results in the same homogenization by dimensional regularization in volume V_β as in references [4]-[9] [15].

In the remainder of the paper we go back to T and T^{-1} in (4) and (5) that yield fully populated $\left[J_{x\beta}\right]$ matrix. Simplification in (31) yielding (32) has been used as an illustration to show that the general mapping T and its inverse T^{-1} for a specific choice considered here yield the same results as in published works. Henceforth, we assume that homogenization by dimensional regularization always exists in the proposed mapping of $V\to V_\beta$.

5. Kinematics of Motion, Deformation and Their Measures

A volume of matter V with material points x_i deforms into a volume $\bar{V}$ with material points at $\bar{x}_i$. Likewise V_β deforms into $\bar{V}_\beta$. Since V_β and $\bar{V}_\beta$ contain only isotropic homogeneous solid matter without pore space, the conservation and balance laws must be derived using volumes V_β and $\bar{V}_\beta$ in x^β- and $\bar{x}^\beta$-spaces, considering micropolar physics. Upon deformation, we have

$$\bar{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}, t) \tag{33}$$

and

$$\bar{\boldsymbol{x}}^\beta = \boldsymbol{x}^\beta + \boldsymbol{u}(\boldsymbol{x}^\beta, t) \tag{34}$$

The material points in x-frame and x^β-frame experience the same displacement $\boldsymbol{u}$, the only difference is that in x-frame (Equation (33)) displacement $\boldsymbol{u}$ is a function of $\boldsymbol{x}$ and t whereas in x^β-frame displacement $\boldsymbol{u}$ is a function of $\boldsymbol{x}^\beta$ and t.

5.1. Deformation Gradient Tensors in x^β- and $\bar{x}^\beta$-Frames

Material points in x^β-space map into $\bar{x}^\beta$-space after deformation. Let $P^\beta(\boldsymbol{x}^\beta)$ and $Q^\beta(\boldsymbol{x}^\beta + \mathrm{d}\boldsymbol{x}^\beta)$ be two material points constituting a line segment $\boldsymbol{P}^\beta\boldsymbol{Q}^\beta$. Upon deformation these occupy $\bar{P}^\beta(\bar{\boldsymbol{x}}^\beta, t)$ and $\bar{Q}^\beta(\bar{\boldsymbol{x}}^\beta + \mathrm{d}\bar{\boldsymbol{x}}^\beta, t)$ positions in $\bar{x}^\beta$-space. Let the motion be given by

$$\bar{\boldsymbol{x}}^\beta = \bar{\boldsymbol{x}}^\beta(\boldsymbol{x}^\beta, t) \tag{35}$$

we assume that $\bar{\boldsymbol{x}}^\beta$ are continuous and differentiable in their arguments, then

$$\bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{Q}}^\beta = \left(\bar{x}_i^\beta\left(x_1^\beta + \mathrm{d}x_1^\beta, x_2^\beta + \mathrm{d}x_2^\beta, x_3^\beta + \mathrm{d}x_3^\beta, t\right) - \bar{x}_i^\beta\left(x_1^\beta, x_2^\beta, x_3^\beta, t\right)\right)\boldsymbol{e}_i^\beta \tag{36}$$

Expanding the first term on the right-hand side of (36) in Taylor series about x^β,

$$\begin{aligned}\bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{Q}}^\beta &= \left(\bar{x}_i^\beta\left(x_1^\beta, x_2^\beta, x_3^\beta, t\right) + \frac{\partial\bar{x}_i^\beta}{\partial x_j^\beta}\mathrm{d}x_j^\beta + \mathcal{O}\left(\left\|\mathrm{d}\boldsymbol{x}^\beta\right\|\right)^2 - \bar{x}_i^\beta\left(x_1^\beta, x_2^\beta, x_3^\beta, t\right)\right)\boldsymbol{e}_i^\beta \\ &= \boldsymbol{e}_i^\beta\mathrm{d}\bar{x}_i^\beta\end{aligned} \tag{37}$$

Neglecting $\mathcal{O}\left(\left\|\mathrm{d}\boldsymbol{x}^\beta\right\|\right)^2$ term, we obtain

$$\mathrm{d}\bar{x}_i^\beta = \frac{\partial\bar{x}_i^\beta}{\partial x_j^\beta}\mathrm{d}x_j^\beta \tag{38}$$

In Murnaghan's notation, we can write (38) as

$$\left\{\mathrm{d}\bar{x}^\beta\right\} = \left[J_\beta\right]\left\{\mathrm{d}x^\beta\right\};\ \left[J_\beta\right] = \left[\frac{\partial\left\{\bar{x}^\beta\right\}}{\partial\left\{x^\beta\right\}}\right] = \left[\frac{\bar{x}_1^{\beta_1}\bar{x}_2^{\beta_2}\bar{x}_3^{\beta_3}}{x_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3}}\right] \tag{39}$$

On the other hand, if we begin with

$$\boldsymbol{P}^\beta\boldsymbol{Q}^\beta = \left(x_i^\beta\left(\bar{x}_1^{\beta_1} + \mathrm{d}\bar{x}_1^{\beta_1}, \bar{x}_2^{\beta_2} + \mathrm{d}\bar{x}_2^{\beta_2}, \bar{x}_3^{\beta_3} + \mathrm{d}\bar{x}_3^{\beta_3}, t\right) - x_i^\beta\left(\bar{x}_1^{\beta_1}, \bar{x}_2^{\beta_2}, \bar{x}_3^{\beta_3}, t\right)\right)\boldsymbol{e}_i^\beta \tag{40}$$

Then, following the same procedure, we can obtain

$$\mathrm{d}x_i^{\beta} = \frac{\partial x_i^{\beta}}{\partial \bar{x}_j^{\beta}} \mathrm{d}\bar{x}_j^{\beta} \tag{41}$$

or

$$\left\{\mathrm{d}x^{\beta}\right\} = \left[\bar{J}_{\beta}\right]\left\{\mathrm{d}\bar{x}^{\beta}\right\};\ \left[\bar{J}_{\beta}\right] = \left[\frac{\partial\left\{x^{\beta}\right\}}{\partial\left\{\bar{x}^{\beta}\right\}}\right] = \left[\frac{x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}}{\bar{x}_1^{\beta_1} \bar{x}_2^{\beta_2} \bar{x}_3^{\beta_3}}\right] \tag{42}$$

$\left[J_{\beta}\right]$ is the deformation gradient tensor in x^{β}-frame and $\left[\bar{J}_{\beta}\right]$ is the inverse deformation gradient tensor in $\bar{x}^{\beta}$-frame.

$$\left[J_{\beta}\right] = \left[\bar{J}_{\beta}\right]^{-1};\ \left[\bar{J}_{\beta}\right] = \left[J_{\beta}\right]^{-1} \tag{43}$$

Thus

$$\left[J_{\beta}\right]\left[\bar{J}_{\beta}\right] = \left[\bar{J}_{\beta}\right]\left[J_{\beta}\right] = [I] \tag{44}$$

Columns of $\left[J_{\beta}\right]$ contain covariant base vectors ${}^{\beta}\tilde{\boldsymbol{g}}_i$, tangent to x^{β}-axes at a material point. Rows of $\left[\bar{J}_{\beta}\right]$ contain contravariant base vectors ${}^{\beta}\tilde{\boldsymbol{g}}^i$. ${}^{\beta}\tilde{\boldsymbol{g}}_i$ and ${}^{\beta}\tilde{\boldsymbol{g}}^i$ form reciprocal nonorthogonal bases. Using (34) we can express $\left[J_{\beta}\right]$ in terms of displacement gradient tensor

$$\left[J_{\beta}\right] = \left[\frac{\partial\left\{\bar{x}^{\beta}\right\}}{\partial\left\{x^{\beta}\right\}}\right] = [I] + \left[\frac{\partial\{u\}}{\partial\left\{x^{\beta}\right\}}\right] = [I] + \left[{}^{d}J_{\beta}\right] \tag{45}$$

$\left[{}^{d}J_{\beta}\right]$ is the deformation gradient tensor in x^{β}-frame. Likewise using $\bar{\boldsymbol{x}}^{\beta}$ in (34) and differentiating with respect to $\bar{\boldsymbol{x}}^{\beta}$ we obtain

$$[I] = \left[\frac{\partial\left\{x^{\beta}\right\}}{\partial\left\{\bar{x}^{\beta}\right\}}\right] + \left[\frac{\partial\{\bar{u}\}}{\partial\left\{\bar{x}^{\beta}\right\}}\right] \tag{46}$$

$$\left[\bar{J}_{\beta}\right] = [I] - \left[{}^{d}\bar{J}_{\beta}\right];\ \left[{}^{d}\bar{J}_{\beta}\right] = \left[\frac{\partial\{\bar{u}\}}{\partial\left\{\bar{x}^{\beta}\right\}}\right] \tag{47}$$

Columns of $\boldsymbol{J}_{\beta}$ are covariant vectors ${}^{\beta}\tilde{\boldsymbol{g}}_i$, tangent to the deformed line $\bar{\boldsymbol{x}}^{\beta}$,

$$\left[J_{\beta}\right] = \left[\left\{{}^{\beta}\tilde{g}_1\right\}, \left\{{}^{\beta}\tilde{g}_2\right\}, \left\{{}^{\beta}\tilde{g}_3\right\}\right] \tag{48}$$

Rows of $\left[\bar{J}_{\beta}\right]$ are contravariant vectors ${}^{\beta}\tilde{\boldsymbol{g}}^i$,

$$\left[\bar{J}_{\beta}\right] = \begin{bmatrix} \left\{{}^{\beta}\tilde{g}^1\right\}^{\mathrm{T}} \\ \left\{{}^{\beta}\tilde{g}^2\right\}^{\mathrm{T}} \\ \left\{{}^{\beta}\tilde{g}^3\right\}^{\mathrm{T}} \end{bmatrix} \tag{49}$$

$\left\{{}^{\beta}\tilde{g}_i\right\}$ and $\left\{{}^{\beta}\tilde{g}^i\right\}$ form nonorthogonal reciprocal bases.

