

Trapped Vlasov-Poisson- Λ Equilibria and Tidal Response

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Abstract

A positive cosmological constant makes the Newtonian particle potential decrease without bound at large radius. The same energy and angular momentum may describe both bounded motion and escape. We prove that a nontrivial global Jeans distribution, which assigns a non-negative density to all states with energy and squared angular momentum, has infinite mass in a class of spherical potentials decreasing as a power of radius. The positive cosmological constant gives the quadratic case, with at least logarithmic mass divergence. We construct instead a smooth distribution of finite mass that occupies only the bounded inner component of each admissible invariant level. The generated gravitational field satisfies the same equations that define particle motion. A smooth radial cutoff removes the exterior component without changing any occupied trajectory. The occupied energies lie a positive distance below the outer radial separatrix, the barrier separating bounded radial motion from escape, and this separation persists under small parameter changes. Linearization about the equilibrium yields a Volterra equation: the density perturbation depends through a time integral on its previous gravitational field and on the imposed tide. On every bounded time interval, we prove existence, uniqueness, convergence of the repeated time-integral series, and continuous dependence on the initial perturbation and tidal history. Rotational symmetry isolates the quadrupole, the trace-free part of the second spatial moment, and reduces its tidal response to a scalar kernel with an exact initial derivative. Orbit formulas express the projected kernels through unperturbed trajectories and their dependence on initial momentum. An explicit equilibrium of uniform density and solutions with one scale factor for all directions provide exact comparisons; deformation about a fixed axis supplies a separate approximation. These results distinguish the global restriction imposed by cosmological repulsion from the local response sustained by self-gravity, without assuming stability or decay over arbitrarily long times.

Keywords

Vlasov-Poisson Equation, Cosmological Constant, Bounded Equilibria, Jeans Distribution, Tidal Response

1. Introduction

Let $\Lambda > 0$ denote the cosmological constant. The Newtonian Vlasov-Poisson model describes a collisionless particle distribution transported by the gravitational field generated by its own mass density; the potential is determined by Poisson's equation. For a spherical distribution of finite mass, the cosmological contribution is a negative quadratic function of radius, so the total particle potential tends to $-\infty$ at large distance. Particle energy and angular momentum can consequently take the same values on a family of bounded trajectories and on a disconnected family extending to infinity. Classical results for the isolated gravitational system establish compactly supported spherical equilibria [1] [2]. Relativistic stationary states with nonzero cosmological constant are constructed in [3]; kinetic descriptions of cosmological structures are considered in [4]. Here the question is Newtonian and localized: how can a time-independent distribution occupy only the bounded trajectories while generating its own gravitational field?

A global Jeans distribution (see Section 3.1) does not distinguish the connected components associated with the same numerical invariants. We prove that every nontrivial non-negative distribution of this form has infinite mass for a class of spherical potentials that decrease as a power of radius. A component-selected equilibrium (see Section 3.2) avoids this obstruction. Its construction uses a smooth radial factor whose variation is confined to an empty annulus, so no force or material wall is added. The occupied trajectories remain below the outer radial separatrix (see Section 3.2), with a strictly positive energy gap. This bounded regime differs from the dispersal of sufficiently small initial distributions under repulsive quadratic potentials [5] [6].

After establishing the existence and confinement properties of the stationary state, we study its linear response to an external tidal field, *i.e.*, to the spatial variation of the external gravitational acceleration relative to the centre of mass. We retain the quadratic potential whose coefficient matrix is symmetric and has zero trace; it therefore describes anisotropic deformation without an isotropic compression term. Linearization retains only terms of first order in the disturbance and tidal amplitude. The energy gap provides a compact invariant set in phase space, the space of particle positions and momenta, on which the linearized equation can be integrated along the stationary trajectories. The resulting Volterra equation is a causal time-integral equation: the density at a given time depends on earlier density values and the earlier imposed tide. General evolution and integral-equation methods [7]-[9] give existence, uniqueness and continuous dependence on each bounded time interval.

These statements differ from conclusions about arbitrarily long times. The spectrum, meaning the complex numbers for which the linearized operator minus that number times the identity has no bounded inverse, and persistent oscillations are studied in [10] [11]. Phase mixing means cancellation in macroscopic observables caused by differing orbital frequencies; its relation to damping is investigated in [12] [13]. Damping here means decay of a density or field perturbation, not necessarily decay of the distribution in a strong norm. Stellar response and weakly damped oscillations are discussed in [14]-[16]. Nonlinear orbital stability would instead require a finite disturbance to remain close to the equilibrium, allowing its symmetries, for all subsequent times. None of these long-time conclusions is inferred from the estimates proved here.

The quadrupole is the trace-free part of the second spatial mass moment. Rotational symmetry reduces its response to one scalar quadrupole susceptibility (see Section 5.3), whose initial derivative follows from an exact moment identity. The associated orbit formulas involve unperturbed trajectories and their derivatives with respect to initial momentum. The uniform-density solution of Kurth [17] [18] and an axisymmetric linear deformation of a spherical distribution [19] [20] provide independent analytical comparisons. Sections 2-5 state the model and prove the equilibrium and response results; Section 6 discusses their mathematical and physical implications.

2. Vlasov-Poisson- Λ Model with an External Tide

2.1. Equations and Normalization

Let $G > 0$ be Newton's gravitational constant, $c > 0$ the speed of light, and t time. A particle position $\mathbf{x}$ and momentum per unit mass $\mathbf{p}$ belong to Euclidean space $\mathbb{R}^3$, with $\dot{\mathbf{x}} = \mathbf{p}$; a dot denotes a time derivative. We write $|\mathbf{x}|$ for Euclidean length and $\mathbf{x} \cdot \mathbf{p}$ for the Euclidean inner product. A phase point is $z = (\mathbf{x}, \mathbf{p}) \in \mathbb{R}^6$, and $dz = d\mathbf{x}d\mathbf{p}$ is Lebesgue phase volume. The non-negative phase-space mass density $F(t, \mathbf{x}, \mathbf{p})$ has spatial density ρ_F and total mass M [1] [21], defined by

$$\rho_F(t, \mathbf{x}) = \int_{\mathbb{R}^3} F(t, \mathbf{x}, \mathbf{p}) d\mathbf{p}, \quad M = \int_{\mathbb{R}^3} \rho_F(t, \mathbf{x}) d\mathbf{x}.$$

Define the cosmological frequency $\Omega_\Lambda > 0$ by $\Omega_\Lambda^2 = c^2 \Lambda / 3$. The total particle potential Φ_F^Λ satisfies the Vlasov-Poisson- Λ equations

$$\partial_t F + \mathbf{p} \cdot \nabla_{\mathbf{x}} F - \nabla_{\mathbf{x}} \Phi_F^\Lambda \cdot \nabla_{\mathbf{p}} F = 0, \quad (1)$$

$$\Delta_{\mathbf{x}} \Phi_F^\Lambda = 4\pi G \rho_F - c^2 \Lambda. \quad (2)$$

Here $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{p}}$ are gradients in position and momentum, and $\Delta_{\mathbf{x}} = \nabla_{\mathbf{x}} \cdot \nabla_{\mathbf{x}}$ is the spatial Laplacian. The matter potential U_F^g is specified by

$$U_F^g(\mathbf{x}) = -G \int_{\mathbb{R}^3} \frac{\rho_F(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}, \quad \Phi_F^\Lambda = U_F^g - \frac{\Omega_\Lambda^2}{2} |\mathbf{x}|^2.$$

Thus the matter acceleration is $-\nabla U_F^g$, whereas the cosmological acceleration is

$+\Omega_\Lambda^2 \mathbf{x}$. For differentiable phase-space functions f, g , the canonical Poisson bracket is [22]

$$\{f, g\} := \nabla_x f \cdot \nabla_p g - \nabla_p f \cdot \nabla_x g.$$

For a spherical state let $r = |\mathbf{x}|$. Finite spherical mass gives $U_F^g(r) = -GM/r + o(r^{-1})$ as $r \rightarrow \infty$, where $o(r^{-1})$ denotes a remainder whose product with r tends to zero. Define $L = |\mathbf{x} \times \mathbf{p}|$, the magnitude of angular momentum per unit mass. Outside the spherical matter support the radial acceleration is

$$\ddot{r} = -\frac{GM}{r^2} + \Omega_\Lambda^2 r + \frac{L^2}{r^3}.$$

No normalization at infinity is imposed on the total potential Φ_F^Λ .

2.2. External Quadrupolar Field

Let $\Phi_{\text{ext}}(t, \mathbf{x})$ be a smooth external gravitational potential near the centre of mass. Its constant term has no effect on motion, and its gradient at the centre is removed by adopting the freely falling frame. Let ∂_i denote differentiation with respect to the i th position coordinate, and let δ_{ij} equal one for $i = j$ and zero otherwise. The symmetric trace-free tidal matrix $\hat{T}_3(t)$ is

$$(\hat{T}_3)_{ij} = \partial_i \partial_j \Phi_{\text{ext}}(t, 0) - \frac{1}{3} \delta_{ij} \Delta_x \Phi_{\text{ext}}(t, 0).$$

For a real matrix A , A^T denotes its transpose and $\text{tr } A$ the sum of its diagonal entries. The tidal matrices form the five-dimensional space

$$\mathbb{T}_2 := \{A \in \mathbb{R}^{3 \times 3} : A = A^T, \text{tr } A = 0\}.$$

The retained external potential and acceleration are

$$\Phi_{\text{tid}}(t, \mathbf{x}) = \frac{1}{2} \mathbf{x}^T \hat{T}_3(t) \mathbf{x}, \quad -\nabla_x \Phi_{\text{tid}} = -\hat{T}_3(t) \mathbf{x}.$$

Consequently the forced kinetic equation is

$$\partial_t F + \mathbf{p} \cdot \nabla_x F - \left[\nabla_x \Phi_F^\Lambda + \hat{T}_3(t) \mathbf{x} \right] \cdot \nabla_p F = 0. \quad (3)$$

The cosmological contribution is included in Φ_F^Λ , not in $\hat{T}_3$.

3. Component-Selected Stationary States

3.1. Global Jeans Obstruction

For a spherical potential $\Phi(r)$, define the particle energy $E = |\mathbf{p}|^2/2 + \Phi(r)$. Energy and L^2 are invariants, that is, constants along a particle trajectory. A global Jeans distribution is $F = \Psi(E, L^2)$ for one non-negative function Ψ , with the same value on every connected component carrying the same invariants. Here “global” requires one function of the numerical invariant values on the entire phase space. This restriction does not exclude stationary distributions assigning different occupations to disconnected components of the same invariant level. We

call the potential anti-confining when $\Phi(r) \rightarrow -\infty$. Measurability and almost-everywhere statements below refer to Lebesgue measure; “almost everywhere” excludes sets of measure zero.

