Journal of Applied Mathematics and Physics, 2026, 14(11), 4305-4311
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Note on Finite Groups Possessing a Normal Series with Factors of Sub-Class Size 1

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How to cite this paper: Liang, Q., Chen, Y. and Li, J. (2026) Note on Finite Groups Possessing a Normal Series with Factors of Sub-Class Size 1. *Journal of Applied Mathematics and Physics*, **14**, 4305-4311.
https://doi.org/10.4236/jamp.2026.1411210

Received: September 20, 2026
Accepted: November 2, 2026
Published: November 5, 2026

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Abstract

For a finite group G and any $x \in G$, G_x is defined to be the unique minimal subnormal subgroup of G that contains the element x. The conjugacy class of x in G_x is called the sub-class of x in G, and its size is called the sub-class size of x in G. This note mainly investigates finite groups that admit a normal series in which every factor group has sub-class size 1. It is proved that if each factor group of some normal series of a finite group G has sub-class size 1, then G is solvable. Conversely, if G is solvable, then there exists a normal series of G such that each of its factor groups has sub-class size 1. In particular, G is a solvable group if and only if each chief factor of G has sub-class size 1.

Keywords

Finite Groups, Sub-Class Sizes, Factor Group, Solvable Group

1. Introduction

The study of group structure has been a central theme in group theory. With the development of this field, it has been recognized that various conjugacy invariants reflect the global structures of finite groups from different perspectives. The conjugacy classes of elements arise directly from the conjugation action of finite groups, and since the 20th century their numerical properties have served as an important tool for studying the structure of finite groups. Extensive research has been devoted to exploring how the sizes of conjugacy classes influence the structure of finite groups [1].

In the study of conjugacy class sizes, two main aspects are typically considered. The first concerns how the set of conjugacy class sizes affects the structure of finite

groups (see [2]-[5]). The second examines the role of conjugacy class sizes satisfying specific arithmetical conditions in determining the structure of finite groups (see [6]-[8]). It should be noted that these two research directions are not completely isolated, researchers often combine them to analyze the nilpotency, solvability, and finer structural properties of finite groups.

When considering subnormal subgroups, it is sometimes insufficient to consider only the arithmetic conditions and set-theoretic characteristics of conjugacy class sizes. Therefore, researchers have begun to refine and generalize the concept of conjugacy class. In 2020, Qian Guohua and Yang Yong proposed the notions of sub-class and sub-class size in [9], which serve as a generalization of conjugacy class at the level of minimal subnormal subgroups. It is known that subnormal subgroups are closed under intersection. Consequently, each element x of a finite group G is contained in precisely one minimal subnormal subgroup of G, denoted by G_x. We define the sub-class of x in G by

$$x^{G^*} \triangleq x^{G_x} = \left\{x^g \mid g \in G_x\right\},$$

and the sub-class size of x in G is defined as the number of elements in x^{G^*}, written as $\left|x^{G^*}\right|$. Furthermore, we define

$$cs^*(G) = \left\{\left|x^{G^*}\right| \mid x \in G\right\},$$

which represents the set of sub-class sizes of the group G.

The definition of a sub-class implies that the set of sub-class sizes forms a divisor set of the conjugacy class sizes. For example, consider the symmetric group S_4, the set of conjugacy class sizes is $cs(S_4) = \{1,3,6,8\}$, while the set of sub-class sizes is $cs^*(S_4) = \{1,4,6\}$. For a finite simple group G, the sub-class sizes coincide with the conjugacy class sizes, for instance,

$$cs(A_5) = cs^*(A_5) = \{1,12,15,20\}.$$

Therefore, sub-class and sub-class size can be regarded as generalizations of conjugacy class and conjugacy class size in some sense.

Some classical results on arithmetical conditions of conjugacy class size have been successfully generalized to sub-class size. For example, a finite group G is abelian precisely when $cs(G) = 1$. This result has been generalized to the statement that a finite group G is nilpotent precisely when $cs^*(G) = 1$ ([9] Corollary 2.3). Other parallel generalizations can be found in Theorems 1.4, 1.5, and 1.7 of [9].