5.2. Areas and Volumes in x^{β}-, $\bar{x}^{\beta}$- and x-, $\bar{x}$-Spaces

5.2.1. Areas

First we consider x^{β}- and $\bar{x}^{\beta}$-spaces. We consider relationships between areas

in x^β - and $\bar{x}^\beta$ -spaces as well as in x - and $\bar{x}$ -spaces. Let $\boldsymbol{P}^\beta\boldsymbol{Q}^\beta$, $\boldsymbol{P}^\beta\boldsymbol{R}^\beta$ and $\boldsymbol{P}^\beta\boldsymbol{S}^\beta$ be line segments in x^β -space emanating from material point P^β and $\bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{Q}}^\beta$, $\bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{R}}^\beta$ and $\bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{S}}^\beta$ be their maps in the $\bar{x}^\beta$ -frame. Then, let the area formed by $\bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{Q}}^\beta$ and $\bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{R}}^\beta$ be $d\bar{\boldsymbol{A}}^\beta$ and the area formed by $\boldsymbol{P}^\beta\boldsymbol{Q}^\beta$ and $\boldsymbol{P}^\beta\boldsymbol{R}^\beta$ be $\mathrm{d}\boldsymbol{A}^\beta$, then

$$\mathrm{d}\bar{\boldsymbol{A}}^\beta = \bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{Q}}^\beta \times \bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{R}}^\beta = \boldsymbol{e}_l^\beta\left(\bar{P}^\beta\bar{Q}^\beta\right)_l \times \boldsymbol{e}_m^\beta\left(\bar{P}^\beta\bar{R}^\beta\right)_m \tag{50}$$

$$\mathrm{d}\bar{\boldsymbol{A}}^\beta = \boldsymbol{e}_n^\beta\epsilon_{lmn}\left(\bar{P}^\beta\bar{Q}^\beta\right)_l\left(\bar{P}^\beta\bar{R}^\beta\right)_m \tag{51}$$

$$\mathrm{d}\bar{A}_n^\beta = \epsilon_{lmn}\left(\bar{P}^\beta\bar{Q}^\beta\right)_l\left(\bar{P}^\beta\bar{R}^\beta\right)_m \tag{52}$$

Similarly

$$\mathrm{d}A_k^\beta = \epsilon_{ijk}\left(P^\beta Q^\beta\right)_i\left(P^\beta R^\beta\right)_j \tag{53}$$

using

$$\left(\bar{P}^\beta\bar{Q}^\beta\right)_l = \left(J_\beta\right)_{lp}\left(P^\beta Q^\beta\right)_p \tag{54}$$

$$\left(\bar{P}^\beta\bar{R}^\beta\right)_m = \left(J_\beta\right)_{mq}\left(P^\beta R^\beta\right)_q \tag{55}$$

using (54) and (55) in (51) and following details in reference [26]

$$\left\{\mathrm{d}\bar{A}^\beta\right\} = \det\left[J_\beta\right]\left[\left[J_\beta\right]^{-1}\right]^{\mathrm{T}}\left\{\mathrm{d}A^\beta\right\} \tag{56}$$

5.2.2. Volumes in x^β - and $\bar{x}^\beta$ -Spaces

$$\begin{aligned}\mathrm{d}\bar{V}_\beta &= \left(\bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{Q}}^\beta \times \bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{R}}^\beta\right)\cdot\bar{\boldsymbol{P}}^\beta\bar{\boldsymbol{S}}^\beta \\ &= \epsilon_{lmn}\left(\bar{P}^\beta\bar{Q}^\beta\right)_l\left(\bar{P}^\beta\bar{R}^\beta\right)_m\left(\bar{P}^\beta\bar{S}^\beta\right)_n\end{aligned} \tag{57}$$

$$\begin{aligned}\mathrm{d}V_\beta &= \left(\boldsymbol{P}^\beta\boldsymbol{Q}^\beta \times \boldsymbol{P}^\beta\boldsymbol{R}^\beta\right)\cdot\boldsymbol{P}^\beta\boldsymbol{S}^\beta \\ &= \epsilon_{ijk}\left(P^\beta Q^\beta\right)_i\left(P^\beta R^\beta\right)_j\left(P^\beta S^\beta\right)_k\end{aligned} \tag{58}$$

using

$$\begin{aligned}\left(\bar{P}^\beta\bar{Q}^\beta\right)_l &= \left(J_\beta\right)_{lp}\left(P^\beta Q^\beta\right)_p \\ \left(\bar{P}^\beta\bar{R}^\beta\right)_m &= \left(J_\beta\right)_{mq}\left(P^\beta R^\beta\right)_q \\ \left(\bar{P}^\beta\bar{S}^\beta\right)_n &= \left(J_\beta\right)_{nr}\left(P^\beta S^\beta\right)_r\end{aligned} \tag{59}$$

Using (59) in (57) and then using (58) we can obtain

$$\mathrm{d}\bar{V}_\beta = \det\left[J_\beta\right]\mathrm{d}V_\beta \tag{60}$$

6. Differential Relations in x^β - and $\bar{x}^\beta$ -Frames

In the following, we derive some differential relations in x^β - and $\bar{x}^\beta$ -frames and express them in x - and $\bar{x}$ -frames. Let

$$\left\{\nabla^\beta\right\}^{\mathrm{T}} = \left[\frac{\partial}{\partial x_1^{\beta_1}} \quad \frac{\partial}{\partial x_2^{\beta_2}} \quad \frac{\partial}{\partial x_3^{\beta_3}}\right] \tag{61}$$

$$\left\{\overline{\nabla}^{\beta}\right\}^{\mathrm{T}}=\left[\begin{array}{ccc}\dfrac{\partial}{\partial \overline{x}_1^{\beta_1}} & \dfrac{\partial}{\partial \overline{x}_2^{\beta_2}} & \dfrac{\partial}{\partial \overline{x}_3^{\beta_3}}\end{array}\right] \tag{62}$$

and

$$\left\{\nabla\right\}^{\mathrm{T}}=\left[\begin{array}{ccc}\dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3}\end{array}\right] \tag{63}$$

be differential operators (gradient operators) in x^{β} -, $\overline{x}^{\beta}$ - and x -frames.

6.1. Gradient of a Scalar Field in x^{β} -Frame

Let α be a scalar field; then the operator ∇^{β} acting on α produces

$$\nabla^{\beta}\alpha\left(\boldsymbol{x}^{\beta}\right)=\boldsymbol{e}_i^{\beta}\frac{\partial\alpha}{\partial x_i^{\beta}}=\boldsymbol{e}_1^{\beta}\frac{\partial\alpha}{\partial x_1^{\beta_1}}+\boldsymbol{e}_2^{\beta}\frac{\partial\alpha}{\partial x_2^{\beta_2}}+\boldsymbol{e}_3^{\beta}\frac{\partial\alpha}{\partial x_3^{\beta_3}} \tag{64}$$

also

$$\begin{aligned}\nabla^{\beta}\alpha&=\boldsymbol{e}_i^{\beta}\left(\frac{\partial\alpha}{\partial x_j}\frac{\partial x_j}{\partial x_i^{\beta}}\right)=\boldsymbol{e}_i^{\beta}\frac{\partial x_j}{\partial x_i^{\beta}}\frac{\partial\alpha\left(\boldsymbol{x}\right)}{\partial x_j}\\&=\boldsymbol{e}_i^{\beta}\left(J_{\beta x}\right)_{ji}\frac{\partial\alpha\left(\boldsymbol{x}\right)}{\partial x_j}=\boldsymbol{e}_i^{\beta}\left(J_{x\beta}^{-1}\right)_{ji}\frac{\partial\alpha\left(\boldsymbol{x}\right)}{\partial x_j}\end{aligned} \tag{65}$$

Alternatively

$$\left\{\nabla^{\beta}\alpha\right\}=\left[J_{\beta x}\right]^{\mathrm{T}}\left\{\nabla\alpha\left(\boldsymbol{x}\right)\right\}=\left[J_{x\beta}^{-1}\right]^{\mathrm{T}}\left\{\nabla\alpha\left(\boldsymbol{x}\right)\right\} \tag{66}$$

6.2. Divergence of a Tensor of Rank One in x^{β} -Frame

Let $\boldsymbol{\phi}\left(\boldsymbol{x}^{\beta}\right)$ be a tensor of rank one, then dot product of ∇^{β} and $\boldsymbol{\phi}$ is called the divergence of the tensor of rank one,

$$\nabla^{\beta}\cdot\boldsymbol{\phi}\left(\boldsymbol{x}^{\beta}\right)=\left(\boldsymbol{e}_i^{\beta}\frac{\partial}{\partial x_i^{\beta}}\right)\cdot\left(\boldsymbol{e}_j^{\beta}\phi_j\left(\boldsymbol{x}^{\beta}\right)\right) \tag{67}$$

$$\nabla^{\beta}\cdot\boldsymbol{\phi}=\boldsymbol{e}_i^{\beta}\cdot\boldsymbol{e}_j^{\beta}\frac{\partial\phi_j\left(\boldsymbol{x}^{\beta}\right)}{\partial x_i^{\beta}} \tag{68}$$

$$\nabla^{\beta}\cdot\boldsymbol{\phi}=\delta_{ij}\frac{\partial\phi_j}{\partial x_i^{\beta}}=\frac{\partial\phi_i\left(\boldsymbol{x}\right)}{\partial x_i^{\beta}}=\frac{\partial\phi_i\left(\boldsymbol{x}\right)}{\partial x_k}\frac{\partial x_k\left(\boldsymbol{x}\right)}{\partial x_i^{\beta}} \tag{69}$$

or

$$\nabla^{\beta}\cdot\boldsymbol{\phi}=\frac{\partial x_k}{\partial x_i^{\beta}}\frac{\partial\phi_i\left(\boldsymbol{x}\right)}{\partial x_k}=\left(J_{\beta x}\right)_{ki}\frac{\partial\phi_i\left(\boldsymbol{x}\right)}{\partial x_k}=\left(J_{x\beta}^{-1}\right)_{ki}\frac{\partial\phi_i\left(\boldsymbol{x}\right)}{\partial x_k} \tag{70}$$

6.3. Gradient of a Tensor of Rank One in x^{β} -Frame

Gradient of $\boldsymbol{\phi}$, a tensor of rank one, is the dyadic product of the differential operator ∇^{β} and $\boldsymbol{\phi}$ and is denoted by $\nabla^{\beta}\boldsymbol{\phi}$ or $\nabla^{\beta}\otimes\boldsymbol{\phi}$.