Theorem 3.1 (Global Jeans states in anti-confining spherical potentials). *Let $a > 0$, $0 < \beta \leq 2$, and let a spherical potential satisfy*

$$\Phi(r) = -ar^\beta + o(r^\beta) \quad (r \rightarrow \infty),$$

where $o(r^\beta)/r^\beta \rightarrow 0$. Let $\Psi: \mathbb{R} \times [0, \infty) \rightarrow [0, \infty]$ be measurable, non-negative, and not almost everywhere zero. Then the spherical phase-space density

$$F(\mathbf{x}, \mathbf{p}) = \Psi(E, L^2), \quad E = \frac{|\mathbf{p}|^2}{2} + \Phi(r), \quad L^2 = |\mathbf{x} \times \mathbf{p}|^2,$$

satisfies, for some $C > 0$ and all sufficiently large r ,

$$\rho_F(r) \geq Cr^{-2-\beta/2}. \quad (4)$$

Consequently its mass is infinite. More precisely, the mass inside radius R grows at least as $R^{1-\beta/2}$ when $0 < \beta < 2$ and at least as $\log R$ when $\beta = 2$.

Proof. Since $\Psi \geq 0$, all integrals below are understood in the extended sense and Tonelli’s theorem allows their order to be interchanged for non-negative integrands. Because Ψ is not almost everywhere zero, its superlevel sets in bounded rectangles give $m > 0$ and a bounded measurable set $A \subset \mathbb{R} \times [0, \infty)$ with positive area $|A|$ such that $\Psi(E, s) \geq m$ almost everywhere on A .

At a fixed radius $r = |\mathbf{x}| > 0$, let $\mathbf{e}_r := \mathbf{x}/r$ be the radial unit vector, write $\mathbf{p} = p_r \mathbf{e}_r + \mathbf{p}_\perp$, where $p_r = \mathbf{p} \cdot \mathbf{e}_r$ and $\mathbf{p}_\perp \cdot \mathbf{e}_r = 0$, and set $s = L^2 = r^2 |\mathbf{p}_\perp|^2$. Then

$$d^3 \mathbf{p} = 2\pi |\mathbf{p}_\perp| d|\mathbf{p}_\perp| dp_r = \frac{\pi}{r^2} ds dp_r.$$

Changing from p_r to E on the two branches $p_r = \pm \sqrt{2[E - \Phi(r) - s/(2r^2)]}$ gives

$$\rho_F(r) = \frac{2\pi}{r^2} \int_0^\infty \int_{\Phi(r)+s/(2r^2)}^\infty \frac{\Psi(E, s)}{\sqrt{2[E - \Phi(r) - s/(2r^2)]}} dE ds. \quad (5)$$

Choose $E_- < E_+$ and $s_+ < \infty$ such that $A \subset [E_-, E_+] \times [0, s_+]$. The asymptotic hypothesis implies that there exist constants $r_0 > 0$ and $C_A > 0$ such that, for every $r \geq r_0$ and every $(E, s) \in A$,

$$0 < E - \Phi(r) - \frac{s}{2r^2} \leq C_A r^\beta.$$

Thus A lies in the integration domain and restriction of Equation (5) to A yields

$$\rho_F(r) \geq \frac{2\pi m |A|}{\sqrt{2C_A}} r^{-2-\beta/2},$$

which is Equation (4). Finally,

$$4\pi \int_{r_0}^R r^2 \rho_F(r) dr \geq C \int_{r_0}^R r^{-\beta/2} dr,$$

and the last integral has the stated power or logarithmic divergence. $\square$

Corollary 3.2 (Global Jeans obstruction for Vlasov-Poisson- Λ). *Let $\Lambda > 0$. No nontrivial spherically symmetric self-consistent Vlasov-Poisson- Λ stationary state of finite total mass can have the global form $F(\mathbf{x}, \mathbf{p}) = \Psi(E, L^2)$ with a continuous non-negative Ψ . Within this ansatz, a finite-mass equilibrium must distinguish the connected invariant components.*

Proof. Finite spherical mass and the Newtonian normalization give $\Phi(r) = -\Omega_\Lambda^2 r^2 / 2 + O(r^{-1})$. Here and below $O(g)$ denotes a remainder bounded in magnitude by a constant multiple of $|g|$ in the stated limit. Continuity and nontriviality of Ψ provide a bounded positive-measure rectangle on which Ψ has a positive lower bound; if the positive point has $s = 0$, take a one-sided rectangle in $s \geq 0$. Theorem 3.1 with $\beta = 2$ gives logarithmic mass divergence. $\square$

Remark 3.3 (Restriction to the inner connected component). The smooth states below occupy the inner bounded component of the admissible phase set. Replacing $\Psi(E, L^2)$ by a smooth global function does not remove the exterior component with the same invariant values and therefore does not restore finite mass.

3.2. A Smooth Stationary Solution on an Inner Bounded Orbit Family

Corollary 3.2 excludes a single global $\Psi(E, L^2)$. We therefore make the connected component part of the stationary-state data.

Definition (Component-selected stationary distribution). Let $H(\mathbf{x}, \mathbf{p}) = |\mathbf{p}|^2 / 2 + \Phi(|\mathbf{x}|)$ be the spherical particle Hamiltonian, the energy as a function on phase space. For invariant values (E, s) , with $s = L^2 \geq 0$, suppose the joint level $\{H = E, L^2 = s\}$ has a bounded inner connected component $\mathcal{C}_{\text{in}}(E, s)$ and an unbounded exterior component $\mathcal{C}_{\text{out}}(E, s)$. A non-negative stationary distribution is component-selected on the inner component when it equals $\Psi(E, s)$ on $\mathcal{C}_{\text{in}}(E, s)$ and zero on $\mathcal{C}_{\text{out}}(E, s)$, for the same invariant values and a non-negative function Ψ . Thus the component, not only the numerical invariants, determines occupation.

The construction below makes this selection without a material boundary. We write C^m for functions with m continuous derivatives, C^∞ for smooth functions, and a subscript c for compact support. The support $\text{supp } f$ is the closure of the set where $f \neq 0$; compact phase-space support bounds both position and momentum. Choose a nondecreasing smooth function $\mathcal{G}: \mathbb{R} \rightarrow [0, 1]$ equal to zero on $(-\infty, 1]$ and one on $[2, \infty)$. Specifically, throughout the construction and numerical illustration, for a real argument u we take

$$\mathcal{G}(u) = \begin{cases} 0, & u \leq 1, \\ \frac{\exp[-1/(u-1)]}{\exp[-1/(u-1)] + \exp[-1/(2-u)]}, & 1 < u < 2, \\ 1, & u \geq 2. \end{cases}$$

Its transition is flat: every derivative of positive order vanishes at both endpoints. For a cutoff parameter $\varepsilon > 0$, define the microscopic weight φ_ε and its isotropic momentum integral g_ε by

$$\varphi_\varepsilon(\eta) = \eta \mathcal{G}(\eta/\varepsilon), \quad g_\varepsilon(y) = 4\pi\sqrt{2} \int_0^y \varphi_\varepsilon(\eta)(y-\eta)^{1/2} d\eta \quad (y > 0), \quad (6)$$

with $g_\varepsilon(y) = 0$ for $y \leq 0$. Here η is the scalar argument of the energy cutoff, and y is the relative potential, the cutoff energy minus the particle potential. For any real u , write $u_+ = \max(u, 0)$.

Lemma 3.4 (Smooth cutoff density and limiting power law). *The function g_ε is C^∞ , vanishes to all orders at $y = \varepsilon$, and on every bounded interval converges in C^1 as $\varepsilon \downarrow 0$ to*

$$g_0(y) = \frac{16\pi\sqrt{2}}{15} (y_+)^{5/2}. \quad (7)$$

Proof. Flatness of $\mathcal{G}$ makes the zero extension of φ_ε smooth. Equation (6) is locally its convolution with $u_+^{1/2}$, the integral of the product after translating one factor by y . More explicitly, for every integer $n \geq 0$,

$$g_\varepsilon^{(n)}(y) = 4\pi\sqrt{2} \int_0^\infty u^{1/2} \varphi_\varepsilon^{(n)}(y-u) du.$$

On a bounded y -interval the integrand vanishes for all sufficiently large u , uniformly in y . Differentiation therefore acts only on the smooth factor φ_ε , while $u^{1/2}$ is integrable at zero. This proves continuity of every displayed derivative, with no endpoint term. For $y > \varepsilon$ the integral is over $0 \leq u \leq y - \varepsilon$; its absolute value is bounded by a constant times $(y - \varepsilon)^{3/2}$ times the supremum of $|\varphi_\varepsilon^{(n)}|$ on $[\varepsilon, y]$. Every derivative thus tends to zero as $y \downarrow \varepsilon$, agreeing with the zero extension.

Moreover $0 \leq \eta_+ - \varphi_\varepsilon(\eta) \leq \eta \mathbf{1}_{[0, 2\varepsilon]}(\eta)$, where $\mathbf{1}_A$ is one on a set A and zero outside. The defining integral and its first y -derivative give the stated C^1 convergence. Direct integration gives Equation (7). $\square$

Let $s \geq 0$ now denote a dimensionless radius, $y_c > 0$ the central relative potential, and $\mu \geq 0$ the dimensionless cosmological parameter. A prime denotes differentiation of a function with respect to its scalar argument. Consider the regular radial equation

$$y'' + \frac{2}{s} y' = \mu - g_\varepsilon(y), \quad y(0) = y_c, \quad y'(0) = 0. \quad (8)$$

At the origin, regularity means a C^2 solution satisfying the displayed central conditions. For $(\varepsilon, \mu) = (0, 0)$, denote the solution by y_0 ; the equation then has a density proportional to $(y_+)^{5/2}$ and is the Lane-Emden equation of index 5/2. Its first zero S_0 is finite and satisfies $y'_0(S_0) < 0$ [2]. A crossing $y(S) = \varepsilon$ is transverse when $y'(S) \neq 0$.

