

# Toward a Unified Controllability Theory for Coupled Degenerate Elliptic-Parabolic and Hyperbolic Systems

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## Abstract

The conclusion of the thesis sets as a long-term objective the unification of its two parts—the theory of renormalized solutions for a degenerate elliptic-parabolic equation (Part I) and the Hilbert Uniqueness Method (HUM) for exact boundary controllability of a system of coupled wave equations (Part II)—within a single methodological framework. This note proposes a poroelastic-type coupled system linking the two problems, precisely identifies the central mathematical obstruction preventing a direct fusion of the two theories—a regularity mismatch between HUM’s finite-energy solutions and the  $L^1$  framework of renormalized solutions—and shows that an appropriate choice of coupling terms, built on objects already present in both parts of the thesis (the truncations  $T_k(u)$  on one side, finite-energy weak solutions on the other), makes it possible to bypass this obstruction at a fixed truncation level. A well-posedness result for the truncated system is established via a fixed-point argument, resting on a precise stability statement for renormalized solutions under right-hand side perturbations (Proposition 3’) whose hypotheses and convergence topology are made explicit. An HUM control-cost estimate for the wave system forced by a fixed interior source is proved (Proposition 4’), giving an explicit dependence of the control norm on that source and an explicit,  $k$ -linear a priori bound relevant to the passage to the limit. Coupled controllability of the truncated system is formulated as a conjecture drawing on Zuazua’s techniques for semilinear systems, with the missing ingredient for a complete proof identified explicitly. A dedicated section discusses the physical scope and the limits of the chosen coupling relative to the classical Biot equations. The passage to the limit  $k \rightarrow +\infty$ —the true heart of the long-term objective—is identified as a second open problem, distinct from and more difficult than the first. Every numbered statement in this note carries an explicit status tag—Established, Established in this note, Conjectured by analogy, or Open—so

that the boundary between what is proved and what remains a research objective is visible at each step, not left to be inferred from the surrounding prose.

## Keywords

Renormalized Solutions, Exact Boundary Controllability, Hilbert Uniqueness Method, Degenerate Elliptic-Parabolic Equations, Coupled Wave Systems, Poroelasticity

## 1. Introduction

The thesis draws on two toolboxes that, taken separately, are well established, but whose synthesis has not, to our knowledge, previously been attempted in this precise setting:

- in Part I, the theory of renormalized solutions (Bénilan, Boccardo, Gallouët, Gariepy, Pierre, Vazquez [1], then Ammar-Wittbold [2] and Carrillo-Wittbold [3]) for the degenerate equation  $\partial_t M(u) - \operatorname{div}(a(u, \nabla u)) = f$ , adapted to data  $f \in L^1(Q)$ ,  $v_0 \in L^1(\Omega)$ ;
- in Part II, J.-L. Lions' HUM method [4] for exact boundary controllability of a linear system of two coupled wave equations, resting on an observability inequality obtained via the multiplier method (Theorem 7.2.1 of the thesis) and on a duality (transposition) argument (Theorems 7.2.2-7.2.3).

Physically, a natural coupling between a (possibly degenerate) diffusion and the propagation of elastic waves is that of poroelasticity, formalized by Biot [5]: the pore pressure (or water content) of a saturated or unsaturated porous medium interacts with the elastic deformations of the solid skeleton, the latter propagating as waves. It is this type of coupling—and not a mere formal juxtaposition—that we adopt here as a test model, sufficiently close to the two systems already studied in the thesis for the tools developed there to be directly applicable. Section 5 below discusses in detail how far this test model can, and cannot, be read as physically faithful to Biot's equations.

The purpose of this note is not to prove an exact controllability theorem for the coupled system—such a result, in our view, is not within reach of currently available techniques and constitutes a research problem in its own right—but to state the problem with the precision needed to identify what is already provable, what amounts to a reasonable conjecture in light of analogous results in the literature, and what constitutes a genuinely open mathematical difficulty.

## 2. A Coupled System for the Unification

Let  $\Omega \subset \mathbb{R}^2$  be a bounded smooth domain with boundary  $\Gamma = \Gamma_0 \cup \Gamma_1$ ,  $\Gamma_0 \cap \Gamma_1 = \emptyset$ , and  $T > 0$ . We use the notation of the thesis:  $M, K$  satisfy hypotheses (H1)-(H2) of Chapter 4,  $a(u, \nabla u) = \nabla u + K(u)e_r$ , and  $(\varphi, y)$  denotes the coupled wave system of Part II (coupling coefficient  $\alpha > 0$ , geometric condi-

tion (CG) of Section 7.2.2). For  $k > 0$  fixed, we consider the coupled system

$$\begin{aligned} (S_k) \quad & \partial M(u)/\partial t - \operatorname{div}(a(u, \nabla u)) = f + \kappa y_t \\ & \varphi'' - \Delta \varphi + \alpha y_t = 0 \\ & y'' - \Delta y - \alpha \varphi_t + \lambda T_k(u) = 0 \\ & u = 0 \text{ on } \partial\Omega \times (0, T), \quad \varphi = 0 \text{ on } \Gamma \times (0, T), \quad y = 0 \text{ on } \Gamma_0 \times (0, T), \quad y = v \text{ on } \Gamma_1 \times (0, T), \\ & M(u(x, 0)) = v_0(x), \quad (\varphi, \varphi_t, y, y_t)(0) = (\varphi_0, \varphi_1, y_0, y_1), \end{aligned} \quad (1)$$

where  $\kappa, \lambda \in \mathbb{R}$  are coupling constants and  $v \in L^2(0, T; L^2(\Gamma_1))$  is the boundary control. The problem ultimately targeted, of which  $(S_k)$  is only a step, is: Given  $T > 0$  large enough and admissible initial data, does there exist a control  $v$  such that the solution of  $(S)$ —the formal limit of  $(S_k)$  as  $k \rightarrow +\infty$ , *i.e.* the coupled system without truncation—satisfies  $(\varphi, \varphi_t, y, y_t)(T) = 0$ , the parabolic component  $u$  then evolving, with no constraint imposed on its final state, as a renormalized solution of the coupled problem?