$$\nabla^{\beta}\boldsymbol{\phi}\left(\boldsymbol{x}^{\beta}\right)=\nabla^{\beta}\otimes\boldsymbol{\phi}=\boldsymbol{e}_i^{\beta}\frac{\partial}{\partial x_i^{\beta}}\otimes\boldsymbol{e}_j^{\beta}\phi_j=\boldsymbol{e}_i^{\beta}\otimes\boldsymbol{e}_j^{\beta}\frac{\partial\phi_j}{\partial x_i^{\beta}} \tag{71}$$

or

$$\nabla^{\beta}\boldsymbol{\phi} = \boldsymbol{e}_i^{\beta} \otimes \boldsymbol{e}_j^{\beta} \frac{\partial \phi_j\left(\boldsymbol{x}^{\beta}\right)}{\partial x_i^{\beta}} = \boldsymbol{e}_i^{\beta} \otimes \boldsymbol{e}_j^{\beta} \frac{\partial \phi_j\left(\boldsymbol{x}^{\beta}\right)}{\partial x_i^{\beta}} \tag{72}$$

or

$$\begin{aligned} \nabla^{\beta}\boldsymbol{\phi} &= \boldsymbol{e}_i^{\beta} \otimes \boldsymbol{e}_j^{\beta} \frac{\partial \phi_j(\boldsymbol{x})}{\partial x_p} \frac{\partial x_p}{\partial x_i^{\beta}} \\ &= \boldsymbol{e}_i^{\beta} \otimes \boldsymbol{e}_j^{\beta} \frac{\partial x_p}{\partial x_i^{\beta}} \frac{\partial \phi_j(\boldsymbol{x})}{\partial x_p} = \boldsymbol{e}_i^{\beta} \otimes \boldsymbol{e}_j^{\beta} \left(J_{\beta x}\right)_{pi} \frac{\partial \phi_j(\boldsymbol{x})}{\partial x_p} \end{aligned} \tag{73}$$

Therefore

$$\nabla^{\beta}\boldsymbol{\phi} = \boldsymbol{e}_i^{\beta} \otimes \boldsymbol{e}_j^{\beta} \left(J_{x\beta}^{-1}\right)_{pi} \frac{\partial \phi_j(\boldsymbol{x})}{\partial x_p} \tag{74}$$

6.4. Cross Product of ∇^{β} with a Tensor of Rank One (Curl of a Tensor of Rank One)

If $\boldsymbol{\phi}\left(\boldsymbol{x}^{\beta}\right)$ is a tensor of rank one, then Curl of $\boldsymbol{\phi}$ in x^{β}-space is

$$\nabla^{\beta} \times \boldsymbol{\phi}\left(\boldsymbol{x}^{\beta}\right) = \left(\boldsymbol{e}_i^{\beta} \frac{\partial}{\partial x_i^{\beta}}\right) \times \left(\boldsymbol{e}_j^{\beta} \phi_j\right) \tag{75}$$

or

$$\nabla^{\beta} \times \boldsymbol{\phi} = \boldsymbol{e}_i^{\beta} \times \boldsymbol{e}_j^{\beta} \frac{\partial \phi_j}{\partial x_i^{\beta}} = \epsilon_{ijk} \boldsymbol{e}_k^{\beta} \frac{\partial \phi_j}{\partial x_i^{\beta}} = \operatorname{Curl} \boldsymbol{\phi} \tag{76}$$

Also

$$\nabla^{\beta} \times \boldsymbol{\phi} = \epsilon_{ijk} \boldsymbol{e}_k^{\beta} \left(\frac{\partial \phi_j(\boldsymbol{x})}{\partial x_p} \frac{\partial x_p}{\partial x_i^{\beta}}\right) = \epsilon_{ijk} \boldsymbol{e}_k^{\beta} \frac{\partial x_p}{\partial x_i^{\beta}} \frac{\partial \phi_j(\boldsymbol{x})}{\partial x_p} \tag{77}$$

Therefore

$$\begin{aligned} \nabla^{\beta} \times \boldsymbol{\phi} &= \epsilon_{ijk} \boldsymbol{e}_k^{\beta} \frac{\partial \phi_j(\boldsymbol{x})}{\partial x_i^{\beta}} \\ &= \epsilon_{ijk} \boldsymbol{e}_k^{\beta} \left(J_{\beta x}\right)_{pi} \frac{\partial \phi_j(\boldsymbol{x})}{\partial x_p} = \epsilon_{ijk} \boldsymbol{e}_k^{\beta} \left(J_{x\beta}^{-1}\right)_{pi} \frac{\partial \phi_j(\boldsymbol{x})}{\partial x_p} \end{aligned} \tag{78}$$

6.5. Divergence of a Tensor of Rank Two in x^{β}-Frame

Let $\boldsymbol{\tau}\left(\boldsymbol{x}^{\beta}\right)$ be a tensor of rank two, then its divergence is denoted by $\nabla^{\beta} \cdot \boldsymbol{\tau}\left(\boldsymbol{x}^{\beta}\right)$ and is defined as

$$\operatorname{div} \boldsymbol{\tau} = \nabla^{\beta} \cdot \boldsymbol{\tau}\left(\boldsymbol{x}^{\beta}\right) = \boldsymbol{e}_i^{\beta} \frac{\partial}{\partial x_i^{\beta}} \cdot \left(\boldsymbol{e}_j^{\beta} \otimes \boldsymbol{e}_k^{\beta}\right) \tau_{jk} \tag{79}$$

$$\operatorname{div} \boldsymbol{\tau} = \left(\boldsymbol{e}_i^{\beta} \cdot \boldsymbol{e}_j^{\beta}\right) \boldsymbol{e}_k^{\beta} \frac{\partial\left(\tau_{jk}\right)}{\partial x_i^{\beta}} = \delta_{ij} \boldsymbol{e}_k^{\beta} \frac{\partial\left(\tau_{jk}\right)}{\partial x_i^{\beta}} \tag{80}$$

$$\operatorname{div} \boldsymbol{\tau} = \boldsymbol{e}_k^{\beta} \frac{\partial\left(\tau_{ik}\right)}{\partial x_i^{\beta}} \tag{81}$$

also

$$\nabla^{\beta}\cdot\boldsymbol{\tau}=\boldsymbol{e}_k^{\beta}\frac{\partial\left(\tau_{ik}\left(\boldsymbol{x}\right)\right)}{\partial x_p}\frac{\partial x_p}{\partial x_i^{\beta}}=\boldsymbol{e}_k^{\beta}\frac{\partial x_p}{\partial x_i^{\beta}}\frac{\partial\left(\tau_{ik}\right)}{\partial x_p} \tag{82}$$

$$\nabla^{\beta}\cdot\boldsymbol{\tau}=\boldsymbol{e}_k^{\beta}\left(J_{\beta x}\right)_{pi}\frac{\partial\left(\tau_{ik}\left(\boldsymbol{x}\right)\right)}{\partial x_p}=\boldsymbol{e}_k^{\beta}\left(J_{x\beta}^{-1}\right)_{pi}\frac{\partial\left(\tau_{ik}\left(\boldsymbol{x}\right)\right)}{\partial x_p} \tag{83}$$

Then we have

$$\begin{aligned}\operatorname{div}\boldsymbol{\tau}=\nabla^{\beta}\cdot\boldsymbol{\tau}\left(\boldsymbol{x}^{\beta}\right)&=\boldsymbol{e}_k^{\beta}\frac{\partial}{\partial x_i^{\beta}}\left(\tau_{ik}\left(\boldsymbol{x}^{\beta}\right)\right)\\&=\boldsymbol{e}_k^{\beta}\left(J_{\beta x}\right)_{pi}\frac{\partial\left(\tau_{ik}\left(\boldsymbol{x}\right)\right)}{\partial x_p}=\boldsymbol{e}_k^{\beta}\left(J_{x\beta}^{-1}\right)_{pi}\frac{\partial\left(\tau_{ik}\left(\boldsymbol{x}\right)\right)}{\partial x_p}\end{aligned} \tag{84}$$

6.6. Laplacian of a Scalar Field

It is defined as the divergence of the gradient of a scalar field. Let $\alpha\left(\boldsymbol{x}^{\beta}\right)$ be a scalar field, then

$$\begin{aligned}\nabla^{\beta}\cdot\left(\nabla^{\beta}\alpha\right)&=\boldsymbol{e}_i^{\beta}\frac{\partial}{\partial x_i^{\beta}}\cdot\boldsymbol{e}_j^{\beta}\frac{\partial\alpha}{\partial x_j^{\beta}}=\boldsymbol{e}_i^{\beta}\cdot\boldsymbol{e}_j^{\beta}\frac{\partial}{\partial x_i^{\beta}}\left(\frac{\partial\alpha}{\partial x_j^{\beta}}\right)\\&=\delta_{ij}\frac{\partial}{\partial x_i^{\beta}}\left(\frac{\partial\alpha}{\partial x_j^{\beta}}\right)=\frac{\partial}{\partial x_i^{\beta}}\left(\frac{\partial\alpha}{\partial x_i^{\beta}}\right)\end{aligned} \tag{85}$$

Therefore

$$\nabla^{\beta}\cdot\nabla^{\beta}\alpha\left(\boldsymbol{x}^{\beta}\right)=\frac{\partial}{\partial x_1^{\beta_1}}\left(\frac{\partial\alpha}{\partial x_1^{\beta_1}}\right)+\frac{\partial}{\partial x_2^{\beta_2}}\left(\frac{\partial\alpha}{\partial x_2^{\beta_2}}\right)+\frac{\partial}{\partial x_3^{\beta_3}}\left(\frac{\partial\alpha}{\partial x_3^{\beta_3}}\right) \tag{86}$$

