

Modelling of Soil Compression Curve with Hysteresis in Void Ratio-Effective Stress Relation

Ippei Iiyama

School of Agriculture, Utsunomiya University, Utsunomiya, Japan
Email: iiyama@a.utsunomiya-u.ac.jp

How to cite this paper: Iiyama, I. (2026) Modelling of Soil Compression Curve with Hysteresis in Void Ratio-Effective Stress Relation. *Journal of Applied Mathematics and Physics*, 14, 4176-4196.
<https://doi.org/10.4236/jamp.2026.1410203>

Received: August 31, 2026

Accepted: October 24, 2026

Published: October 27, 2026

Copyright © 2026 by author(s) and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

Soil compression curve is used for predicting land subsidence associated with a consolidation process. It is often described as a relationship between void ratio e [$\text{m}^3 \cdot \text{m}^{-3}$] of the soil and effective stress σ' [$\text{N} \cdot \text{m}^{-2}$] exerted on the soil, and the trajectory of it depends on stress history. A mathematical model of soil compression curve is useful for reproducing and predicting purposes when it is concise and able to formulate hysteretic behaviours in e - σ' relation. This study aimed at developing a mathematical model to describe a compression curve and enabling the model to predict hysteretic behaviours in e - σ' relation. The model was developed based on the geometrical features of a compression curve commonly observed on the e - $\log_{10} \sigma'$ relation. The expansion and recompression processes were also modelled for predicting hysteresis of e - σ' relations by introducing the concept of potential expansion curve and assuming the geometrical similarity between compression curves and expansion curves on the e - $\log_{10} \sigma'$ plane. The proposed model performed well for reproducing and predicting various kinds of measured e - σ' relations, and enables to predict any hysteretic behaviours found along repetitions of expansion-compression processes.

Keywords

Compressibility Index, Consolidation Test, Normal Consolidation, Overconsolidation, Preconsolidation Stress

1. Introduction

Compressibility of a cohesive water-saturated soil layer is the knowledge required for predicting possible land subsidence associated with the developments in wa-

terfront areas or river deltas where human populations tend to grow. For instance, waterfront developments inevitably produce large amounts of dredged clays, causing problems of how to get rid of them that need the quantification of effective capacity of disposal ponds based on subsidence prediction [1]. Knowledge about soil compressibility is also required for mitigating potential threats of liquefaction induced by an earthquake event because groundwater lowering is a promising countermeasure for preventing liquefaction in alluvially deposited soil layers [2], inevitably causing consolidation of the soil layers.

Soil compressibility is generally expressed by a compression curve obtained from consolidation test for a soil body. A compression curve is often described as a relationship between void ratio e [$\text{m}^3 \cdot \text{m}^{-3}$] of the soil and effective stress σ' [$\text{N} \cdot \text{m}^{-2}$] exerted on the soil, pairs of which are sequentially observed through the consolidation test. The test commonly progresses with step-by-step increases in σ' [$\text{N} \cdot \text{m}^{-2}$] accompanied with gradual decreases in e [$\text{m}^3 \cdot \text{m}^{-3}$], and a resultant data set is often plotted on the e - $\log_{10}\sigma'$ plane, drawing the compression curve.

When a compression curve is made along a monotonous increase in σ' [$\text{N} \cdot \text{m}^{-2}$], the e - $\log_{10}\sigma'$ curve starts with relatively a gentle slope, becomes steeper distinctly around a certain effective stress, and joins an almost-straight line. This threshold effective stress is called the yielding stress or the preconsolidation stress σ'_p [$\text{N} \cdot \text{m}^{-2}$], and the quasi-linear part of the e - $\log_{10}\sigma'$ curve is called the virgin compression curve.

Phenomenologically, σ'_p [$\text{N} \cdot \text{m}^{-2}$] is interpreted as the maximum stress that the soil has ever experienced, meaning that any e - σ' state in the domain of $\sigma' < \sigma'_p$ [$\text{N} \cdot \text{m}^{-2}$] cannot take place after $\sigma' = \sigma'_p$ [$\text{N} \cdot \text{m}^{-2}$] is exerted on the soil unless the soil experiences some releases from overburdens like melting of ice sheets or erosion of overlying soil layers at some stages in its history and even might have experienced an expansion process caused by the loss of overburdens.

Although it is always difficult to estimate how many repetitions of expansion and recompression have occurred in a soil layer, the phenomenological interpretation about σ'_p [$\text{N} \cdot \text{m}^{-2}$] is reflected on the names of the processes of “overconsolidation” and “normal consolidation”, as the inclination of the e - $\log_{10}\sigma'$ curve in the domain of $\sigma' > \sigma'_p$ [$\text{N} \cdot \text{m}^{-2}$] is called the compression index C_c [–] for the normal consolidation process while another inclination in the domain of $\sigma' < \sigma'_p$ [$\text{N} \cdot \text{m}^{-2}$] is called the recompression index C_r [–] for the overconsolidation process.

Generally, C_r [–] is less steep than C_c [–] regardless of the size of σ'_p [$\text{N} \cdot \text{m}^{-2}$] while the virgin compression line with the inclination of C_c [–] is likely to be an instinct feature of the soil layer, meaning that the trajectory of a compression curve depends on stress history and tends to return to the virgin compression curve when σ' [$\text{N} \cdot \text{m}^{-2}$] becomes larger.

Because the consolidation test is commonly laborious and costliness, ways that allow to predict or easily estimate the compression parameters are always useful in practical engineering cases in relation to land subsidence prediction. Therefore,

many efforts have been made to develop concise and objective ways to evaluate the compressibility parameters or to investigate the relation between each of the compressibility parameters and more easily-obtainable soil properties.

A possible way for mitigating the labor and cost of consolidation tests is the development of a mathematical model that can typify common features of a compression curve. This approach can make it possible not only to evaluate relevant compressibility factors like C_c [–], C_r [–], and σ'_p [N·m⁻²] but to describe an entire consolidation process. One of widely used models of compression curve is expressed as two half lines which connect with each other at a point on the e - $\log_{10}\sigma'$ plane so that the point of the connection indicates approximately the pre-consolidation stress σ'_p [N·m⁻²] and the two half lines represent an overconsolidation process and a normal consolidation process bounded by $\sigma' = \sigma'_p$ [N·m⁻²]. In this model, C_c [–] and C_r [–] represent the inclinations of the half lines in $\sigma' > \sigma'_p$ [N·m⁻²] and $\sigma' < \sigma'_p$ [N·m⁻²], respectively.

The ways for predicting the inclination of a compression curve by using basic soil properties have also been concerned. The soil properties that may explain the compression index C_c [–] include the initial void ratio e_0 [m³·m⁻³] [3]–[5], liquid limit w_L [kg·kg⁻¹] [4]–[6], and plastic limit w_p [kg·kg⁻¹] [5] [6]. The recompression index C_r [–] is also seemingly affected by e_0 [m³·m⁻³] [3] [7] while it is often related to C_c [–] [3] [4].

However, there seem to be few models that can formulate a sequence of over-consolidation and normal consolidation processes with a single function. Besides, hysteretic behaviours in e - $\log_{10}\sigma'$ relation have rarely been formulated presumably because of relatively small amounts of measured facts about expansion and recompression processes. Therefore, this study aimed at developing a mathematical model to describe a compression curve as a series of an overconsolidation process and a normal consolidation process, and enabling the model to predict hysteretic behaviours in e - $\log_{10}\sigma'$ relation.

2. Modelling of a Monotonous Compression Process

Figure 1 is a schematic diagram of a compression curve of a soil body as a series of e - σ' plots. Because the domain of effective stress σ' [N·m⁻²] often spans several orders in magnitude, the common logarithmic scale is assigned to the horizontal σ' axis. The compression curve as shown in **Figure 1** can be obtained mainly for a fully saturated soil of low permeability like a clayey soil through a one-dimensional consolidation drainage process. During that process, it is normally assumed that water can escape from the soil through both its upper and lower ends. The stress range in which the main features of the curve are observed is often $\sigma' < 1000$ [kN·m⁻²]. However, for a sandy soil, the stress range including $\sigma' > 1000$ [kN·m⁻²] is likely to enable to draw the typical compression curve [8].

In this figure, the preconsolidation stress was denoted σ'_{p0} [N·m⁻²], which represents the maximum stress that the soil body has ever experienced. The compression curve is divided by $\sigma' = \sigma'_{p0}$ [N·m⁻²] into the overconsolidation part in

$\sigma' < \sigma'_{p0}$ [$\text{N}\cdot\text{m}^{-2}$] and the normal consolidation part in $\sigma' > \sigma'_{p0}$ [$\text{N}\cdot\text{m}^{-2}$]. The normal consolidation part is often called the virgin compression curve and is approximated as an almost linear curve on the e - $\log_{10}\sigma'$ plane.

