

Two Dimensional Schrödinger-Poisson Equation: Existence and Qualitative Properties

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Abstract

We consider the following two dimensional Schrödinger-Poisson equation

$$\begin{cases} -\Delta u + |x|^2 u + \varphi u = \lambda u + a|u|^{p-2} u, \\ \Delta \varphi = u^2, \end{cases} \quad (0.1)$$

where $a, \lambda \in \mathbb{R}_+$ and $p > 4$. We prove that problem (0.1) admits a local minimizer u_p . Moreover, we investigate the asymptotic behaviour of u_p as $p \rightarrow 4^+$, and establish the stability of the solution.

Keywords

Normalized Solution, Schrödinger-Poisson Equation, Mass-Supercritical, Qualitative Properties

1. Introduction

In this paper, we study the Schrödinger-Poisson system

$$\begin{cases} i\psi_t - \Delta \psi + |x|^2 \psi + \varphi \psi = a|\psi|^{p-2} \psi & \text{in } \mathbb{R}^2 \times \mathbb{R}, \\ \Delta \varphi = |\psi|^2 & \text{in } \mathbb{R}^2 \times \mathbb{R} \end{cases} \quad (1.1)$$

in the dimension two, where $\psi: \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{C}$ is the time-dependent wave function, $a \in \mathbb{R}_+$ and $p > 4$. We seek for the standing wave solution of (1.1) of the form

$$\psi(x, t) = e^{i\lambda t} u(x).$$

The function u then satisfies

$$\begin{cases} -\Delta u + |x|^2 u + \varphi u = \lambda u + a|u|^{p-2} u, \\ \Delta \varphi = u^2. \end{cases} \quad (1.2)$$

Solving the second equation in (1.2) via the Newtonian potential

$$\mathcal{N}(x) = \frac{1}{2\pi} \ln|x|,$$

problem (1.2) can be reduced to the single equation

$$-\Delta u + |x|^2 u + \left(\int_{\mathbb{R}^2} \ln|x-y| u^2(y) dy \right) u = \lambda u + a|u|^{p-2} u \quad \text{in } \mathbb{R}^2. \quad (1.3)$$

The study of the three-dimensional Schrödinger-Poisson system (1.2), also known as the Schrödinger-Newton system, has a long history. With appropriate positive coupling constants, it first emerged in 1954 in the context of the quantum mechanics of a polaron at rest [1]. Subsequently it was employed in [2] to model an electron trapped in its own hole in 1976, and later in 1996, it appeared in the framework of self-gravitating matter [3]. In particular, the existence of a unique ground state of (1.2) in the three-dimensional case was established in [2] by solving a suitable minimization problem; this ground state is positive, spherically symmetric, and strictly decreasing. Since then, the system has been extensively studied for $N \geq 3$. Variational methods have been employed to derive existence and multiplicity results for standing wave solutions [1]-[7], as well as for standing wave solutions with prescribed L^2 -norm [8]-[11].

Compared with the higher-dimensional counterparts, problem (1.3) exhibits several new features, which originate from the Newtonian potential $\mathcal{N}(x)$. Moreover, this potential is sign-changing, and has singularities both at the origin and at infinity, which introduce additional difficulties in the analysis. One such difficulty is that the energy functional

$$\begin{aligned} E_p(u) = & \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 |u|^2 dx \\ & + \frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u^2(x) u^2(y) dy dx - \frac{a}{p} \int_{\mathbb{R}^2} |u|^p dx \end{aligned} \quad (1.4)$$

is not well-defined on $H^1(\mathbb{R}^2)$ as noted in [12]. To overcome this difficulty, Stubbe [12] introduced a new Hilbert space

$$\mathcal{X} = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \ln(1+|x|) u^2(x) dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{\mathcal{X}} = \left(\int_{\mathbb{R}^2} \left[|\nabla u|^2 + u^2 + \ln(1+|x|) u^2 \right] dx \right)^{\frac{1}{2}}.$$

It can be verified that the functional $E_p(u)$ is C^1 on the space $\mathcal{X}$.

Using rearrangement theory, Stubbe [12] proved the existence and uniqueness of ground state solutions for (1.3). This work opened up a new approach to the study of two-dimensional Schrödinger-Poisson equations and has inspired many subsequent investigations, see [13]-[18] and the references therein. In particular, in [19] the authors employed a closed subgroup of the $O(2)$ to prove that the Cerami sequence of the associated invariant functional has a convergent subsequence. For both critical and subcritical problems, various existence results have been obtained for (1.3) under different potential functions.