Lemma 3.5 (Persistence of a transverse matter boundary). *For sufficiently small $\varepsilon, \mu > 0$, the regular solution of Equation (8) is strictly decreasing up to a unique first crossing $y(S_{\varepsilon, \mu}) = \varepsilon$ with $y'(S_{\varepsilon, \mu}) < 0$. Moreover $S_{\varepsilon, \mu} \rightarrow S_0$ and $y'(S_{\varepsilon, \mu}) \rightarrow y'_0(S_0) < 0$ as $(\varepsilon, \mu) \rightarrow (0, 0)$.*

Proof. The regular integral equation is

$$y(s) = y_c + \int_0^s \tau^{-2} \int_0^\tau \sigma^2 [\mu - g_\varepsilon(y(\sigma))] d\sigma d\tau.$$

Choose $d > 0$ small enough that $y'_0 < 0$ on $(0, S_0 + d]$. Lemma 3.4 gives a common Lipschitz bound for g_ε on a bounded interval containing the range of y_0 . Gronwall's inequality for the integral equation and continuation on that interval yield $y \rightarrow y_0$ in $C^1([0, S_0 + d])$.

Since $g_0(y_c) > 0$, sufficiently small parameters give $g_\varepsilon(y_c) > \mu$ and $y''(0) = [\mu - g_\varepsilon(y_c)]/3 < 0$. By continuity there are $s_c > 0$ and $\kappa > 0$, independent of these parameters, with $g_\varepsilon(y(s)) - \mu \geq \kappa$ on $[0, s_c]$. Hence

$$s^2 y'(s) = \int_0^s \sigma^2 [\mu - g_\varepsilon(y(\sigma))] d\sigma \leq -\kappa s^3/3,$$

so $y' < 0$ near the centre. On $[s_c, S_0 + d]$, y'_0 is bounded above by a negative constant; C^1 convergence preserves its sign. The inequalities $y(S_0 - d) > \varepsilon$ and $y(S_0 + d) < \varepsilon$ hold for small parameters. Strict monotonicity gives one transverse crossing between them. Convergence of its radius and derivative follows from C^1 convergence and $y'_0(S_0) < 0$. $\square$

Theorem 3.6 (Smooth finite-mass stationary solution on an inner bounded orbit family). *Let $\Lambda > 0$. There exist a positive density scale C_ρ , a cutoff energy $E_c \in \mathbb{R}$, a smooth spherical potential Φ_* , and radii $0 < R < r_1 < r_2 < r_\Lambda$, where $r_\Lambda = (GM/\Omega_\Lambda^2)^{1/3}$ and M is the resulting total mass. Let ζ be a smooth radial function equal to one for $r \leq r_1$ and zero for $r \geq r_2$. The nonzero spherical distribution*

$$F_{*,\varepsilon}(\mathbf{x}, \mathbf{p}) = C_\rho \zeta(|\mathbf{x}|) \varphi_\varepsilon \left(E_c - \frac{|\mathbf{p}|^2}{2} - \Phi_*(|\mathbf{x}|) \right) \in C_c^\infty(\mathbb{R}^6), \quad (9)$$

together with $\Phi_ \in C^\infty(\mathbb{R}^3)$, is a classical self-consistent stationary Vlasov-Poisson- Λ state, meaning that both equations hold pointwise with its own generated potential. Its matter support lies in the closed ball $\overline{B_R} = \{\mathbf{x} : |\mathbf{x}| \leq R\}$. Define $H_*(\mathbf{x}, \mathbf{p}) = |\mathbf{p}|^2/2 + \Phi_*(|\mathbf{x}|)$ and*

$$r_\Lambda = \left(\frac{GM}{\Omega_\Lambda^2} \right)^{1/3}, \quad E_{\max} = \Phi_*(R), \quad E_{\text{sep}} = \Phi_*(r_\Lambda).$$

The exterior radial potential has its maximum at r_Λ . The energy E_{sep} is the outer radial separatrix level, separating bounded radial motion from the exterior escape branch. Every occupied trajectory remains in the inner bounded component, and

$$\sup_{\text{supp } F_{*,\varepsilon}} H_* \leq E_{\max} < E_{\text{sep}}, \quad \Delta_{\text{sep}} := E_{\text{sep}} - E_{\max} > 0. \quad (10)$$

Proof. Choose the parameters from Lemma 3.5, put $\alpha = 3\Omega_\Lambda^2/\mu$, $C_\rho = \alpha/(4\pi G)$, $R = S_{\varepsilon,\mu}/\sqrt{\alpha}$, and $y_*(r) = y(\sqrt{\alpha}r)$. The density $\rho_* = C_\rho g_\varepsilon(y_*)$ on $r < R$ and zero outside is C_c^∞ because the crossing is transverse and g_ε is flat. Define $U_*(\mathbf{x}) = -G \int_{\mathbb{R}^3} \rho_*(\mathbf{y})/|\mathbf{x} - \mathbf{y}| d\mathbf{y}$ and $\Phi_* = U_* - \Omega_\Lambda^2 r^2/2$. Elliptic regularity gives smoothness [23]; outside the support,

$\Phi_*(r) = -GM/r - \Omega_\Lambda^2 r^2/2$ by Newton's theorem [21]. Integrating the radial equation at R gives $R^2 y'(R) = \Omega_\Lambda^2 R^3 - GM < 0$, hence $R < r_\Lambda$. With $E_c = \varepsilon + \Phi_*(R)$, the radial equations imply $E_c - \Phi_* = y_*$ on $[0, R]$.

Choose $R < r_1 < r_2 < r_\Lambda$ and a smooth selector ζ equal to one on $[0, r_1]$ and zero on $[r_2, \infty)$. On $\text{supp } \nabla \zeta$ the argument of φ_ε is below ε , so the microscopic cutoff is already zero. Hence Equation (9) has density ρ_* and

$$\{F_{*,\varepsilon}, H_*\} = C_\rho \zeta'(r) \frac{\mathbf{x} \cdot \mathbf{p}}{r} \varphi_\varepsilon(E_c - H_*) = 0.$$

Thus the selector introduces neither a wall nor a force. Finally, with $\xi = R/r_\Lambda$,

$$\Delta_{\text{sep}} = \Omega_\Lambda^2 r_\Lambda^2 \frac{(1-\xi)^2 (\xi+2)}{2\xi} > 0. \quad (11)$$

Nonzero occupation requires $E_c - H_* > \varepsilon$, so continuity gives

$\sup_{\text{supp } F_{*,\varepsilon}} H_* \leq E_c - \varepsilon = E_{\text{max}}$. For fixed angular momentum the effective radial potential is $\Phi_*(r) + L^2/(2r^2)$. Any trajectory reaching $r = r_\Lambda$ must satisfy

$$E = \frac{p_r^2}{2} + \Phi_*(r_\Lambda) + \frac{L^2}{2r_\Lambda^2} \geq E_{\text{sep}}.$$

No occupied trajectory can therefore cross that sphere. The non-negative angular-momentum term strengthens this bound; no assertion about a global maximum of the effective potential is needed. $\square$

For a fixed ε , we henceforth abbreviate this equilibrium by $F_* := F_{*,\varepsilon}$.

Remark 3.7 (Role of the radial cut-off ζ). The derivative of ζ is supported where the microscopic energy cutoff vanishes; it removes the disconnected exterior component without changing occupied characteristics or the self-consistent density.

Remark 3.8 (Comparison with dispersing solutions). The state remains confined by its self-gravity. This differs from the dispersing solutions in [5] [6], whose trajectories enter the region dominated by the outward quadratic force.

Corollary 3.9 (Local persistence of the component-selected family). *If one member of the radial family has a transverse first crossing and $g_\varepsilon(y_c) > \mu$, then the same is true in a parameter neighbourhood. There R , M , r_Λ , and Δ_{sep} vary continuously, remain strictly ordered by $R < r_\Lambda$ and $\Delta_{\text{sep}} > 0$, and the same selector radii $r_1 < r_2$ may be used after shrinking the neighbourhood.*

Proof. The integral equation above gives C^1 continuous dependence on (y_c, ε, μ) , so the nonzero crossing derivative and strict monotonicity persist for sufficiently small parameter changes. Writing $S = S_{\varepsilon, \mu}$, continuity of the mass integral and $GM - \Omega_\Lambda^2 R^3 = -\alpha^{-1/2} S^2 y'(S) > 0$ gives continuity and the strict ordering; compactness of a smaller neighbourhood permits fixed selector radii. $\square$

3.3. An Illustrative Equilibrium

In dimensionless units $G=1$ and $4\pi G C_\rho = 1$, the choice $y_c = 1$, $\varepsilon = 0.05$, $\mu = 0.005$ gives

$$R = 2.273801, \quad M = 0.974642, \quad r_\Lambda = 8.362422, \quad \Delta_{\text{sep}} = 0.258123.$$

Figure 1 illustrates the smooth density and the separatrix gap.

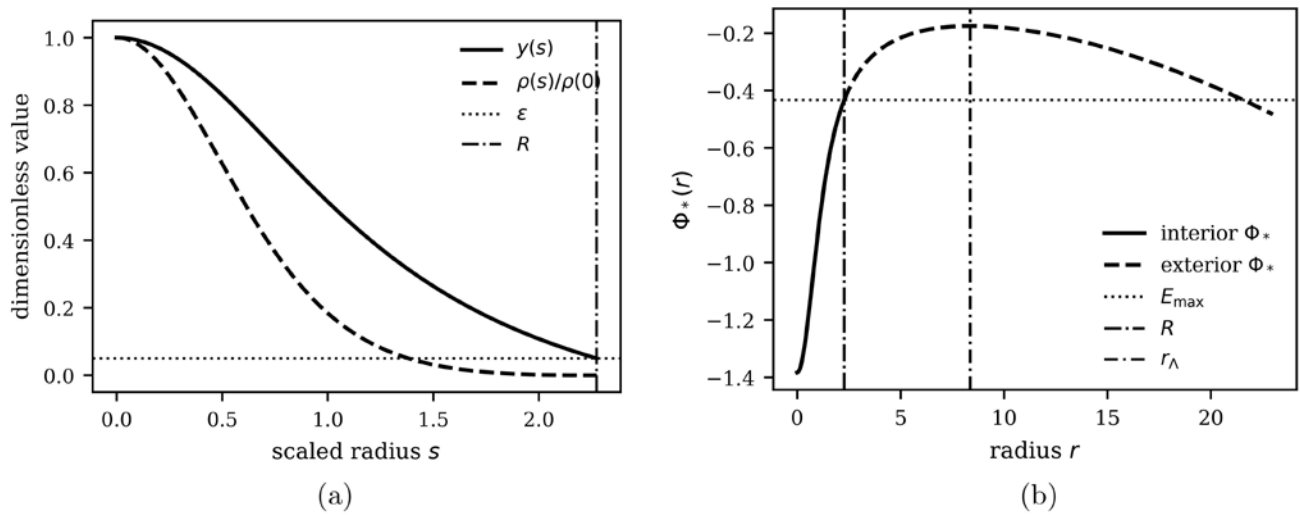


Figure 1. Representative smooth trapped equilibrium. The matter boundary is $y(R) = \varepsilon$ and $E_{\text{max}} < E_{\text{sep}}$. (a) Relative potential and normalized density. (b) Total potential and outer separatrix.