The shape of the compression curve on the e - $\log_{10}\sigma'$ plane suggests that the compression curve can be modelled by characterizing how the slope $de/d(\log_{10}\sigma')$ behaves along a series of e - $\log_{10}\sigma'$ plots. One of the most obvious features about the slope $de/d(\log_{10}\sigma')$ is that $de/d(\log_{10}\sigma')$ should take a negative value for any σ' [$\text{N}\cdot\text{m}^{-2}$] such as:

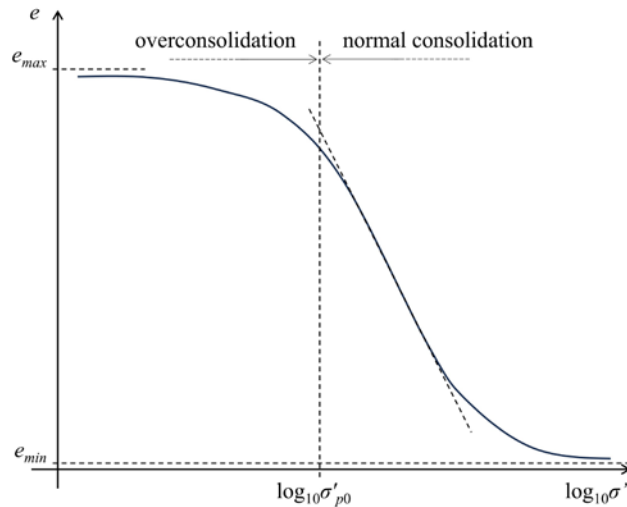


Figure 1. A schematic diagram of a compression curve in the form of void ratio-effective stress relation (e - σ' relation). The horizontal axis indicates the common logarithmic scale of the effective stress σ' [$\text{N}\cdot\text{m}^{-2}$] while the vertical axis indicates the void ratio e [$\text{m}^3\cdot\text{m}^{-3}$]. The domain of σ' [$\text{N}\cdot\text{m}^{-2}$] is divided by the preconsolidation stress σ'_{p0} [$\text{N}\cdot\text{m}^{-2}$]. It is assumed that the void ratio e [$\text{m}^3\cdot\text{m}^{-3}$] of the compression curve approaches its maximum e_{max} [$\text{m}^3\cdot\text{m}^{-3}$] as σ' [$\text{N}\cdot\text{m}^{-2}$] reduces in the overconsolidation process ($\sigma' < \sigma'_{p0}$ [$\text{N}\cdot\text{m}^{-2}$]), while it approaches its minimum e_{min} [$\text{m}^3\cdot\text{m}^{-3}$] as σ' [$\text{N}\cdot\text{m}^{-2}$] increases because a soil body may or may not be compressed into a complete solid with no void. The mid slope of the curve is often approximated as a linear e - $\log_{10}\sigma'$ relation.

$$\frac{de}{d(\log_{10}\sigma')} < 0 \quad (1)$$

Then, it was assumed that the void ratio e [$\text{m}^3\cdot\text{m}^{-3}$] approaches its maximum e_{max} [$\text{m}^3\cdot\text{m}^{-3}$] when σ' [$\text{N}\cdot\text{m}^{-2}$] is ideally reduced toward 0 [$\text{N}\cdot\text{m}^{-2}$], while it approaches its minimum e_{min} [$\text{m}^3\cdot\text{m}^{-3}$] when σ' [$\text{N}\cdot\text{m}^{-2}$] becomes very large because any soil body may or may not be compressed into a complete solid with no void. On this assumption, the slope $de/d(\log_{10}\sigma')$ should asymptotically get close to zero toward either of the two extremities of σ' [$\text{N}\cdot\text{m}^{-2}$]. Therefore, by using a positive constant μ'' [-], a possible candidate of the negative slope $de/d(\log_{10}\sigma')$ is as below:

$$\frac{de}{d(\log_{10}\sigma')} = -\mu''(e - e_{min})(e_{max} - e) \quad (2)$$

Introducing a new positive constant $\mu' = \mu'/\log_e 10$ [-] makes this equation transform into:

$$\frac{de}{d(\ln \sigma')} = -\mu'(e - e_{min})(e_{max} - e) \quad (3)$$

This differential equation can be solved by the separation of variables as follows:

$$e = e_{min} + \frac{e_{max} - e_{min}}{1 + C\sigma'^{\mu'(e_{max} - e_{min})}} \quad (4)$$

where C is an arbitrary positive constant stemming from the indefinite integral of Equation (3). By introducing two arbitrary positive constants $\mu = \mu'(e_{max} - e_{min})$ [-] and $\gamma = C^{1/\mu}$ [$m^2 \cdot N^{-1}$], this equation gives the model of the e - σ' relation under a consolidation process of monotonous compression such as:

$$e = e_{min} + \frac{e_{max} - e_{min}}{1 + (\gamma\sigma')^\mu} \quad (5)$$

In consequence, this model can work for reproducing or predicting a compression curve, once certain values are assigned to the four parameters e_{max} [$m^3 \cdot m^{-3}$], e_{min} [$m^3 \cdot m^{-3}$], γ [$m^2 \cdot N^{-1}$], and μ [-]. The parameters e_{max} [$m^3 \cdot m^{-3}$] and e_{min} [$m^3 \cdot m^{-3}$] define the range of void ratio that the compression curve can span. The parameter γ [$m^2 \cdot N^{-1}$] controls the size of the domain where the slope $de/d(\log_{10} \sigma')$ is less steep than that of the linear e - $\log_{10} \sigma'$ part so that the smaller γ [$m^2 \cdot N^{-1}$] makes the linear e - $\log_{10} \sigma'$ part shift toward the domain of larger σ' [$N \cdot m^{-2}$]. The parameter μ [-] controls the steepness of the curve.

3. Compression Index and Virgin Compression Curve

The compression index C_c [-] for the compression curve is an essential parameter for land subsidence prediction. The definition of C_c [-] is expressed as below:

$$C_c = -\frac{de}{d(\log_{10} \sigma')} \quad (6)$$

When the compression curve is modelled by Equation (5), C_c [-] can be expressed as follows:

$$C_c = \frac{\mu \ln 10}{e_{max} - e_{min}} (e - e_{min})(e_{max} - e) \quad (7)$$

Equation (7) means that C_c [-] becomes 0 as its minimum when e [$m^3 \cdot m^{-3}$] takes either end of e [$m^3 \cdot m^{-3}$], while it becomes its maximum $C_{c \max}$ [-] for $e = (e_{max} + e_{min})/2$ [$m^3 \cdot m^{-3}$]. $C_{c \max}$ [-] can represent the inclination of the virgin compression curve on the e - $\log_{10} \sigma'$ plane because it occurs in the mid slope of the modelled compression curve of Equation (5). For denoting the average of e_{min} [$m^3 \cdot m^{-3}$] and e_{max} [$m^3 \cdot m^{-3}$], a new parameter e_v [$m^3 \cdot m^{-3}$] is defined as below:

$$e_v = \frac{e_{max} + e_{min}}{2} \quad (8)$$

Then, $C_{c \max}$ [-] is determined by substituting e_v [$m^3 \cdot m^{-3}$] into Equation (7):

$$C_{c\max} = \frac{\mu \ln 10}{4} (e_{\max} - e_{\min}) \quad (9)$$

When the effective stress for e_v [$\text{m}^3 \cdot \text{m}^{-3}$] is expressed as σ'_v [$\text{N} \cdot \text{m}^{-2}$], it can be determined by substituting e_v [$\text{m}^3 \cdot \text{m}^{-3}$] into Equation (5) as below:

$$\sigma'_v = \frac{1}{\gamma} \quad (10)$$

As a result, the linear part of the modelled virgin compression curve can be approximated as the line passing through the point (σ'_v, e_v) with the slope $-C_{c\max}$ [-] such as:

$$e = e_v - C_{c\max} \log_{10} \frac{\sigma'}{\sigma'_v} \quad (11)$$

Since the modelled curve is of point symmetric on the e - $\log_{10} \sigma'$ plane and its symmetric center is the point (σ'_v, e_v) , the conjugate of the preconsolidation stress σ'_{p0} [$\text{N} \cdot \text{m}^{-2}$], denoted as σ'_{p0*} [$\text{N} \cdot \text{m}^{-2}$], can be defined by using σ'_v [$\text{N} \cdot \text{m}^{-2}$] and σ'_{p0} [$\text{N} \cdot \text{m}^{-2}$] as below:

$$\sigma'_{p0*} = \left(\frac{\sigma'_v}{\sigma'_{p0}} \right)^2 \sigma'_{p0} \quad (12)$$

This conjugate preconsolidation stress σ'_{p0*} [$\text{N} \cdot \text{m}^{-2}$] is used for modelling an expansion process as explained in the next section.

Consequently, the relations between the model compression curve of Equation (5) and the parameters defined above are described in **Figure 2**.

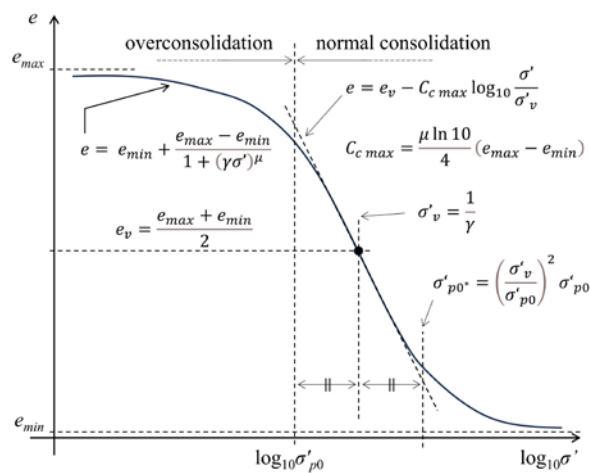


Figure 2. The relevant parameters that characterize the modelled compression curve of Equation (5). The compression index C_c [-] for any σ' [$\text{N} \cdot \text{m}^{-2}$] can be calculated as $-de/d(\log_{10} \sigma')$ and its maximum $C_{c\max}$ [-] is given for the point (σ'_v, e_v) where $\sigma'_v = 1/\gamma$ [$\text{N} \cdot \text{m}^{-2}$] and $e_v = (e_{\max} + e_{\min})/2$ [$\text{m}^3 \cdot \text{m}^{-3}$]. By using these parameters, the linear part of the normal consolidation curve can be approximated by the line $e = e_v - C_{c\max} \log_{10} (\sigma'/\sigma'_v)$. Since the modelled curve is of point symmetric on the e - $\log_{10} \sigma'$ plane and its symmetric center is the point (σ'_v, e_v) , the preconsolidation stress σ'_{p0} [$\text{N} \cdot \text{m}^{-2}$] has its conjugate $\sigma'_{p0*} = (\sigma'_v/\sigma'_{p0})^2 \sigma'_{p0}$ [$\text{N} \cdot \text{m}^{-2}$].

