

Optimal Control of Dengue-Malaria Co-Infection Dynamics

Eyaran Wilson Ewesit*, Mary Wainaina, Damian Maingi

Department of Mathematics and Actuarial Science, Catholic University of Eastern Africa, Nairobi, Kenya
Email: *ewesitwilson@gmail.com

How to cite this paper: Ewesit, E.W., Wainaina, M. and Maingi, D. (2026) Optimal Control of Dengue-Malaria Co-Infection Dynamics. *Journal of Applied Mathematics and Physics*, 14, 3777-3795.
<https://doi.org/10.4236/jamp.2026.1410186>

Received: September 2, 2026

Accepted: October 6, 2026

Published: October 9, 2026

Copyright © 2026 by author(s) and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

Malaria-dengue fever coinfection is an emerging problem of the tropical and subtropical regions of the world. The coinfection is spreading because of the shared geographical regions, shared symptoms, and the shared vectors, that is mosquitoes. This study explored the optimal control of malaria-dengue fever coinfection through mathematical epidemiological modelling. A mathematical epidemiological model for malaria-dengue fever spread dynamics was developed and its sensitivity was calculated using the normalised forward sensitivity index. The objective of sensitivity analysis was to determine the candidate parameters for optimal control. Numerical sensitivity indices were determined. The optimal control problem was designed using retooling of healthcare workers, sensitization of communities, and administration of anti-malaria drugs as the control measures. The objective functional was developed using the Bolza form of the objective function which incorporated all the measures and the target compartments. The Pontryagin maximum principle was used to determine the Hamiltonian of the problem. The study developed four strategies on the implementation of the control measure with each strategy developed from a combination of various control measures. Each strategy was analysed numerically and compared to the others to determine the most effective combination. It was determined that the combination of all the three control parameters was the strategy with the most desirable outcome.

Keywords

Coinfection Dynamics, Malaria-Dengue Fever Co-Infection, Optimal Control, Pontryagin Maximum Principle, Numerical Simulations

1. Introduction

Malaria-dengue fever coinfection is a major public health concern that requires optimized control in sub-Saharan Africa. The incidence of the two diseases has increased due to share geographical areas of endemicity. The diseases are more pronounced in the tropical and subtropical regions where the sub-Saharan region of Africa lies. The two diseases are both spread by mosquitoes and have almost similar aetiology thus making misdiagnosis a common phenomenon in low-resource economies. In sub-Saharan Africa, malaria is more prevalent when compared to dengue fever, as a consequence, dengue fever is more often misdiagnosed as malaria when compared to malaria as dengue [1]. Furthermore, coinfection is seldom detected due to shared signs and symptoms and scarcity of accurate resources to isolate dengue fever from malaria [2]. Therefore, coinfection is often treated as a mono-infection as opposed to a coinfection of two diseases. The treatment regimens of coinfections significantly differ from the regimens for mono infection. Therefore, delayed correct diagnosis leads to deterioration of the health of the patients and sometimes can lead to fatalities [3].

Optimal control is one of the key interventional measures that can be implemented to curb the adverse effects of coinfection spread dynamics. This study focused on the development of an epidemiological mathematical model that incorporates the control measures to optimally control the spread dynamics of dengue fever coinfection. Retool healthcare services providers to correctly and effectively isolate the two diseases and coinfection for the administration of the correct treatment regimens [4] [5]. The communities in the geographical areas where the two diseases are co-endemic need to be sensitized on the aetiology of the two diseases and how to prevent them. Furthermore, since both diseases are spread by mosquitoes, the communities need to be sensitized on the effective ways of managing mosquitoes to prevent the spread of the two diseases [6]. Lastly, since malaria is more dominant than dengue fever, there should be efforts to increase treatment rates so as to bring down the incidence of coinfection in the communities [7]. This study focused on the above control measures to develop an optimal control model for the spread of malaria-dengue fever coinfection.

2. Model Formulation

To achieve the optimal control of malaria and dengue spread dynamics, we considered the the implementation of the following control parameters: Retooling of human resources in the health sector on how to actively differentiate and handle malaria and dengue u_1 , to conduct public sensitization on effective methods of malaria and dengue prevention measures u_2 , and administering of anti-malaria drugs u_3 . The optimized malaria-dengue fever compartmental model was developed as shown in **Figure 1** shown below.

The resulting system of differential equations was expressed as follows.

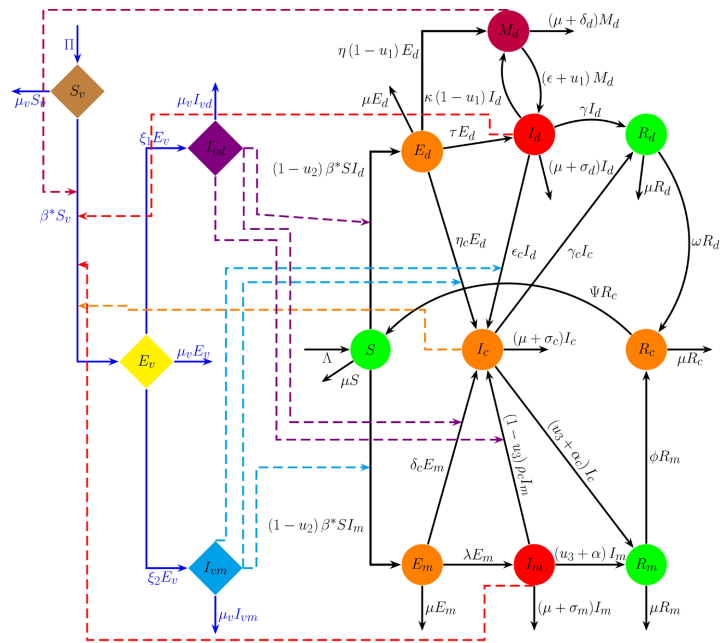


Figure 1. Coinfection model flow chart with control parameters.

$$\begin{cases}
 \frac{dS}{dt} = \Lambda - (1-u_2)\beta^*SI_d - (1-u_2)\beta^*SI_m - \mu S + \Psi R_c \\
 \frac{dE_d}{dt} = (1-u_2)\beta^*SI_d - (\mu + \eta_c + (1-u_1)\eta + \tau)E_d \\
 \frac{dM_d}{dt} = (1-u_1)\eta E_d + (1-u_1)\kappa I_d - (u_1 + \varepsilon)M_d - (\mu + \delta_d)M_d \\
 \frac{dI_d}{dt} = \tau E_d + (u_1 + \varepsilon)M_d - (1-u_1)\kappa I_d - (\mu + \sigma_d)I_d - \varepsilon_c I_d - \gamma I_d \\
 \frac{dR_d}{dt} = \gamma I_d + \gamma_c I_c - \mu R_d - \omega R_d \\
 \frac{dE_m}{dt} = (1-u_2)\beta^*SI_m - \mu E_m - \lambda E_m - \delta_c E_m \\
 \frac{dI_m}{dt} = \lambda E_m - \rho_c I_m - (\mu + \sigma_m)I_m - (u_3 + \alpha)I_m \\
 \frac{dR_m}{dt} = (u_3 + \alpha)I_m + (u_3 + \alpha_c)I_c - \Phi R_m - \mu R_m \\
 \frac{dI_c}{dt} = \delta_c E_m + \rho_c I_m + \varepsilon_c I_d + \eta_c E_d - (\mu + \delta_c)I_c - (u_3 + \alpha_c)I_c - \gamma_c I_c \\
 \frac{dR_c}{dt} = \phi R_m + \omega R_d - \mu R_c - \Psi R_c \\
 \frac{dS_v}{dt} = \Pi - \mu_v S_v - \beta^* S_v \\
 \frac{dE_v}{dt} = \beta_v^* S_v - \xi_1 E_v - \xi_2 E_v - \mu_v E_v \\
 \frac{dI_{vm}}{dt} = \xi_2 E_v - \mu_v I_{vm} \\
 \frac{dI_{vd}}{dt} = \xi_1 E_v - \mu_v I_{vd}
 \end{cases} \quad (1)$$

Subject to the initial conditions

$$\begin{aligned} S(0) > 0, E_d(0) \geq 0, M_d(0) \geq 0, I_d(0) > 0, R_d(0) \geq 0, E_m(0) \geq 0, I_m(0) > 0, \\ R_m(0) \geq 0, I_c(0) \geq 0, R_c(0) \geq 0, I_c(0) \geq 0, R_c(0) \geq 0 \end{aligned} \quad (2)$$