3.4. An Explicit Kurth- Λ Equilibrium

For the explicit comparison state, fix its mass $M > 0$ and radius $R > 0$. Define the gravitational frequency ω_g , the net interior frequency ω_* , and the exterior radius of zero radial acceleration r_Λ [24] by

$$\omega_g^2 = \frac{GM}{R^3}, \quad \omega_*^2 = \omega_g^2 - \Omega_\Lambda^2 > 0, \quad R < r_\Lambda = \left(\frac{GM}{\Omega_\Lambda^2} \right)^{1/3} \quad (12)$$

With $\mathbf{X} = \mathbf{x}/R$, $\mathbf{V} = \mathbf{p}/(\omega_* R)$, $\mathcal{Q} = 1 - |\mathbf{X}|^2 - |\mathbf{V}|^2 + |\mathbf{X} \times \mathbf{V}|^2$, and $\mathcal{D} = \{\mathcal{Q} > 0, |\mathbf{X} \times \mathbf{V}| < 1\}$, set

$$F_{0,\Lambda}(\mathbf{x}, \mathbf{p}) = \frac{3M}{4\pi^3 \omega_*^3 R^6} \frac{\mathbf{1}_{\mathcal{D}}(\mathbf{X}, \mathbf{V})}{\sqrt{\mathcal{Q}}}. \quad (13)$$

The quotient is evaluated on $\mathcal{D}$ and the distribution is zero elsewhere. The velocity integral is $\int \mathbf{1}_{\mathcal{D}}/\sqrt{\mathcal{Q}} d\mathbf{V} = \pi^2 \mathbf{1}_{\{|\mathbf{X}| < 1\}}$, so

$$\rho_0 = \frac{3M}{4\pi R^3} \mathbf{1}_{\{r < R\}}, \quad \Phi_0^\Lambda(r) = \begin{cases} \omega_*^2 r^2/2 - 3GM/(2R), & r \leq R, \\ -GM/r - \Omega_\Lambda^2 r^2/2, & r \geq R. \end{cases} \quad (14)$$

A stationary weak solution has a locally integrable distribution, satisfies Poisson's equation after integration against smooth compactly supported test functions, and obeys

$$\iint_{\mathbb{R}^6} F \left[\mathbf{p} \cdot \nabla_{\mathbf{x}} \psi - \nabla_{\mathbf{x}} \Phi \cdot \nabla_{\mathbf{p}} \psi \right] d\mathbf{z} = 0 \quad \text{for every } \psi \in C_c^1(\mathbb{R}^6).$$

This definition does not require derivatives of F at its support boundary.

Theorem 3.10 (Finite-mass Kurth- Λ equilibrium) Under Equation (12), $(F_{0,\Lambda}, \Phi_0^\Lambda)$ is a finite-mass stationary weak Vlasov-Poisson- Λ solution with com-

pact support. The potential is C^1 and Poisson's equation has no surface mass concentration at $r = R$.

Proof. Equation (14) follows from the velocity integral and its potential and radial derivative match at R . On the support, the scaled characteristics are harmonic, $\mathbf{X}(t) = \mathbf{X}_0 \cos(\omega_* t) + \mathbf{V}_0 \sin(\omega_* t)$ and $\mathbf{V}(t) = \mathbf{V}_0 \cos(\omega_* t) - \mathbf{X}_0 \sin(\omega_* t)$, preserving $\mathcal{Q}$ and $\mathcal{D}$; testing this invariant flow against C_c^1 functions gives the weak transport identity. $\square$

Remark 3.11 (Boundary regularity). The singularity $F_{0,\Lambda} \sim \mathcal{Q}^{-1/2}$ is integrable but not classically differentiable. It is used as an exact weak solution for comparison; the classical linearization in Section 5 uses the smooth state F_* .

3.5. Exact Isotropic Scaling Solutions

Let $a \in C^2(J)$ be a positive scale factor on a time interval J . A homothetic motion changes all spatial lengths by this same factor. The transformation below preserves the phase volume dz : its position and momentum determinants cancel. Define

$$F_a(t, \mathbf{x}, \mathbf{p}) = F_{0,\Lambda} \left(\frac{\mathbf{x}}{a}, a\mathbf{p} - \dot{a}\mathbf{x} \right). \quad (15)$$

Theorem 3.12 (Homothetic Vlasov-Poisson- Λ family). *The function (15) is a weak solution exactly when*

$$\ddot{a} = -\frac{\omega_g^2}{a^2} + \frac{\omega_*^2}{a^3} + \Omega_\Lambda^2 a. \quad (16)$$

Let $\mathcal{E}[F]$ denote the physical energy: the kinetic energy plus one half of the matter-potential integral and the full cosmological-potential integral. Thus

$$\mathcal{E}[F] = \iint \frac{|\mathbf{p}|^2}{2} F dz + \frac{1}{2} \int \rho_F U_F^g d\mathbf{x} - \frac{\Omega_\Lambda^2}{2} \int |\mathbf{x}|^2 \rho_F d\mathbf{x}.$$

For this equilibrium define its second spatial moment $I_0 = \iint |\mathbf{x}|^2 F_{0,\Lambda} dz$. Along the scaling family,

$$\mathcal{E}[F_a] = I_0 \left[\frac{1}{2} \dot{a}^2 - \frac{\omega_g^2}{a} + \frac{\omega_*^2}{2a^2} - \frac{\Omega_\Lambda^2 a^2}{2} \right], \quad I_0 = \frac{3MR^2}{5}. \quad (17)$$

Proof. With $X = \mathbf{x}/a$ and $P = a\mathbf{p} - \dot{a}\mathbf{x}$, the transformed characteristics become $\dot{X} = P/a^2$ and $\dot{P} = -\omega_*^2 X/a^2$ precisely under Equation (16); the time change $d\tau = a^{-2}dt$ recovers the harmonic Kurth flow. The energy formula follows by scaling. $\square$

The homothetic family is isotropic; the tidal response below belongs to the trace-free degree-two sector.

4. Tensor Moment Equations and the Initial Tidal Response

4.1. Moment Matrices

For a solution of Equation (3), define the inertia matrix $\hat{\mathbf{I}}_3$, the mixed position-momentum matrix $\hat{\mathbf{J}}_3$, and the kinetic second-moment matrix $\hat{\mathbf{K}}_3$ by

$$\begin{aligned}(\hat{\mathbf{I}}_3)_{ij} &= \iint x_i x_j F dz, \\(\hat{\mathbf{J}}_3)_{ij} &= \iint (x_i p_j + x_j p_i) F dz, \\(\hat{\mathbf{K}}_3)_{ij} &= \iint p_i p_j F dz.\end{aligned}$$

All unmarked phase-space integrals extend over $\mathbb{R}^6$. Let $\hat{\mathbf{Id}}_3$ be the 3×3 identity matrix. The quadrupole matrix and total angular momentum are, respectively,

$$\hat{\mathbf{Q}}_3 = \hat{\mathbf{I}}_3 - \frac{\text{tr} \hat{\mathbf{I}}_3}{3} \hat{\mathbf{Id}}_3, \quad \mathbf{L}(t) = \iint \mathbf{x} \times \mathbf{p} F(t, \mathbf{x}, \mathbf{p}) dz.$$

The vector $\mathbf{L}$ is the total angular momentum of the distribution, distinct from the one-particle magnitude L . Repeated component indices are summed from one to three.

Proposition 4.1 (Exact second-moment identities). *Assume the displayed moments are finite and may be differentiated, with no boundary contributions in integration by parts; these conditions hold for classical solutions of compact phase-space support. Write $\mathbf{a}_F = -\nabla_{\mathbf{x}} \Phi_F^\Lambda - \hat{\mathbf{T}}_3 \mathbf{x}$ for the acceleration. Then*

$$\dot{\hat{\mathbf{I}}}_3 = \hat{\mathbf{J}}_3, \quad \left(\dot{\hat{\mathbf{J}}}_3 \right)_{ij} = 2 \left(\hat{\mathbf{K}}_3 \right)_{ij} + \iint (x_i a_{F,j} + x_j a_{F,i}) F dz.$$

The external tide contributes

$$\left. \dot{\hat{\mathbf{I}}}_3 \right|_{\text{tid}} = -(\hat{\mathbf{T}}_3 \hat{\mathbf{I}}_3 + \hat{\mathbf{I}}_3 \hat{\mathbf{T}}_3). \quad (18)$$

Let ε_{ijk} be the alternating symbol, equal to the sign of a permutation of $(1, 2, 3)$ and zero when indices repeat. The tidal torque is

$$\left. \dot{\mathbf{L}}_i \right|_{\text{tid}} = -\varepsilon_{ijk} \left(\hat{\mathbf{T}}_3 \right)_{kl} \left(\hat{\mathbf{I}}_3 \right)_{jl}.$$

Proof. Multiply Equation (3) successively by $x_i x_j$, $x_i p_j + x_j p_i$, and $(\mathbf{x} \times \mathbf{p})_i$, and integrate by parts in position and momentum. The external acceleration $-\hat{\mathbf{T}}_3 \mathbf{x}$ gives the stated matrix product and torque. $\square$

4.2. Preservation of Spherical Symmetry

Theorem 4.2 (Preservation of spherical symmetry). *If the unforced problem has a unique solution in a rotation-invariant class and the initial state is spherical, then it remains spherical and $\hat{\mathbf{Q}}_3(t) = 0$. The same holds for the homothetic family of Theorem 3.12.*

Proof. Rotating position and momentum produces a solution with the same initial data; uniqueness gives rotational invariance, hence $\hat{\mathbf{I}}_3$ is proportional to the identity. $\square$

4.3. Initial Quadrupole Identity

The mass current of a distribution F is $\mathbf{j}_F(t, \mathbf{x}) = \int_{\mathbb{R}^3} \mathbf{p} F(t, \mathbf{x}, \mathbf{p}) d\mathbf{p}$.

Theorem 4.3 (Initial quadrupole response). *Let F_s be a spherical stationary*

state with vanishing mass current and finite second spatial moment

$I_0 = \iint |\mathbf{x}|^2 F_s dz$. If the forced solution has twice differentiable second moments and the preceding integrations by parts are valid, then

$$\ddot{\hat{Q}}_3(0) = -\frac{2I_0}{3} \hat{T}_3(0). \quad (19)$$

For the uniform Kurth ball this is $\ddot{\hat{Q}}_3(0) = -(2MR^2/5) \hat{T}_3(0)$.