4. Modelling of an Expansion Process

When a saturated soil body experiences a series of compression and expansion processes, the e - σ' relation shows hysteretic behaviours along the stress history. If the soil body is placed under an expansion process after it reaches some e - σ' point on the normal consolidation process, the e - σ' relation may show an expansion curve with a slope $de/d(\log_{10} \sigma')$ that is often less steep than that of the original compression curve.

Suppose that the original compression curve is modelled by using Equation (5) as follows:

$$e_{oc} = e_{oc\ min} + \frac{e_{oc\ max} - e_{oc\ min}}{1 + (\gamma_{oc} \sigma')^{\mu_{oc}}} \quad (13)$$

where the subscript “oc” is attached to the expressions that are defined for the original compression curve. Because it is expected that an expansion process is likely to have similar characteristics to the original compression process in terms of the e - $\log_{10} \sigma'$ relation, it was assumed that an expansion curve can be expressed by a similitude of Equation (13). On this assumption, this study introduced the concept of the potential expansion curve e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] to formulate an expansion process that starts from an arbitrary point P_1 (σ'_{p1} , e_{p1}) with $\sigma'_{p1} > \sigma'_{p0}$ [$\text{N} \cdot \text{m}^{-2}$] on the original compression curve e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$]. In this concept, e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] indicates the void ratio for any σ' [$\text{N} \cdot \text{m}^{-2}$] that may be lost by a very strong additional effective stress σ'_∞ [$\text{N} \cdot \text{m}^{-2}$] but will be recovered when the strong stress σ'_∞ [$\text{N} \cdot \text{m}^{-2}$] disappears again. This means that the inclination $|de_{pe}/d(\log_{10} \sigma')|$ indicates a potential expansion rate at any σ' [$\text{N} \cdot \text{m}^{-2}$] while e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] approaches zero as σ' [$\text{N} \cdot \text{m}^{-2}$] increases.

When the potential expansion curve e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] is modelled by using a similitude of Equation (13) and the parameters specific for the curve e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] are denoted in the similar manner to those for the curve e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$], the parameters for e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] can be expressed as $e_{pe\ max}$ [$\text{m}^3 \cdot \text{m}^{-3}$], $e_{pe\ min}$ [$\text{m}^3 \cdot \text{m}^{-3}$], γ_{pe} [$\text{m}^2 \cdot \text{N}^{-1}$], and μ_{pe} [-]. And the curve e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] is described as below:

$$e_{pe} = e_{pe\ min} + \frac{e_{pe\ max} - e_{pe\ min}}{1 + (\gamma_{pe} \sigma')^{\mu_{pe}}} \quad (14)$$

In Equation (14), $e_{pe\ min}$ should be 0 [$\text{m}^3 \cdot \text{m}^{-3}$], because the amount of potentially reversible compression should reduce as the stress imposed on a soil body becomes stronger and the soil body becomes more compacted, and recoverable voids may no longer remain with an infinitesimal $|de_{pe}/d(\log_{10} \sigma')|$ when a very strong stress σ'_∞ [$\text{N} \cdot \text{m}^{-2}$] is imposed. Thus,

$$e_{pe\ min} = 0 \quad (15)$$

The parameter $e_{pe\ max}$ [$\text{m}^3 \cdot \text{m}^{-3}$] is hardly formulated because it is often difficult to predict the total amount of reversible compression of a soil body before the beginning of compression-expansion history. Therefore, this study assumed that $e_{pe\ max}$ [$\text{m}^3 \cdot \text{m}^{-3}$] can be determined as a fitting parameter and is a constant through-

out compression-expansion history. Future studies may uncover the ways for predicting the amount of reversible compression a soil body potentially has by relating it to more easily obtainable quantities that characterize the soil body.

The parameters γ_{pe} [$\text{m}^2 \cdot \text{N}^{-1}$] and μ_{pe} [–] can be identified when considering a possible change in inclination $|de_{pe}/d(\log_{10}\sigma')|$ with the decrease in σ' [$\text{N} \cdot \text{m}^{-2}$]. It is expected that the inclination $|de_{pe}/d(\log_{10}\sigma')|$ increases very gradually when σ' [$\text{N} \cdot \text{m}^{-2}$] reduces in the domain $\sigma' > \sigma'_{p0}$ [$\text{N} \cdot \text{m}^{-2}$] and becomes progressively greater as σ' [$\text{N} \cdot \text{m}^{-2}$] becomes smaller in the domain $\sigma' < \sigma'_{p0}$ [$\text{N} \cdot \text{m}^{-2}$], because the amount of compression that has occurred in $\sigma' > \sigma'_{p0}$ [$\text{N} \cdot \text{m}^{-2}$] is likely to be due to irreversible compression while reversible compression has taken place mainly in $\sigma' < \sigma'_{p0}$ [$\text{N} \cdot \text{m}^{-2}$]. To reflect this expectation on the shape of the curve e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$], the two parameters γ_{pe} [$\text{m}^2 \cdot \text{N}^{-1}$] and μ_{pe} [–] are regulated so that the slope foot of the curve e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] on the e - $\log_{10}\sigma'$ plane is found in the domain of σ' [$\text{N} \cdot \text{m}^{-2}$] where the shoulder of the curve e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$] is found (**Figure 3(a)**). Considering the expression of the denominator in the second term in the right side of Equation (5), one of concise examples about setting γ_{pe} [$\text{m}^2 \cdot \text{N}^{-1}$] and μ_{pe} [–] for realizing this expectation is expressed as:

$$\gamma_{pe} \sigma'_{p0} = \gamma_{oc} \sigma'_{p0*} \quad (16)$$

$$\mu_{pe} = \mu_{oc} \quad (17)$$

where the subscript “ pe ” indicates that the terms with it are defined for the curve e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$]. Then, Equation (16) gives:

$$\gamma_{pe} = \frac{\sigma'_{p0*}}{\sigma'_{p0}} \gamma_{oc} \quad (18)$$

As shown above, the potential expansion curve e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] can be formulated by using Equation (14) with the parameters defined in Equations (15), (17) and (18). All the parameters necessary for predicting e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] can be obtained by the curve e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$] and σ'_{p0} [$\text{N} \cdot \text{m}^{-2}$], except that $e_{pe \max}$ [$\text{m}^3 \cdot \text{m}^{-3}$] is a fitting parameter.

Then, by using e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$], the expansion curve e_e [$\text{m}^3 \cdot \text{m}^{-3}$] that starts from the point P_1 (σ'_{p1} , e_{p1}) on the curve e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$] can be obtained by parallel translation of e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] (**Figure 3(b)**), and is described as below:

$$e_e = e_{pe} + (e_{p1} - e_{pe}(\sigma'_{p1})) \quad (19)$$

$$e_{p1} = e_{oc}(\sigma'_{p1}) \quad (20)$$

In summation, once the original compression curve e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$], the preconsolidation stress for the original compression curve σ'_{p0} [$\text{N} \cdot \text{m}^{-2}$], and the potential expansion curve e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] are given, the expansion curve e_e [$\text{m}^3 \cdot \text{m}^{-3}$] can be predicted for a reversal point $P_1(\sigma'_{p1}, e_{p1})$ that can arbitrarily be selected in the domain of $\sigma' > \sigma'_{p0}$ [$\text{N} \cdot \text{m}^{-2}$] on e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$]. Promising ways for identifying the curves e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$] and e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] include fitting the models to measured data sets for obtaining the parameters $e_{oc \max}$ [$\text{m}^3 \cdot \text{m}^{-3}$], $e_{oc \min}$ [$\text{m}^3 \cdot \text{m}^{-3}$], γ_{oc} [$\text{m}^2 \cdot \text{N}^{-1}$], μ_{oc}

$[-]$, and $e_{pe\ max}$ $[m^3 \cdot m^{-3}]$ in Equation (13) and (14). And the best way to have a value of σ'_{p0} $[N \cdot m^{-2}]$ is to have known a stress history about a soil layer of interest, while an alternative way for the case without a known value of σ'_{p0} $[N \cdot m^{-2}]$ is to evaluate an approximate of σ'_{p0} $[N \cdot m^{-2}]$ in the way described in a later section.