**Remark 1.** The choice of coupling terms  $\kappa y_t$  and  $\lambda T_k(u)$ —rather than, for instance,  $\kappa \operatorname{div}(y_t)$  and  $\lambda \nabla u$ , which would be the direct analogues of the classical Biot equations—is not arbitrary: it is dictated by the regularity that each of the two theories can already produce on its own side (Section 4). It is precisely this choice that makes the truncated problem (1) tractable. Section 5 gives the physical reading of this choice and states plainly what it costs in terms of fidelity to Biot's equations.

### 3. Recap of the Two Theoretical Frameworks

#### 3.1. Renormalized Solutions (Part I)

We recall Definition 5.2.1 of the thesis:  $u$  is a renormalized solution of the parabolic equation associated with  $(g, v_0)$  if  $M(u) \in L^1(Q)$ ,

$T_k(u) \in L^2(0, T; H_0^1(\Omega))$  for every  $k > 0$ , and if for every  $h \in C_c^1(\mathbb{R})$ ,  $\varphi \in C_c^\infty([0, T] \times \Omega)$ ,

$$\begin{aligned} & \iint \varphi \cdot \left( \int_{u(0,x)}^u h(r) dM(r) \right) dx dt + \iint \varphi g h(u) dx dt \\ & = \iint a(u, \nabla u) \cdot \nabla (h(u) \varphi) dx dt \end{aligned}$$

with the energy condition: the integral of  $a(u, \nabla u) \cdot \nabla u$  over the set

$Q \cap \{k \leq |u| \leq k+1\}$  tends to 0 as  $k \rightarrow +\infty$ . Theorem 5.2.1 of the thesis ensures the existence of such a solution as soon as  $g \in L^1(Q)$  and  $v_0 \in L^1(\Omega)$ , with no additional regularity hypothesis on  $g$ .

#### 3.2. HUM and Exact Controllability (Part II)

We recall the energy space  $\mathcal{E} = (H_0^1(\Omega) \times L^2(\Omega))^2$  and the observability inequality of Theorem 7.2.1: under the geometric condition (CG) and for  $0 < \alpha < \alpha_0$ ,

there exist  $T_0 > 0$  and constants  $C_2, C_3 > 0$  such that, for every  $T > T_0$  and every initial datum  $\Phi_0 \in \mathcal{E}$  of the homogeneous system (7.24) of the thesis,

$$C_2 \cdot \iint_e |\partial y / \partial v|^2 d\sigma dt \leq E_0 \leq C_3 \cdot \iint_e |\partial y / \partial v|^2 d\sigma dt \quad (2)$$

(the double-integral sign  $\iint_e$  denoting integration over  $(0, T) \times \Gamma_1$ ). Theorem 7.2.3 deduces from this, via the Lax-Milgram isomorphism applied to the operator  $\Lambda$  (eqs. 7.87-7.89), the existence of a boundary control  $v \in L^2(0, T; L^2(\Gamma_1))$  driving the homogeneous controlled system to the null state, for every initial datum  $U_0 \in (L^2(\Omega) \times H^{-1}(\Omega))^2$ .

A point that will be crucial below: the observability constants  $C_2, C_3$  depend only on the geometry of  $\Omega$ , on  $\alpha$  and on  $T$ —not on any right-hand side that might be added to the controlled system.

## 4. The Central Regularity Obstruction

[Established—thesis/cited literature]

A “naïve” coupling, modeled on the classical Biot equations, would force the parabolic equation by  $\kappa \operatorname{div}(y_t)$  and the wave equation by  $\lambda \nabla u$ . Let us examine why such a choice is, as things stand, out of reach of the two theories being combined.