6.7. Material Derivative

While working in x^{β} - and $\bar{x}^{\beta}$ -spaces we often need the material derivative of a quantity defined in $\bar{x}^{\beta}$ -space. Let $\bar{\theta}\left(\bar{\boldsymbol{x}}^{\beta},t\right)$ be some quantity of interest, then we need to determine $\dfrac{D\bar{\theta}\left(\bar{\boldsymbol{x}}^{\beta},t\right)}{Dt}$

$$\frac{D\bar{\theta}\left(\bar{\boldsymbol{x}}^{\beta},t\right)}{Dt}=\frac{\partial\bar{\theta}\left(\bar{\boldsymbol{x}}^{\beta},t\right)}{\partial t}+\frac{\partial\bar{\theta}\left(\bar{\boldsymbol{x}}^{\beta},t\right)}{\partial\bar{\boldsymbol{x}}^{\beta}}\cdot\frac{\partial\bar{\boldsymbol{x}}^{\beta}}{\partial t} \tag{87}$$

$$\frac{D\bar{\theta}\left(\bar{\boldsymbol{x}}^{\beta},t\right)}{Dt}=\frac{\partial\bar{\theta}\left(\bar{\boldsymbol{x}}^{\beta},t\right)}{\partial t}+\frac{\partial\bar{\theta}\left(\bar{\boldsymbol{x}}^{\beta},t\right)}{\partial\bar{\boldsymbol{x}}}\cdot\frac{\partial\bar{\boldsymbol{x}}}{\partial\bar{\boldsymbol{x}}^{\beta}}\frac{\partial\bar{\boldsymbol{x}}^{\beta}}{\partial\bar{\boldsymbol{x}}}\frac{\partial\bar{\boldsymbol{x}}}{\partial t} \tag{88}$$

$$\frac{D\bar{\theta}\left(\bar{\boldsymbol{x}}^{\beta},t\right)}{Dt}=\frac{\partial\bar{\theta}\left(\bar{\boldsymbol{x}},t\right)}{\partial t}+\frac{\partial\bar{\theta}\left(\bar{\boldsymbol{x}},t\right)}{\partial\bar{\boldsymbol{x}}}\cdot\bar{\boldsymbol{v}}\left(\bar{\boldsymbol{x}},t\right) \tag{89}$$

or

$$\frac{D\bar{\theta}\left(\bar{\boldsymbol{x}}^{\beta},t\right)}{Dt}=\frac{\partial\bar{\theta}\left(\bar{\boldsymbol{x}},t\right)}{\partial t}+\left(\bar{\nabla}\cdot\bar{\boldsymbol{v}}\right)\bar{\theta}\left(\bar{\boldsymbol{x}},t\right) \tag{90}$$

From (90) we conclude that the material derivative of a quantity defined in $\bar{x}^{\beta}$ -frame is the same as the material derivative of the quantity defined in the $\bar{x}$ -frame. When $\theta=\theta\left(\boldsymbol{x}^{\beta},t\right)$, that is in Lagrangian description, then

$$\frac{D\theta\left(\boldsymbol{x}^{\beta},t\right)}{Dt}=\frac{\partial\theta\left(\boldsymbol{x}^{\beta},t\right)}{\partial t}\bigg|_{x^{\beta}\text{ fixed}}=\frac{\partial\theta\left(\boldsymbol{x}^{\beta},t\right)}{\partial t} \tag{91}$$

7. Strain Measures

In this section we derive nonlinear strain measures (finite deformation, finite strain) in Lagrangian and Eulerian descriptions in x^{β} - and $\bar{x}^{\beta}$ -frames. V_{β} and $\bar{V}_{\beta}$ are undeformed and deformed volumes with x^{β} and $\bar{x}^{\beta}$ coordinates of the material points.

7.1. Covariant Cauchy Strain $\left[{}^{\beta}C_{[0]}\right]$

Consider lengths $\mathrm{d}s^{\beta}$ and $\mathrm{d}\bar{s}^{\beta}$ in x^{β} - and $\bar{x}^{\beta}$ -frames. Then

$$\frac{\left(\mathrm{d}\bar{s}^{\beta}\right)^2}{\left(\mathrm{d}s^{\beta}\right)^2}=\frac{\left\{\mathrm{d}\bar{x}^{\beta}\right\}^{\mathrm{T}}\left\{\mathrm{d}\bar{x}^{\beta}\right\}}{\left\{\mathrm{d}x^{\beta}\right\}^{\mathrm{T}}\left\{\mathrm{d}x^{\beta}\right\}}=\frac{\left\{\mathrm{d}x^{\beta}\right\}^{\mathrm{T}}\left[J_{\beta}\right]^{\mathrm{T}}\left[J_{\beta}\right]\left\{\mathrm{d}x^{\beta}\right\}}{\left\{\mathrm{d}x^{\beta}\right\}^{\mathrm{T}}\left\{\mathrm{d}x^{\beta}\right\}} \tag{92}$$

Since $\left\{\mathrm{d}x^{\beta}\right\}$ is fixed, the measure of deformation must be in the term that does not contain $\left\{dx^{\beta}\right\}$ and $\left\{\mathrm{d}x^{\beta}\right\}^{\mathrm{T}}$. We define Cauchy strain $\left[{}^{\beta}C_{[0]}\right]$ as

$$\left[{}^{\beta}C_{[0]}\right]=\left[J_{\beta}\right]^{\mathrm{T}}\left[J_{\beta}\right] \tag{93}$$

Since

$$\left[J_{\beta}\right]=\left[I\right]+\left[{}^{d}J_{\beta}\right] \tag{94}$$

$$\left[{}^{\beta}C_{[0]}\right]=\left[{}^{d}J_{\beta}\right]+\left[{}^{d}J_{\beta}\right]^{\mathrm{T}}+\left[{}^{d}J_{\beta}\right]^{\mathrm{T}}\left[{}^{d}J_{\beta}\right]+\left[I\right] \tag{95}$$

$\left[{}^{\beta}C_{[0]}\right]$ in (95) can be expressed in terms of $\left[{}^{d}J\right]$.

$$\left[{}^{d}J_{\beta}\right]=\left[\frac{\partial\{u\}}{\partial\left\{x^{\beta}\right\}}\right]=\left[\frac{\partial\{u\}}{\partial\{x\}}\right]\left[\frac{\partial\{x\}}{\partial\left\{x^{\beta}\right\}}\right]=\left[{}^{d}J\right]\left[J_{\beta x}\right] \tag{96}$$

Therefore

$$\left[{}^{\beta}C_{[0]}\right]=\left[{}^{d}J\right]\left[J_{\beta x}\right]+\left[J_{\beta x}\right]^{\mathrm{T}}\left[{}^{d}J\right]^{\mathrm{T}}+\left[J_{\beta x}\right]^{\mathrm{T}}\left[{}^{d}J\right]^{\mathrm{T}}\left[{}^{d}J\right]\left[J_{\beta x}\right]+\left[I\right] \tag{97}$$

Definition of $\left[{}^{\beta}C_{[0]}\right]$ contains $\left[{}^{d}J\right]$, gradients of $\{u\}$ with respect to $\{x\}$, hence in x -frame. We clearly see that $\left[{}^{\beta}C_{[0]}\right]$ is symmetric (from (93)). We note $\left[{}^{d}J\right]$ derivation in x -frame is not possible but components of $\left[{}^{d}J\right]$ that are gradients of u_i with respect to x_j do exist.

7.2. Covariant Green's Strain Tensor $\left[{}^{\beta}\varepsilon_{[0]}\right]$

Consider

$$\frac{\left(\mathrm{d}\bar{s}^{\beta}\right)^2-\left(\mathrm{d}s^{\beta}\right)^2}{2\left(\mathrm{d}s^{\beta}\right)^2}=\frac{\left\{\mathrm{d}\bar{x}^{\beta}\right\}^{\mathrm{T}}\left\{\mathrm{d}\bar{x}^{\beta}\right\}-\left\{\mathrm{d}x^{\beta}\right\}^{\mathrm{T}}\left\{\mathrm{d}x^{\beta}\right\}}{2\left\{\mathrm{d}x^{\beta}\right\}^{\mathrm{T}}\left\{\mathrm{d}x^{\beta}\right\}} \tag{98}$$

$$\frac{\left(\mathrm{d}\bar{s}^{\beta}\right)^2-\left(\mathrm{d}s^{\beta}\right)^2}{2\left(\mathrm{d}s^{\beta}\right)^2}=\frac{\left\{\mathrm{d}x^{\beta}\right\}^{\mathrm{T}}\left[J_{\beta}\right]^{\mathrm{T}}\left[J_{\beta}\right]\left\{\mathrm{d}x^{\beta}\right\}-\left\{\mathrm{d}x^{\beta}\right\}^{\mathrm{T}}\left[I\right]\left\{\mathrm{d}x^{\beta}\right\}}{2\left\{\mathrm{d}x^{\beta}\right\}^{\mathrm{T}}\left\{\mathrm{d}x^{\beta}\right\}} \tag{99}$$

Since $\left\{\mathrm{d}x^{\beta}\right\}$ are fixed, all the deformation is contained in the terms that do not have $\left\{\mathrm{d}x^{\beta}\right\}$ and $\left\{\mathrm{d}x^{\beta}\right\}^{\mathrm{T}}$, we define Green's strain $\left[{}^{\beta}\varepsilon_{[0]}\right]$ as

$$\left[{}^{\beta}\varepsilon_{[0]}\right]=\frac{1}{2}\left(\left[J_{\beta}\right]^{\mathrm{T}}\left[J_{\beta}\right]-\left[I\right]\right) \tag{100}$$

Since $\left[J_{\beta}\right]=\left[I\right]+\left[{}^{d}J_{\beta}\right]$, (100) can be written as

$$\left[{}^{\beta}\varepsilon_{[0]}\right]=\frac{1}{2}\left(\left[{}^{d}J_{\beta}\right]+\left[{}^{d}J_{\beta}\right]^{\mathrm{T}}+\left[{}^{d}J_{\beta}\right]^{\mathrm{T}}\left[{}^{d}J_{\beta}\right]\right) \tag{101}$$

Using (96) we can write (101) as

$$\left[{}^{\beta}\varepsilon_{[0]}\right]=\frac{1}{2}\left(\left[{}^{d}J\right]\left[J_{\beta x}\right]+\left[J_{\beta x}\right]^{\mathrm{T}}\left[{}^{d}J\right]^{\mathrm{T}}+\left[J_{\beta x}\right]^{\mathrm{T}}\left[{}^{d}J\right]^{\mathrm{T}}\left[{}^{d}J\right]\left[J_{\beta x}\right]\right) \tag{102}$$

Green's strain can be expressed in terms of Cauchy strain using (97).