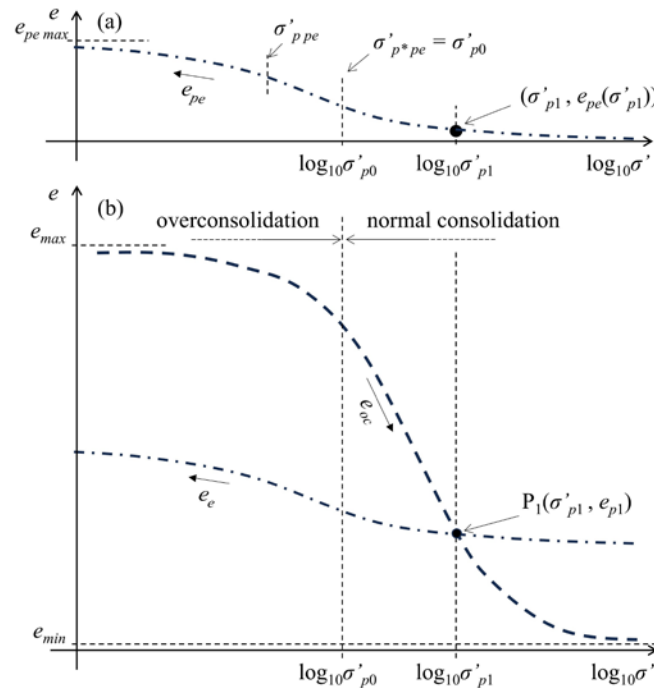


Figure 3. A schematic diagram of the potential expansion curve e_{pe} $[m^3 \cdot m^{-3}]$ (dashed line in the sub-graph (a)) in relation to the original compression curve e_{oc} $[m^3 \cdot m^{-3}]$ (thick dotted line in the sub-graph (b)) and the expansion curve e_e $[m^3 \cdot m^{-3}]$ (dashed line in the sub-graph (b)). For defining the curve e_{pe} $[m^3 \cdot m^{-3}]$, this study assumed that (i) the inclination $|de_{pe}/d(\log_{10} \sigma')|$ indicates a potential expansion rate at any σ' $[N \cdot m^{-2}]$, (ii) e_{pe} $[m^3 \cdot m^{-3}]$ is a similitude of e_{oc} $[m^3 \cdot m^{-3}]$ on the e - $\log_{10} \sigma'$ plane, (iii) e_{pe} $[m^3 \cdot m^{-3}]$ asymptotically gets close to 0 $[m^3 \cdot m^{-3}]$ as σ' $[N \cdot m^{-2}]$ increases, (iv) the apparent “conjugate preconsolidation stress” for e_{pe} $[m^3 \cdot m^{-3}]$ is equivalent to the preconsolidation stress for e_{oc} $[m^3 \cdot m^{-3}]$ so that $\sigma'_{p^*pe} = \sigma'_{p0}$ $[N \cdot m^{-2}]$. Then, the expansion curve e_e $[m^3 \cdot m^{-3}]$ is obtained by parallel translation of e_{pe} $[m^3 \cdot m^{-3}]$ along e -axis so that e_e $[m^3 \cdot m^{-3}]$ passes through the reversal point P_1 (σ'_{p1}, e_{p1}) at which the original compression process turns into the expansion process.

5. Modelling of a Recompression Process

Suppose that the expansion process ends and the new compression process starts at the point R_1 (σ'_{r1}, e_{r1}) on the curve e_e $[m^3 \cdot m^{-3}]$ (Figure 4). This recompression process can be formulated by the same mathematical function as the original compression curve e_{oc} $[m^3 \cdot m^{-3}]$ because this recompression process should be a self-similarity of the original compression process. When the recompression curve e_{rc} $[m^3 \cdot m^{-3}]$ is described by a similitude of Equation (13) and the parameters specific for the curve e_{rc} $[m^3 \cdot m^{-3}]$ are denoted in the similar manner to those for the curve e_{oc} $[m^3 \cdot m^{-3}]$, the parameters for e_{rc} $[m^3 \cdot m^{-3}]$ are expressed as $e_{rc\ max}$ $[m^3 \cdot m^{-3}]$, $e_{rc\ min}$

$[m^3 \cdot m^{-3}]$, $\gamma_{rc} [m^2 \cdot N^{-1}]$, and $\mu_{rc} [-]$. Then, the curve $e_{rc} [m^3 \cdot m^{-3}]$ is described as below:

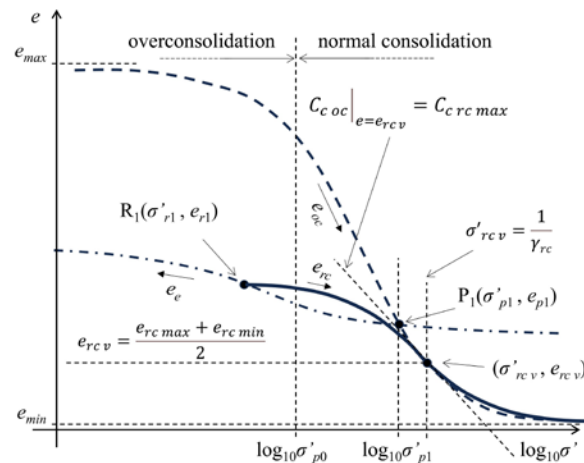


Figure 4. A schematic diagram of the recompression curve $e_{rc} [m^3 \cdot m^{-3}]$ (thick line) in relation to the expansion curve $e_e [m^3 \cdot m^{-3}]$ (dashed line) and the original compression curve $e_{oc} [m^3 \cdot m^{-3}]$ (thick dotted line). Considering that $e_{rc} [m^3 \cdot m^{-3}]$ is a self-similarity of $e_{oc} [m^3 \cdot m^{-3}]$, this study assumed that after $e_{rc} [m^3 \cdot m^{-3}]$ starts from the point $R_1 (\sigma'_{r1}, e_{r1})$ on the curve $e_e [m^3 \cdot m^{-3}]$, $e_{rc} [m^3 \cdot m^{-3}]$ approaches $e_{oc} [m^3 \cdot m^{-3}]$ as $\sigma' [N \cdot m^{-2}]$ increases. To satisfy this assumption, $e_{rc \min} [m^3 \cdot m^{-3}]$ was set to be equal to $e_{oc \min} [m^3 \cdot m^{-3}]$, and $C_{c \max} [-]$ of $e_{rc} [m^3 \cdot m^{-3}]$ was set to be equal to $C_c [-]$ of $e_{oc} [m^3 \cdot m^{-3}]$ at the point $(\sigma'_{rcv}, e_{rcv}) = (1/\gamma_{rc}, (e_{rc \max} + e_{rc \min})/2)$, while the curve $e_{oc} [m^3 \cdot m^{-3}]$ passes through the point (σ'_{rcv}, e_{rcv}) .

$$e_{rc} = e_{rc \min} + \frac{e_{rc \max} - e_{rc \min}}{1 + (\gamma_{rc} \sigma')^{\mu_{rc}}} \quad (21)$$

Four equations are required to identify the four parameters in Equation (21). Because the curve $e_{rc} [m^3 \cdot m^{-3}]$ passes through the point $R_1 (\sigma'_{r1}, e_{r1})$ and should approach the original compression curve $e_{oc} [m^3 \cdot m^{-3}]$ as $\sigma' [N \cdot m^{-2}]$ increases, the following four conditions can be ideated: (i) the point $R_1 (\sigma'_{r1}, e_{r1})$ satisfies Equation (21), (ii) the minimum void ratio of the recompression curve is equal to that of the original compression curve, (iii) the point of reflection for the curve $e_{rc} [m^3 \cdot m^{-3}]$, namely $(\sigma'_{rcv}, e_{rcv}) = (1/\gamma_{rc}, (e_{rc \max} + e_{rc \min})/2)$, is found on the curve $e_{oc} [m^3 \cdot m^{-3}]$, and (iv) the slope of the curve $e_{rc} [m^3 \cdot m^{-3}]$ and that of the curve $e_{oc} [m^3 \cdot m^{-3}]$ is similar to each other at the point (σ'_{rcv}, e_{rcv}) .

The condition (i) and (ii) are expressed as below:

$$e_{r1} = e_{rc \min} + \frac{e_{rc \max} - e_{rc \min}}{1 + (\gamma_{rc} \sigma'_{r1})^{\mu_{rc}}} \quad (22)$$

$$e_{rc \min} = e_{oc \min} \quad (23)$$

The condition (iii) is formulated as below:

$$e_{rcv} = e_{oc \min} + \frac{e_{oc \max} - e_{oc \min}}{1 + (\gamma_{oc} \sigma'_{rcv})^{\mu_{oc}}} \quad (24)$$

where σ'_{rcv} [$\text{N}\cdot\text{m}^{-2}$] and e_{rcv} [$\text{m}^3\cdot\text{m}^{-3}$] are derived by Equations (8) and (10) such as:

$$e_{rcv} = \frac{e_{rc\max} + e_{rc\min}}{2} \quad (25)$$

$$\sigma'_{rcv} = \frac{1}{\gamma_{rc}} \quad (26)$$

Formulation of the condition (iv) requires both the maximum inclination of the recompression curve $C_{c\ rc\max}$ [–] and the inclination of the original compression curve $C_{c\ oc}$ [–] for $e_{rcv} = (e_{rc\max} + e_{rc\min})/2$ [$\text{m}^3\cdot\text{m}^{-3}$]. According to Equations (7) and (9), they can be expressed as:

$$C_{c\ oc} \Big|_{e=e_{rcv}} = \frac{\mu_{oc} \ln 10}{e_{oc\max} - e_{oc\min}} (e_{rcv} - e_{oc\min}) (e_{oc\max} - e_{rcv}) \quad (27)$$

$$C_{c\ rc\max} = \frac{\mu_{rc} \ln 10}{4} (e_{rc\max} - e_{rc\min}) \quad (28)$$

Therefore, when $C_{c\ oc}$ [–] for $e = e_{rcv}$ [$\text{m}^3\cdot\text{m}^{-3}$] equals $C_{c\ rc\max}$ [–], the right sides of Equations (27) and (28) are equal:

$$\frac{\mu_{rc} \ln 10}{4} (e_{rc\max} - e_{rc\min}) = \frac{\mu_{oc} \ln 10}{e_{oc\max} - e_{oc\min}} (e_{rcv} - e_{oc\min}) (e_{oc\max} - e_{rcv})$$

And solving this equation for μ_{rc} [–] with Equations (23) and (25) gives:

$$\mu_{rc} = \mu_{oc} \left(1 + \frac{e_{oc\max} - e_{rc\max}}{e_{oc\max} - e_{oc\min}} \right) \quad (29)$$

Thus, the condition (iv) can be represented by Equation (29).