The force of infection for the vectors in the coinfection model 0 was taken to be $\beta_v^* = \beta_{vd}^* + \beta_{vm}^*$ where

$$\beta_{vd}^* = \frac{b_h \beta_{vhd} I_{hd}}{N_h}$$

where

b_h is the number of mosquito bites that one human has per unit time

β_{vhd} is the dengue transmission rate from mosquitoes to human beings

I_{hd} infected human beings with dengue fever

and

$$\beta_{vm}^* = \frac{b_v T \beta_{hvm} I_{hm}}{N_h}$$

where

b_v is the number of human bites that one mosquito has per unit time

β_{hvm} is the malaria transmission rate from mosquitoes to human beings

T is the temperature coefficient of biting rate effectiveness given by $0 \leq T \leq 1$

While the force of infection of the human population $\beta_h^* = \beta_{hm}^* + \beta_{hd}^*$, defined as

$$\beta_{hm}^* = \frac{b_h T \beta_{hvm} I_{vm}}{N_v}$$

where

b_h is the number of mosquito bites that one human has per unit time

β_{hvm} is the malaria transmission rate from mosquitoes to human beings

T is the temperature coefficient of biting rate effectiveness given by $0 \leq T \leq 1$

and

$$\beta_{hd}^* = \frac{b_h \beta_{hvd} I_{vd}}{N_v} + \frac{b_h \theta M_{hd}}{N_h}$$

where

b_h is the number of mosquito bites that one human has per unit time

β_{hvd} is the dengue transmission rate from human beings to mosquitoes

θ infectiousness modifiers for misdiagnosis ($0 < \theta \leq 1$)

I_{vd} dengue infected mosquitoes population

M_{hd} Misdiagnosed human population

Sensitivity Analysis

Sensitivity analysis of the basic reproduction number give information of the parameters which are sensitive to changes in the basic reproduction number [8]. Such parameters form the best candidates for control. This study utilized the normalized forward sensitivity index which was defined as

$$\Upsilon_p^{R_{0c}} = \frac{\partial R_{0c}}{\partial p} \times \frac{p}{R_{0c}} \quad (3)$$

where the basic reproduction number was calculated using the next generation matrix and explained in [9]; and obtained as

$$R_{0c} = \frac{-\frac{b_h \theta \Lambda (\eta k_3 + \kappa \tau)}{\mu k_1 (\epsilon \kappa - k_2 k_3)} + \sqrt{\left(\frac{b_h \theta \Lambda (\eta k_3 + \kappa \tau)}{\mu k_1 (\epsilon \kappa - k_2 k_3)} \right)^2 + 4 \left(\frac{T b_h \beta_{hm} \Lambda \xi_2}{\mu k_7 k_8} + \frac{b_h \beta_{hvd} \Lambda \xi_1}{\mu k_7 k_9} \right) \left(\frac{b_h \beta_{vhd} \Pi (\epsilon \eta + \tau k_2)}{\mu_v k_1 (\epsilon \kappa - k_2 k_3)} \right)}}{2} \quad (4)$$

where $k_1 = (\mu + \eta_c + \eta + \tau)$ $k_2 = (\epsilon + \mu + \delta_d)$ $k_3 = (\kappa + \mu + \sigma_d + \epsilon_c + \gamma)$
 $k_4 = (\mu + \lambda + \delta_c)$ $k_5 = (\rho_c + \mu + \sigma_m + \alpha)$ $k_6 = (\mu + \delta_c + \alpha_c)$ $k_7 = (\xi_1 + \xi_2 + \mu)$
 $k_8 = \mu_v$ $k_9 = \mu_v$ The spread dynamics of malaria and dengue fever were further analyzed numerically to determine the spread dynamics patterns. The numerical values of the parameters were derived from secondary sources and were summarized in the table below.

Table 1. Numerical values for malaria-dengue co-infection model parameters.

Param	Description	Value (day ⁻¹)	Reference
Λ	Human recruitment rate	0.00011	[10]
β^*	Force of infection rate	0.35000	[11]
η	rate of exposed dengue individuals transfer to misdiagnosed	0.05000	[7]
ϵ	rate of misdiagnosed individuals move to infectious human population	0.02000	[12]
λ	rate at which individuals exposed to malaria move to infectious malaria population	0.10000	[13]
τ	rate at which individual exposed to dengue move infectious dengue population	0.14200	[14]
κ	rate at which infectious dengue individuals move to misdiagnosis	0.01000	[7]
ϵ_c	rate of infectious dengue individuals moving to infectious co-infection	0.00200	[Estimated]
ρ_c	rate of infectious malaria individuals moving to infectious coinfection	0.00300	[Estimated]
η_c	rate of exposed dengue individuals move to infectious co-infection	0.00100	[Estimated]
δ_c	rate of exposed malaria individuals move to infectious co-infection	0.00100	[15]
γ	rate of recovered dengue individual from infectious dengue population	0.14200	[16]
α	rate of recovered malaria individuals from infectious malaria population	0.07100	[17]
γ_c	rate of dengue recovered individuals from co-infection	0.12500	[18]
α_c	rate of malaria recovered individuals from co-infection	0.05000	[19]
ω	rate of individual moving to co-infection recovered from dengue recovered	0.00100	[19]
ϕ	rate of individual moving to co-infection recovered from malaria recovered	0.00100	[18]
μ	rate of natural death rate	0.00004	[14] [20]
δ_d	induced death rate for misdiagnosed	0.00200	[Estimated]
σ_d	disease induced death rate for dengue	0.00040	[21]
σ_c	diseases induced death rate for co-infection	0.00500	[22]
σ_m	disease induced death rate for malaria	0.00100	[21]

This section focused on determining the most sensitive parameter of the basic reproduction number. It utilised the numerical values of the simulation described in **Table 1** above. The sensitivity indices were summarized in the sensitivity graph below. The positive indices implies that such parameters have a direct proportionality with the basic reproduction number [23], and are coloured navy blue in the graph. The negative sensitivity indices imply that such parameters have an inverse proportionality with the basic reproduction number and are coloured maroon in the graph in **Figure 2** below.

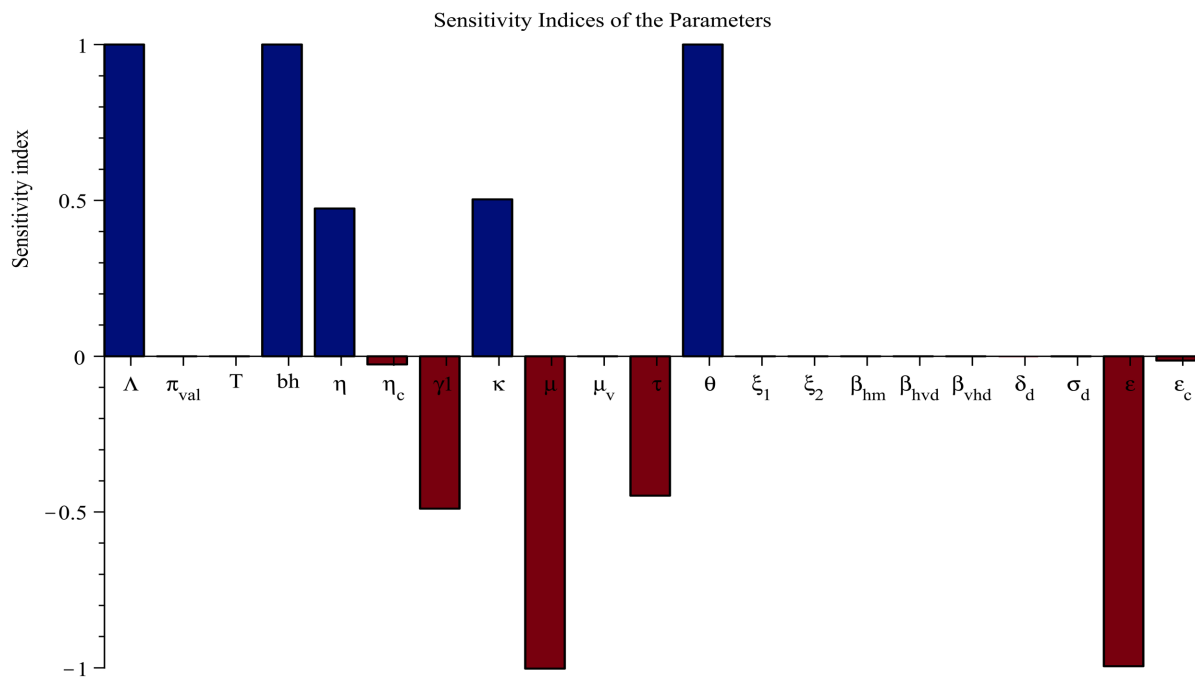


Figure 2. Sensitivity analysis of malaria dengue coinfection spread dynamics.