- **Waves  $\rightarrow$  parabolic.** Theorem 7.2.2 provides, for a minimal control  $v \in L^2(0, T; L^2(\Gamma_1))$ , only a solution by transposition  $U = (\varphi_t, -\varphi, y_t, -y) \in C^0([0, T]; \mathcal{E}')$  with  $\mathcal{E}' = (H^{-1}(\Omega) \times L^2(\Omega))^2$ : this is the “natural” regularity level of HUM. At this level,  $y_t \in L^2(\Omega)$  only (no extra spatial regularity), and  $\operatorname{div}(y_t)$  is then defined only as a distribution in  $H^{-1}(\Omega)$ —not as a function, let alone one in  $L^1(\Omega)$ . Yet Definition 5.2.1 requires  $g = f + \kappa \operatorname{div}(y_t) \in L^1(Q)$ : the datum coming out of HUM at the minimal  $L^2$ -regularity level does not satisfy this hypothesis. Obtaining  $y_t \in H^1(\Omega)$  (so that  $\operatorname{div}(y_t) \in L^2(\Omega) \subset L^1(\Omega)$ ,  $\Omega$  being bounded) would require strong solutions ( $\Phi_0 \in D(A)$ , Theorem 7.1.2), *i.e.* controls and initial data significantly more regular than those HUM naturally produces in  $L^2$  norm.
- **Parabolic  $\rightarrow$  waves.** Conversely, forcing the wave equation by  $\lambda \nabla u$  would require  $\nabla u \in L^1(0, T; L^2(\Omega))$  (or comparable regularity) to make sense of the solution via Duhamel’s formula. But—and this can be established precisely—the Brooks-Corey function  $M$  is bounded ( $\theta_r \leq M \leq \theta_s$ ), hence not coercive in  $u$ : the sole information  $M(u) \in L^1(Q)$ , guaranteed by Theorem 5.2.1, provides no bound on  $u$  itself, let alone on  $\nabla u$ . Only the truncations  $T_k(u)$  enjoy an  $L^2(0, T; H_0^1(\Omega))$  estimate (property (ii) of Definition 5.2.1). This is precisely why the thesis itself never works with  $\nabla u$  directly, but systematically with  $\nabla T_k(u)$  (Chapter 5, proof of Theorem 5.2.1).

**Remark 2.** These two obstructions are of a different nature. The first (waves  $\rightarrow$  parabolic) is a problem of spatial regularity of HUM solutions. The second (parabolic  $\rightarrow$  waves) is a problem of lack of coercivity of  $M$ , intrinsic to the Brooks-

Corey model and hence structural to the problem studied in Part I—it would not disappear even with strong solutions.

System (1) bypasses both obstructions simultaneously, by placing the couplings not at the level of gradients/divergences but at the level of quantities already controlled by each of the two theories:

- $\kappa y_t$  (rather than  $\kappa \operatorname{div}(y_t)$ ): a finite-energy weak solution (Theorem 7.2.2, second assertion, or Proposition 7.2.2) gives  $y_t \in C^0([0, T]; L^2(\Omega))$ , hence  $y_t \in L^\infty(0, T; L^2(\Omega)) \subset L^2(Q) \subset L^1(Q)$  ( $\Omega$  bounded,  $T$  finite): Definition 5.2.1 applies with no additional regularity hypothesis on the wave solution.
- $\lambda T_k(u)$  (rather than  $\lambda u$  or  $\lambda \nabla u$ ): by property (ii) of Definition 5.2.1,  $T_k(u) \in L^2(0, T; H_0^1(\Omega)) \subset L^2(Q) \subset L^1(0, T; L^2(\Omega))$ , which is enough to give a classical (mild/semigroup) meaning to the forcing term in the wave equation, for every fixed  $k > 0$ .

## 5. Physical Interpretation of the Modified Coupling Terms

[Established in this note]—discussion, not a derivation

This section responds directly to the concern that the couplings  $\kappa y_t$  and  $\lambda T_k(u)$  might be chosen for mathematical convenience alone. We state plainly what is lost relative to Biot's equations, then identify what remains physically legible, and finally mark the limit of the analogy without overstatement.

### 5.1. What Is Lost

The classical Biot couplings [5] are of direct mechanical origin: the pressure gradient  $\nabla p$  forces the momentum balance of the solid skeleton, and the divergence of the skeleton's strain rate  $\operatorname{div}(\partial_t u)$  forces the fluid mass balance. Replacing these by  $\kappa y_t$  and  $\lambda T_k(u)$  abandons this direct stress-strain reading: neither term is derived from a coupled elastic-poroelastic energy potential in Biot's sense. This must be stated without qualification: system  $(S_k)$  is not a Biot model. It is a test system, chosen for its analytical compatibility with the two theories of the thesis, and its physical content must be assessed on that basis rather than assumed by association with poroelasticity.

### 5.2. What Nonetheless Remains Physically Legible

Two features prevent the choice of couplings from being a purely formal convenience devoid of physical content.

**Rate-type coupling.** The term  $\kappa y_t$  (a multiple of the wave velocity, rather than  $\kappa \operatorname{div}(y_t)$ , a multiple of the strain-rate divergence) corresponds to the regime of Kelvin-Voigt-type, poro-viscoelastic extensions of Biot's theory, in which the fluid-solid coupling includes a term proportional to a rate of deformation rather than to the deformation itself—a standard, non ad hoc ingredient of the attenuative (viscoelastic) extensions of Biot's equations used, for instance, in seismology of dissipative media.  $\kappa y_t$  can then be read as isolating the rate-dependent part of a poroelastic coupling, with its rate-independent (gradient) part absent.

**Saturation-capped feedback.** The term  $\lambda T_k(u)$  (rather than  $\lambda u$  or  $\lambda \nabla u$ ) caps the feedback of pore pressure onto the elastic skeleton at a saturation level  $k$ . This is not physically arbitrary for a partially saturated medium governed, as in Part I, by a Brooks-Corey law: the effective mechanical response of a partially saturated soil itself saturates as the capillary pressure becomes very negative or very positive (capillary trapping, bounded effective stress).  $T_k(u)$  can therefore be read as a physically motivated regularization of the effective stress beyond a saturation threshold  $k$ , rather than as a purely analytical truncation device.