Proof. Sphericity gives $\hat{I}_3(0) = I_0 \hat{I}_{d3}/3$ and makes the unforced internal term isotropic. The trace-free part of Equation (18) yields the result; Equation (17) gives the uniform-ball coefficient. $\square$

Corollary 4.4 (Short-time tidal expansion). If $\hat{T}_3$ is C^1 near zero and the second moments have a third time derivative that is bounded near zero, then $\hat{Q}_3(t) = -(I_0/3) \hat{T}_3(0) t^2 + O(t^3)$; under twice differentiability the remainder is $o(t^2)$.

Proof. Use Taylor's formula together with $\hat{Q}_3(0) = \dot{\hat{Q}}_3(0) = 0$ and Theorem 4.3. $\square$

5. Causal Volterra Response

5.1. Function Spaces and Characteristic Flow

We linearize about F_* with Hamiltonian H_* and use $I_0 = \iint |\mathbf{x}|^2 F_* dz$ for this equilibrium's second spatial moment. Fix $0 < \delta < \Delta_{\text{sep}}$. Let $\mathcal{K}_\delta$ be the closed inner connected component of $\{H_* \leq E_{\text{max}} + \delta\}$ containing $\text{supp } F_*$. The energy bound makes it compact and invariant under the stationary particle motion. Define the projections $\pi_x(\mathbf{x}, \mathbf{p}) = \mathbf{x}$ and $\pi_p(\mathbf{x}, \mathbf{p}) = \mathbf{p}$.

For an integer $k \geq 1$ and $0 < \alpha < 1$, $C^{k,\alpha}(\mathbb{R}^n)$ is the normed space of functions with bounded continuous derivatives through order k whose derivatives of order k satisfy an α -Hölder bound: the difference between values at two points is bounded by a constant times the α th power of their distance. The norm includes the supremum of each derivative and these difference bounds. Define the little Hölder space $c_0^{k,\alpha}(\mathbb{R}^n)$ as the closure of $C_c^\infty(\mathbb{R}^n)$ in that norm. The fixed-support spaces are

$$\begin{aligned} \mathcal{H}_\delta^{k,\alpha} &= \{h \in c_0^{k,\alpha}(\mathbb{R}^6) : \text{supp } h \subset \mathcal{K}_\delta\}, \\ \mathcal{X}_\delta^{k,\alpha} &= \{\rho \in c_0^{k,\alpha}(\mathbb{R}^3) : \text{supp } \rho \subset \pi_x \mathcal{K}_\delta\}, \quad \mathcal{H}_{\delta,+}^{k,\alpha} := \mathcal{H}_\delta^{k+1,\alpha}. \end{aligned}$$

Thus $\mathcal{H}_{\delta,+}^{k,\alpha}$ has one additional spatial and momentum derivative. For normed spaces X, Y , let $\mathcal{B}(X, Y)$ denote the bounded linear maps with their operator norm, and write $\mathcal{B}(X) = \mathcal{B}(X, X)$.

Let $\Phi_t z$ denote the stationary Hamiltonian flow, the solution at time t issued from z of $\dot{\mathbf{x}} = \mathbf{p}$, $\dot{\mathbf{p}} = -\nabla_x \Phi_*(\mathbf{x})$. Define the transport and momentum-integration operators

$$(\mathcal{U}_0(t)h)(z) = h(\Phi_{-t}z), \quad (\mathfrak{P}h)(\mathbf{x}) = \int_{\mathbb{R}^3} h(\mathbf{x}, \mathbf{p}) d\mathbf{p}.$$

For $\sigma \in \mathcal{X}_\delta^{k,\alpha}$, its Newtonian perturbation potential is

$\phi_\sigma(\mathbf{x}) = -G \int \sigma(\mathbf{y}) / |\mathbf{x} - \mathbf{y}| d\mathbf{y}$. The sources due to the perturbed gravitational field and the imposed tide are

$$\mathfrak{S}\sigma = \nabla_x \phi_\sigma \cdot \nabla_p F_*, \quad \mathfrak{S}_{\text{tid}} \left[\hat{\mathbf{T}}_3 \right] = \left(\hat{\mathbf{T}}_3 \mathbf{x} \right) \cdot \nabla_p F_*.$$

The family $\mathfrak{U}_0$ is called a strongly continuous group when $\mathfrak{U}_0(t+s) = \mathfrak{U}_0(t)\mathfrak{U}_0(s)$, $\mathfrak{U}_0(0)$ is the identity, and $\|\mathfrak{U}_0(t)h - h\| \rightarrow 0$ as $t \rightarrow 0$ for every h .

Lemma 5.1 (Invariant support and the composition group). *For every $T > 0$, $\mathfrak{U}_0$ is a strongly continuous group on $\mathcal{H}_\delta^{k,\alpha}$ and there is a finite constant C_T such that*

$$\sup_{|t| \leq T} \|\mathfrak{U}_0(t)\| < \infty, \quad \|\mathfrak{U}_0(t) - \mathfrak{U}_0(s)\|_{\mathcal{B}(\mathcal{H}_\delta^{k,\alpha}, \mathcal{H}_\delta^{k,\alpha})} \leq C_T |t - s|.$$

Proof. Energy conservation makes $\mathcal{K}_\delta$ invariant and the smooth flow has uniformly bounded derivatives on $[-T, T] \times \mathcal{K}_\delta$. Standard composition estimates and smooth approximation give strong continuity [7] [8]. Define the transport differential operator $\mathfrak{L}_0 f = \{f, H_*\}$. For $f \in \mathcal{H}_{\delta,+}^{k,\alpha}$, $(\mathfrak{U}_0(t) - \mathfrak{U}_0(s))f = -\int_s^t \mathfrak{U}_0(r) \mathfrak{L}_0 f dr$. The coefficients of $\mathfrak{L}_0$ are bounded on $\mathcal{K}_\delta$, giving the Lipschitz bound. $\square$

Lemma 5.2 (Momentum projection). *The map $\mathfrak{P} : \mathcal{H}_\delta^{k,\alpha} \rightarrow \mathcal{X}_\delta^{k,\alpha}$ is bounded.*

Proof. All momentum supports lie in the fixed compact set $\pi_p \mathcal{K}_\delta$; differentiate under the integral and integrate the corresponding Hölder seminorm. For little Hölder membership, approximate h by smooth functions multiplied by one fixed compactly supported cutoff equal to one near $\mathcal{K}_\delta$. The same estimate on their common momentum support passes to the limit. $\square$

Lemma 5.3 (Fixed-support Newtonian map and source operators). *The maps $\mathfrak{S} : \mathcal{X}_\delta^{k,\alpha} \rightarrow \mathcal{H}_{\delta,+}^{k,\alpha}$ and $\mathfrak{S}_{\text{tid}} : \mathbb{T}_2 \rightarrow \mathcal{H}_{\delta,+}^{k,\alpha}$ are bounded.*

Proof. Put $K_x = \pi_x \mathcal{K}_\delta$. Choose a smooth compactly supported spatial function χ equal to one near $\pi_x(\text{supp } \nabla_p F_*)$, and a bounded open neighbourhood $\mathcal{O}$ containing $\text{supp } \chi$ with positive distance to its boundary. Writing $\|\cdot\|_{L^\infty(\mathcal{O})}$ for the essential-supremum norm, local integrability of $|\mathbf{x} - \mathbf{y}|^{-1}$ gives

$$\|\phi_\sigma\|_{L^\infty(\mathcal{O})} \leq G \|\sigma\|_{C^0} \sup_{\mathbf{x} \in \mathcal{O}} \int_{K_x} |\mathbf{x} - \mathbf{y}|^{-1} d\mathbf{y} \leq C \|\sigma\|_{C^0}.$$

Interior Schauder estimates [23], which bound potential derivatives by the density's Hölder norm, then yield

$$\|\chi \nabla \phi_\sigma\|_{C^{k+1,\alpha}(\mathbb{R}^3)} \leq C \|\sigma\|_{C^{k,\alpha}(\mathbb{R}^3)}.$$

The constants depend only on G, k, α and the fixed supports, neighbourhood and cutoff, not on σ .

By the definition of $\mathcal{C}_0^{k,\alpha}$, choose smooth compactly supported $\sigma_n \rightarrow \sigma$ in $C^{k,\alpha}$. Multiplying by a fixed cutoff equal to one near K_x makes their supports lie in one compact set without changing the limit. The preceding estimate, with this common support, shows that $\chi \nabla \phi_{\sigma_n} \rightarrow \chi \nabla \phi_\sigma$ in $C^{k+1,\alpha}$. Each approximant is smooth and compactly supported, proving little Hölder membership. Multipli-

cation by $\nabla_p F_*$ gives the bound for $\mathfrak{S}$, with phase-space support inside $\mathcal{K}_\delta$. The tidal bound follows directly from the smooth fixed-support product. $\square$

We write $C^{0,1}([0, T]; X)$ for Lipschitz-continuous paths with values in a normed space X .

Proposition 5.4 (Finite-time kernel regularity) *The kernels*

$$\mathfrak{K}(t) = \mathfrak{P}\mathfrak{U}_0(t)\mathfrak{S}, \quad \mathfrak{R}(t) = \mathfrak{P}\mathfrak{U}_0(t)\mathfrak{S}_{\text{tid}}$$

satisfy

$$\mathfrak{K} \in C^{0,1}([0, T]; \mathcal{B}(\mathcal{X}_\delta^{k, \alpha})), \quad \mathfrak{R} \in C^{0,1}([0, T]; \mathcal{B}(\mathbb{T}_2, \mathcal{X}_\delta^{k, \alpha})). \quad (20)$$

Proof. Combine Lemmas 5.1-5.3 and the one-derivative Lipschitz estimate for $\mathfrak{U}_0$. $\square$

5.2. Integration along Characteristics and the Volterra Equation

Write $F = F_* + h$, where h is the perturbation, and retain terms linear in h and $\hat{\mathbb{T}}_3$. Let $h_{\text{in}} = h(0)$ be the initial perturbation and let $\rho = \mathfrak{P}h$ be the spatial density of h . Then $\partial_t h + \{h, H_*\} = \mathfrak{S}\rho + \mathfrak{S}_{\text{tid}}[\hat{\mathbb{T}}_3]$. For compatible operator or vector valued functions f, g , define causal convolution by $(f * g)(t) = \int_0^t f(t-s)g(s)ds$. All time integrals are Bochner integrals, obtained as norm limits of integrals of step functions; the integrands here are continuous. The integrated evolution formula below is Duhamel's formula.