The parameters with moderate positive sensitivity include the rate of misdiagnosis of dengue infected patients, and the rate at which patients in the incubation period confuse dengue with malaria. Therefore, these two parameters also form good candidates for control in the control measures. The parameters with the most negative sensitivity indices include the rate at which misdiagnosis is corrected and the natural death rate of human beings. Therefore, the control measures should target these parameters to enhance them so as to contain the spread of dengue and malaria. These parameters formed the basis of the optimal control solution where control measures such as retooling of health care providers u_1 , public sensitization on how to prevent the two mosquito-borne diseases u_1 , and administration of antimalarial drugs u_1 as the some of the key control measures that could contain the spread of the disease.

3. Optimal Control Theory

Optimal control theory entails finding controls for dynamical systems over period

of time to obtain an optimum solution of an objective function to achieve the best possible performance of the dynamical system [24] [25]. That is, in epidemic modeling, the best solution is obtained by minimizing the infection in spread dynamics. This study sought to develop an optimum solution for the malaria-dengue co-infection spread dynamics to determine the most effective approaches to managing the co-infection of malaria-dengue in a resource-scarce environment. This study aimed at reducing the number of individuals coinfecting with malaria and dengue fevers through interventions that target the individual diseases. This objective was utilized to develop the following objective functional.

$$J(u_1, u_2, u_3) = \int_0^T \left(A_1 E_d + A_2 I_d + A_3 E_m + A_4 I_m + A_5 I_c + \frac{1}{2} (C_1 u_1^2 + C_2 u_2^2 + C_3 u_3^2) \right) dt \quad (5)$$

This study utilized the Bolza form of the objective functional which utilizes the quadratic approach of implementing the control variables [26]. The terms A_1, A_2, A_3, A_4, A_5 are the weighing factors of the infectious compartments detailing the relevant importance of each compartment in the spread dynamics. On the other hand, the terms C_1, C_2, C_3 are the weight factors of the control parameters in the objective functional. The main task was to identify a solution $J(u_1^*, u_2^*, u_3^*)$ where u_1^*, u_2^*, u_3^* are the control parameters u_1, u_2, u_3 in their optimized form such that Equation (5) is minimized. The above information gave the optimality condition described in Equation (6)

$$J(u_1^*, u_2^*, u_3^*) = \min_{u_1, u_2, u_3} \{ J(u_1, u_2, u_3) : u_1, u_2, u_3 \in U \} \quad (6)$$

where U is the control set of the optimal control problem defined as

$$U = \{ u_1(t), u_2(t), u_3(t) : 0 \leq u_1(t) \leq 1, 0 \leq u_2(t) \leq 1, 0 \leq u_3(t) \leq 1, 0 \leq t \leq T \} \quad (7)$$

3.1. The Pontryagin's Maximum Principle

The pontryagin maximum principle served as the bridge between the spread dynamics and the optimized control of the coinfection by introducing the Hamiltonian [27] [28]. It converts the system of differential equation in Equation (1) and Equation (6) into a minimizing point-wise problem using the Hamiltonian $\mathbb{H}$. The Hamiltonian of the problem was defined as

$$\begin{aligned} \mathbb{H}(S, E_d, M_d, I_d, R_d, E_m, I_m, R_m, I_c, R_c, S_v, E_v, I_{vm}, I_{vd}, t) \\ = A_1 E_d + A_2 I_d + A_3 E_m + A_4 I_m + A_5 I_c + \frac{1}{2} (C_1 u_1^2 + C_2 u_2^2 + C_3 u_3^2) \\ + \lambda_1 \frac{dS}{dt} + \lambda_2 \frac{dE_d}{dt} + \lambda_3 \frac{dM_d}{dt} + \lambda_4 \frac{dI_d}{dt} + \lambda_5 \frac{dR_d}{dt} + \lambda_6 \frac{dE_m}{dt} + \lambda_7 \frac{dI_m}{dt} \\ + \lambda_8 \frac{dR_m}{dt} + \lambda_9 \frac{dI_c}{dt} + \lambda_{10} \frac{dR_c}{dt} + \lambda_{11} \frac{dS_v}{dt} + \lambda_{12} \frac{dE_v}{dt} + \lambda_{13} \frac{dI_{vm}}{dt} + \lambda_{14} \frac{dI_{vd}}{dt} \end{aligned} \quad (8)$$

Equation (8) was expanded to

$$\begin{aligned}
\mathbb{H} = & A_1 E_d + A_2 I_d + A_3 E_m + A_4 I_m + A_5 I_c + \frac{1}{2} (C_1 u_1^2 + C_2 u_2^2 + C_3 u_3^2) \\
& + \lambda_1 [\Lambda - (1-u_2) \beta^* S (I_d + I_m) - \mu S + \Psi R_c] \\
& + \lambda_2 [(1-u_2) \beta^* S I_d - (\mu + \eta_c + (1-u_1) \eta + \tau) E_d] \\
& + \lambda_3 [(1-u_1) \eta E_d + (1-u_1) \kappa I_d - (u_1 + \varepsilon) M_d - (\mu + \delta_d) M_d] \\
& + \lambda_4 [\tau E_d + (u_1 + \varepsilon) M_d - (1-u_1) \kappa I_d - (\mu + \sigma_d) I_d - \varepsilon_c I_d - \gamma I_d] \\
& + \lambda_5 [\gamma I_d + \gamma_c I_c - \mu R_d - \omega R_d] \\
& + \lambda_6 [(1-u_2) \beta^* S I_m - (\mu + \lambda + \delta_c) E_m] \\
& + \lambda_7 [\lambda E_m - (\rho_c + \mu + \sigma_m + (u_3 + \alpha)) I_m] \\
& + \lambda_8 [(u_3 + \alpha) I_m + (u_3 + \alpha_c) I_c - (\Phi + \mu) R_m] \\
& + \lambda_9 [\delta_c E_m + \rho_c I_m + \varepsilon_c I_d + \eta_c E_d - (\mu + \delta_c + (u_3 + \alpha_c) + \gamma_c) I_c] \\
& + \lambda_{10} [\phi R_m + \omega R_d - (\mu + \Psi) R_c] \\
& + \lambda_{11} [\Pi - \mu_v S_v - \beta^* S_v] \\
& + \lambda_{12} [\beta_v^* S_v - \xi_1 E_v - \xi_2 E_v - \mu_v E_v] \\
& + \lambda_{13} [\xi_2 E_v - \mu_v I_{vm}] \\
& + \lambda_{14} [\xi_1 E_v - \mu_v I_{vd}]
\end{aligned} \tag{9}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}$ are the adjoint variables associated with each compartment. The optimality of the coinfection requires that

$$\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x(t)} \tag{10}$$

where the function $x(t) = (S, E_d, M_d, I_d, R_d, E_m, I_m, R_m, I_c, R_c, t)$. The result of the calculations was obtained as

$$\frac{d\lambda_1}{dt} = \mu \lambda_1 + (1-u_2) \beta^* [I_d (\lambda_1 - \lambda_2) + I_m (\lambda_1 - \lambda_6)] \tag{11}$$

$$\frac{d\lambda_2}{dt} = -A_1 + \lambda_2 (\mu + \eta_c + (1-u_1) \eta + \tau) - \lambda_3 (1-u_1) \eta - \lambda_4 \tau - \lambda_9 \eta_c \tag{12}$$

$$\frac{d\lambda_3}{dt} = \lambda_3 ((u_1 + \varepsilon) + \mu + \delta_d) - \lambda_4 (u_1 + \varepsilon) \tag{13}$$