A standard result states that a finite group G is solvable if and only if it possesses a normal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_s = 1,$$

in which each factor G_i/G_{i+1} is abelian, *i.e.*, $cs(G_i/G_{i+1}) = 1$ for $i = 0, \cdots, s-1$. A natural question is whether this classical result admits a parallel extension to sub-class. Obviously, for a finite solvable group, the sub-class sizes of its chief factors are all 1. Conversely, what is the structure of a finite group possessing a chief series each of whose chief factors has sub-class size 1? Or more generally,

what is the structure of a finite group possessing a normal series each of whose factors has sub-class size 1? In this note, we investigate these questions, and while achieving such a parallel generalization, also obtain some new characterizations of solvability for finite groups.

For convenience, all groups considered in this paper are finite. For a group G, we define the Fitting subgroup $F(G)$ as the subgroup generated by all nilpotent normal subgroups of G. Define

$$F_0(G)=1,\ F_1(G)=F(G),\ F_{k+1}(G)/F_k(G)=F\left(G/F_k(G)\right),$$

where k is a nonnegative integer. Since G is finite, $F_k(G)$ eventually stabilizes. Let s be the least nonnegative integer satisfying

$$F_s(G)=F_{s+1}(G).$$

This gives rise to a characteristic series of G

$$1=F_0(G)\le F_1(G)=F(G)\le F_2(G)\le\cdots\le F_k(G)\le\cdots\le F_s(G),$$

which is called the upper Fitting series of G, and s is called the *stabilization index* of the upper Fitting series of G. In particular, s is referred to as the Fitting length of G if G is solvable. The rest of the notation and terminology follows standard conventions and can be found in [10] [11].

2. Preliminary

In this section, we present several necessary lemmas needed in the proofs of our theorems.

Lemma 2.1 ([9] *Remark* 1.8; *Remark* 1.10) *Let G be a group and N be any normal (or subnormal) subgroup of G. Then for every $x\in N$, we have $N_x=G_x$, $x^{N_x}=x^{G_x}$, and $\left|x^{N^*}\right|\mid\left|x^G\right|$.*

Lemma 2.2 ([10] *Theorem* 2.2) *If N is a nilpotent subnormal subgroup of a group G, then $N\subseteq F(G)$.*

Lemma 2.3 ([12] *Chapter* 3, *Theorem* 2; *Remark* 2) *Let G be a group. Then $F(G)=\left\{x\in G\mid\langle x\rangle\lhd\lhd G\right\}$. Furthermore, G is nilpotent if and only if $\langle x\rangle\lhd\lhd G$ for all $x\in G$.*

3. Finite Groups with Sub-Class Size 1

Theorem 3.1 *Let G be a group. Then $x\in F(G)$ if and only if $\left|x^{G^*}\right|=1$. Moreover, $F(G)=\left\{x\in G\mid\left|x^{G^*}\right|=1\right\}$.*

Proof. First we prove necessity. By Lemma 2.3, $\langle x\rangle\lhd\lhd G$ for each $x\in F(G)$. Hence $\langle x\rangle$ is a minimal subnormal subgroup of G containing x, and thus $\left|x^{G^*}\right|=\left|x^{\langle x\rangle}\right|=1$.

Now we prove sufficiency. Suppose $x\in G$ and $\left|x^{G^*}\right|=1$. Then

$\left|x^{G^*}\right| = \left|G_x : C_{G_x}(x)\right| = 1$, so $C_{G_x}(x) = G_x$, meaning $x \in Z(G_x)$. Consequently, $\langle x\rangle \trianglelefteq Z(G_x) \trianglelefteq G_x \trianglelefteq\trianglelefteq G$, which implies $\langle x\rangle \trianglelefteq\trianglelefteq G$. Then by the minimality of G_x and Lemma 2.3, we obtain $G_x = \langle x\rangle$ and $x \in F(G)$.