### 5.3. Limits of the Analogy

These two readings give a local physical meaning to the retained couplings, but they do not amount to a derivation of system  $(S_k)$  from a poroelastic variational principle under an identified simplifying hypothesis. We do not claim that  $(S_k)$  follows from Biot's theory; only that it is a structurally close relative, in the spirit of the rate-type and saturation-capped regularizations already present in the poroelastic literature. Restoring the full physical coupling  $(\kappa \operatorname{div}(y_t), \lambda \nabla u)$  remains the reference objective, and is explicitly the subject of the "Duality adapted to  $L^1$  data" direction discussed in Section 9: the hierarchy—a physically legible but non-derived test model here, versus an unattained physical reference objective there—should be read as a single, deliberately incomplete, research trajectory rather than as two unrelated remarks.

## 6. Well-Posedness of the Truncated System

**Proposition 3** (Existence for the forced parabolic part—direct extension of Theorem 5.2.1). [*Established—thesis/cited literature*]

Let  $y_t \in L^2(Q)$  be fixed and  $f \in L^1(Q)$ ,  $v_0 \in L^1(\Omega)$ . Then there exists a renormalized solution  $u$  of  $\partial_t M(u) - \operatorname{div}(a(u, \nabla u)) = f + \kappa y_t$ ,  $M(u(0, \cdot)) = v_0$ ,  $u = 0$  on  $\partial\Omega$ , in the sense of Definition 5.2.1.

*Proof sketch.* Since  $L^2(Q) \subset L^1(Q)$  on the bounded domain  $Q = ]0, T[ \times \Omega$ , we have  $f + \kappa y_t \in L^1(Q)$ , and the proof of Theorem 5.2.1—approximation  $f_{m,n} = \max(\min(f + \kappa y_t, m), -n)$ , perturbation  $\psi_{m,n}$ , passage to the limit  $m, n \rightarrow +\infty$  via the five integrals  $J_1, \dots, J_5$  (Chapter 5, pp. 45-49 of the thesis)—applies verbatim, the right-hand side  $g := f + \kappa y_t$  playing exactly the role of  $f$  in the original statement.

**Proposition 3'** (Stability of renormalized solutions under perturbation of the right-hand side). [*Established in this note*]

Let  $v_0 \in L^1(\Omega)$  be fixed and let  $M$ , a satisfy hypotheses (H1)-(H2) of Chapter 4 together with the structure  $a(u, \nabla u) = \nabla u + K(u)e_r$ ,  $K$  bounded and continuous. Let  $(g_n) \subset L^1(Q)$  with  $g_n \rightarrow g$  strongly in  $L^1(Q)$ , and let  $u_n$  be a renormalized solution (Definition 5.2.1) of

$\partial_t M(u_n) - \operatorname{div}(a(u_n, \nabla u_n)) = g_n$ ,  $M(u_n(0)) = v_0$ . Then, along a subsequence: (i)  $u_n \rightarrow u$  a.e. in  $Q$ ; (ii)  $M(u_n) \rightarrow M(u)$  in  $C([0, T]; L^1(\Omega))$ ; (iii) for every  $k > 0$ ,  $T_k(u_n) \rightarrow T_k(u)$  strongly in  $L^2(0, T; H_0^1(\Omega))$  and in  $L^2(Q)$ ; (iv)  $u$  is

a renormalized solution associated with  $(g, v_0)$ .

If, in addition, uniqueness of the renormalized solution holds for data  $(g, v_0)$ —e.g. under the extra structural ( $L^1$ -contraction/monotonicity) condition of Carrillo-Wittbold [3]—then the whole sequence converges and the solution map  $g \mapsto u$  is continuous from  $(L^1(Q), \text{strong})$  into the topology defined by (ii)-(iii).

*Proof sketch.* The energy estimate obtained by testing against a truncation of  $u_n$  (Chapter 5, proof of Theorem 5.2.1) gives, for every  $k$ , a bound on

$\|T_k(u_n)\|_{L^2(0,T;H_0^1(\Omega))}$  depending only on  $k$  and on  $\|g_n\|_{L^4(Q)}$ , hence uniform in

$n$  since  $g_n \rightarrow g$  in  $L^1(Q)$  is in particular bounded. The compact embedding  $L^2(0,T;H_0^1(\Omega)) \subset L^2(Q)$  and a diagonal extraction over  $k$  give (i) and the weak part of (iii); a.e. convergence and an argument on  $M(u_n)$  using only that  $(g_n)$  is bounded in  $L^1$  (as in the existence proof, not that it is equi-integrable) upgrade this to (ii). Passing to the limit in the weak formulation

$$\iint \varphi \left( \int_{u(0,x)}^u h(r) dM(r) \right) dx dt + \iint \varphi g_n h(u_n) dx dt = \iint a(u_n, \nabla u_n) \cdot \nabla (h(u_n) \varphi) dx dt$$

uses  $g_n \rightarrow g$  strongly in  $L^1(Q)$  together with  $h(u_n)$  bounded and

$h(u_n) \rightarrow h(u)$  a.e. (dominated convergence): this is exactly the point where strong convergence of the data is used, and where mere weak- $L^1$  convergence would instead require  $g_n$  equi-integrable (Dunford-Pettis) to control

$\iint \varphi g_n h(u_n)$ —equi-integrability substitutes for strong convergence only in that one step, and is not needed here. The energy condition of Definition 5.2.1 passes to the limit by lower semicontinuity, giving (iv). Strong (not merely weak) convergence of  $T_k(u_n)$  in (iii) follows from strong convergence of  $M(u_n)$  combined with the strict monotonicity of  $a$  on the truncated range, as in the uniqueness argument of [3].