$$\begin{aligned}
\frac{d\lambda_4}{dt} = & -A_2 + (1-u_2) \beta^* S (\lambda_1 - \lambda_2) - (1-u_1) \kappa \lambda_3 \\
& + \lambda_4 ((1-u_1) \kappa + \mu + \sigma_d + \varepsilon_c + \gamma) - \gamma \lambda_5 - \varepsilon_c \lambda_9
\end{aligned} \tag{14}$$

$$\frac{d\lambda_5}{dt} = \lambda_5 (\mu + \omega) - \lambda_{10} \omega \tag{15}$$

$$\frac{d\lambda_6}{dt} = -A_3 + \lambda_6 (\mu + \lambda + \delta_c) - \lambda_7 \lambda - \lambda_9 \delta_c \tag{16}$$

$$\begin{aligned}
\frac{d\lambda_7}{dt} = & -A_4 + (1-u_2) \beta^* S (\lambda_1 - \lambda_6) + \lambda_7 (\rho_c + \mu + \sigma_m + (u_3 + \alpha)) \\
& - \lambda_8 (u_3 + \alpha) - \lambda_9 \rho_c
\end{aligned} \tag{17}$$

$$\frac{d\lambda_8}{dt} = \lambda_8(\Phi + \mu) - \lambda_{10}\phi \quad (18)$$

$$\frac{d\lambda_9}{dt} = -A_5 - \lambda_5\gamma_c - \lambda_8(u_3 + \alpha_c) + \lambda_9(\mu + \delta_c + (u_3 + \alpha_c) + \gamma_c) \quad (19)$$

$$\frac{d\lambda_{10}}{dt} = -\lambda_1\Psi + \lambda_{10}(\mu + \Psi) \quad (20)$$

$$\frac{d\lambda_{11}}{dt} = \lambda_{11}(\mu_v + \beta^*) - \lambda_{12}\beta_v^* \quad (21)$$

$$\frac{d\lambda_{11}}{dt} = -\lambda_{12}(\xi_1 + \xi_2 + \mu_v) + \lambda_{13}\xi_2 + \lambda_{14}\xi_1 \quad (22)$$

$$\frac{d\lambda_{12}}{dt} = -\lambda_{13}\mu_v \quad (23)$$

$$\frac{d\lambda_{13}}{dt} = -\lambda_{14}\mu_v \quad (24)$$

To solve the system of differential equations for optimality we needed to determine the transversality conditions which give the terminal conditions at $t = T$. The transversality conditions were given as

$$\begin{aligned} \lambda_1(T) = 0, \lambda_2(T) = 0, \lambda_3(T) = 0, \lambda_4(T) = 0, \lambda_5(T) = 0, \\ \lambda_6(T) = 0, \lambda_7(T) = 0, \lambda_8(T) = 0, \lambda_9(T) = 0, \lambda_{10}(T) = 0, \\ \lambda_{11}(T) = 0, \lambda_{12}(T) = 0, \lambda_{13}(T) = 0, \lambda_{14}(T) = 0 \end{aligned} \quad (25)$$

The condition in Equation (25) guarantees the existence of an optimal solution that encompass control variables u_1^*, u_2^*, u_3^* that justifies the functional $J(u_1^*, u_2^*, u_3^*) = \min_{u_1, u_2, u_3} \{J(u_1, u_2, u_3) : u_1, u_2, u_3 \in U\}$. The stationary points of the hamiltonian with to control variables give the extremal points that fulfill the Pontryagin's maximum principle. The function of the formula was defined as

$$\frac{\partial \mathbb{H}}{\partial u_i^*} = 0 \quad (26)$$

The result of the differentiation with respect to u_1 is given by

$$\frac{\partial \mathbb{H}}{\partial u_1} = C_1 u_1 + \eta E_d(\lambda_2 - \lambda_3) + (\kappa I_d + \varepsilon M_d)(\lambda_4 - \lambda_3) = 0 \quad (27)$$

Making u_1 the subject to obtain the optimal control characterization at $u_1 = u_1^*$, we obtained

$$u_1^* = \frac{1}{C_1} [\eta E_d(\lambda_3 - \lambda_2) + (\kappa I_d + \varepsilon M_d)(\lambda_3 - \lambda_4)] \quad (28)$$

The process was repeated again by differentiating the hamiltonian with respect to u_2 to obtain the optimal control characterization at $u_2 = u_2^*$ and obtained. The differentiation of the hamiltonian with respect to u_2 was obtained as

$$\frac{\partial \mathbb{H}}{\partial u_2} = C_2 u_2 + \lambda_1 [\beta^* S(I_d + I_m)] - \lambda_2 [\beta^* S I_d] - \lambda_6 [\beta^* S I_m] = 0 \quad (29)$$

The optimal control characterization was obtained by making u_2 the subject of the equation at the point $u_2 = u_2^*$ and the result was obtained as

$$u_2^* = \frac{\beta^* S}{C_2} [I_d (\lambda_2 - \lambda_1) + I_m (\lambda_6 - \lambda_1)] \quad (30)$$

The third optimal control characterization was obtained by differentiating the hamiltonian with respect to u_3 and obtained

$$\frac{\partial \mathbb{H}}{\partial u_3} = C_3 u_3 - \lambda_7 (\alpha I_m) + \lambda_8 (\alpha I_m + \alpha_c I_c) - \lambda_9 (\alpha_c I_c) = 0 \quad (31)$$

making u_3 the subject of the equation and obtaining the optimal control characterization at $u_3 = u_3^*$ as

$$u_3^* = \frac{1}{C_3} [\alpha I_m (\lambda_7 - \lambda_8) + \alpha_c I_c (\lambda_9 - \lambda_8)] \quad (32)$$

Theorem

Considering the state variable solutions given by

$S^{**}, E_d^{**}, M_d^{**}, I_d^{**}, R_d^{**}, E_m^{**}, I_m^{**}, R_m^{**}, I_c^{**}, R_c^{**}, S_v^{**}, E_v^{**}, I_{vm}^{**}, I_{vd}^{**}$ whose optimal solutions are contained in the control set $U^* = (u_1^*, u_2^*, u_3^*)$ which are the optimized solution of the optimal control problem 1. The costate functions are given by

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}$ which are the solutions of the Equations (11)-(24). The full characterization equation of the optimization

u_1^*, u_2^*, u_3^* together with bonds was given by

$$\begin{aligned} u_1^* &= \max \left(0, \min \left(1, \frac{1}{C_1} [\eta E_d (\lambda_3 - \lambda_2) + (\kappa I_d + \varepsilon M_d) (\lambda_3 - \lambda_4)] \right) \right) \\ u_2^* &= \max \left\{ 0, \min \left(1, \frac{\beta^* S}{C_2} [I_d (\lambda_2 - \lambda_1) + I_m (\lambda_6 - \lambda_1)] \right) \right\} \\ u_3^* &= \max \left\{ 0, \min \left(1, \frac{\alpha I_m (\lambda_7 - \lambda_8) + \alpha_c I_c (\lambda_9 - \lambda_8)}{C_3} \right) \right\} \end{aligned}$$

Proof

The existence of an optimal control solution of 1 over the control and convex set $U = u_1^*, u_2^*, u_3^*$ is guaranteed by the convexity of the integral of the functional $J(u_1^*, u_2^*, u_3^*)$. The state condition are satisfied by the boundedness that existed prior leading to the satisfaction of the Lipschitz property of the state system. The boundedness was described as

$$\begin{aligned} u_1^* &= \tilde{u}_1 = \frac{1}{C_1} [\eta E_d^{**} (\lambda_3 - \lambda_2) + (\kappa I_d^{**} + \varepsilon M_d^{**}) (\lambda_3 - \lambda_4)] \\ u_2^* &= \tilde{u}_2 = \frac{\beta^* S^{**}}{C_2} [I_d^{**} (\lambda_2 - \lambda_1) + I_m^{**} (\lambda_6 - \lambda_1)] \\ u_3^* &= \tilde{u}_3 = \frac{\alpha I_m^{**} (\lambda_7 - \lambda_8) + \alpha_c I_c^{**} (\lambda_9 - \lambda_8)}{C_3} \end{aligned}$$

whose boundedness was given as follows

$$u_1^* = \begin{cases} 0, & \text{if } \tilde{u}_1 \leq 0 \\ \tilde{u}_1, & \text{if } 0 \leq \tilde{u}_1 \leq 0.5 \\ 1, & \text{if } \tilde{u}_1 \geq 1 \end{cases} \quad u_2^* = \begin{cases} 0, & \text{if } \tilde{u}_2 \leq 0 \\ \tilde{u}_2, & \text{if } 0 \leq \tilde{u}_2 \leq 1 \\ 1, & \text{if } \tilde{u}_2 \geq 1 \end{cases} \quad u_3^* = \begin{cases} 0, & \text{if } \tilde{u}_3 \leq 0 \\ \tilde{u}_3, & \text{if } 0 \leq \tilde{u}_3 \leq 1 \\ 1, & \text{if } \tilde{u}_3 \geq 1 \end{cases}$$