As a result, solving Equations (22), (23), (24), and (29) simultaneously makes $e_{rc\max}$ [$\text{m}^3\cdot\text{m}^{-3}$], $e_{rc\min}$ [$\text{m}^3\cdot\text{m}^{-3}$], γ_{rc} [$\text{m}^2\cdot\text{N}^{-1}$], and μ_{rc} [–] identified. An example of this solving process is comprised of the following steps: (r1) $e_{rc\max} = e_{r1}$ [$\text{m}^3\cdot\text{m}^{-3}$], $e_{rc\min} = e_{oc\min}$ [$\text{m}^3\cdot\text{m}^{-3}$], and $\gamma_{rc} = \gamma_{oc}$ [$\text{m}^2\cdot\text{N}^{-1}$] are assigned as the initial approximations, (r2) μ_{rc} [–] is determined by Equation (29) with the latest value of $e_{rc\max}$ [$\text{m}^3\cdot\text{m}^{-3}$], (r3) Equation (22) is solved explicitly for $e_{rc\max}$ [$\text{m}^3\cdot\text{m}^{-3}$] by using the latest parameter values, (r4) Equation (24) with Equation (26) is solved explicitly for γ_{rc} [$\text{m}^2\cdot\text{N}^{-1}$] by using the latest parameter values, (r5) return to (r2). In many cases, several times of iterating the steps (r2) through (r5) are enough for converging the approximates of $e_{rc\max}$ [$\text{m}^3\cdot\text{m}^{-3}$], $e_{rc\min}$ [$\text{m}^3\cdot\text{m}^{-3}$], γ_{rc} [$\text{m}^2\cdot\text{N}^{-1}$], and μ_{rc} [–].

As shown above, the recompression curve e_{rc} [$\text{m}^3\cdot\text{m}^{-3}$] can be derived from the original compression curve e_{oc} [$\text{m}^3\cdot\text{m}^{-3}$] and the reversal point R_1 (σ'_{r1} , e_{r1}), the latter of which is found on the expansion curve e_e [$\text{m}^3\cdot\text{m}^{-3}$]. And, because the curve e_{rc} [$\text{m}^3\cdot\text{m}^{-3}$] is a self-similarity of the original compression curve e_{oc} [$\text{m}^3\cdot\text{m}^{-3}$], the procedure through which the curve e_{rc} [$\text{m}^3\cdot\text{m}^{-3}$] is identified can be repeatedly used for predicting the repetitions of compression and expansion processes by updating the parameter set of the model and the coordinates of the reversal points on the e - $\log_{10}\sigma'$ plane.

6. Approximate of a Preconsolidation Stress

When the preconsolidation stress σ'_{p0} [$\text{N}\cdot\text{m}^{-2}$] is unknown for a given compression curve, Equations (12) and (18) cannot work and, in turn, the expansion curve of Equation (19) and the recompression curve of Equation (21) cannot be identified. Therefore, in such cases, an approximate value of σ'_{p0} [$\text{N}\cdot\text{m}^{-2}$] should be prepared for making the model work effectively to describe a series of expansion and recompression processes.

The classical Casagrande's method has been broadly used on the e - $\log_{10}\sigma'$ chart. Although this method is seemingly ad-hoc, it can be mathematically defined when a compression curve is modelled by Equation (5). Thus, this section explains a procedure for deriving an approximate of σ'_{p0} [$\text{N}\cdot\text{m}^{-2}$] for the modelled compression curve of Equation (5) based on the Casagrande's method.

The Casagrande's method runs through the following steps: (c1) extend the almost linear part of the compression curve (line a in **Figure 5**), (c2) determine the point of the maximum curvature on the compression curve (point M in **Figure 5**), (c3) draw the tangential line of the compression curve at the point M (line b in **Figure 5**), (c4) draw the horizontal line through the point M (line c in **Figure 5**), (c5) draw the bisector between the lines b and c (line d in **Figure 5**), (c6) determine the point P_0 as the intersection of the lines a and d, (c7) the vertical line through P_0 gives the approximate of σ'_{p0} [$\text{N}\cdot\text{m}^{-2}$].

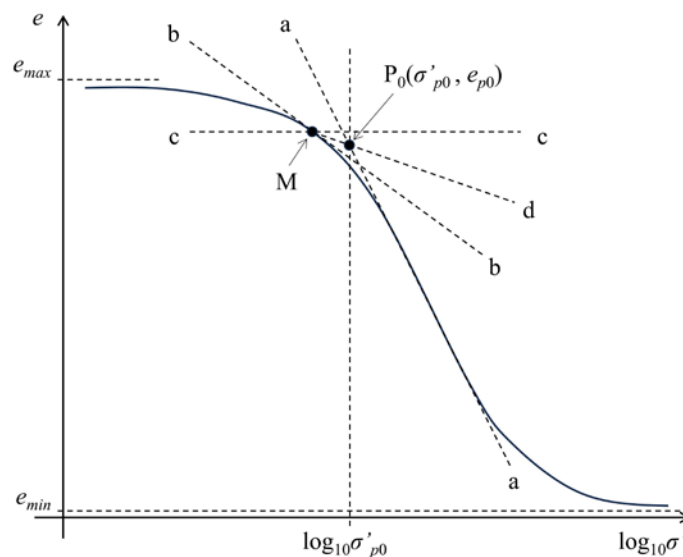


Figure 5. A schematic diagram of the Casagrande's method for determining the approximate of the preconsolidation stress σ'_{p0} [$\text{N}\cdot\text{m}^{-2}$]. The solid line indicates a compression curve. The thin-dotted lines a, b, c, and d denote the slope of the virgin compression curve, the tangential of the compression curve at the point M where the maximum curvature of the compression curve is found, the horizontal line passing through M, and the bisector line between the lines b and c, respectively. The point P_0 is obtained as the intersection of the lines a and d, telling the approximate of the preconsolidation stress σ'_{p0} [$\text{N}\cdot\text{m}^{-2}$]. The void ratio corresponding to σ'_{p0} [$\text{N}\cdot\text{m}^{-2}$] is denoted as e_{p0} [$\text{m}^3\cdot\text{m}^{-3}$].

As mentioned in these steps, the point of maximum curvature on the compression curve should be identified. According to the algebraic knowledge, when a curve on the x - y plane is expressed as $y = f(x)$, the curvature κ of the curve $f(x)$ is expressed as below:

$$\kappa = \frac{f''(x)}{(1 + f'(x)^2)^{\frac{3}{2}}} \quad (30)$$

where $f(x)$ and $f'(x)$ are the first and second order differentials of the curve $f(x)$ with respect to x . The size of the curvature $|\kappa|$ should take maxima for certain values of x while it may take zero as minima when x indicates points of inflection on $f(x)$. Therefore, a value of x that corresponds to the maximum curvature on $f(x)$ should make the differential $d\kappa/dx$ take zero with $\kappa > 0$ or $\kappa < 0$. The differential $d\kappa/dx$ is derived from Equation (30) such as:

$$\frac{d\kappa}{dx} = \frac{f'''(x)(1 + f'(x)^2) - 3f''(x)^2 f'(x)}{(1 + f'(x)^2)^{\frac{5}{2}}} \quad (31)$$

Equation (31) indicates that a value of x for the maximum curvature on $f(x)$ should make the numerator in the left side of Equation (31) zero as follows:

$$f'''(x)(1 + f'(x)^2) - 3f''(x)^2 f'(x) = 0 \quad (32)$$

The equations corresponding to Equations (30) and (32) for the compression curve on the e - $\log_{10}\sigma'$ plane are as below:

$$\kappa = \frac{d^2e}{d(\log_{10}\sigma')^2} \left(1 + \left(\frac{de}{d(\log_{10}\sigma')} \right)^2 \right)^{-\frac{3}{2}} \quad (33)$$

$$\frac{d^3e}{d(\log_{10}\sigma')^3} \left(1 + \left(\frac{de}{d(\log_{10}\sigma')} \right)^2 \right) - 3 \left(\frac{d^2e}{d(\log_{10}\sigma')^2} \right)^2 \frac{de}{d(\log_{10}\sigma')} = 0 \quad (34)$$

The first, second, and third orders of differential in these equations are derived by differentiating Equation (5) with respect to $\log_{10}\sigma'$ as below:

$$\frac{de}{d(\log_{10}\sigma')} = -\mu \ln 10 (e_{\max} - e_{\min}) \frac{(\gamma\sigma')^\mu}{(1 + (\gamma\sigma')^\mu)^2} \quad (35)$$

$$\frac{d^2e}{d(\log_{10}\sigma')^2} = \mu \ln 10 \frac{1 - (\gamma\sigma')^\mu}{1 + (\gamma\sigma')^\mu} \frac{de}{d(\log_{10}\sigma')} \quad (36)$$

$$\frac{d^3e}{d(\log_{10}\sigma')^3} = (\mu \ln 10)^2 \frac{(1 - (\gamma\sigma')^\mu)^2 - 2(\gamma\sigma')^\mu}{(1 + (\gamma\sigma')^\mu)^2} \frac{de}{d(\log_{10}\sigma')} \quad (37)$$

Because it is hard to solve Equation (34) explicitly for σ' [$\text{N}\cdot\text{m}^{-2}$], the effective stress for the point of the maximum curvature was obtained by applying the New-

ton-Raphson iterative scheme to Equation (34) with Equations (35) through (37).