**Remark 3.** In the fixed-point loop of Theorem 5 below, the datum submitted to Proposition 3' is  $g_n = f + \kappa y_{t,n}$ , where  $y_{t,n} = \partial_t y[w_n]$  is the image of the iterate  $w_n$  under the linear continuous map of Proposition 4 (waves  $\rightarrow$  parabolic right-hand side). The iterates  $w_n = T_k(u[y_t[w_{n-1}]])$  remain, by construction (point 1 of the proof of Theorem 5), in the ball  $B_R = \{\|w\|_{L^2(Q)} \leq R\}$  with  $R = k \cdot \sqrt{|Q|}$ , and the compactness argument already invoked at point 2 of that same proof (compact embedding  $L^2(0,T;H_0^1(\Omega)) \subset L^2(Q)$  for the component  $T_k(u)$ ) shows that any weakly convergent subsequence of  $(w_n)$  in fact converges strongly in  $L^2(Q)$ . By continuity of Proposition 4,  $y_{t,n}$  then converges strongly in  $L^2(Q) \subset L^1(Q)$  ( $\Omega, Q$  bounded). It is therefore the strong-convergence hypothesis of Proposition 3'—not a weaker, equi-integrable, weak- $L^1$  hypothesis—that is actually available, and needed, in the proof of Theorem 5.

**Proposition 4** (Existence for the forced wave part—extension by superposition of Theorem 7.2.2). [*Established—thesis/cited literature*]

Let  $w \in L^2(Q)$  be fixed (in particular  $w = T_k(u)$  for a given measurable  $u$ ),  $v \in L^2(0,T;L^2(\Gamma_1))$  and  $U_0 \in (L^2(\Omega) \times H^{-1}(\Omega))^2$ . Then the system  $\varphi'' - \Delta \varphi + \alpha y_t = 0$ ,  $y'' - \Delta y - \alpha \varphi_t + \lambda w = 0$ , with the boundary and initial condi-

tions of (1), admits a unique solution by transposition

$U \in C^0\left([0, T]; \left(L^2(\Omega) \times H^{-1}(\Omega)\right)^2\right)$ , and the linear map  $(U_0, v, w) \mapsto U$  is continuous.

**Proof sketch.** By linearity,  $U = U_{\text{hom}} + U_{\text{part}}$ , where  $U_{\text{hom}}$  solves the problem without the  $\lambda w$  term (Theorem 7.2.2 of the thesis, applicable as is) and  $U_{\text{part}}$  is the mild solution, in the sense of the semigroup generated by the  $m$ -dissipative operator  $A$  (Proposition 7.2.1), of the equation  $\Phi_t = A\Phi + (0, \lambda w, 0, 0)$  with zero initial datum and  $v = 0$ . Since  $w \in L^2(Q) = L^2(0, T; L^2(\Omega)) \subset L^1(0, T; L^2(\Omega))$ , Duhamel's formula  $U_{\text{part}}(t) = \int_0^t e^{(t-s)A} (0, \lambda w(s), 0, 0) \, ds$  defines an element of  $C^0([0, T]; \mathcal{E}) \subset C^0([0, T]; \mathcal{E}')$ , by standard properties of contraction semigroups (Pazy [6], Thm 1.2, Chap. 4). Continuity of the map  $(U_0, v, w) \mapsto U$  follows from estimate (7.84) of the thesis for  $U_{\text{hom}}$  and from the semigroup's contraction estimate for  $U_{\text{part}}$ .

**Theorem 5** (Well-posedness of the truncated system  $(S_k)$ ,  $v$  fixed). [Established in this note]

For every  $k > 0$ , every  $v \in L^2(0, T; L^2(\Gamma_1))$  and all admissible data  $(f, v_0, \Phi_0) \in L^1(Q) \times L^1(\Omega) \times \left(L^2(\Omega) \times H^{-1}(\Omega)\right)^2$ , system  $(S_k)$  admits at least one solution  $(u, \varphi, y)$ , with  $u$  a renormalized solution and  $(\varphi, \varphi_t, y, y_t) \in C^0([0, T]; \mathcal{E}')$  a solution by transposition.

*Proof.* Fixed-point argument. Let  $B_R = \{w \in L^2(Q) : \|w\|_{L^2(Q)} \leq R\}$  for  $R > 0$  to be specified, and let  $\Psi : B_R \rightarrow L^2(Q)$  be defined by  $w \rightarrow (\text{Prop. 4}) \rightarrow y_t[w] \rightarrow (\text{Prop. 3}) \rightarrow u[y_t[w]] \rightarrow (T_k) \rightarrow \Psi(w) := T_k(u[y_t[w]])$ . Three observations make this fixed point rigorous:

1)  $\|\Psi(w)\|_{L^2(Q)} \leq \sqrt{|Q|} \cdot k$  for every  $w$ , since  $|T_k(\cdot)| \leq k$  pointwise—the truncation bounds the image of  $\Psi$  a priori, independently of the size of  $w$ . It therefore suffices to choose  $R = k\sqrt{|Q|}$  for  $\Psi(B_R) \subset B_R$ .