To establish the uniqueness of the optimal control solution, traditional analytical frameworks can be employed as long as the simulation timeframes remain sufficiently small. This specific methodology was adopted for this study because both the state and costate equations are characterized by Lipschitz continuous functions. Furthermore, since the optimal control system is inherently bounded, the costate equations utilized linear bounded coefficients, which ensured that the costate variables themselves possessed defined upper bounds. Restricting the duration of the time interval served as a vital condition to formally guarantee that the system's optimal control is unique. This small-time requirement is fundamentally driven by the opposing temporal directions of the optimality system; while the state problem is defined by initial conditions and solved forward, the adjoint problem is defined by terminal conditions and solved backward, creating a complex two-point boundary value structure.

Numerical Results

The section focused on presenting a numerical solution of the optimal control problem when compared with the non-optimized problem. Forward iteration method was utilized to solve the optimal control differential equations in Equation (1). The system of differential equation 1 incorporates the control measures u_1, u_2 and u_3 . The backward iteration technique was used to solve the co-state system of differential equations with the control bounds and initial conditions as the boundary conditions. The controls variables were updated progressively using the weighted convex combination that supported both backward and forward iteration until the optimum solution was reached. The numerical scheme was implemented in the maple software by utilizing both runge-kutta order four method and the forward-backward sweep methods. The numerical values of the parameters were taken from the numerical values listed in **Table 1**.

The initial conditions of the optimal control problem were taken at the 100th day of the general spread dynamics [9]. At this point, $E_d(t) = 12384.258970$, $E_m(t) = 3132.981741$, $I_c(t) = 1218.741262$, $I_d(t) = 11536.904329$, $I_m(t) = 3659.732227$, $M_d(t) = 3676.038969$, $R_c(t) = 8138.870086$, $R_d(t) = 7689.097503$, $R_m(t) = 1298.592250$, $S(t) = 47251.178561$ [29]. To contain the spread of malaria and dengue fever, the number of exposed, infectious and coinfecting compartments has to be minimized. The relative importance of the compartments to be minimized was assumed to be equal, that is

$A_1 = A_2 = A_3 = A_4 = 1$. The unit cost of retooling healthcare workers was taken to be 200 shillings thus the weighting $C_1 = 200$ [29], the unit cost public sensitization was taken to be 100 shillings thus the weighting $C_2 = 100$, and the unit cost of treating malaria through administering antimalaria drugs was taken to be 800

shillings thus making the weighting component $C_3 = 800$ [30]. The above numerical values were incorporated into the following strategies of implementing the control parameters.

1. **Strategy 1:** The implementation of retooling and public sensitization, that is $u_1 \neq 0$, $u_2 \neq 0$, and $u_3 = 0$.
2. **Strategy 2:** The administration of antimalarial drugs and retooling of health care workers, that is $u_1 \neq 0$, $u_2 = 0$, and $u_3 \neq 0$.
3. **Strategy 3:** The implementation of all the three strategies, that is, retooling health care workers, conducting public sensitization campaigns and administering the antimalarial drugs i.e $u_1 \neq 0$, $u_2 \neq 0$, and $u_3 \neq 0$.

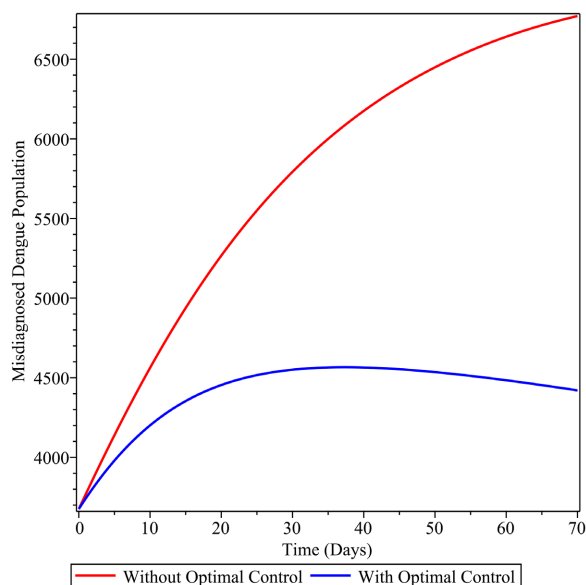
3.2. The Implementation of Retooling and Public Sensitization, That Is $u_1 \neq 0$, $u_2 \neq 0$, and $u_3 = 0$

This strategy reduces the misdiagnosed population progressively from the first day implementing the control measures. The control measures stabilizes at 4500 people per day while the uncontrolled disease dynamics increase exponentially as illustrated in **Figure 3(a)**. **Figure 3(b)** illustrates the impact of application of the two control values, the implementation of the control variables reduces the number of infected individuals daily. The strategy also reduced the infected malaria population limiting the numbers below 4000 effectively creating an upper limit to the spread dynamics of the disease under controlled environment. In contrast, the the uncontrolled spread dynamics of individuals with malaria increase consistently leading to uncontrolled spread of the disease as illustrated in **Figure 3(c)**. The coinfecting spread dynamics also show robust response to the control measures by breaking the increasing population and flattening the curve as illustrated in **Figure 3(d)**.

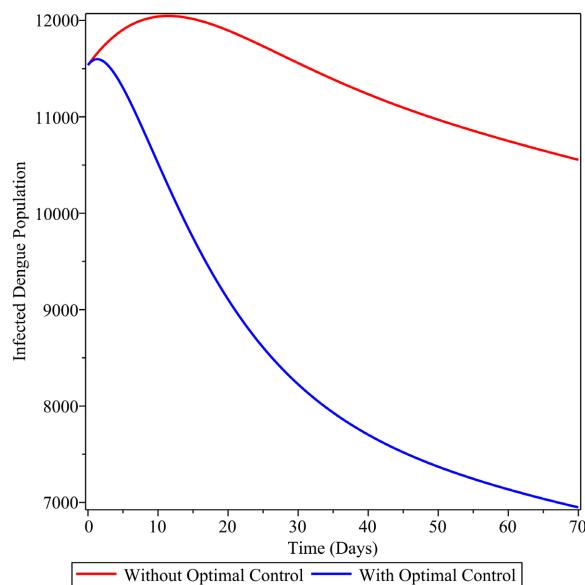
3.3. The Administration of Antimalarial Drugs and Retooling of Health Care Workers, That Is $u_1 \neq 0$, $u_2 = 0$, and $u_3 \neq 0$

This strategy targets training the of workers to better equip and the administration of antimalarial drugs. The combination of the two control measures in this strategy are illustrated in **Figure 4**. The strategy takes time before gaining traction in controlling the number of misdiagnosed people per day. The strategy fails to show any meaningful difference between the controlled spread dynamics and the uncontrolled spread dynamics in the first 15 days of the simulation. However, there is a marked difference between the controlled and uncontrolled spread dynamics. The strategy has little impact on the spread dynamics as shown in **Figure 4(a)**. The dengue fever spread dynamics show an increasing in the controlled spread dynamics when compared to the uncontrolled spread dynamics as shown in **Figure 4(b)**. This unconventional behaviour rises from focus of the strategy on the control of malaria at the expense of dengue fever. The number of people infected with malaria under this strategy shows progressive decline to values near zero as

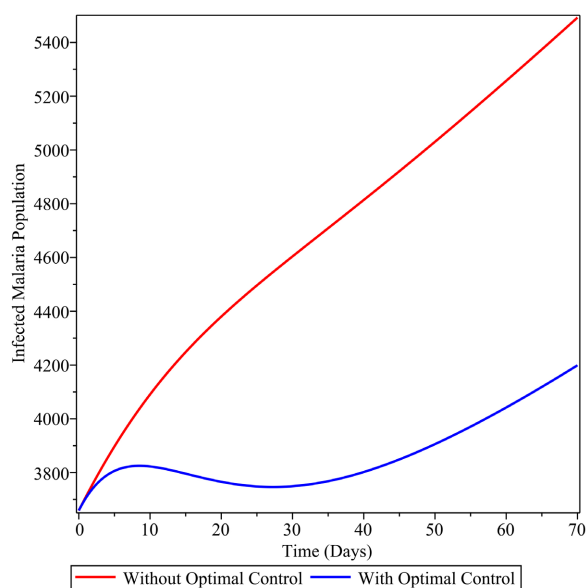
shown in **Figure 4(c)**. The impact of the strategy on the coinfectd people per day was effective in reducing the number of coinfectd people to a minimum of around 200 people per people in the first 15 days of the control simulation. The number of coinfectd people per day starts to increase slowly after the first 15 days but still remains in low levels. The uncontrolled spread dynamics remains high compared to the controlled spread dynamics as shown in **Figure 4(d)** below.