On the $e\text{-}\log_{10}\sigma'$ curve modelled by Equation (5), there are two values of σ' [N·m⁻²] for which the size of κ becomes maximum. One can be found in the slope shoulder of the curve and the other in the foot slope of the curve. The former is for the point of the maximum curvature for identifying the approximate of the preconsolidation stress σ'_{p0} [N·m⁻²] and, thus, is denoted σ'_M [N·m⁻²] as the point is labeled M on **Figure 5**. The latter can be regarded as the conjugate of σ'_M [N·m⁻²]. This study denoted this conjugate σ'_{M*} [N·m⁻²], and the point at which σ'_{M*} [N·m⁻²] is found on the $e\text{-}\log_{10}\sigma'$ plane was denoted M*. These two effective stresses σ'_M [N·m⁻²] and σ'_{M*} [N·m⁻²] can be distinguished from each other by confirming the sign of κ for each, because κ should take a negative value when the curvature is convex like at the point M and become positive when the curvature is concave like at the point M*. Since the left side of Equation (34) takes negative values in the domains of $\sigma' < \sigma'_M$ [N·m⁻²] and $\sigma' > \sigma'_{M*}$ [N·m⁻²] and approaches zero toward both extremities of the domain of σ' [N·m⁻²], it is recommended to find at first the sub-domain of σ' [N·m⁻²] in which the sign of the left side of Equation (34) turns from negative to positive with the increase in σ' [N·m⁻²] and, then, the Newton-Raphson iterative scheme is implemented in the obtained sub-domain.

For identifying the points M and M* on the $e\text{-}\log_{10}\sigma'$ plane, the void ratios for σ'_M [N·m⁻²] and σ'_{M*} [N·m⁻²] were called e_M [m³·m⁻³] and e_{M*} [m³·m⁻³], respectively, which are evaluated by substituting σ'_M [N·m⁻²] and σ'_{M*} [N·m⁻²] into Equation (5).

For determining the tangential line at the point M in **Figure 5**, the slope of the tangential line at M should be evaluated. Substituting the values of σ'_M [N·m⁻²] and e_M [m³·m⁻³] into Equation (35) gives the inclination of the tangential line C_{cM} [-] as follows:

$$C_{cM} = - \left. \frac{de}{d(\log_{10} \sigma')} \right|_{\sigma'=\sigma'_M} = \mu \ln 10 (e_{max} - e_{min}) \frac{(\gamma \sigma'_M)^\mu}{(1 + (\gamma \sigma'_M)^\mu)^2} \quad (38)$$

Then, the line that passes through the point M with this inclination is expressed as below:

$$e = e_M - C_{cM} (\log_{10} \sigma' - \log_{10} \sigma'_M) \quad (39)$$

When the angle between the line of Equation (39) and the horizontal line $e = e_M$ [m³·m⁻³] is denoted as α in radian, C_{cM} equals $\tan(\alpha)$. In this case, the angle between the line $e = e_M$ [m³·m⁻³] and the bisector line between these two lines should be $\alpha/2$ and, thus, the inclination of the bisector line can be expressed as $\tan(\alpha/2)$. Because $\tan(\alpha) = 2\tan(\alpha/2)/(1 - \tan^2(\alpha/2))$, $\tan(\alpha)$ can be approximated by $2\tan(\alpha/2)$ when the angle α is so acute as that found between the lines b and c in **Figure 5**. Thus, the inclination of the bisector between the line of Equation (39) and the horizontal line can be approximated as $C_{cM}/2$ and the approximate of the bisector line can be expressed below:

$$e = e_M - \frac{1}{2} C_{cM} (\log_{10} \sigma' - \log_{10} \sigma'_M) \quad (40)$$

According to the Casagrande's method, an approximate of the preconsolidation stress σ'_{p0} [N·m⁻²] is found at the intersection of the virgin compression line of Equation (11) and the bisector line of Equation (40) on the e - $\log_{10} \sigma'$ plane. Therefore, identifying σ' [N·m⁻²] that satisfies both Equations (11) and (40) gives the approximate of σ'_{p0} [N·m⁻²] as below:

$$\sigma'_{p0} \approx \left(10^{e_v - e_M} \cdot \sigma_v^{C_{cmax}} \cdot \sigma_M^{\frac{1}{2} C_{cM}} \right)^{\frac{2}{2C_{cmax} - C_{cM}}} \quad (41)$$

The approximate of the conjugate preconsolidation stress, namely σ'_{p0^*} [N·m⁻²], can also be derived in the same manner as Equation (41). Because the slope of the tangential line of the compression curve at M* should be equal to that at M, the conjugate of the bisector line of Equation (40) can be expressed as below:

$$e = e_{M^*} - \frac{1}{2} C_{cM} (\log_{10} \sigma' - \log_{10} \sigma'_{M^*}) \quad (42)$$

And identifying σ' [N·m⁻²] that satisfies both Equations (11) and (42) gives the approximate of σ'_{p0^*} [N·m⁻²] as below:

$$\sigma'_{p0^*} \approx \left(10^{e_v - e_{M^*}} \cdot \sigma_v^{C_{cmax}} \cdot \sigma_{M^*}^{\frac{1}{2} C_{cM}} \right)^{\frac{2}{2C_{cmax} - C_{cM}}} \quad (43)$$

This conjugate σ'_{p0^*} [N·m⁻²] can also be obtained by Equation (12). And all the parameters in Equations (41) and (43) can be known by the four parameters of the modelled compression curve of Equation (5).

When the void ratio that corresponds to σ'_{p0} [N·m⁻²] is denoted e_{p0} [m³·m⁻³], the approximate of e_{p0} [m³·m⁻³] can also be derived by substituting the approximate of σ'_{p0} [N·m⁻²] into Equation (11) as below:

$$e_{p0} = e_v - C_{cmax} (\log_{10} \sigma'_{p0} - \log_{10} \sigma'_v) \quad (44)$$

And e_{p0^*} [m³·m⁻³], as the conjugate of e_{p0} [m³·m⁻³], can also be approximated as:

$$e_{p0^*} = e_v - C_{cmax} (\log_{10} \sigma'_{p0^*} - \log_{10} \sigma'_v) \quad (45)$$

Finally, it should be noticed that although the Casagrande's method is concise and popular, its effectiveness for determining σ'_{p0} [N·m⁻²] has also been debated, meaning that the preconsolidation stress derived from this method is a geometric estimate rather than necessarily the soil's actual historical maximum effective stress. Some studies recommended that several different methods including the Casagrande's method are to be used for elaborately determining σ'_{p0} [N·m⁻²] [9] [10]. Sometimes, even the types of charts for determining σ'_{p0} [N·m⁻²] have been questioned, and several studies reported that the bi-logarithmic chart, instead of the semi-logarithmic chart, for describing e - σ' relations is effective for determining σ'_{p0} [N·m⁻²] [11]-[13]. Thus, it is recommended that the method mentioned in this section is used as an alternative way for the case that a

value of σ'_{p0} [$\text{N}\cdot\text{m}^{-2}$] is uncertain for a given compression curve. And because the model-derived σ'_{p0} [$\text{N}\cdot\text{m}^{-2}$] is used only in the determination of the modelled curve of the first expansion process while the inclination of any expansion curve is generally far less steep than that of compression curve, it is expected that the uncertainty on σ'_{p0} [$\text{N}\cdot\text{m}^{-2}$] has a limited influence on the predicted e - $\log_{10}\sigma'$ hysteresis path.

7. Numerical Examples

Figure 6 shows the data sets of monotonous compression curves obtained from peats, organic soils, and ordinary clays with wide range of natural water contents [14], and the modelled curves fitted to the measured data sets. The parameters for the modelled curves were tabulated on **Table 1**. The source data sets for the fitting process were manually extracted from the printed references. Then, the parameter values of Equation (5) were determined by the method of least squares. The solver add-in of Microsoft Excel was used for minimizing the sum of the squared residuals. Each of the residuals was calculated for each of the measured effective stress values as the difference between the measured and modelled values of void ratio e [$\text{m}^3\cdot\text{m}^{-3}$]. The determination coefficient of any of the fitting results was more than 0.99, implying that the model can perform well for reproducing various types of e - $\log_{10}\sigma'$ relations.