In summary, $x \in F(G)$ if and only if $\left|x^{G^*}\right| = 1$, and further we obtain $F(G) = \left\{x \in G \,|\, \left|x^{G^*}\right| = 1\right\}$. Thus the conclusion is proved.

Combining the results of Lemma 2.3 and Theorem 3.1, we can give two characterizations of elements in the Fitting subgroup.

Corollary 3.2 *For a group G, the conditions below are equivalent.*

1. $x \in F(G)$;
2. $\langle x\rangle \trianglelefteq\trianglelefteq G$;
3. $\left|x^{G^*}\right| = 1$.

Using Theorem 3.1, we can provide a direct proof of Corollary 2.3 in [9] below.

Corollary 3.3 *Let G be a group. Then G is nilpotent if and only if* $cs^*(G) = 1$.

Proof. Assume first that G is nilpotent, then $G = F(G)$. By Theorem 3.1, for every $x \in G$, we have $\left|x^{G^*}\right| = 1$, which means $cs^*(G) = 1$.

Conversely, suppose that $cs^*(G) = 1$, then $\left|x^{G^*}\right| = 1$ for all $x \in G$. By Theorem 3.1, which implies $x \in F(G)$ for every $x \in G$, so $G \subseteq F(G)$. Therefore, $G = F(G)$, *i.e.*, G is nilpotent. This establishes the desired equivalence.

Combining Lemma 2.3 with Corollary 3.3, we obtain the following two characterizations of nilpotent groups.

Corollary 3.4 *For a group G, the statements below are equivalent.*

1. G *is nilpotent*;
2. $\langle x\rangle \trianglelefteq\trianglelefteq G$ *for all* $x \in G$;
3. $cs^*(G) = 1$.

4. Finite Groups with All Factor Groups Having Sub-Class Size 1

Theorem 4.1 *Let $1 = F_0(G) \le F_1(G) \le F_2(G) \le \cdots \le F_s(G)$ be the upper* Fitting *series of a group G. Then $F_s(G)$ is solvable and $G/F_s(G)$ has no nontrivial solvable normal subgroup.*

Proof. By the definition of the upper Fitting series, each factor $F_{i+1}(G)/F_i(G)$ for $i = 1, 2, \cdots, s-1$ is nilpotent, and hence solvable. Since $F_1(G) = F(G)$ is nilpotent and $F_2(G)/F_1(G)$ is solvable, it follows that $F_2(G)$ is solvable. Proceeding in this manner, $F_3(G)$ is solvable, then $F_4(G)$ is solvable, and continuing inductively we conclude that $F_s(G)$ is solvable.

Assume that $G/F_s(G)$ contains a nontrivial solvable normal subgroup. Choose a minimal solvable normal subgroup N in $G/F_s(G)$, which is abelian. Hence $G/F_s(G)$ contains a nontrivial nilpotent normal subgroup. This would

force the stabilization index of G greater than s, a contradiction.

Theorem 4.1 directly yields the following equivalent condition for the solvability of finite groups.

Corollary 4.2 *Let* $1=F_0(G)\le F_1(G)\le F_2(G)\le\cdots\le F_k(G)\le\cdots\le F_s(G)$ *be the upper* Fitting *series of a group* G. *Then* G *is solvable if and only if* $F_s(G)=G$.

Corollary 4.2 implies that a solvable group G always possesses a normal series

$$1<F_1(G)<F_2(G)<\cdots<F_k(G)<\cdots<F_s(G)=G,$$

in which all its factor groups are nilpotent. Consequently, by Corollary 3.3, the sub-class sizes of all its factor groups are equal to 1. Conversely, if G admits a normal series

$$G=G_0\trianglerighteq G_1\trianglerighteq\cdots\trianglerighteq G_m=1$$

with $cs^*(G_i/G_{i+1})=1$ for all $i=0,1,\cdots,m-1$, then is G solvable, or even supersolvable? Note that it is not necessarily supersolvable. For instance, the symmetric group S_4 has a chief series

$$1\trianglelefteq K_4\trianglelefteq A_4\trianglelefteq S_4,$$

whose chief factors are $K_4\cong C_2\times C_2$, $A_4/K_4\cong C_3$, and $S_4/A_4\cong C_2$, all of which are abelian. Hence, the sub-class sizes of all chief factors are 1, yet S_4 is solvable but not supersolvable. In fact, the group G is solvable, as stated in the following theorem.