Theorem 5.5 (Finite-time Volterra representation). *Duhamel's formula gives*

$$h(t) = \mathfrak{U}_0(t)h_{\text{in}} + \int_0^t \mathfrak{U}_0(t-s) \left[\mathfrak{S}\rho(s) + \mathfrak{S}_{\text{tid}}[\hat{\mathbb{T}}_3(s)] \right] ds, \quad (21)$$

$$\rho(t) = \mathfrak{B}(t)h_{\text{in}} + (\mathfrak{K} * \rho)(t) + (\mathfrak{R} * \hat{\mathbb{T}}_3)(t), \quad \mathfrak{B}(t) = \mathfrak{P}\mathfrak{U}_0(t). \quad (22)$$

Define $\mathfrak{K}^1 = \mathfrak{K}$ and $\mathfrak{K}^{*(n+1)} = \mathfrak{K} * \mathfrak{K}^{*n}$ for integers $n \geq 1$. If

$M_T = \sup_{0 \leq t \leq T} \|\mathfrak{K}(t)\|$, the resolvent kernel, which incorporates repeated gravitational responses, is $\mathfrak{G} = \sum_{n \geq 1} \mathfrak{K}^{*n}$. The series converges uniformly and

$$\|\mathfrak{G}(t)\| \leq M_T e^{M_T t}, \quad \rho = f + \mathfrak{G} * f, \quad f = \mathfrak{B}h_{\text{in}} + \mathfrak{R} * \hat{\mathbb{T}}_3. \quad (23)$$

Proof. Apply $\mathfrak{P}$ to Duhamel's formula [7]. Since $\|\mathfrak{K}^{*n}(t)\| \leq M_T^n t^{n-1}/(n-1)!$, standard Volterra theory [9] gives uniform convergence, the resolvent identity, and uniqueness. $\square$

A mild solution is a continuous pair (h, ρ) satisfying Equations (21)-(22). For a normed space X , write $C_T(X) = C([0, T]; X)$ and

$\|f\|_{C_T(X)} = \sup_{0 \leq t \leq T} \|f(t)\|_X$. In the next estimates abbreviate $\mathcal{H} = \mathcal{H}_\delta^{k, \alpha}$ and $\mathcal{X} = \mathcal{X}_\delta^{k, \alpha}$.

Theorem 5.6 (Finite-time well-posedness and continuous dependence). *For $h_{\text{in}} \in \mathcal{H}_\delta^{k, \alpha}$ and $\hat{\mathbb{T}}_3 \in C([0, T]; \mathbb{T}_2)$ there is a unique mild solution*

$$(h, \rho) \in C_T(\mathcal{H}) \times C_T(\mathcal{X}). \text{ Set } B_T = \sup_{0 \leq t \leq T} \|\mathfrak{B}(t)\|_{\mathcal{B}(\mathcal{H}, \mathcal{X})},$$

$$R_T = \sup_{0 \leq t \leq T} \|\mathfrak{R}(t)\|_{\mathcal{B}(\mathbb{T}_2, \mathcal{X})}, \text{ and } U_T = \sup_{0 \leq t \leq T} \|\mathfrak{U}_0(t)\|_{\mathcal{B}(\mathcal{H})}. \text{ Then}$$

$$\|\rho\|_{C_T(\mathcal{X})} \leq e^{M_T T} \left(B_T \|h_{\text{in}}\|_{\mathcal{H}} + TR_T \|\hat{T}_3\|_{C_T(\mathbb{T}_2)} \right), \quad (24)$$

$$\|h\|_{C_T(\mathcal{H})} \leq U_T \left(\|h_{\text{in}}\|_{\mathcal{H}} + T \|\mathfrak{S}\| \|\rho\|_{C_T(\mathcal{X})} + T \|\mathfrak{S}_{\text{tid}}\| \|\hat{T}_3\|_{C_T(\mathbb{T}_2)} \right). \quad (25)$$

The same bounds hold for differences of solutions.

Proof. Theorem 5.5 gives existence and uniqueness of ρ and then h . Gronwall's inequality yields Equation (24); Equation (25) follows directly from Duhamel's formula. $\square$

5.3. Decomposition into Spherical Harmonics

The rotation group is $SO(3) = \{R \in \mathbb{R}^{3 \times 3} : R^T R = \hat{\text{Id}}_3, \det R = 1\}$, where $\det R$ is the determinant. A rotation $\hat{R}_3 \in SO(3)$ acts on a phase-space function and a spatial density by

$$(\mathfrak{U}_{\hat{R}_3} h)(\mathbf{x}, \mathbf{p}) = h(\hat{R}_3^{-1} \mathbf{x}, \hat{R}_3^{-1} \mathbf{p}), \quad (\mathcal{V}_{\hat{R}_3} \rho)(\mathbf{x}) = \rho(\hat{R}_3^{-1} \mathbf{x}).$$

These actions compose as rotations do and are continuous in the little Hölder norms. For an integer $\ell \geq 0$, let V_ℓ be the $(2\ell+1)$ -dimensional space of real spherical harmonics of degree ℓ : restrictions to the unit sphere of homogeneous polynomials of degree ℓ whose Laplacian is zero. Choose a basis $Y_{\ell m}$, $m = -\ell, \dots, \ell$, orthonormal for the usual surface-area measure. Each degree is irreducible under rotations, meaning it has no smaller nonzero rotation-invariant subspace.

Let $\chi_\ell(\hat{R}_3)$ be the trace of the rotation acting on V_ℓ , called its character. Let $d\hat{R}_3$ denote Haar probability measure, the rotation-invariant measure on $SO(3)$ normalized to total mass one. The bounded projection

$$\Pi_\ell \rho = (2\ell+1) \int_{SO(3)} \chi_\ell(\hat{R}_3^{-1}) \mathcal{V}_{\hat{R}_3} \rho d\hat{R}_3 \quad (26)$$

selects all density functions of degree ℓ . Its range $\mathcal{X}_\ell = \Pi_\ell \mathcal{X}_s^{k,\alpha}$ is the closed degree- ℓ subspace [25]. A density in this space has the finite form

$\sum_{m=-\ell}^{\ell} \rho_{\ell m}(r) Y_{\ell m}(\mathbf{x}/r)$ for $r > 0$; the scalar functions $\rho_{\ell m}$ are its radial coefficients. Write $b_{\ell m}(t)$ for the same coefficients of $\mathfrak{B}(t)h_{\text{in}}$.

Let $\mathcal{X}_\ell^{\text{mult}}$ denote the radial coefficient space with the norm induced from $\mathcal{X}_\ell$. The notation $\mathcal{X}_\ell^{\text{mult}} \hat{\otimes} V_\ell$ means the finite sum of copies of this radial space, one for each harmonic, completed in that induced norm. No convergence of a series over all degrees is assumed.

Theorem 5.7 (Separation of angular degrees). *For every $\ell \geq 0$, the kernels $\mathfrak{K}(t)$ and $\mathfrak{G}(t)$ commute with Π_ℓ and preserve the closed subspace $\mathcal{X}_\ell$. This subspace admits a radial coefficient representation*

$$\mathcal{X}_\ell \simeq \mathcal{X}_\ell^{\text{mult}} \hat{\otimes} V_\ell, \quad \mathfrak{K}(t)|_{\mathcal{X}_\ell} = \mathcal{A}_\ell(t) \otimes \text{I}_{V_\ell},$$

where $\mathcal{X}_\ell^{\text{mult}}$ is the radial coefficient space, $\mathcal{A}_\ell(t)$ is bounded on that space, and I_{V_ℓ} is the identity on V_ℓ . Thus the same radial gravitational-response operator acts for every $m = -\ell, \dots, \ell$. For $\ell \neq 2$ the block equation contains no tidal forc-

ing.

$$\rho_{\ell m}(t) = b_{\ell m}(t) + \int_0^t \mathcal{A}_\ell(t-s) \rho_{\ell m}(s) ds.$$

The forced degree-two block is

$$\Pi_2 \rho(t) = \Pi_2 \mathfrak{B}(t) h_{\text{in}} + \int_0^t \mathfrak{K}_2(t-s) \Pi_2 \rho(s) ds + \int_0^t \mathfrak{R}_2(t-s) \hat{\mathbf{T}}_3(s) ds,$$

where $\mathfrak{K}_\ell$ is the restriction of $\mathfrak{K}$ to $\mathcal{X}_\ell$ and $\mathfrak{R}_2(t) := \Pi_2 \mathfrak{R}(t) \in \mathcal{B}(\mathbb{T}_2, \mathcal{X}_2)$ is the degree-two tidal forcing operator. A spherical initial perturbation contributes only to $\ell = 0$, whereas a trace-free quadratic tide contributes only to $\ell = 2$. Consequently, these two inputs generate no degrees outside $\{0, 2\}$ at first order. In particular, if $h_{\text{in}} = 0$, the forced response lies entirely in the $\ell = 2$ block.

Proof. Sphericity makes the transport, Poisson, source, and projection operators rotation-equivariant, meaning that applying a rotation before or after each operator gives the same result, so $\mathfrak{K}(t)$, every convolution power, and $\mathfrak{G}(t)$ commute with the projections Π_ℓ . The decomposition of the rotation action then gives the displayed radial coefficient form and identical radial action for all m [25] [26]. A trace-free homogeneous quadratic polynomial is harmonic of degree two; hence $\Pi_\ell \mathfrak{R}(t) = 0$ for $\ell \neq 2$. $\square$

Define the bounded quadrupole functional

$$\mathcal{M}_2 \rho := \int_{\mathbb{R}^3} \left(\mathbf{x} \mathbf{x}^T - \frac{|\mathbf{x}|^2}{3} \text{Id}_3 \right) \rho(\mathbf{x}) d\mathbf{x} \in \mathbb{T}_2.$$

A superscript (1) denotes the part linear in the tidal amplitude. By the *tensor quadrupole susceptibility* we mean the causal kernel that maps a tidal-matrix history $\hat{\mathbf{T}}_3$ to the linear quadrupole $\hat{\mathbf{Q}}_3^{(1)}$. A subscript 2 on an operator denotes its restriction to the $\ell = 2$ block, and Π_2 is the bounded projection in Equation (26).