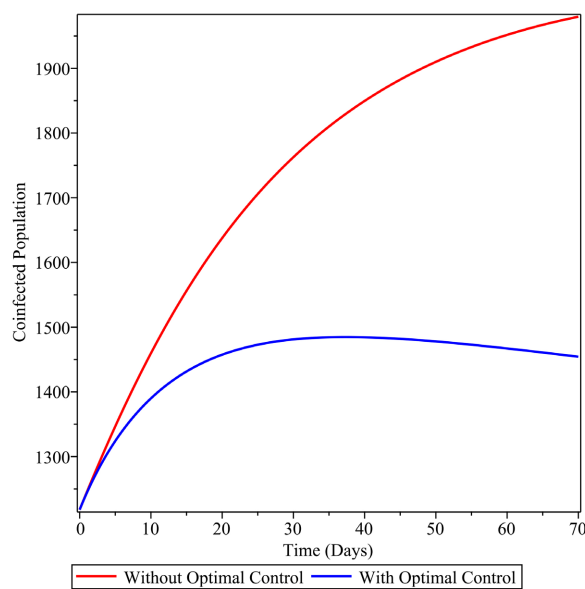
(a) Population of Misdiagnosed with dengue fever



(b) Population of people infected with dengue fever

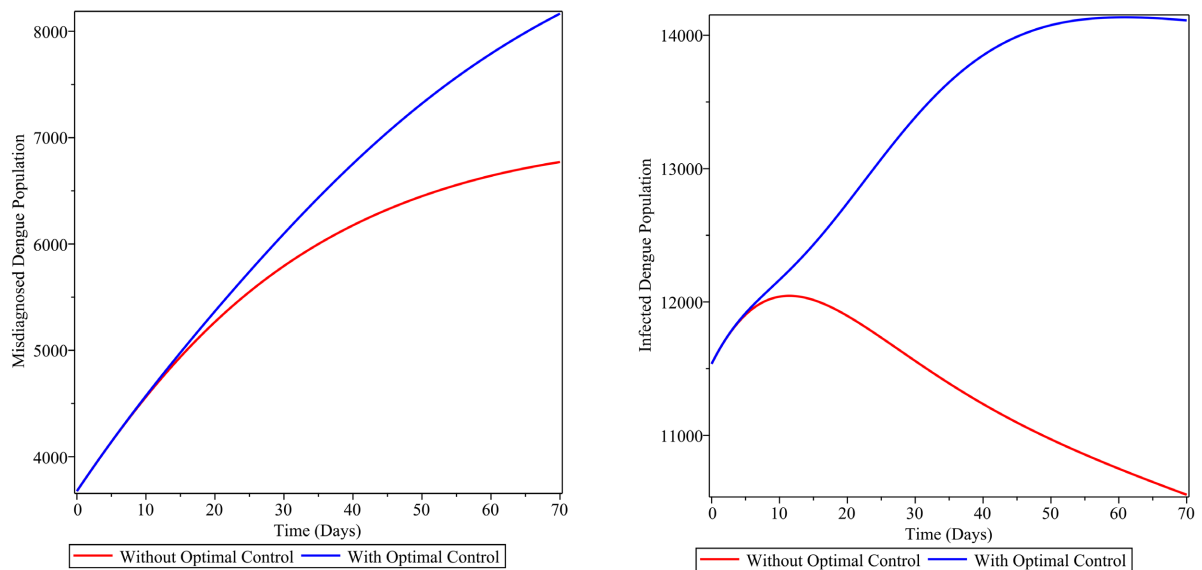


(c) Population of people infected with malaria



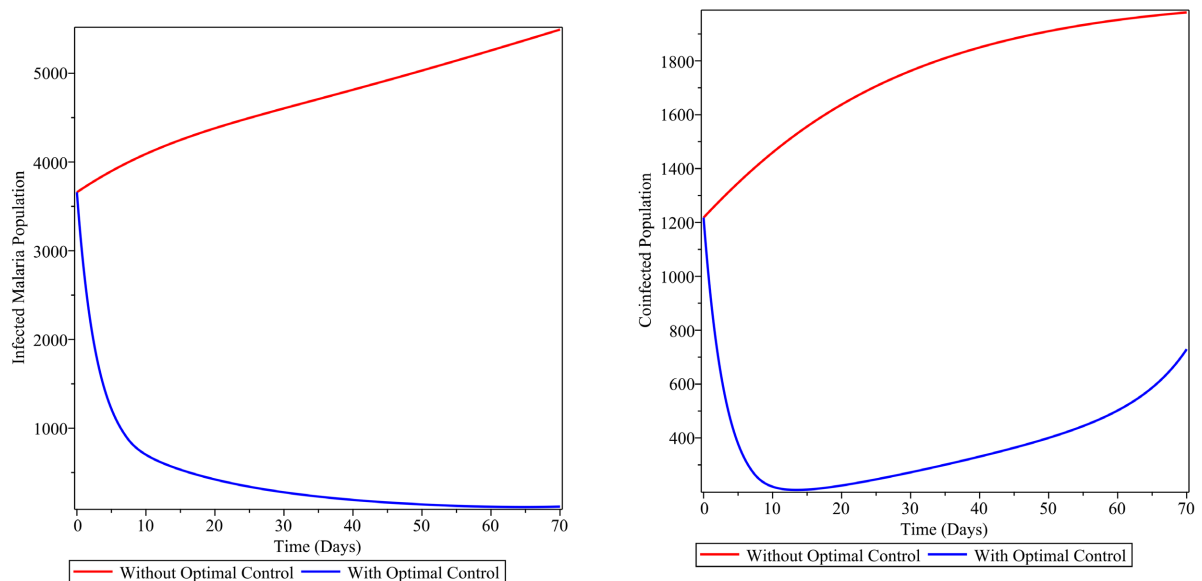
(d) Population of people coinfectd with malaria and dengue

Figure 3. Comparative graphs of different infectious groups under strategy 1 optimal control.



(a) Population of Misdiagnosed with dengue fever

(b) Population of people infected with dengue fever



(c) Population of people infected with malaria

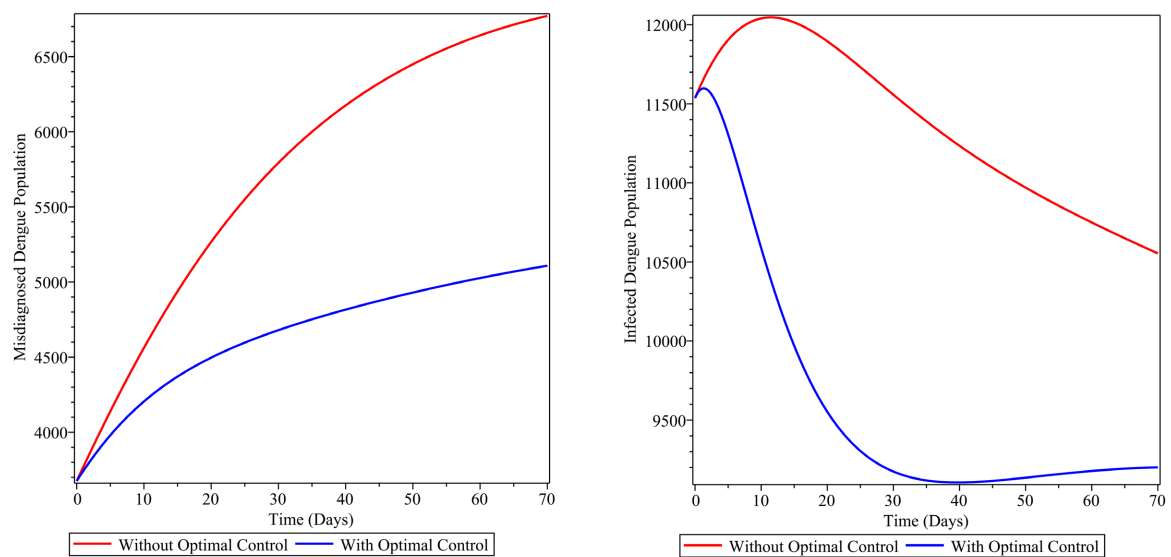
(d) Population of people coinfected with malaria and dengue

Figure 4. Comparative graphs of different infectious groups under strategy 3 optimal control.