$$\left[{}^{\beta}\varepsilon_{[0]}\right]=\frac{1}{2}\left(\left[{}^{\beta}C_{[0]}\right]-\left[I\right]\right) \tag{103}$$

Remarks

For small strain and small deformation, the nonlinear terms in the strain measures can be omitted, and we obtain

1) Cauchy strain $\left[{}^{\beta}C\right]$

$$\left[{}^{\beta}C\right]=\left[{}^{d}J_{\beta}\right]+\left[{}^{d}J_{\beta}\right]^{\mathrm{T}}+\left[I\right] \tag{104}$$

or

$$\left[{}^{\beta}C\right]=\left[{}^{d}J\right]\left[J_{\beta x}\right]+\left[J_{\beta x}\right]^{\mathrm{T}}\left[{}^{d}J\right]^{\mathrm{T}}+\left[I\right] \tag{105}$$

2) Green's strain $\left[{}^{\beta}\varepsilon\right]$

$$\left[{}^{\beta}\varepsilon\right]=\frac{1}{2}\left(\left[{}^{d}J_{\beta}\right]+\left[{}^{d}J_{\beta}\right]^{\mathrm{T}}\right) \tag{106}$$

or

$$\left[{}^{\beta}\varepsilon\right]=\frac{1}{2}\left(\left[{}^{d}J\right]\left[J_{\beta x}\right]+\left[J_{\beta x}\right]^{\mathrm{T}}\left[{}^{d}J\right]^{\mathrm{T}}\right) \tag{107}$$

8. Stress Tensors

8.1. Cauchy Stress Tensor

Consider volume V_{β} and tetrahedron ${}^{\beta}T$ such that the oblique plane of the tetrahedron has area A_n^{β}. Upon deformation, the tetrahedron is distorted into ${}^{\beta}\bar{T}$ in $\bar{x}^{\beta}$-frame. Let ${}^{\beta}\tilde{\boldsymbol{g}}_i$ be covariant base vectors to the deformed lines $\bar{x}^{\beta}$ at a material point, where ${}^{\beta}\tilde{\boldsymbol{g}}_i$ are defined by the columns of deformation gradient tensor $\left[J_{\beta}\right]$. Let ${}^{\beta}\tilde{\boldsymbol{g}}_i$ be the edge of the deformed tetrahedron (covariant directions); then the vectors normal to the faces of the tetrahedron ${}^{\beta}\bar{T}$ are con-

travariant base vectors ${}^{\beta}\tilde{\boldsymbol{g}}^{i}$. We have the following

$${}^{\beta}\tilde{\boldsymbol{g}}_{i} = \frac{\partial \overline{x}_{j}^{\beta}}{\partial x_{i}^{\beta}} \boldsymbol{e}_{j}^{\beta} = \left(J_{\beta}\right)_{ji} \boldsymbol{e}_{j}^{\beta} \tag{108}$$

$${}^{\beta}\tilde{\boldsymbol{g}}^{i} = \frac{\partial x_{i}^{\beta}}{\partial \overline{x}_{j}^{\beta}} \boldsymbol{e}_{j}^{\beta} = \left(\overline{J}_{\beta}\right)_{ij} \boldsymbol{e}_{j}^{\beta} \tag{109}$$

The contravariant stress components ${}^{\beta}\underset{\sim}{\sigma}_{ij}^{(0)}$ acting on the faces of ${}^{\beta}\overline{T}$ require covariant bases in x^{β}-frame. The components of ${}^{\beta}\underset{\sim}{\sigma}_{ij}^{(0)}$ in the basis of x^{β}-frame are given by

$${}^{\beta}\boldsymbol{\sigma}^{0} = {}^{\beta}\tilde{\boldsymbol{g}}_{i} \otimes {}^{\beta}\tilde{\boldsymbol{g}}_{j} \left({}^{\beta}\underset{\sim}{\sigma}_{ij}^{(0)}\right) \tag{110}$$

Substituting from (108) in (110) we obtain

$${}^{\beta}\boldsymbol{\sigma}^{0} = \frac{\partial \overline{x}_{m}^{\beta}}{\partial x_{i}^{\beta}} \boldsymbol{e}_{m}^{\beta} \otimes \frac{\partial \overline{x}_{n}^{\beta}}{\partial x_{j}^{\beta}} \boldsymbol{e}_{n}^{\beta} \left({}^{\beta}\underset{\sim}{\sigma}_{ij}^{(0)}\right) \tag{111}$$

$${}^{\beta}\boldsymbol{\sigma}^{0} = \left(\boldsymbol{e}_{m}^{\beta} \otimes \boldsymbol{e}_{n}^{\beta}\right) \left(\frac{\partial \overline{x}_{m}^{\beta}}{\partial x_{i}^{\beta}} \frac{\partial \overline{x}_{n}^{\beta}}{\partial x_{j}^{\beta}} \left({}^{\beta}\underset{\sim}{\sigma}_{ij}^{(0)}\right) \right) \tag{112}$$

$${}^{\beta}\boldsymbol{\sigma}^{0} = \left(\boldsymbol{e}_{m}^{\beta} \otimes \boldsymbol{e}_{n}^{\beta}\right) \left(J_{\beta}\right)_{mi} \left(J_{\beta}\right)_{nj} \left({}^{\beta}\underset{\sim}{\sigma}_{ij}^{(0)}\right) \tag{113}$$

or

$${}^{\beta}\boldsymbol{\sigma}^{0} = \boldsymbol{e}_{m}^{\beta} \otimes \boldsymbol{e}_{n}^{\beta} \left({}^{\beta}\sigma_{mn}^{0}\right) \tag{114}$$

in which

$$\left({}^{\beta}\sigma_{mn}^{0}\right) = \left(J_{\beta}\right)_{mi} \left({}^{\beta}\underset{\sim}{\sigma}_{ij}^{(0)}\right) \left(J_{\beta}\right)_{nj} \quad (\text{def.}) \tag{115}$$

We note that

$$\boldsymbol{e}_{m}^{\beta} = \left(J_{x\beta}\right)_{mp} \boldsymbol{e}_{p}; \quad \boldsymbol{e}_{n}^{\beta} = \left(J_{x\beta}\right)_{nq} \boldsymbol{e}_{q} \tag{116}$$

Therefore,

$${}^{\beta}\boldsymbol{\sigma}^{(0)} = \left(J_{x\beta}\right)_{mp} \boldsymbol{e}_{p} \otimes \left(J_{x\beta}\right)_{nq} \boldsymbol{e}_{q} \left({}^{\beta}\sigma_{mn}^{0}\right) \tag{117}$$

or

$${}^{\beta}\boldsymbol{\sigma}^{(0)} = \boldsymbol{e}_{p} \otimes \boldsymbol{e}_{q} \left(J_{x\beta}\right)_{mp} \left(J_{x\beta}\right)_{nq} \left({}^{\beta}\sigma_{mn}^{0}\right) \tag{118}$$

or

$${}^{\beta}\boldsymbol{\sigma}^{(0)} = \boldsymbol{e}_{p} \otimes \boldsymbol{e}_{q} \, {}^{\beta}\sigma_{pq}^{(0)} \tag{119}$$

${}^{\beta}\boldsymbol{\sigma}^{0}$ is a stress tensor with dyads in x^{β}-frame. ${}^{\beta}\boldsymbol{\sigma}^{(0)}$ is contravariant Cauchy stress tensor. It acts on the flat faces of a tetrahedron; the faces are orthogonal to each other and are parallel to the faces of the x-frame, but the oblique plane of this new tetrahedron is the same as that of the actual deformed tetrahedron ${}^{\beta}\overline{T}$.

Cauchy principle can be applied to ${}^{\beta}\overline{\boldsymbol{P}}$, force per unit area on the oblique

plane of ${}^{\beta}\bar{T}$ and the stress tensor ${}^{\beta}\boldsymbol{\sigma}^{(0)}$. We note Cauchy principle is not applicable to the stress tensor ${}^{\beta}\boldsymbol{\sigma}^{0}$. Furthermore, the areas of the faces of the new tetrahedron are assumed to be the same as those of the original deformed tetrahedron ${}^{\beta}\bar{T}$. Thus

$${}^{\beta}\bar{\boldsymbol{P}} = \left({}^{\beta}\bar{\boldsymbol{\sigma}}^{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}}^{\beta} \tag{120}$$

Following a similar procedure, we can also define the covariant Cauchy stress tensor ${}^{\beta}\boldsymbol{\sigma}_{(0)}$ for which Cauchy principle similar to (120) holds. As discussed in the references [26] [27], ${}^{\beta}\boldsymbol{\sigma}_{(0)}$ is only valid for infinitesimal deformations, hence not useful in nonlinear theories.

8.2. Contravariant Cauchy Stress Tensor in x-Frame

This is useful when transforming the mathematical model from x^{β}-frame to x-frame in Lagrangian description or from $\bar{x}^{\beta}$-frame to $\bar{x}$-frame in Eulerian description. Consider the integral of the average stress ${}^{\beta}\bar{\boldsymbol{P}}$ on the oblique plane of the deformed tetrahedron with area $\bar{A}_n^{\beta}$ and use Cauchy principle.

$$\int_{\bar{A}_n^{\beta}} {}^{\beta}\bar{\boldsymbol{P}}\mathrm{d}\bar{A}_n^{\beta} = \int_{\bar{A}_n^{\beta}} \left({}^{\beta}\bar{\boldsymbol{\sigma}}^{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}}^{\beta}\mathrm{d}\bar{A}_n^{\beta} = \int_{A_n^{\beta}} \left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{121}$$

Using

$$\mathrm{d}\boldsymbol{A}_n^{\beta} = \left|\boldsymbol{J}_{x\beta}\right|\left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A} \tag{122}$$

Substituting from (122) in (121)

$$\int_{\bar{A}_n^{\beta}} {}^{\beta}\bar{\boldsymbol{P}}\mathrm{d}\bar{A}_n^{\beta} = \int_{A_n} \left|\boldsymbol{J}_{x\beta}\right|\left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A} = \int_{A_n} \left(\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A} \tag{123}$$

Thus,

$$\left(\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} = \left|\boldsymbol{J}_{x\beta}\right|\left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \tag{124}$$

or

$$\boldsymbol{\sigma}^{(0)} = \left|\boldsymbol{J}_{x\beta}\right|\left(\boldsymbol{J}_{x\beta}\right)^{-1} \cdot \left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right) \tag{125}$$

Equation (125) shows that Cauchy stress tensor in x-frame is not symmetric, even when $\left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right)$ is symmetric.