Theorem 5.8 (Tensor quadrupole susceptibility). Assume $h_{\text{in}} = 0$. For every finite $T > 0$ there is a unique continuous scalar kernel $\chi_2^Q : [0, T] \rightarrow \mathbb{R}$ such that, for every continuous tidal history,

$$\hat{\mathbf{Q}}_3^{(1)}(t) = \int_0^t \chi_2^Q(t-s) \hat{\mathbf{T}}_3(s) ds. \quad (27)$$

Moreover,

$$\|\hat{\mathbf{Q}}_3^{(1)}\|_{C_T(\mathbb{T}_2)} \leq T \|\chi_2^Q\|_{C([0, T])} \|\hat{\mathbf{T}}_3\|_{C_T(\mathbb{T}_2)}.$$

The kernel is right differentiable at the origin and

$$(\chi_2^Q)'(0+) = -\frac{2I_0}{3}. \quad (28)$$

Proof. In the degree-two block, $\rho = (\mathcal{H}_2 * \hat{\mathbf{T}}_3)$ with $\mathcal{H}_2 = \mathfrak{R}_2 + \mathfrak{G}_2 * \mathfrak{R}_2$. The map $\mathcal{M}_2 \mathcal{H}_2(t) : \mathbb{T}_2 \rightarrow \mathbb{T}_2$ commutes with rotations; the real degree-two representation has only scalar maps commuting with every rotation (Schur's lemma), which gives $\mathcal{M}_2 \mathcal{H}_2(t) = \chi_2^Q(t) \text{I}_{\mathbb{T}_2}$. Continuity follows from Proposition 5.4 and Theorem 5.5.

Since $\mathfrak{P}\left[\left(\hat{T}_3 \mathbf{x}\right) \cdot \nabla_p F^*\right]=0$, one has $\mathfrak{R}_2(0)=0$. The one-derivative source bound and the generator identity from Lemma 5.1 imply, in operator norm on the finite-dimensional space $\mathbb{T}_2$,

$$\mathfrak{R}_2'(0+)\hat{T}_3=-\Pi_2 \mathfrak{P} \mathfrak{L}_0 \mathfrak{S}_{\text{tid}}\left[\hat{T}_3\right] .$$

Thus $\mathfrak{R}_2(t)=O(t)$, while continuity of $\mathfrak{S}_2$ gives $(\mathfrak{S}_2 * \mathfrak{R}_2)(t)=O\left(t^2\right)$; hence χ_2^Q is right differentiable and $\chi_2^Q(0)=0$.

Let $t_0>0$ and fix $\hat{T}_3^{(0)} \in \mathbb{T}_2$. For a tide equal to $\hat{T}_3^{(0)}$ on $[0, t_0]$, the combined source $g(t)=\mathfrak{S} \rho(t)+\mathfrak{S}_{\text{tid}}\left[\hat{T}_3^{(0)}\right]$ is continuous with values in $\mathcal{H}_{\delta,+}^{k, \alpha}$. Differentiating the Duhamel formula by the same generator identity gives $h \in C^1\left([0, t_0] ; \mathcal{H}_{\delta}^{k, \alpha}\right)$ and $\dot{h}(0)=\mathfrak{S}_{\text{tid}}\left[\hat{T}_3^{(0)}\right]$. Because $k \geq 1$ and the support is compact, the inertia and mixed-moment maps are continuously differentiable in time and integration by parts is valid. Their relation $\hat{I}_3^h=\hat{J}_3^h$ then makes $\hat{I}_3^h$ twice differentiable; the superscript h denotes moments evaluated with the perturbation h in place of F . The linearized moment identity therefore gives $\ddot{Q}_3^h(0)=-(2 I_0 / 3) \hat{T}_3^{(0)}$. Comparing with $\hat{Q}_3^{(1)}(t)=\hat{T}_3^{(0)} \int_0^t \chi_2^Q(u) d u$ proves $\left(\chi_2^Q\right)'(0+)=-2 I_0 / 3$. $\square$

5.4. Quadrupole Projection and Orbit Formulas

For real 3×3 matrices, the notation

$$\hat{A}_3: \hat{B}_3:=\operatorname{tr}\left(\hat{A}_3^T \hat{B}_3\right)$$

denotes the Frobenius inner product [27]. For an axisymmetric tide with a fixed axis, choose that axis as the third coordinate. With diag denoting a diagonal matrix, write

$$\hat{S}_3:=\operatorname{diag}\left(-\frac{1}{2},-\frac{1}{2}, 1\right), \quad \hat{S}_3: \hat{S}_3=\frac{3}{2},$$

as

$$\hat{T}_3(t)=\tau(t) \hat{S}_3, \quad (29)$$

where $\tau(t)$ is the scalar tidal amplitude. Define

$$q_Q(t):=\frac{\hat{Q}_3(t): \hat{S}_3}{\hat{S}_3: \hat{S}_3}, \quad q(t):=\frac{3 q_Q(t)}{2 I_0} . \quad (30)$$

Write q_l for the part of q that is linear in the tidal amplitude. Then

$$\hat{Q}_3^{(1)}(t)=\frac{2 I_0}{3} q_l(t) \hat{S}_3, \quad (31)$$

and Equation (19) gives $\ddot{q}(0)=-\tau(0)$.

Corollary 5.9 (Fixed-axis scalar response). Assume $h_{\text{in}}=0$ and Equation (29). Then the linear quadrupole has the form (31), with

$$q_l=\chi_2 * \tau, \quad \chi_2:=\frac{3}{2 I_0} \chi_2^Q, \quad \chi_2'(0+)=-1 .$$

The kernel retains the orbital phases and the self-consistent gravitational contribution; this statement does not assert decay of the response [15].

The degree-two equation retains all radial and orbital variables. For an illustrative scalar approximation, restrict the density to multiples of one prescribed quadrupole density d_Q [15], defined by

$$d_Q(\mathbf{x}) := -\frac{3}{2I_0} \nabla_{\mathbf{x}} \cdot (\rho_*(|\mathbf{x}|) \hat{\mathbf{S}}_3 \mathbf{x}) = -\frac{3\rho'_*(r)}{2I_0 r} \mathbf{x}^T \hat{\mathbf{S}}_3 \mathbf{x}, \quad r = |\mathbf{x}|.$$

It is normalized by

$$\int_{\mathbb{R}^3} \left(\mathbf{x} \mathbf{x}^T - \frac{r^2}{3} \hat{\mathbf{I}}_3 \right) d_Q(\mathbf{x}) d\mathbf{x} = \hat{\mathbf{S}}_3.$$

Let ϕ_Q solve $\Delta_{\mathbf{x}} \phi_Q = 4\pi G d_Q$. Spherical symmetry gives $\phi_Q(\mathbf{x}) = f_2(r) \mathbf{x}^T \hat{\mathbf{S}}_3 \mathbf{x}$. With $D_2(r) := -(3/2I_0) r \rho'_*(r)$,

$$f_2(r) = -\frac{4\pi G}{5} \left[r^{-5} \int_0^r s^4 D_2(s) ds + \int_r^\infty \frac{D_2(s)}{s} ds \right].$$

In this approximation the density equals $q_Q(t) d_Q$. Applying the normalized quadrupole moment to the evolution gives the scalar projected equation

$$q_Q(t) = \int_0^t k_2(t-s) q_Q(s) ds + \int_0^t r_2(t-s) \tau(s) ds. \quad (32)$$

For the dimensionless q , the forcing kernel is $\tilde{r}_2 := 3r_2/(2I_0)$, while the self-gravity kernel remains k_2 . Equation (32) is the single-density-profile approximation of the degree-two Volterra equation.

Proposition 5.10 (Orbit formulas without derivatives of F_*). Let

$\mathbf{z}_0 := (\mathbf{x}_0, \mathbf{p}_0) \in \mathbb{R}^6$, $\mathbf{x}_t := \mathbf{x}(t; \mathbf{z}_0)$ be the unperturbed orbit, and

$$\hat{\mathbf{A}}_3(t; \mathbf{z}_0) := \partial_{\mathbf{p}_0} \mathbf{x}(t; \mathbf{z}_0) \in \mathbb{R}^{3 \times 3}.$$

Then

$$r_2(t) = -\frac{2}{\hat{\mathbf{S}}_3 : \hat{\mathbf{S}}_3} \iint F_*(\mathbf{z}_0) (\hat{\mathbf{S}}_3 \mathbf{x}_0)^T \hat{\mathbf{A}}_3(t; \mathbf{z}_0)^T \hat{\mathbf{S}}_3 \mathbf{x}_t d\mathbf{z}_0, \quad (33)$$

$$k_2(t) = -\frac{2}{\hat{\mathbf{S}}_3 : \hat{\mathbf{S}}_3} \iint F_*(\mathbf{z}_0) \nabla_{\mathbf{x}} \phi_Q(\mathbf{x}_0)^T \hat{\mathbf{A}}_3(t; \mathbf{z}_0)^T \hat{\mathbf{S}}_3 \mathbf{x}_t d\mathbf{z}_0. \quad (34)$$

In particular,

$$r_2(0) = 0, \quad r'_2(0) = -\frac{2I_0}{3}, \quad \tilde{r}'_2(0) = -1. \quad (35)$$

Proof. Apply the quadrupole functional to the transported tidal and self-consistent sources, change from the current phase point to $\mathbf{z}_0$, and integrate by parts in $\mathbf{p}_0$. Since

$$\partial_{\mathbf{p}_0} (\mathbf{x}_t^T \hat{\mathbf{S}}_3 \mathbf{x}_t) = 2\hat{\mathbf{A}}_3(t; \mathbf{z}_0)^T \hat{\mathbf{S}}_3 \mathbf{x}_t,$$

no derivative of F_* remains. At $t = 0$, $\hat{\mathbf{A}}_3(t; \mathbf{z}_0) = t\hat{\mathbf{I}}_3 + O(t^2)$; isotropy gives $\iint \mathbf{x}_0^T \hat{\mathbf{S}}_3^2 \mathbf{x}_0 F_* d\mathbf{z}_0 = (I_0/3) (\hat{\mathbf{S}}_3 : \hat{\mathbf{S}}_3)$. $\square$

5.5. Illustrative Response and Affine Comparison

Rotation averaging reduces the orbit integrals to three scalar variables. If $\mathbf{x} = \mathbf{x}_0$, $\mathbf{y} = \mathbf{x}_t$, $A = \partial_{p_0} \mathbf{x}_t$, $r = |\mathbf{x}|$, and $\mathbf{n} = \mathbf{x}/r$, define

$$f_A = \frac{1}{2} \mathbf{x}^T A \mathbf{y} + \frac{1}{2} (\mathbf{x} \cdot \mathbf{y}) \operatorname{tr} A - \frac{1}{3} \mathbf{x}^T A^T \mathbf{y},$$

$$f_B = (\mathbf{x} \cdot \mathbf{y}) \mathbf{n}^T A^T \mathbf{x} - \frac{r^2}{3} \mathbf{n}^T A^T \mathbf{y}.$$

Proposition 5.11 (Integration over the rotation group). Write $\mathbb{E}$ for expectation under the probability measure $F_* \mathrm{d}z / M$, after averaging the common orientation of position and momentum with Haar probability measure. This averaging [25] gives

$$r_2(t) = -\frac{2M}{5} \mathbb{E} f_A, \quad k_2(t) = -\frac{2M}{5} \mathbb{E} [2f_2(r) f_A + f_2'(r) f_B]. \quad (36)$$

Proof. Use the isotropic second-moment identity for the rotated trace-free matrix $R^T \hat{S}_3 R$ and contract it with the tensors in the orbit formulas. $\square$

For the smooth equilibrium define the relative energy $\eta = E_c - H_*$ and the cosine $\chi_v = (\mathbf{x} \cdot \mathbf{p}) / (r|\mathbf{p}|)$ of the angle between position and momentum. The remaining integration variables satisfy $0 < r < R$, $\varepsilon < \eta < y_*(r)$ and $-1 < \chi_v < 1$, with mass-measure weight $8\pi^2 C_\rho \sqrt{2} r^2 \varphi_\varepsilon(\eta) \sqrt{y_*(r) - \eta} \mathrm{d}r \mathrm{d}\eta \mathrm{d}\chi_v$. The orbit derivatives in $\hat{A}_3$ are evaluated by differentiating the unperturbed equations with respect to initial momentum.