Theorem 4.3 *Suppose* G *is a group. Then* G *is solvable if and only if* G *admits a normal series*

$$G=G_0\trianglerighteq G_1\trianglerighteq\cdots\trianglerighteq G_m=1,$$

satisfying $cs^*(G_i/G_{i+1})=1$ *for* $i=0,\cdots,m-1$.

Proof. The necessity is clear, so it remains to prove the sufficiency. The sufficiency can also be proved below by induction on $|G|$, or by a similar method used in the proof of Theorem 4.1.

When $|G|=1$, the conclusion is clearly true. Assume $|G|>1$, all groups of order less than $|G|$ hold. Since $cs^*(G/G_1)=1$, Corollary 3.3 implies that G/G_1 is nilpotent, hence solvable. Since G_1 is a proper normal subgroup of G, its order is less than $|G|$. Consider the normal series of G_1,

$$G_1\trianglerighteq G_2\trianglerighteq\cdots\trianglerighteq G_m=1.$$

By the induction hypothesis, G_1 is solvable, and therefore G is solvable.

Corollary 4.4 *Suppose that a group* G *possesses a chief series* $G=G_0\trianglerighteq G_1\trianglerighteq\cdots\trianglerighteq G_t=1$. *Then* G *is solvable if and only if* $cs^*(G_i/G_{i+1})=1$ *for* $i=0,1,\cdots,t-1$.

Proof. Indeed, Theorem 4.3 directly yields the sufficiency. To establish necessity, assume that G is solvable, then all chief factors of G are abelian. Hence, by Corollary 3.3,

$$cs^*(G_i/G_{i+1})=1,\ i=0,1,\cdots,t-1.$$ □

Remark 4.5 *Corollary* 3.3 *characterizes nilpotency from the point of view of the whole group, using the condition* $cs^*(G)=1$. *In contrast, Theorem* 4.3 *and Corollary* 4.4 *turn the viewpoint to the factors of a normal series and to chief factors, characterizing solvability by the sub-class sizes of these factors. The former is the basic result, while the latter are its extensions and refinements.*

Remark 4.6 *By the method of Theorem* 4.3, *we obtain the subsequent conclusion: for a fixed prime* p *dividing* $|G|$, G *is* p*-solvable if and only if every* p*-chief factor of* G *has sub-class size* 1.

Combining Theorem 4.3 with the conclusion of Remark 4.6, we obtain two characterizations of solvable groups below.

Corollary 4.7 *For a group* G, *the conditions below are equivalent.*

1. G *is solvable.*
2. *For every prime divisor* p *of* $|G|$, *all* p*-chief factors of* G *have sub-class size* 1.
3. G *admits a normal series*

$$G=G_0 \rhd G_1 \rhd \cdots \rhd G_m=1,$$

satisfying $cs^*(G_i/G_{i+1})=1$ *for* $i=0,\cdots,m-1$.

5. Conclusion

In this note, we studied the structure of finite groups by means of sub-class sizes. We characterized elements of the Fitting subgroup in terms of sub-class size 1. Our main result gives an equivalent condition for the solvability of a finite group: a finite group is solvable if and only if it has a normal series each of whose factors has sub-class size 1. Moreover, this condition can be further reduced to chief factors. Some related results for p-solvable groups are also obtained. Next, we will study the structural characterization of finite groups with a given set of sub-class sizes.

Acknowledgement

This work was supported by the Natural Science Foundation Project of Chongqing (CSTB2022NSCQ-MSX0238).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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