2) The regularity gained at the wave step ( $y_t \in C^0([0, T]; L^2(\Omega))$ , Proposition 4) and the compactness of the embedding  $L^2(0, T; H_0^1(\Omega)) \subset L^2(Q)$  for the component  $T_k(u)$  (bounded by construction) give the compactness of  $\Psi$ , which is proved directly: any bounded sequence  $(w_n) \subset B_R$  produces, via Proposition 4, a sequence  $y_{t,n}$  bounded in  $C^0([0, T]; L^2(\Omega))$ ; by the compact embedding just cited, a subsequence of the resulting  $T_k(u_n)$  converges strongly in  $L^2(Q)$ . This opens the way to a Schauder fixed-point theorem rather than a Banach one—which dispenses with a strict contraction estimate, generally out of reach given the nonlinearity of  $u \mapsto M(u)$ .

3) Continuity of  $\Psi$  follows from Proposition 3' applied to the sequence of right-hand sides  $g_n = f + \kappa y_{t,n}$  generated along the iteration. As shown in Remark 3 above, the mode of convergence available here is strong convergence in  $L^2(Q) \subset L^1(Q)$ , not merely weak convergence in  $L^1(Q)$ ; this is precisely the regime covered by Proposition 3', and it removes the need for a separate equi-integrability argument.

### Roadmap of the research program (Figure 1)

Part I: renormalized solutions  $(f \in L^1(Q)) \leftrightarrow$  [Prop. 3, 3' + Prop. 4: coupling via  $y_t$  and  $T_k(u)$ ]  $\leftrightarrow$  Part II: HUM (finite-energy solutions)  
 $\downarrow$  Theorem 5: fixed point,  $(S_k)$  well-posed,  $v$  fixed [Established in this note]  
 $\downarrow$  Proposition 4': HUM control-cost estimate for a fixed source  $w$  [Established in this note]  
 $\downarrow$  Conjecture 6: coupled controllability of  $(S_k)$  [Conjectured by analogy—unproven]  
 $\downarrow$  Open Problem 7: passage to the limit  $k \rightarrow +\infty$  (uniform observability, compactness of controls  $v_k$ ) [Open]

**Figure 1.** Roadmap of the research program—from the well-posedness of the truncated system (rigorous) to the removal of the truncation (open problem).

## 7. Toward Controllability of the Truncated System

For  $k$  fixed, Theorem 5 gives a well-posed system for every control  $v$ . It remains to choose  $v$  so that  $(\varphi, \varphi_t, y, y_t)(T) = 0$ . The difficulty here is no longer one of regularity but of coupling in the choice of control: the HUM control built in Theorem 7.2.3 for the homogeneous wave system does not account for the right-hand side  $\lambda T_k(u)$ , which itself depends, through  $u$ , on the very wave solution one is trying to steer. Proposition 4' below makes this dependence quantitative.

**Proposition 4'** (Control-cost estimate for the forced wave system). [*Established in this note*]

Under the hypotheses of Proposition 4 and Theorem 7.2.1 (geometric condition (CG),  $0 < \alpha < \alpha_0$ ,  $T > T_0$ ), for every  $w \in L^2(Q)$  and every  $U_0 \in \mathcal{E}' = (L^2(\Omega) \times H^{-1}(\Omega))^2$  there exists a control  $v \in L^2(0, T; L^2(\Gamma_1))$  such that the solution  $U$  of the forced system of Proposition 4 with this control satisfies  $U(T) = 0$ , and

$$\|v\|_{L^2(0, T; L^2(\Gamma_1))} \leq C'_2 \|U_0\|_{\mathcal{E}'} + C_4 \|w\|_{L^2(Q)}$$

where  $C'_2$  is the HUM constant of Theorem 7.2.3 (depending only on  $\Omega$ ,  $\alpha$  and  $T$ —not on  $w$  or  $U_0$ ) and  $C_4 = C'_2 \cdot C_{\text{pazy}}$ ,  $C_{\text{pazy}}$  being the contraction-semigroup constant used in the proof of Proposition 4 to bound

$$\|U_{\text{part}}(T)\|_{\mathcal{E}'} \leq C_{\text{pazy}} \|w\|_{L^2(Q)}.$$

*Proof.* By linearity,  $U = U_{\text{hom}}[U_0, v] + U_{\text{part}}[w]$  (Proposition 4). Take  $v = v_0 + v_1$ :  $v_0$  is the HUM null-control of Theorem 7.2.3 steering  $U_{\text{hom}}[U_0, \cdot]$  from  $U_0$  to 0, with  $\|v_0\| \leq C'_2 \|U_0\|_{\mathcal{E}'}$ ;  $v_1$  is the control given by the same isomorphism  $\Lambda^{-1} - \Lambda$  depends only on the geometry,  $\alpha$  and  $T$ , not on the target (eqs. 7.87-7.89)—steering the homogeneous system from 0 to the target  $-U_{\text{part}}[w](T) \in \mathcal{E}'$ , with  $\|v_1\| \leq C'_2 \|U_{\text{part}}(T)\|_{\mathcal{E}'}$ . The bound  $\|U_{\text{part}}(T)\|_{\mathcal{E}'} \leq C_{\text{pazy}} \|w\|_{L^2(Q)}$  is the estimate already established in the proof of Proposition 4 via Duhamel's formula and the contraction property of the semi-

group generated by  $A$ . Setting  $v = v_0 + v_1$ ,

$U(T) = U_{\text{hom}}[U_0, v_0 + v_1] + U_{\text{part}}[w] = 0 + (-U_{\text{part}}[w](T)) + U_{\text{part}}[w](T) = 0$ , and the triangle inequality gives the claim with  $C_4 = C'_2 C_{\text{Pazy}}$ .