8.3. First Piola-Kirchhoff Stress Tensor ${}^{\beta}\boldsymbol{\sigma}^{*}$ in x^{β}-Frame

Consider x^{β}-space. The correspondence rule is

$$\mathrm{d}\bar{\boldsymbol{F}}^{\beta} = \mathrm{d}\boldsymbol{F}^{\beta} \tag{126}$$

$$\mathrm{d}\bar{\boldsymbol{F}}^{\beta} = \int_{\bar{A}_n^{\beta}} {}^{\beta}\bar{\boldsymbol{P}}\mathrm{d}\bar{A}_n^{\beta} = \int_{\bar{A}_n^{\beta}} \left({}^{\beta}\bar{\boldsymbol{\sigma}}^{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}}^{\beta}\mathrm{d}\bar{A}_n^{\beta} = \int_{\bar{A}_n^{\beta}} \left({}^{\beta}\bar{\boldsymbol{\sigma}}^{(0)}\right)^{\mathrm{T}} \cdot \mathrm{d}\bar{\boldsymbol{A}}_n^{\beta} \tag{127}$$

$$\mathrm{d}\bar{\boldsymbol{A}}_n^{\beta} = \left|\boldsymbol{J}_{\beta}\right|\left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{128}$$

Thus,

$$\mathrm{d}\bar{\boldsymbol{F}}^{\beta} = \int_{A_n^{\beta}} \left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} \cdot \left|\boldsymbol{J}_{\beta}\right|\left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{129}$$

$$\mathrm{d}\boldsymbol{F}^{\beta} = \int_{A_n^{\beta}} \left({}^{\beta}\boldsymbol{\sigma}^{*}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{130}$$

Using (129) and (130) in (126),

$$\int_{A_n^{\beta}} \left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} \left|\boldsymbol{J}_{\beta}\right| \left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} = \int_{A_n^{\beta}} \left({}^{\beta}\boldsymbol{\sigma}^{*}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{131}$$

Therefore,

$$\left({}^{\beta}\boldsymbol{\sigma}^{*}\right)^{\mathrm{T}} = \left|\boldsymbol{J}_{\beta}\right| \left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \tag{132}$$

Thus,

$${}^{\beta}\boldsymbol{\sigma}^{*} = \left|\boldsymbol{J}_{\beta}\right| \left(\boldsymbol{J}_{\beta}^{-1}\right) \cdot \left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right) \tag{133}$$

${}^{\beta}\boldsymbol{\sigma}^{*}$ is clearly nonsymmetric, even when ${}^{\beta}\boldsymbol{\sigma}^{(0)}$ is symmetric.

8.4. Second Piola-Kirchhoff Stress Tensor in x^{β}-Frame

In this case, we use the following correspondence rule

$$\mathrm{d}\bar{\boldsymbol{F}}^{\beta} = \boldsymbol{J}_{\beta} \cdot \mathrm{d}\boldsymbol{F}^{\beta} \tag{134}$$

$$\mathrm{d}\bar{\boldsymbol{F}}^{\beta} = \int_{\bar{A}_n^{\beta}} {}^{\beta}\bar{\boldsymbol{P}}\,\mathrm{d}\bar{A}_n^{\beta} = \int_{\bar{A}_n^{\beta}} \left({}^{\beta}\bar{\boldsymbol{\sigma}}^{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}}^{\beta}\,\mathrm{d}\bar{A}_n^{\beta} = \int_{\bar{A}_n^{\beta}} \left({}^{\beta}\bar{\boldsymbol{\sigma}}^{(0)}\right)^{\mathrm{T}} \cdot \mathrm{d}\bar{\boldsymbol{A}}_n^{\beta} \tag{135}$$

Using (128) in (135),

$$\mathrm{d}\bar{\boldsymbol{F}}^{\beta} = \int_{A_n^{\beta}} \left|\boldsymbol{J}_{\beta}\right| \left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{136}$$

and

$$\boldsymbol{J}_{\beta} \cdot \mathrm{d}\boldsymbol{F}^{\beta} = \int_{A_n^{\beta}} \boldsymbol{J}_{\beta} \cdot \left({}^{\beta}\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{137}$$

Equating (136) and (137),

$$\int_{A_n^{\beta}} \left|\boldsymbol{J}_{\beta}\right| \left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} = \int_{A_n^{\beta}} \boldsymbol{J}_{\beta} \cdot \left({}^{\beta}\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{138}$$

Therefore,

$$\boldsymbol{J}_{\beta} \cdot \left({}^{\beta}\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} = \left|\boldsymbol{J}_{\beta}\right| \left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \tag{139}$$

Thus,

$$\left({}^{\beta}\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} = \left|\boldsymbol{J}_{\beta}\right| \left(\boldsymbol{J}_{\beta}^{-1}\right) \cdot \left({}^{\beta}\boldsymbol{\sigma}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \tag{140}$$

When ${}^{\beta}\boldsymbol{\sigma}^{(0)}$ is symmetric, ${}^{\beta}\boldsymbol{\sigma}^{[0]}$ is also symmetric.

8.5. First Piola-Kirchhoff Stress Tensor σ^{*} in x-Frame

Between x^{β}-frame and x-frame we have

$$\int_{A_n^{\beta}} \left({}^{\beta}\boldsymbol{\sigma}^{*}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} = \int_{A_n} \left(\boldsymbol{\sigma}^{*}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n \tag{141}$$

$$\mathrm{d}\boldsymbol{A}_n^{\beta} = \left|\boldsymbol{J}_{x\beta}\right| \left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n \tag{142}$$

Substituting from (142) in (141),

$$\int_{A_n^\beta} \left({}^\beta\boldsymbol{\sigma}^*\right)^{\mathrm{T}} \left|\boldsymbol{J}_{x\beta}\right| \left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n = \int_{A_n} \left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n \tag{143}$$

Therefore,

$$\left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}} = \left|\boldsymbol{J}_{x\beta}\right| \left({}^\beta\boldsymbol{\sigma}^*\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \tag{144}$$

or

$$\left({}^\beta\boldsymbol{\sigma}^*\right)^{\mathrm{T}} = \frac{1}{\left|\boldsymbol{J}_{x\beta}\right|} \left(\boldsymbol{\sigma}^*\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{x\beta}^{\mathrm{T}}\right) \tag{145}$$

8.6. Second Piola-Kirchhoff Stress Tensor $\left[\sigma^{[0]}\right]$ in x-Frame

Between x^β- and x-frames we have

$$\int_{A_n^\beta} \left({}^\beta\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^\beta = \int_{A_n} \left(\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n \tag{146}$$

and

$$\mathrm{d}\boldsymbol{A}_n^\beta = \left|\boldsymbol{J}_{x\beta}\right| \left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n \tag{147}$$

Using (147) in (146),

$$\int_{A_n^\beta} \left({}^\beta\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} \left|\boldsymbol{J}_{x\beta}\right| \left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n = \int_{A_n} \left(\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n \tag{148}$$

Therefore,

$$\left(\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} = \left|\boldsymbol{J}_{x\beta}\right| \left({}^\beta\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \tag{149}$$

or

$$\left({}^\beta\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} = \frac{1}{\left|\boldsymbol{J}_{x\beta}\right|} \left(\boldsymbol{\sigma}^{[0]}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{x\beta}^{\mathrm{T}}\right) \tag{150}$$

9. Moment Tensors

Since the map of the porous material in the curvilinear fractional frame is micropolar, in addition to the stress tensors, we also have moments and moment tensors independent of the forces and stresses. For defining the moment measures, we also consider the same deformed tetrahedron ${}^\beta\overline{T}$ as considered for stress measures, with covariant vectors ${}^\beta\tilde{\boldsymbol{g}}_i$ forming the edges and the contravariant vectors ${}^\beta\tilde{\boldsymbol{g}}^i$ being normal to the flat faces of the tetrahedron ${}^\beta\overline{T}$ formed by the covariant pairs ${}^\beta\tilde{\boldsymbol{g}}_2\,{}^\beta\tilde{\boldsymbol{g}}_3$, ${}^\beta\tilde{\boldsymbol{g}}_3\,{}^\beta\tilde{\boldsymbol{g}}_1$ and ${}^\beta\tilde{\boldsymbol{g}}_1\,{}^\beta\tilde{\boldsymbol{g}}_2$.

9.1. Contravariant and Covariant Cauchy Moment Tensors in x^β-Frame

The contravariant Cauchy moment tensor ${}^\beta\boldsymbol{m}^{(0)}$ or ${}^\beta\overline{\boldsymbol{m}}^{(0)}$ is defined using tetrahedron ${}^\beta\overline{T}$. Let ${}^\beta m_{ij}^{(0)}$ or ${}^\beta\overline{m}_{ij}^{(0)}$ be the contravariant moment components defined using contravariant directions ${}^\beta\tilde{\boldsymbol{g}}^i$; then we can express a moment tensor ${}^\beta\boldsymbol{m}^0$ using the covariant vectors in $\boldsymbol{e}_i^\beta$ basis of the x^β-frame.

$$ {}^{\beta}\boldsymbol{m}^{0} = {}^{\beta}\tilde{\boldsymbol{g}}_{i} \otimes {}^{\beta}\tilde{\boldsymbol{g}}_{j} \left({}^{\beta}\underset{\sim}{m}_{ij}^{(0)} \right) \tag{151} $$