For the equilibrium of **Figure 1**, $M = 0.974642$ and $I_0 = 0.944457$. Evaluating the orbit integrals gives the normalized initial slope $\tilde{r}_2'(0) = -1.00002$, compared with the exact value -1 in Proposition 5.10. The displayed integrals are evaluated by deterministic quadrature, meaning a weighted sum at prescribed integration points, and their time convolutions by the trapezoidal rule [28]. These values illustrate the formulas and do not enter any proof. The supplementary PDF adds an independent check of the equilibrium quantities and explicit radial-field formulas. The response curves are not precision benchmarks: the largest reported coarse-to-fine kernel difference is about 3.9%.

For an affine comparison, transform reference position and momentum $(\mathbf{X}, \mathbf{P})$ by $\mathbf{x} = \hat{A}_3^{\text{aff}} \mathbf{X}$ and $\mathbf{p} = (\hat{A}_3^{\text{aff}})^{-T} \mathbf{P} + \hat{A}_3^{\text{aff}} \mathbf{X}$. Here $\hat{A}_3^{\text{aff}} = \operatorname{diag}(a_\perp, a_\perp, a_\parallel)$, with positive transverse and axial scale factors. Parameterize them by the isotropic change η_{aff} and anisotropic change q_{aff} : $a_\perp = 1 + \eta_{\text{aff}} - q_{\text{aff}}/2$ and $a_\parallel = 1 + \eta_{\text{aff}} + q_{\text{aff}}$. The physical quadrupole of Equation (30) is

$$q = \frac{a_\parallel^2 - a_\perp^2}{3} = q_{\text{aff}} (1 + \eta_{\text{aff}}) + \frac{1}{4} q_{\text{aff}}^2. \quad (37)$$

Define the equilibrium gravitational potential energy and its normalized magnitude by

$$W_0 = -\frac{G}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho_*(\mathbf{x}) \rho_*(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{y} < 0, \quad \gamma = -W_0 / I_0.$$

The linear anisotropic equation in this family is $\ddot{q}_{\text{aff}} + \nu_2^2 q_{\text{aff}} = -\tau$, where $\nu_2^2 = 14\gamma/5 - 4\Omega_\Lambda^2$ is its squared oscillation coefficient. This finite-dimensional equation is an approximation, not an exact restriction of the general kinetic evolution. The supplementary PDF gives its variational derivation and an energy estimate excluding vanishing principal scales under bounded second moment and finite tidal work. **Figure 2** compares the scalar projected kinetic response with this separate approximation for zero initial perturbation and a unit step, $\tau(t) = 1$ for $t \geq 0$, in the same dimensionless time and with $\Omega_\Lambda^2 = 1/600$.

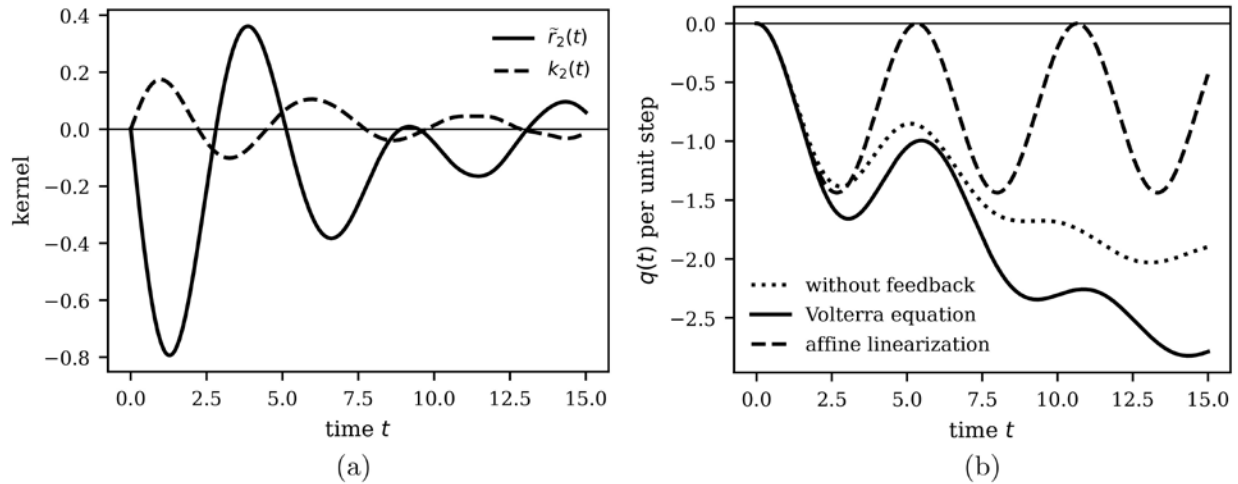


Figure 2. Illustration of the analytically derived response. The kernel calculation uses 3920 quadrature nodes and time step 0.05; panel (b) uses the normalized quadrupole q . The curve labelled “without feedback” omits the gravitational field induced by the density perturbation. (a) Normalized forcing and self-gravity kernels. (b) Unit-step response and affine comparison.

The projected kinetic and affine curves have the same initial quadratic law but need not have the same subsequent phase or amplitude: one retains integrals over the equilibrium orbits, whereas the other restricts the density to an axisymmetric linear deformation. Their difference is therefore a difference between approximations, not evidence for a change in the exact initial moment identity.

6. Conclusions

A positive cosmological constant affects both the global admissibility of a stationary distribution and the local response of a bounded gravitating system. These effects must be distinguished. At large radius, the quadratic cosmological term drives the particle potential to $-\infty$. Consequently, identical energy and angular-momentum values can occur on disconnected bounded and unbounded components. A global Jeans distribution assigns the same occupation to both. The lower bound $\rho_F(r) \geq Cr^{-2-\beta/2}$ proves infinite total mass for the stated class of decreasing power-law potentials. In the quadratic case relevant to $\Lambda > 0$, the enclosed mass diverges at least logarithmically. Thus, an energy cutoff alone cannot specify a nontrivial localized equilibrium in this geometry.

The component-selected construction removes the unbounded component

while preserving the physical equations. The radial function ζ varies only where the microscopic distribution already vanishes. Its derivative therefore contributes nothing to the stationary transport equation, and it changes neither occupied trajectories nor the density generating the gravitational field. The construction produces a smooth compactly supported stationary distribution, not a system confined by a reflecting boundary or an additional force. The transverse matter boundary persists under small parameter changes, as do the matter radius, total mass and strict separation from r_Λ .

The energy gap $\Delta_{\text{sep}} > 0$ has a direct dynamical meaning. An occupied particle cannot reach r_Λ : its energy there would be at least $E_{\text{sep}} + L^2/(2r_\Lambda^2)$, so angular momentum can only strengthen this obstruction. The same gap provides a compact invariant neighbourhood on which the unperturbed flow and its derivatives are bounded for each finite time interval. This statement concerns the stationary Hamiltonian used in the linearization; it does not assert that an arbitrary finite disturbance preserves confinement under its evolving self-consistent potential.

On this invariant set, the kinetic and density formulations are linked by the exact Volterra equation. Its self-gravity kernel $\mathfrak{K}$ retains the Newtonian field induced by earlier density perturbations. The uniformly convergent series defining $\mathfrak{G}$ accounts for repeated gravitational response without imposing a finite-dimensional approximation. The resulting existence, uniqueness and continuous-dependence estimates describe a causal linear response to both the initial perturbation and the entire prescribed tidal history. They are statements about the full linearized kinetic problem in the specified function spaces, rather than about the illustrative scalar projection.

Rotational symmetry determines which part of the response is observed by the quadrupole. With zero initial perturbation, a symmetric trace-free quadratic tide excites only angular degree two. The induced tensor map is consequently represented by one scalar susceptibility χ_2^Q . The normalization $(\chi_2^Q)'(0+) = -2I_0/3$ follows from the second-moment balance and does not depend on an approximation to the radial density. It relates the initial anisotropic acceleration directly to the equilibrium's second spatial mass moment. The cosmological term influences this law through the equilibrium and its occupied trajectories; the trace-free external field specifies the anisotropic forcing and does not directly force the isotropic moment at linear order.

The orbit formulas express the projected response using unperturbed motion and its derivatives with respect to initial momentum. Integration by parts removes derivatives of the equilibrium distribution from these formulas. The explicit uniform-density equilibrium and its exact isotropic scaling motion provide a separate check on the kinetic moments, while the affine family isolates an approximate anisotropic deformation. Neither comparison replaces the full Volterra equation or extends the exactness of the isotropic scaling solution to general deformations.

Finally, the time horizon matters. The constants in Theorem 5.6 may depend on the terminal time, so these estimates do not prove damping, rule out persistent

oscillations, or establish nonlinear orbital stability. Phase mixing requires quantitative information about the variation of orbital frequencies; sustained oscillations require information about the linearized spectrum. A nonlinear stability argument must additionally control the evolving distribution and its gravitational field, including possible loss of the energy gap. The present results establish the equilibrium geometry and a normalized finite-time tidal response without assigning an unproved long-time fate to the system.

Data Availability

No external datasets were used. The figures are illustrative evaluations of the formulas stated in this analytical study. The material accompanying the article is supplied solely as a supplementary PDF.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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