**Corollary** (order of magnitude in  $k$ ). [*Established in this note*]

Applying Proposition 4' with  $w = T_k(u)$  and using  $|T_k(\cdot)| \leq k$  pointwise, hence  $\|T_k(u)\|_{L^2(Q)} \leq k\sqrt{|Q|}$ , gives, for the control  $v_k$  of Conjecture 6:

$$\|v_k\|_{L^2(0,T;L^2(\Gamma_1))} \leq C'_2 \|U_0\|_{\mathcal{E}'} + C_4 k\sqrt{|Q|}.$$

**Remark 4.** This estimate answers the request for an explicit control-cost dependence on the source, and simultaneously clarifies what is missing to close the control-side fixed point of Conjecture 6: Proposition 4' controls the cost of the control as a function of a source  $w$  assumed known and fixed; what remains is to control, in a stability sense, the dependence of  $w = T_k(u[y_t[v]])$  on the control itself. It is also, as noted in Section 8, an explicitly  $k$ -linear bound: it shows that the naive superposition estimate does not by itself give a uniform-in- $k$  bound on  $(v_k)_k$ , so that any compactness of this sequence, if it holds, must come from a finer mechanism than Proposition 4' alone.

**Conjecture 6** (Exact coupled controllability of the truncated system). [*Conjectured by analogy—unproven*]

Under the hypotheses of Theorem 7.2.1 (geometric condition (CG),  $0 < \alpha < \alpha_0$ ) and for  $T > T_0$ , there exists, for every  $k > 0$  and all admissible initial data, a control  $v \in L^2(0, T; L^2(\Gamma_1))$  such that system  $(S_k)$ , equipped with this control, admits a solution satisfying  $(\varphi, \varphi_t, y, y_t)(T) = (0, 0, 0, 0)$ .

This conjecture rests on an analogy with the fixed-point method for controllability of semilinear equations introduced by Zuazua [7] and developed by Fabre, Puel and Zuazua [8] for globally Lipschitz nonlinearities: one controls the linearized equation by HUM with the nonlinearity “frozen,” then closes a fixed point between the resulting control and the trajectory it generates. Two elements argue in favor of its accessibility:

- the observability constant  $C_2$ ,  $C_3$  in (2), and hence the constant  $C'_2$  of Proposition 4', depends neither on  $k$  nor on the right-hand side  $\lambda T_k(u)$ : it is a purely geometric property of the homogeneous adjoint system, which allows the same operator  $\Lambda$  as in Theorem 7.2.3 to be used for every  $k$  and every trajectory  $u$ —this is now made explicit by Proposition 4' rather than asserted;
- the truncation  $T_k$  is globally bounded ( $|T_k| \leq k$ ) and 1-Lipschitz, which is precisely the “mild” nonlinearity setting under which the Zuazua-Fabre-Puel fixed-point arguments apply most directly, without any smallness assumption on the data.

We do not establish this conjecture here. Proposition 4' supplies the first of the two ingredients required by the Zuazua-Fabre-Puel argument—an estimate of the control cost as a function of the frozen source. The second ingredient—a stability estimate of the renormalized solution  $u$  with respect to the control  $v$ , uniform over a set of admissible trajectories (via  $y_t$ , Proposition 3)—is not established in this

text and constitutes, together with the compactness question of Section 8, the principal technical gap separating Conjecture 6 from a theorem.

## 8. The Central Open Problem: Beyond the Truncation

The true objective stated in the conclusion of the thesis—controlling the coupled system without truncation—requires letting  $k \rightarrow +\infty$  in Conjecture 6. This raises two cumulative difficulties, of a different nature, it should be recalled, from the regularity obstruction already resolved in Section 4:

**Open Problem 7** (Passage to the limit  $k \rightarrow +\infty$  in the controllability problem). [*Open*]

1) Compactness of the sequence of controls  $(v_k)_k$ . Nothing in the construction of  $v_k$  (Conjecture 6) guarantees a priori a uniform-in- $k$  bound on

$\|v_k\|_{L^2(0,T;L^2(\Gamma_1))}$ : the constant  $C_2'$  of Proposition 4' is independent of  $k$  (a good point, now quantified), but the corollary to Proposition 4' shows that the naive bound obtained from it grows like  $k\sqrt{|Q|}$ , which is not by itself summable/bounded information—a genuinely finer argument, exploiting the fixed-point structure of Theorem 5 rather than the superposition estimate alone, would be needed to obtain a uniform bound, if one exists at all. This difficulty is structurally reminiscent of the one already encountered—and resolved—in Chapter 5 for existence (passage to the limit  $m, n \rightarrow +\infty$ , uniform estimates of Lemma 5.2.1 and following); its transposition to the controllability setting, where the datum  $u_k$  itself depends on the control one is trying to make converge, is new and not covered by Proposition 4'.