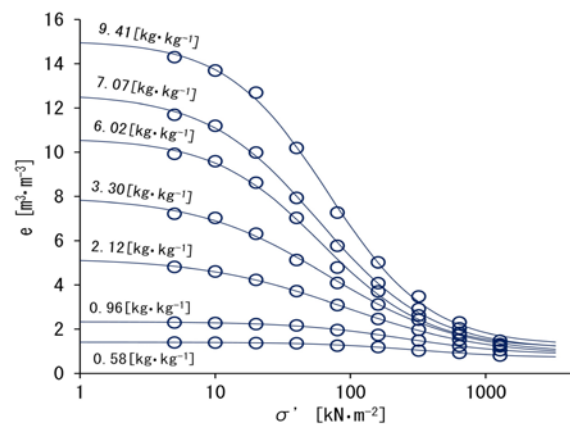


Figure 6. The modelled curves for reproducing various kinds of the e - $\log_{10}\sigma'$ relations. The data sets (circles) were plotted after [14]. The numerals in the plot area indicate the natural water contents of the measured data sets. Each curve was drawn by fitting the model of Equation (5) to each of the measured data sets by minimizing the sum of squared residuals between the model and the measured data set.

Figure 7(a) shows the modelled compression and expansion curves, which were obtained by fitting Equations (13) and (14) to the measured data set reported in [11], and the predicted compression-expansion-compression curve.

At first, the entire monotonous compression process was modelled as the original compression curve e_{oc} [$\text{m}^3\cdot\text{m}^{-3}$] of Equation (13), whose parameters were obtained in the same manner as those for the modelled curves in **Figure 6**. Then, the

Table 1. The parameter values of Equation (5) for drawing the curves on **Figure 6**, obtained by minimizing the sum of the squared residuals between the model and the measured data plots in [14], and were classified by the natural water contents w_n [kg·kg⁻¹] of the soil samples.

w_n [kg·kg ⁻¹]	9.41	7.07	6.02	3.3	2.12	0.96	0.58
e_{max} [m ³ ·m ⁻³]	15.050	12.645	10.633	7.981	5.203	2.333	1.409
e_{min} [m ³ ·m ⁻³]	1.266	1.076	1.132	0.873	0.797	0.823	0.702
γ [m ² ·kN ⁻¹]	0.015	0.017	0.016	0.015	0.011	0.004	0.003
μ [-]	1.165	1.052	1.108	0.894	0.831	1.043	1.150
σ'_{p0} [-]	14.025	10.519	12.095	10.176	13.790	71.682	112.522

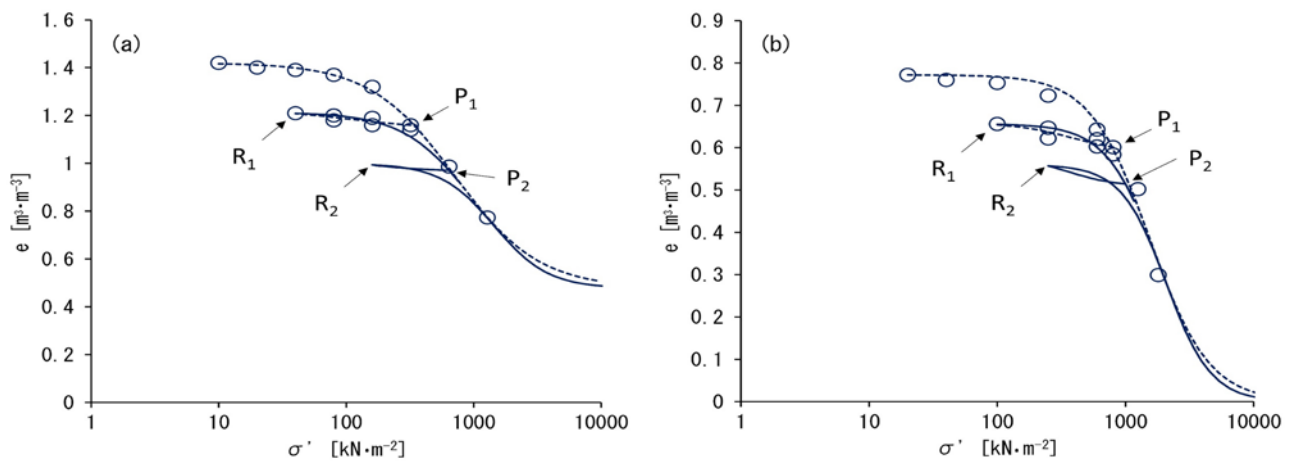


Figure 7. (a) The modelled compression and expansion curves (dotted line), which were obtained by fitting Equations (13) and (14) to the measured data set reported in [11] (circles), and the predicted compression-expansion-compression curve (solid line). The measured expansion curve was found between the two reversal points of P_1 (320 [kN·m⁻²], 1.16 [m³·m⁻³]) and R_1 (40 [kN·m⁻²], 1.21 [m³·m⁻³]). The approximate of the preconsolidation stress for the modelled compression curve was 275.8 [kN·m⁻²], derived from Equation (41), and was used for modelling the expansion process (P_1R_1). The predicted compression-expansion-compression curve was derived from the parameters for the modelled curves. The reversal points of P_2 (640 [kN·m⁻²], 0.97 [m³·m⁻³]) and R_2 (160 [kN·m⁻²], 0.99 [m³·m⁻³]) were arbitrarily selected for testing the model performance in the prediction of the processes of “recompression (1)” (R_1P_2), “expansion (1)” (P_2R_2), and “recompression (2)” (from R_2). (b) The modelled compression and expansion curves (dotted line) reproducing the measured data set reported in [4] (circles) with the predicted curves (solid line) of “recompression (1)” (R_1P_2), “expansion (1)” (P_2R_2), and “recompression (2)” (from R_2), both of which were drawn with the same procedures as in the sub-graph (a). The measured expansion process was found between the two reversal points of P_1 (800 [kN·m⁻²], 0.60 [m³·m⁻³]) and R_1 (100 [kN·m⁻²], 0.66 [m³·m⁻³]), and was modelled with the approximate of the preconsolidation stress for the original compression curve (280.3 [kN·m⁻²]), while the arbitrarily-selected reversal points for the predicted curves were P_2 (1000 [kN·m⁻²], 0.51 [m³·m⁻³]) and R_2 (250 [kN·m⁻²], 0.56 [m³·m⁻³]).

measured expansion process, found between the reversal points of P_1 (320 [kN·m⁻²], 1.16 [m³·m⁻³]) and R_1 (40 [kN·m⁻²], 1.21 [m³·m⁻³]), was also modelled by Equation (19) after the parameter $e_{pe\ max}$ [m³·m⁻³] of the potential expansion curve e_{pe} [m³·m⁻³] of Equation (14) was determined by the least squared procedure for the measured data set. The preconsolidation stress for the original compression curve σ'_{p0} [N·m⁻²] was approximately determined by Equation (41), and the resultant value of 275.8 [kN·m⁻²] was used for identifying the modelled expansion curve. These two curves followed so closely the measured data set as shown

in **Figure 7(a)** with the parameter sets displayed in the first and second columns on **Table 2**.

After the original compression and expansion curves were parameterized, the possible series of recompression-expansion-recompression process was predicted by using the parameters $e_{oc\ max}$ [$\text{m}^3 \cdot \text{m}^{-3}$], $e_{oc\ min}$ [$\text{m}^3 \cdot \text{m}^{-3}$], γ_{oc} [$\text{m}^2 \cdot \text{N}^{-1}$], μ_{oc} [-], and $e_{pe\ max}$ [$\text{m}^3 \cdot \text{m}^{-3}$] with the reversal point at the end of the original expansion process R_1 (40 [$\text{kN} \cdot \text{m}^{-2}$], 1.21 [$\text{m}^3 \cdot \text{m}^{-3}$]).

Table 2. The parameter sets obtained and used for drawing the modelled curves in **Figure 7(a)**.

	obtained by fitting to the measured data set		predicted		
	e_{oc}	e_{pe}	e_{rc1}	e_{pe1}	e_{rc2}
e_{max} [$\text{m}^3 \cdot \text{m}^{-3}$]	1.420	0.080	1.213	0.080	0.999
e_{min} [$\text{m}^3 \cdot \text{m}^{-3}$]	0.478	0.000	0.478	0.000	0.478
γ [$\text{m}^2 \cdot \text{N}^{-1}$]	0.001414	0.009295	0.001004	0.009730	0.000690
μ [-]	1.315	1.315	1.605	1.605	2.073

The new reversal points of P_2 (640 [$\text{kN} \cdot \text{m}^{-2}$], 0.97 [$\text{m}^3 \cdot \text{m}^{-3}$]) and R_2 (160 [$\text{kN} \cdot \text{m}^{-2}$], 0.99 [$\text{m}^3 \cdot \text{m}^{-3}$]) were arbitrarily selected on the trajectory of the predicted curves for testing the performance of the model. The predicted recompression-expansion-recompression process was divided into three sub-processes, namely “recompression (1)”, “expansion (1)”, and “recompression (2)” in order.

The “recompression (1)” curve e_{rc1} [$\text{m}^3 \cdot \text{m}^{-3}$] was obtained by the procedure through which Equation (21) is parameterized by using the original compression curve e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$] and R_1 .

The “expansion (1)” curve e_{e1} [$\text{m}^3 \cdot \text{m}^{-3}$] was made of the “potential expansion (1)” curve e_{pe1} [$\text{m}^3 \cdot \text{m}^{-3}$]. The curve e_{pe1} [$\text{m}^3 \cdot \text{m}^{-3}$] was derived from e_{rc1} [$\text{m}^3 \cdot \text{m}^{-3}$], P_1 , and P_2 with the same procedure as the former e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] was from e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$], the approximate preconsolidation stress of e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$], and P_1 . In this procedure, $e_{pe\ max}$ [$\text{m}^3 \cdot \text{m}^{-3}$] of the curve e_{pe} [$\text{m}^3 \cdot \text{m}^{-3}$] was assigned to the maximum void ratio of the curve e_{pe1} [$\text{m}^3 \cdot \text{m}^{-3}$].