3.4. The Implementation of All the Three Strategies, That Is, Retooling Health Care Workers, Conducting Public Sensitization Campaigns and Administering the Antimalarial Drugs *i.e.* $u_1 \neq 0$, $u_2 \neq 0$, and $u_3 \neq 0$,

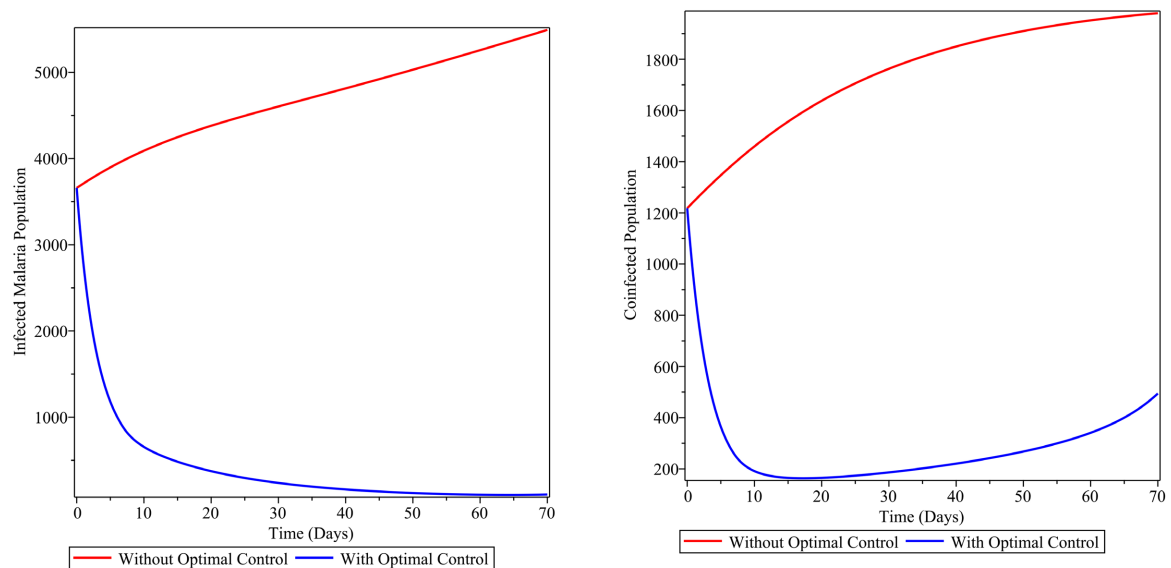
This strategy is a combination of all the controlled variables under consideration; that is retooling of human resources of health care providers, conducting public sensitization campaigns, and administration of antimalarial drugs. The impact of the spread dynamics is illustrated in **Figure 5** below.

Figure 5(a) illustrates the spread dynamics of misdiagnosed population under the application of the three control variables. The control variables have an immediate impact on the people misdiagnosed with dengue with the control spread dynamics having lower levels and the uncontrolled spread dynamics having high values. The impact of the strategy on the compartment of infected dengue population starts immediately leading to the decrease to the number infected people with dengue fever as illustrated in **Figure 5(b)**. The impact of the strategy on the people infected with malaria is illustrated in **Figure 5(c)** below. The number of infected people with malaria decreases drastically to numbers near zero within the



(a) Population of Misdiagnosed with dengue fever

(b) Population of people infected with dengue fever



(c) Population of people infected with malaria

(d) Population of people confectioned with malaria and dengue

Figure 5. Comparative graphs of different infectious groups under strategy 4 optimal control.

70 days of simulation when the simulation is under control. On the other hand, the uncontrolled spread dynamics remain high when compared to the controlled spread dynamics as shown in **Figure 5(c)**. The spread dynamics of coinfecting people with malaria and dengue fever under control is illustrated in **Figure 5(d)**. The number of people coinfecting people per day decreases rapidly in the first 10 days of the simulation to around 200 people per day. The number of coinfecting people starts to gradually increase after the 15th day to a maximum of about 400 people per day. The uncontrolled spread dynamics increase consistently to values of about 1900 people per day. This strategy is the most effective since it guarantees a reduction in all the infected compartments of the spread dynamics of malaria and dengue fever.

4. Conclusion

A coinfection spread dynamics epidemiological model was developed that incorporated the identified control measures. The identified control measure included retooling of healthcare service providers, public sensitization on how to prevent the two mosquito-borne diseases and administration of antimalarial drugs. An optimal control problem was developed and solved using the Hamiltonian and the Pontryagin's maximum principle under four strategies that utilized different combinations of the control parameters. The four strategies formed the backbone of the analysis of the optimal control outcomes and as a measure of the strategy with the best outcomes in their combinations. The study found that the combined utilization of the three control parameters: retooling of healthcare workers, conducting public sensitization, and administering antimalarial drugs, had the greatest impact on the reduction of the coinfection spread dynamics.

5. Recommendations

The study considered the following recommendations supported by the findings and the conclusions of the study. The recommendations are as follows:

- The identified sensitive parameters can be developed into functions that could be used to simulate the critical levels for which these parameters offer the cheapest controls. This aspect is particularly important during outbreaks when time is limited for each sensitive parameter to be investigated independently.
- The study recommends the investigation of the less sensitive parameters especially in low endemic spread of the diseases and the associated cost implications. This approach will help in providing comparative analysis of different control measures while in action during different spread dynamics regimes.
- The study recommends the utilization of the application of retooling of healthcare workers, conducting public sensitization, administering antimalarial drugs since it was able to demonstrate reduced populations in all the four infectious compartments.
- The study recommends further investigation of the cost effectiveness of the control measures explored in the application of the control measures. This in-

vestigation will be able to expose the different health economics involved in the implementation of these control measures.

Author Contributions

Conceptualization, Ewesit EW; methodology, Ewesit EW; software, Ewesit EW; validation, Ewesit EW, Wainaina M and Maingi D; formal analysis, Ewesit EW; investigation, Ewesit EW; resources, Ewesit EW; data curation, Ewesit EW; writing-original draft preparation, Ewesit EW; writing-review and editing, Ewesit EW; visualization, Ewesit EW; supervision, Wainaina M and Maingi D; project administration, Wainaina M and Maingi D; funding acquisition, Ewesit EW. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