Using (108), we can write (151) as

$$ {}^{\beta}\boldsymbol{m}^{0} = \frac{\partial \overline{x}_{m}^{\beta}}{\partial x_{i}^{\beta}} \boldsymbol{e}_{m}^{\beta} \otimes \frac{\partial \overline{x}_{n}^{\beta}}{\partial x_{j}^{\beta}} \boldsymbol{e}_{n}^{\beta} \left({}^{\beta}\underset{\sim}{m}_{ij}^{(0)} \right) \tag{152} $$

or

$$ {}^{\beta}\boldsymbol{m}^{0} = \boldsymbol{e}_{m}^{\beta} \otimes \boldsymbol{e}_{n}^{\beta} \left(\frac{\partial \overline{x}_{m}^{\beta}}{\partial x_{i}^{\beta}} \frac{\partial \overline{x}_{n}^{\beta}}{\partial x_{j}^{\beta}} \left({}^{\beta}\underset{\sim}{m}_{ij}^{(0)} \right) \right) \tag{153} $$

or

$$ {}^{\beta}\boldsymbol{m}^{0} = \boldsymbol{e}_{m}^{\beta} \otimes \boldsymbol{e}_{n}^{\beta} \left(J_{\beta} \right)_{mi} \left(J_{\beta} \right)_{nj} \left({}^{\beta}\underset{\sim}{m}_{ij}^{(0)} \right) \tag{154} $$

or

$$ {}^{\beta}\boldsymbol{m}^{0} = \boldsymbol{e}_{m}^{\beta} \otimes \boldsymbol{e}_{n}^{\beta} \left({}^{\beta}m_{mn}^{0} \right) \tag{155} $$

in which

$$ {}^{\beta}m_{mn}^{0} = \left(J_{\beta} \right)_{mi} \left({}^{\beta}\underset{\sim}{m}_{ij}^{(0)} \right) \left(J_{\beta} \right)_{nj} \tag{156} $$

We note that

$$ \boldsymbol{e}_{m}^{\beta} = \left(J_{x\beta} \right)_{mp} \boldsymbol{e}_{p};\ \boldsymbol{e}_{n}^{\beta} = \left(J_{x\beta} \right)_{nq} \boldsymbol{e}_{q} \tag{157} $$

Using (157) in (156),

$$ {}^{\beta}\boldsymbol{m}^{(0)} = \left(J_{x\beta} \right)_{mp} \boldsymbol{e}_{p} \otimes \left(J_{x\beta} \right)_{nq} \boldsymbol{e}_{q} \left({}^{\beta}m_{mn}^{0} \right) \tag{158} $$

or

$$ {}^{\beta}\boldsymbol{m}^{(0)} = \boldsymbol{e}_{p} \otimes \boldsymbol{e}_{q} \left(J_{x\beta} \right)_{mp} \left(J_{x\beta} \right)_{nq} \left({}^{\beta}m_{mn}^{0} \right) \tag{159} $$

or

$$ {}^{\beta}\boldsymbol{m}^{(0)} = \boldsymbol{e}_{p} \otimes \boldsymbol{e}_{q} \left({}^{\beta}m_{pq}^{(0)} \right) \tag{160} $$

${}^{\beta}\boldsymbol{m}^{(0)}$ is the contravariant Cauchy moment tensor acting on the faces of a tetrahedron in x^{β}-frame where faces are flat and orthogonal to each other and are parallel to the planes of the x-frame. The oblique plane of ${}^{\beta}\overline{T}$ and this new tetrahedron remains the same. Furthermore the areas of the faces of ${}^{\beta}\overline{T}$ and those of the new tetrahedron remain the same. Cauchy principle holds for ${}^{\beta}\boldsymbol{m}^{(0)}$ and ${}^{\beta}\overline{\boldsymbol{m}}^{(0)}$ moment tensors.

$$ {}^{\beta}\overline{\boldsymbol{M}} = \left({}^{\beta}\overline{\boldsymbol{m}}^{(0)} \right)^{\mathrm{T}} \cdot \overline{\boldsymbol{n}}^{\beta} \quad \text{or} \quad {}^{\beta}\boldsymbol{M} = \left({}^{\beta}\boldsymbol{m}^{(0)} \right)^{\mathrm{T}} \cdot \boldsymbol{n}^{\beta} \tag{161} $$

where ${}^{\beta}\boldsymbol{M}$ or ${}^{\beta}\overline{\boldsymbol{M}}$ is the average moment per unit area on the oblique plane of the tetrahedron ${}^{\beta}\overline{T}$. Just like ${}^{\beta}\boldsymbol{\sigma}_{(0)}$, the covariant Cauchy moment tensor ${}^{\beta}\boldsymbol{m}_{(0)}$ is also not valid for nonlinear deformation.

9.2. Contravariant Moment Tensor in x-Frame

This is useful when transforming the mathematical model from x^{β}-frame to x

-frame in Lagrangian description. Consider the integral of the average moment ${}^{\beta}\bar{\boldsymbol{M}}$ on the oblique plane of the tetrahedron with area $\mathrm{d}\bar{A}_n^{\beta}$ and we use Cauchy principle

$$\int_{\bar{A}_n^{\beta}} {}^{\beta}\bar{\boldsymbol{M}}\mathrm{d}\bar{A}_n^{\beta} = \int_{\bar{A}_n^{\beta}} \left({}^{\beta}\bar{\boldsymbol{m}}^{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}}^{\beta}\mathrm{d}\bar{A}_n^{\beta} = \int_{A_n^{\beta}} \left({}^{\beta}\boldsymbol{m}^{(0)}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{162}$$

Using

$$\mathrm{d}\boldsymbol{A}^{\beta} = \left|\boldsymbol{J}_{x\beta}\right| \left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A} \tag{163}$$

Using (163) in (162) and equating the corresponding integral in x-frame,

$$\int_{A_n} \left|\boldsymbol{J}_{x\beta}\right| \left({}^{\beta}\boldsymbol{m}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A} = \int_{A_n} \left(\boldsymbol{m}^{(0)}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A} \tag{164}$$

Therefore,

$$\left(\boldsymbol{m}^{(0)}\right)^{\mathrm{T}} = \left|\boldsymbol{J}_{x\beta}\right| \left({}^{\beta}\boldsymbol{m}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \tag{165}$$

and

$$\left({}^{\beta}\boldsymbol{m}^{(0)}\right)^{\mathrm{T}} = \frac{1}{\left|\boldsymbol{J}_{x\beta}\right|} \left(\boldsymbol{m}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{x\beta}^{\mathrm{T}}\right) \tag{166}$$

9.3. First Piola-Kirchhoff Moment Tensor in x^{β}-Frame

Consider x^{β}- and $\bar{x}^{\beta}$-frames and the correspondence rule

$$\mathrm{d}\left({}^{\beta}\bar{\boldsymbol{M}}\right) = \mathrm{d}\left({}^{\beta}\boldsymbol{M}\right) \tag{167}$$

$$\mathrm{d}\left({}^{\beta}\bar{\boldsymbol{M}}\right) = \int_{\bar{A}_n^{\beta}} {}^{\beta}\bar{\boldsymbol{M}}\mathrm{d}\bar{A}_n^{\beta} = \int_{\bar{A}_n^{\beta}} \left({}^{\beta}\bar{\boldsymbol{m}}^{(0)}\right)^{\mathrm{T}} \cdot \bar{\boldsymbol{n}}^{\beta}\mathrm{d}\bar{A}_n^{\beta} = \int_{\bar{A}_n^{\beta}} \left({}^{\beta}\bar{\boldsymbol{m}}^{(0)}\right)^{\mathrm{T}} \cdot \mathrm{d}\bar{\boldsymbol{A}}_n^{\beta} \tag{168}$$

$$\mathrm{d}\bar{\boldsymbol{A}}_n^{\beta} = \left|\boldsymbol{J}_{\beta}\right| \left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{169}$$

$$\mathrm{d}\left({}^{\beta}\boldsymbol{M}\right) = \int_{A_n^{\beta}} {}^{\beta}\boldsymbol{m}^{*} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{170}$$

Using (167) and (168),

$$\mathrm{d}\left({}^{\beta}\bar{\boldsymbol{M}}\right) = \int_{A_n^{\beta}} \left({}^{\beta}\bar{\boldsymbol{m}}^{(0)}\right)^{\mathrm{T}} \cdot \left|\boldsymbol{J}_{\beta}\right| \left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{171}$$

Using (170) and (171) in (167),

$$\int_{A_n^{\beta}} \left|\boldsymbol{J}_{\beta}\right| \left({}^{\beta}\bar{\boldsymbol{m}}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} = \int_{A_n^{\beta}} \left({}^{\beta}\boldsymbol{m}^{*}\right)^{\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}_n^{\beta} \tag{172}$$

Therefore,

$$\left({}^{\beta}\boldsymbol{m}^{*}\right)^{\mathrm{T}} = \left|\boldsymbol{J}_{\beta}\right| \left({}^{\beta}\boldsymbol{m}^{(0)}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \tag{173}$$

Thus,

$${}^{\beta}\boldsymbol{m}^{*} = \left|\boldsymbol{J}_{\beta}\right| \left(\boldsymbol{J}_{\beta}\right)^{-1} \cdot \left({}^{\beta}\boldsymbol{m}^{(0)}\right) \tag{174}$$

${}^{\beta}\boldsymbol{m}^{*}$ is non-symmetric even when ${}^{\beta}\boldsymbol{m}^{(0)}$ is symmetric.