The “recompression (2)” curve e_{rc2} [$\text{m}^3 \cdot \text{m}^{-3}$] was obtained by using the “recompression (1)” curve e_{rc1} [$\text{m}^3 \cdot \text{m}^{-3}$] and R_2 through the same procedure as e_{rc1} [$\text{m}^3 \cdot \text{m}^{-3}$] was obtained by using the original compression curve e_{oc} [$\text{m}^3 \cdot \text{m}^{-3}$] and R_1 .

The parameter sets of the curves e_{rc1} [$\text{m}^3 \cdot \text{m}^{-3}$], e_{pe1} [$\text{m}^3 \cdot \text{m}^{-3}$], and e_{rc2} [$\text{m}^3 \cdot \text{m}^{-3}$] were displayed in the third, fourth, and fifth columns on **Table 2**. The resultant curve of the “recompression (1)” agreed well with the corresponding measured plots, while the curve of the “recompression (2)” approached the original compression curve like commonly-observed e - $\log_{10} \sigma'$ relations.

Figure 7(b) shows the modelled compression and expansion curves fitted to the measured data set reported in [4] with the predicted curves for the processes of “recompression (1)”, “expansion (1)”, and “recompression (2)”, drawn with the

same procedures as the curves in **Figure 7(a)**.

The measured expansion process was found between the two reversal points of P_1 ($800 \text{ [kN}\cdot\text{m}^{-2}]$, $0.60 \text{ [m}^3\cdot\text{m}^{-3}]$) and R_1 ($100 \text{ [kN}\cdot\text{m}^{-2}]$, $0.66 \text{ [m}^3\cdot\text{m}^{-3}]$), and was modelled with the approximate of the preconsolidation stress for the original compression curve ($280.3 \text{ [kN}\cdot\text{m}^{-2}]$), while the arbitrarily-selected reversal points for the predicted curves were P_2 ($1000 \text{ [kN}\cdot\text{m}^{-2}]$, $0.51 \text{ [m}^3\cdot\text{m}^{-3}]$) and R_2 ($250 \text{ [kN}\cdot\text{m}^{-2}]$, $0.56 \text{ [m}^3\cdot\text{m}^{-3}]$). The resultant parameter sets were tabulated on **Table 3**. The predicted curves in **Figure 7(b)** also imitated possible series of recompression-expansion iterations as well as those in **Figure 7(a)**.

Table 3. The parameter sets obtained and used for drawing the modelled curves in **Figure 7(b)**.

	obtained by fitting to the measured data set		predicted		
	e_{oc}	e_{pe}	e_{rc1}	e_{pe1}	e_{rc2}
$e_{max} \text{ [m}^3\cdot\text{m}^{-3}]$	0.772	0.073	0.656	0.073	0.560
$e_{min} \text{ [m}^3\cdot\text{m}^{-3}]$	0.000	0.000	0.000	0.000	0.000
$\gamma \text{ [m}^2\cdot\text{N}^{-1}]$	0.000649	0.002530	0.000552	0.002831	0.000481
$\mu \text{ [-]}$	1.881	1.881	2.162	2.162	2.480

As explained in the section 5, because a recompression curve is a self-similarity of its former compression curve, the procedure through which a recompression curve is identified can be repeatedly used for predicting any hysteretic behaviour along the repetitions of compression and expansion processes by updating the parameter set of the model and the coordinates of the reversal points on the $e\text{-}\log_{10}\sigma'$ plane.

8. Conclusions

This study aimed at developing a mathematical model to describe a compression curve of a soil body and enabling the model to predict hysteretic behaviours in void ratio $e \text{ [m}^3\cdot\text{m}^{-3}]$ versus effective stress $\sigma' \text{ [N}\cdot\text{m}^{-2}]$ relation. As a result, the mathematical function of Equation (5) was derived from geometrical features of a compression curve commonly observed on the $e\text{-}\log_{10}\sigma'$ plane. The derived model performed well for reproducing various kinds of measured $e\text{-}\log_{10}\sigma'$ relations that had frequently been described by logarithmic linear relations between $e \text{ [m}^3\cdot\text{m}^{-3}]$ and $\sigma' \text{ [N}\cdot\text{m}^{-2}]$ like those adopted in the classical constitutive Cam-Clay model, implying that it can work effectively when incorporated into existing constitutive models to describe mechanical behaviours of various types of undrained compressible soils.

This study introduced the concept of potential expansion curve, and proposed the procedure through which an expansion curve is modelled when a reversal point from a compression process to the expansion process is arbitrarily given on a normal consolidation process. This study also proposed the procedure for pre-

dicting a recompression curve by using its former compression curve and a reversal point at which the expansion process turns to the recompression process.

Because any recompression process should be a self-similarity of its former compression process, the procedure that makes the model predict a recompression curve can be repeatedly applied to the prediction of any hysteretic behaviour found along repetitions of expansion-compression processes, once an original compression curve and a potential expansion curve are formulated. This feature of the model is an advantage for land subsidence prediction under complex stress history. Several numerical examples showed successful reproductions and predictions of hysteretic behaviour of measured e - $\log_{10}\sigma'$ relations.

This study regarded the parameter $e_{pe\ max}$ [$\text{m}^3\cdot\text{m}^{-3}$] in the potential expansion curve as a fitting parameter and a constant throughout compression-expansion history, since it is often difficult to predict a total amount of reversible compression of a soil body. Further studies are required for making it possible to predict the amount of reversible compression a soil body potentially has.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- [1] Okada, H., Oshima, A. and Shibuya, S. (2017) Prediction of Settlement of Dredged Clay Containing Prefabricated Vertical Drains and the History of Reclamation in the Disposal Pond at Kobe Airport. *Japanese Geotechnical Journal*, **12**, 19-31. <https://doi.org/10.3208/jgs.12.19>
- [2] Kasugai, M., Oshima, A. and Yamaguchi, T. (2022) Prediction of Ground Settlement and Liquefaction Countermeasure by Groundwater Lowering Using Shallow Subsurface Model in Osaka and Kobe Areas. *Japanese Geotechnical Journal*, **17**, 537-550. <https://doi.org/10.3208/jgs.17.537>
- [3] Gunduz, Z. and Arman, H. (2007) Possible Relationships between Compression and Recompression Indices of a Low-Plasticity Clayey Soil. *Arabian Journal for Science and Engineering*, **32**, 179-190.
- [4] Kifae, A.A., Hassan, A.R. and Shanbara, H.K. (2018) Empirical Relationships between Index Properties and Compression Indices of Clayey Soils in Al-Nasiriya City. *Journal of University of Babylon, Engineering Sciences*, **26**, 348-356.
- [5] Kim, M., Senturk, M.A. and Li, L. (2024) Compression Index Regression of Fine-Grained Soils with Machine Learning Algorithms. *Applied Sciences*, **14**, Article 8695. <https://doi.org/10.3390/app14198695>
- [6] Habibbeygi, F., Nikraz, H. and Verheyde, F. (2017) Determination of the Compression Index of Reconstituted Clays Using Intrinsic Concept and Normalized Void Ratio. *International Journal of Geomate*, **13**, 54-60. <https://doi.org/10.21660/2017.39.98271>
- [7] Kordnaeij, A., Kalantary, F., Kordtabar, B. and Mola-Abasi, H. (2015) Prediction of Recompression Index Using GMDH-Type Neural Network Based on Geotechnical Soil Properties. *Soils and Foundations*, **55**, 1335-1345. <https://doi.org/10.1016/j.sandf.2015.10.001>
- [8] Coop, M.R. (1990) The Mechanics of Uncemented Carbonate Sands. *Géotechnique*,

- 40, 607-626. <https://doi.org/10.1680/geot.1990.40.4.607>
- [9] Ural, N. and Küçüker, U. (2021) Determination of Pre-Consolidation Pressure by Different Method. *Journal of Civil Engineering and Construction*, **10**, 36-43. <https://doi.org/10.32732/jcec.2021.10.1.36>
- [10] Olek, B.S. and Moskal, M. (2024) Comparison between Methods for Determining the Effective Vertical Yield Stress of Intermediate Fine-Grained Soils. *Scientific Reports*, **14**, Article No. 131. <https://doi.org/10.1038/s41598-023-50026-2>
- [11] Gwak, D., Park, K., Jo, B. and Baek, S. (2026) Interpretation of the Preconsolidation Stress in Soft Clay Using the One-Dimensional Consolidation Test. *Journal of Marine Science and Engineering*, **14**, Article 740. <https://doi.org/10.3390/jmse14080740>
- [12] Okada, H. and Oshima, A. (2015) Reconsideration on Expression of Consolidation Characteristics of Clay by Logarithm Volume Ratio *f*. *Japanese Geotechnical Journal*, **10**, 187-199. <https://doi.org/10.3208/jgs.10.187>
- [13] Umar, M. and Sadrekarimi, A. (2017) Accuracy of Determining Pre-Consolidation Pressure from Laboratory Tests. *Canadian Geotechnical Journal*, **54**, 441-450. <https://doi.org/10.1139/cgj-2016-0203>
- [14] Oikawa, H. (1987) Compression Curve of Soft Soils. *Soils and Foundations*, **27**, 99-104. https://doi.org/10.3208/sandf1972.27.3_99