- [1] Wiwanitkit, V. (2011) Concurrent Malaria and Dengue Infection: A Brief Summary and Comment. *Asian Pacific Journal of Tropical Biomedicine*, **1**, 326-327. [https://doi.org/10.1016/s2221-1691\(11\)60053-1](https://doi.org/10.1016/s2221-1691(11)60053-1)
- [2] Katzelnick, L.C., Gresh, L., Halloran, M.E., Mercado, J.C., Kuan, G., Gordon, A., *et al.* (2017) Antibody-Dependent Enhancement of Severe Dengue Disease in Humans. *Science*, **358**, 929-932. <https://doi.org/10.1126/science.aan6836>
- [3] Gebremariam, T.T., Schallig, H.D.F.H., Kurmane, Z.M. and Danquah, J.B. (2023) Increasing Prevalence of Malaria and Acute Dengue Virus Coinfection in Africa: A Meta-Analysis and Meta-Regression of Cross-Sectional Studies. *Malaria Journal*, **22**, Article No. 300. <https://doi.org/10.1186/s12936-023-04723-y>
- [4] Sufi Aiman Sabrina, R., Muhammad Azami, N.A. and Yap, W.B. (2025) Dengue and Flavivirus Co-Infections: Challenges in Diagnosis, Treatment, and Disease Management. *International Journal of Molecular Sciences*, **26**, Article No. 6609. <https://doi.org/10.3390/ijms26146609>
- [5] Jal, S., Chhotaray, S., Pattnaik, G., Mishra, S., Mohapatra, R.K., Kandi, V., *et al.* (2024) Dengue Fever Coinfection in COVID-19 Era: A Public Health Concern. *Health Science Reports*, **7**, e2089. <https://doi.org/10.1002/hsr2.2089>
- [6] Garba, K.M., Rabi, I., Muhammad, S.I., Baba, Z.A. and Idris, F.Z. (2026) Comparative Evaluations on Dynamics of Malaria and Dengue Co-Infections among Patients in North-West Geopolitical Zones of Nigeria. *Federal University Gusau Faculty of Education Journal*, **6**, 141-149. <https://doi.org/10.64348/zije.2026389>
- [7] Butt, A.I.K., Imran, M., McKinney, B.A., Batool, S. and Aftab, H. (2023) Mathematical and Stability Analysis of Dengue-Malaria Co-Infection with Disease Control Strategies. *Mathematics*, **11**, Article No. 4600. <https://doi.org/10.3390/math11224600>
- [8] Nyasagare, B.N., Osman, S. and Wainaina, M. (2019) Modelling and Analysis of Campylobacteriosis in Human and Animal Populations. *Global Journal of Pure and Applied Mathematics*, **15**, 551-567.
- [9] Ewesit, E.W., Wainaina, M. and Maingi, D. (2026) Mathematical Modelling of the Impact of Misdiagnosis on Dengue-Malaria Co-Infection Dynamics. *Journal of Applied Mathematics and Physics*, **14**, 3311-3330.

- <https://doi.org/10.4236/jamp.2026.149165>
- [10] Aguiar, M., Anam, V., Blyuss, K.B., Estadilla, C.D.S., Guerrero, B.V., Knopoff, D., et al. (2022) Mathematical Models for Dengue Fever Epidemiology: A 10-Year Systematic Review. *Physics of Life Reviews*, **40**, 65-92. <https://doi.org/10.1016/j.plrev.2022.02.001>
 - [11] Puspita, J.W., Fakhruddin, M., Nuraini, N. and Soewono, E. (2022) Time-Dependent Force of Infection and Effective Reproduction Ratio in an Age-Structure Dengue Transmission Model in Bandung City, Indonesia. *Infectious Disease Modelling*, **7**, 430-447. <https://doi.org/10.1016/j.idm.2022.07.001>
 - [12] Adom-Konadu, A., Yankson, E., Naandam, S.M. and Dwomoh, D. (2022) A Mathematical Model for Effective Control and Possible Eradication of Malaria. *Journal of Mathematics*, **2022**, Article ID: 6165581. <https://doi.org/10.1155/2022/6165581>
 - [13] Alhaj, M.S. (2023) Mathematical Model for Malaria Disease Transmission. *Journal of Mathematical Analysis and Modeling*, **4**, 1-16. <https://doi.org/10.48185/jmam.v4i1.688>
 - [14] Khan, M.A. and Fatmawati, (2021) Dengue Infection Modeling and Its Optimal Control Analysis in East Java, Indonesia. *Heliyon*, **7**, e06023. <https://doi.org/10.1016/j.heliyon.2021.e06023>
 - [15] Duru, E.C., Mbah, G.C.E., Anyanwu, M.C. and Nnamani, N.T. (2024) Modelling the Co-Infection of Malaria and Zika Virus Disease. *Journal of the Nigerian Society of Physical Sciences*, **6**, Article No. 1938. <https://doi.org/10.46481/jnsps.2024.1938>
 - [16] Tan, L.K., Low, S.L., Sun, H., Shi, Y., Liu, L., Lam, S., et al. (2019) Force of Infection and True Infection Rate of Dengue in Singapore: Implications for Dengue Control and Management. *American Journal of Epidemiology*, **188**, 1529-1538. <https://doi.org/10.1093/aje/kwz110>
 - [17] Duve, P., Charles, S., Munyakazi, J., Lühken, R. and Witbooi, P. (2024) A Mathematical Model for Malaria Disease Dynamics with Vaccination and Infected Immigrants. *Mathematical Biosciences and Engineering*, **21**, 1082-1109. <https://doi.org/10.3934/mbe.2024045>
 - [18] Imran, M., McKinney, B.A., Butt, A.I.K., Palumbo, P., Batool, S. and Aftab, H. (2024) Optimal Control Strategies for Dengue and Malaria Co-Infection Disease Model. *Mathematics*, **13**, Article No. 43. <https://doi.org/10.3390/math13010043>
 - [19] Prakash Raj, M., Venkatesh, A., Arun Kumar, K. and Manivel, M. (2025) Mathematical Modeling of the Co-Infection Dynamics of Dengue and Malaria Using Delay Differential Equations. *Advanced Theory and Simulations*, **8**, Article ID: 2400609. <https://doi.org/10.1002/adts.202400609>
 - [20] Onsongo, W.M., Mwini, E.D., Nyanaro, B.N. and Osman, S. (2022) Stability Analysis and Modelling the Dynamics of Psittacosis in Human and Poultry Populations. *Communications in Mathematical Biology and Neuroscience*, **2022**, Article No. 9.
 - [21] Harshit and Harjule, P. (2024) Mathematical Modelling and Analysis of Dengue Transmission Dynamics. *Procedia Computer Science*, **235**, 539-548. <https://doi.org/10.1016/j.procs.2024.04.053>
 - [22] Kotepui, M., Kotepui, K.U., Milanez, G.D.J. and Masangkay, F.R. (2020) Prevalence of and Risk Factors for Severe Malaria Caused by Plasmodium and Dengue Virus Co-Infection: A Systematic Review and Meta-Analysis. *Infectious Diseases of Poverty*, **9**, Article No. 134. <https://doi.org/10.1186/s40249-020-00741-z>
 - [23] Nyanaro, B., Kimathi, G. and Wainaina, M. (2024) Mathematical Modelling of Dengue Fever Transmission Dynamics in Kenya. *Journal of AppliedMath*, **2**, Article No. 1807. <https://doi.org/10.59400/jam1807>

- [24] Osman, S., Otoo, D. and Makinde, O.D. (2020) Modeling Anthrax with Optimal Control and Cost Effectiveness Analysis. *Applied Mathematics*, **11**, 255-275. <https://doi.org/10.4236/am.2020.113020>
- [25] Onsongo, W.M., Mwini, E.D., Nyanaro, B.N. and Osman, S. (2021) The Dynamics of Psittacosis in Human and Poultry Populations: A Mathematical Modelling Perspective. *Journal of Mathematics and Computer Science*, **11**, 8472-8505.
- [26] Bolzoni, L. and Della Marca, R. (2025) On the Optimal Vaccination Control of SIR Model with Erlang-Distributed Infectious Period. *Journal of Optimization Theory and Applications*, **205**, Article No. 39. <https://doi.org/10.1007/s10957-025-02645-0>
- [27] Nyanaro, B., Kimathi, G. and Wainaina, M. (2025) Optimal Control of Dengue Fever Transmission Dynamics in Kenya. *Journal of AppliedMath*, **3**, Article No. 2353. <https://doi.org/10.59400/jam2353>
- [28] Osman, S., Otoo, D. and Sebil, C. (2020) Analysis of Listeriosis Transmission Dynamics with Optimal Control. *Applied Mathematics*, **11**, 712-737. <https://doi.org/10.4236/am.2020.117048>
- [29] Awerigiya, R., Adom-Konadu, A., Bonyah, E. and Adam, B. (2026) Mathematical Approach to Optimal and Cost-Effective Strategies for Malaria Control in Ghana. *Results in Control and Optimization*, **23**, Article ID: 100712. <https://doi.org/10.1016/j.rico.2026.100712>
- [30] Xie, K., Otuko, R., Marathe, A., Gutierrez, A.S., Kariuki, M., Musyoka, L., *et al.* (2026) Cost and Cost-Effectiveness of Ivermectin Mass Drug Administration for Malaria Control in Kwale County, Kenya: A Modelling Analysis of a Cluster-Randomised Trial. *The Lancet Global Health*, **14**, e435-e443. [https://doi.org/10.1016/s2214-109x\(25\)00470-x](https://doi.org/10.1016/s2214-109x(25)00470-x)