9.4. Second Piola-Kirchhoff Moment Tensor in x^{β}-Frame

In this case, we use the correspondence rule

$$\mathrm{d}\left({}^{\beta}\bar{\boldsymbol{M}}\right)=\boldsymbol{J}_{\beta}\cdot\mathrm{d}\left({}^{\beta}\boldsymbol{M}\right) \tag{175}$$

$$\mathrm{d}\left({}^{\beta}\bar{\boldsymbol{M}}\right)=\int_{\bar{A}_n^{\beta}}{}^{\beta}\bar{\boldsymbol{M}}\mathrm{d}\bar{A}_n^{\beta}=\int_{\bar{A}_n^{\beta}}\left({}^{\beta}\bar{\boldsymbol{m}}^{(0)}\right)^{\mathrm{T}}\cdot\bar{\boldsymbol{n}}^{\beta}\mathrm{d}\bar{A}_n^{\beta}=\int_{\bar{A}_n^{\beta}}\left({}^{\beta}\bar{\boldsymbol{m}}^{(0)}\right)^{\mathrm{T}}\cdot\mathrm{d}\bar{\boldsymbol{A}}_n^{\beta} \tag{176}$$

$$\boldsymbol{J}_{\beta}\cdot\mathrm{d}\left({}^{\beta}\boldsymbol{M}\right)=\int_{A_n^{\beta}}\left(\boldsymbol{J}_{\beta}\right)\cdot\left({}^{\beta}\boldsymbol{m}^{[0]}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n^{\beta} \tag{177}$$

Using (169) in (176),

$$d\left({}^{\beta}\bar{\boldsymbol{M}}\right)=\int_{A_n^{\beta}}\left({}^{\beta}\boldsymbol{m}^{(0)}\right)^{\mathrm{T}}\cdot\left|\boldsymbol{J}_{\beta}\right|\left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n^{\beta} \tag{178}$$

Equating (177) and (178),

$$\int_{A_n^{\beta}}\left({}^{\beta}\boldsymbol{m}^{(0)}\right)^{\mathrm{T}}\cdot\left|\boldsymbol{J}_{\beta}\right|\left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n^{\beta}=\int_{A_n^{\beta}}\left(\boldsymbol{J}_{\beta}\right)\cdot\left({}^{\beta}\boldsymbol{m}^{[0]}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n^{\beta} \tag{179}$$

Therefore,

$$\left(\boldsymbol{J}_{\beta}\right)\cdot\left({}^{\beta}\boldsymbol{m}^{[0]}\right)^{\mathrm{T}}=\left|\boldsymbol{J}_{\beta}\right|\left({}^{\beta}\boldsymbol{m}^{(0)}\right)^{\mathrm{T}}\cdot\left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \tag{180}$$

Thus,

$$\left({}^{\beta}\boldsymbol{m}^{[0]}\right)^{\mathrm{T}}=\left|\boldsymbol{J}_{\beta}\right|\left(\boldsymbol{J}_{\beta}^{-1}\right)\cdot\left({}^{\beta}\boldsymbol{m}^{(0)}\right)^{\mathrm{T}}\cdot\left(\boldsymbol{J}_{\beta}^{-1}\right)^{\mathrm{T}} \tag{181}$$

If ${}^{\beta}\boldsymbol{m}^{(0)}$ is symmetric, then ${}^{\beta}\boldsymbol{m}^{[0]}$ is also symmetric.

9.5. First Piola-Kirchhoff Moment Tensor in x-Frame

Between x^{β}-frame and x-frame we have

$$\int_{A_n^{\beta}}\left({}^{\beta}\boldsymbol{m}^{*}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n^{\beta}=\int_{A_n}\left(\boldsymbol{m}^{*}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n \tag{182}$$

$$\mathrm{d}\boldsymbol{A}_n^{\beta}=\left|\boldsymbol{J}_{x\beta}\right|\left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n \tag{183}$$

Substituting from (183) into (182),

$$\int_{A_n}\left({}^{\beta}\boldsymbol{m}^{*}\right)^{\mathrm{T}}\cdot\left|\boldsymbol{J}_{x\beta}\right|\left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n=\int_{A_n}\left(\boldsymbol{m}^{*}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n \tag{184}$$

Therefore,

$$\left(\boldsymbol{m}^{*}\right)^{\mathrm{T}}=\left|\boldsymbol{J}_{x\beta}\right|\left({}^{\beta}\boldsymbol{m}^{*}\right)^{\mathrm{T}}\cdot\left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \tag{185}$$

or

$$\boldsymbol{m}^{*}=\left|\boldsymbol{J}_{x\beta}\right|\left(\boldsymbol{J}_{x\beta}\right)^{-1}\cdot\left({}^{\beta}\boldsymbol{m}^{*}\right) \tag{186}$$

and

$${}^{\beta}\boldsymbol{m}^{*}=\frac{1}{\left|\boldsymbol{J}_{x\beta}\right|}\left(\boldsymbol{J}_{x\beta}\right)\cdot\left(\boldsymbol{m}^{*}\right) \tag{187}$$

9.6. Second Piola-Kirchhoff Moment Tensor in x-Frame

Between x^{β}- and x-frames we have

$$\int_{A_n^{\beta}}\left({}^{\beta}\boldsymbol{m}^{[0]}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n^{\beta}=\int_{A_n}\left(\boldsymbol{m}^{[0]}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n \tag{188}$$

Using

$$\mathrm{d}\boldsymbol{A}_n^{\beta} = \left|\boldsymbol{J}_{x\beta}\right|\left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n \tag{189}$$

in (188) we obtain

$$\int_{A_n}\left({}^{\beta}\boldsymbol{m}^{[0]}\right)^{\mathrm{T}}\cdot\left|\boldsymbol{J}_{x\beta}\right|\left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n = \int_{A_n}\left(\boldsymbol{m}^{[0]}\right)^{\mathrm{T}}\cdot\mathrm{d}\boldsymbol{A}_n \tag{190}$$

Therefore,

$$\left(\boldsymbol{m}^{[0]}\right)^{\mathrm{T}} = \left|\boldsymbol{J}_{x\beta}\right|\left({}^{\beta}\boldsymbol{m}^{[0]}\right)^{\mathrm{T}}\cdot\left(\boldsymbol{J}_{x\beta}^{-1}\right)^{\mathrm{T}} \tag{191}$$

or

$$\boldsymbol{m}^{[0]} = \left|\boldsymbol{J}_{x\beta}\right|\left(\boldsymbol{J}_{x\beta}\right)^{-1}\cdot\left({}^{\beta}\boldsymbol{m}^{[0]}\right) \tag{192}$$

and

$${}^{\beta}\boldsymbol{m}^{[0]} = \frac{1}{\left|\boldsymbol{J}_{x\beta}\right|}\left(\boldsymbol{J}_{x\beta}\right)\cdot\left(\boldsymbol{m}^{[0]}\right) \tag{193}$$

10. Summary and Conclusions

In this paper, we have shown that additive or porous matter volume V in x-space can be mapped into a volume V_β in the fractional curvilinear coordinate space x^β with nonorthogonal curvilinear coordinate axes $x_1^{\beta_1}$, $x_2^{\beta_2}$, $x_3^{\beta_3}$ such that the volume V_β contains no pore space and results in homogenization by dimensional regularization [4] [5] [15]. We have considered the most general case of the mapping $T: x_1^{\beta_1} = x_1^{\beta_1}\left(x_1,x_2,x_3\right)$, $x_2^{\beta_2} = x_2^{\beta_2}\left(x_1,x_2,x_3\right)$ and $x_3^{\beta_3} = x_3^{\beta_3}\left(x_1,x_2,x_3\right)$, in which the gradient tensor $\boldsymbol{J}_{x\beta}$ mapping elemental lengths dx_1, dx_2, dx_3 from x-space to the elemental lengths $dx_1^{\beta_1}$, $dx_2^{\beta_2}$, $dx_3^{\beta_3}$ in x^β-space is fully populated, permitting any desired choice of T based on the porosity of the solid media. It is shown that $\beta_1,\beta_2,\beta_3 \in (0,1]$ in which a value of 1 for all β_i implies a nonporous medium. Progressively decreasing values of each β_i from one imply progressively increasing porosity. We have shown that within the limiting choice of mapping T as used in references [4]-[9] [15], the approach presented in this paper is in complete agreement with reported findings.

Besides porosity and homogenization by dimensional regularization, another aspect of the physics of porous media is that when a symmetric Cauchy stress tensor in volume V_β containing nonpolar, isotropic, and homogeneous matter in the x^β-frame is mapped or transformed into x-space, it becomes non-symmetric. In the volume V in the x-frame, the symmetric components of the stress tensor are self-equilibrating, while the antisymmetric components generate a moment field in the volume V (or $\bar{V}$) which must be balanced by an equal and opposite moment field from some other source. Anisotropy of porous matter and its interaction with the classical rotation field (or a similar rotation field) in the deforming volume V or $\bar{V}$ generates a moment field that balances the moment field due to antisymmetric stress components. This physics is exactly parallel

to the definition of physics in micropolar microcontinua.

Thus, when mapping $V \to V_\beta$ in x^β-frame two things must happen: solid matter in V is mapped in V_β without porosity and is homogenized by dimensional regularization and secondly the micropolar physics of volume V or $\bar{V}$ must also be mapped in the volume V_β. This is easily accomplished by assuming that volume V_β contains isotropic, homogeneous solid matter with rigid microconstituents that only experience rigid body rotations in volume V_β *i.e.* solid matter mapped in volume V_β is micropolar. Interaction of the rigid rotations of the microconstituents and the classical rotation field in V_β must generate the same moment field as in volume V or $\bar{V}$. We derive conservation and balance laws for micropolar solid matter for V_β in the x^β-frame or $\bar{V}_\beta$ in the $\bar{x}^\beta$-frame using conservation and balance laws of classical thermodynamics with appropriate modification due to new micropolar physics. We may also need to introduce new conservation and/or balance laws. This is all permissible as long as these are supported by classical thermodynamics, because only then the new thermodynamic framework is thermodynamically consistent, *i.e.*, supported by classical thermodynamics, which is essential for its validity. Derivations of conservation and balance laws for micropolar solid media in the x^β-frame for compressible nonlinear deformation are presented in the follow-up papers. We remark that the assumption of microconstituents in volume V_β is only used to create the necessary micropolar physics in V_β or $\bar{V}_\beta$, which is a map of micropolar physics in volumes V and $\bar{V}$.

In this paper, basic material in x^β-, $\bar{x}^\beta$- and x-frames that is necessary for deriving conservation and the balance laws and the constitutive theories has been presented. Mapping of points, lengths, areas and volumes in x- and x^β-spaces; kinematics of motion, deformation and measures; differential relations in x^β- and $\bar{x}^\beta$-frames; nonlinear Cauchy and Green's strain measures in x^β-frame; contravariant Cauchy stress tensor ${}^\beta\boldsymbol{\sigma}^{(0)}$ and Cauchy principle in x^β-frame; contravariant Cauchy stress tensor $\boldsymbol{\sigma}^{(0)}$ in x-frame; first and second Piola-Kirchhoff stress tensors in x^β-frame based on contravariant Cauchy stress tensor ${}^\beta\bar{\boldsymbol{\sigma}}^{(0)}$; and the first and second Piola-Kirchhoff stress tensors in x-frame are presented. Similar material has also been presented for the moment tensors that exist due to micropolar physics.

Derivations of the Cauchy moment tensor, first and second Piola-Kirchhoff moment tensors in x^β-frame are presented followed by their transformation or mapping in the x-frame. The material presented in this paper consists of basic core definitions and derivations that are essential in the derivation of conservation and the balance laws in $\bar{x}^\beta$- and x^β-frames and the constitutive theories addressed in subsequent papers.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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