2) Stability of the controllability property under weak passage to the limit. Even assuming  $v_k \rightharpoonup v$  weakly in  $L^2(0,T;L^2(\Gamma_1))$  and  $u_k \rightarrow u$  in the sense of convergence of renormalized solutions (a.e. convergence, strong convergence of truncations in  $L^2(0,T;H_0^1(\Omega))$ , in the sense of Proposition 3'), one still needs to show that the limit  $v$  exactly controls the untruncated system—which requires passing to the limit in the terminal condition  $(\varphi_k, \varphi_{k,t}, y_k, y_{k,t})(T) = 0$ , a property that is pointwise in time, and considerably more delicate to transport under weak convergence than an identity integrated in time and space, of the kind handled in Chapter 5.

**Remark 5.** It is not excluded that the exact, time-pointwise controllability property may fail to survive the passage to the limit  $k \rightarrow +\infty$ , and that only a weaker form—approximate controllability ( $\|(\varphi, \varphi_t, y, y_t)(T)\|_{\mathcal{E}'} < \varepsilon$  for every  $\varepsilon > 0$ , with a control  $v_\varepsilon$  depending on  $\varepsilon$ )—might be attainable for the complete, untruncated system. This would already, in itself, be a non-trivial result, faithful to the spirit of the objective stated in the thesis's conclusion.

## 9. Alternative Strategies and Complementary Directions

Three directions, not explored in this note, would deserve to be examined alongside the program above:

- **Duality adapted to  $L^1$  data.** Rather than coupling HUM (built for states in  $\mathcal{E}$

or  $\mathcal{E}'$ ) to an  $L^1$  theory, one could seek to reformulate the transposition method of Section 3.2 directly within an  $(L^\infty, L^1)$  duality scale, drawing on the techniques for controllability with low-regularity data developed for the heat equation by Fursikov-Imanuvilov [9] via Carleman inequalities—techniques which, however, typically assume a non-degenerate equation, which would exclude precisely the non-coercive M case treated in Chapter 5. This is also the direction along which the physically faithful Biot couplings of Remark 1 and Section 5.1  $(\kappa \operatorname{div}(y_t), \lambda \nabla u)$  could eventually be restored.

- **Linearized/frozen system.** A realistic intermediate step consists in establishing controllability of the system linearized around a fixed reference trajectory  $\bar{u}$  (*i.e.* replacing  $T_k(u)$  by  $T_k(\bar{u})$ , known), which reduces the problem to a linear wave system with a fixed right-hand side—exactly Proposition 4, with cost given by Proposition 4'—and would provide a local controllability result around  $\bar{u}$  via a perturbation argument, without requiring the full fixed point of Conjecture 6.
- **Application to saltwater intrusion.** As suggested by the thesis's conclusion, controlling saline intrusion in a coastal aquifer via boundary water releases would provide a concrete application setting to numerically test (in the sense of Chapter 6) the conjectures formulated here, before any attempt at a complete proof: a numerical validation of Conjecture 6, for instance by observing whether a numerical fixed-point algorithm (numerical HUM alternated with a renormalized time step) converges in practice for moderate  $k$ , would provide valuable evidence for the plausibility of the program before investing in its full proof.

## 10. Synthesis

**Table 1** summarizes, for each numbered statement introduced in this note, its current status, ranging from results already established in the thesis or the cited literature to the open problem that defines the program's long-term objective.

**Table 1.** Status of the statements introduced in this note.

| Statement                                                                                                           | Status                                                                  |
|---------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------|
| Choice of couplings— $\kappa y_t$ (waves $\rightarrow$ parabolic), $\lambda T_k(u)$ (parabolic $\rightarrow$ waves) | Established (Section 4); physical scope discussed in Section 5          |
| Stability of the RHS-perturbed parabolic problem—Proposition 3'                                                     | Established in this note                                                |
| Existence, $v$ fixed—Theorem 5, Schauder fixed point                                                                | Established in this note                                                |
| Control cost for a fixed source $w$ —Proposition 4' and Corollary                                                   | Established in this note                                                |
| Controllability, $k$ fixed—Conjecture 6, Zuazua-Fabre-Puel method                                                   | Conjectured by analogy—unproven (missing ingredient named in Section 7) |
| Limit $k \rightarrow +\infty$ —Open Problem 7, compactness of controls and stability of the terminal property       | Open                                                                    |

## 11. Conclusions

This note does not claim to resolve the long-term objective stated in the conclusion of the thesis [10], but gives it a precise and hierarchized mathematical form, now with an explicit status attached to every numbered statement. The main result, in the proper sense of the term, is negative: a “natural” coupling between the two parts (via gradients and divergences, in the manner of the classical Biot equations) is structurally incompatible with the level of regularity that each of the two theories can produce for the other, as shown in Section 4. Section 5 states, without qualification, that the alternative coupling adopted here departs from that physical model, while identifying the specific, non-arbitrary sense in which it remains physically legible.

The positive results are that a different choice of coupling—energy against energy, truncation against truncation—restores compatibility and makes the truncated system accessible to the methods already developed in the thesis (Theorem 5, resting on the precise stability statement of Proposition 3'), and that the cost of controlling this truncated system admits an explicit bound in terms of the frozen source (Proposition 4'). Controllability of the truncated system and, a fortiori, the passage to the limit that would recover the complete system, remain open, for reasons now stated explicitly rather than left implicit: a missing stability-of-the-trajectory-in-the-control estimate for Conjecture 6, and a missing uniform-in- $k$  compactness argument, sharper than the linear bound of the Corollary to Proposition 4', for Open Problem 7. This is, quite concretely, what the expression “long-term objective” in the thesis’s conclusion covers.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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