The existence and concentration of normalized solutions of (1.3) were investi-

gated in [14] [16] [18] for mass-subcritical and mass-critical cases. In particular, for the mass-subcritical case $p \in (2, 4)$, Cingolani and Jeanjean [14] established existence and multiplicity results for normalized solutions of (1.3).

Physically, $p = 4$ is the critical point separating different regimes. In particular, in [20] the authors studied Bose-Einstein condensates with attractive interactions in two dimensions, described by the Gross-Pitaevskii energy functional. They considered the case where the condensate is confined to a trapping potential $V(x)$, including the harmonic potential $V(x) = |x|^2$. In the mass critical case $p = 4$, it was shown in [20] that there exists a critical particle number $a^* > 0$ for collapse of the Bose-Einstein condensate, even when $2 < p < 4$. They proved that the condensate survives by showing that the corresponding minimization problem is attained for $0 < a < a^*$. However, for the supercritical case, that is $p > 4$, the minimization problem admits no minimizer for any $a > 0$. Furthermore, in [21], the authors studied the evolution of the condensate as $p \rightarrow 4^-$, and proved that the minimizers blow up in this limit.

In the present paper, we study the existence, asymptotic behavior and stability of solutions to problem (1.3) under the supercritical condition $p > 4$, subject to the normalization constraint

$$\int_{\mathbb{R}^2} |u(x)|^2 dx = 1. \quad (1.5)$$

Such solutions are called normalized solutions. The variational framework on the space $\mathcal{X}$ provides a fundamental tool for the study of problem (1.3). However, the analysis still requires many new ideas and delicate estimates because the norm on $\mathcal{X}$ is not translation-invariant, and the quadratic part of E_p is not coercive.

Unlike the mass-critical exponent $p = 4$, the case $p > 4$ is a mass-supercritical. The key difference is that the corresponding energy functional E_p , when constrained on the sphere

$$S(1) = \left\{ u \in \mathcal{H} : \int_{\mathbb{R}^2} |u|^2 dx = 1 \right\},$$

is neither bounded above nor below, where

$$\mathcal{H} = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |x|^2 |u|^2 dx < \infty \right\}.$$

To overcome this difficulty, we consider the local minimization problem

$$\mathcal{M}_{p,k} = \inf_{u \in A_k} E_p(u), \quad (1.6)$$

where $k > 0$ to be determined and

$$A_k = \left\{ u \in S(1) : \int_{\mathbb{R}^2} |\nabla u|^2 dx \leq k \right\}.$$

We note that the space $\mathcal{H}$ is a closed subspace of $\mathcal{X}$, since the inequality

$$\int_{\mathbb{R}^2} \ln(1+|x|) u^2(x) dx \leq C \int_{\mathbb{R}^2} |x|^2 u^2(x) dx$$

holds. Therefore, we can define a norm

$$\|u\|_{\mathcal{H}} = \left(\int_{\mathbb{R}^2} (|\nabla u|^2 + |x|^2 u) dx \right)^{\frac{1}{2}}$$

on $\mathcal{H}$, and then the functional E_p is well-defined on $\mathcal{H}$.

To state our results, we recall that the positive ground state solution $\varphi_p(x)$ of the scalar field equation

$$-\Delta u + \frac{2}{p-2}u = \frac{2}{p-2}|u|^{p-2}u \text{ in } \mathbb{R}^2 \quad (1.7)$$

is unique and radially symmetric [22] [23]. It satisfies the Pohozaev identity

$$\int_{\mathbb{R}^2} |\nabla \varphi_p|^2 dx = \int_{\mathbb{R}^2} |\varphi_p|^2 dx = \frac{2}{p} \int_{\mathbb{R}^2} |\varphi_p|^p dx, \quad (1.8)$$

see for instance [22] [24].

Let

$$a_p^* = \|\varphi_p\|_2^{p-2}. \quad (1.9)$$

Theorem 1.1 Suppose that $0 < a < a^*$. There exists a $\delta > 0$ such that, for any $p \in (4, 4 + \delta)$, the minimization problem $\mathcal{M}_{p, \tau_p^2}$ has a minimizer

$u_p \in S(1) \cap A_{\tau_p^2}$, where

$$\tau_p^2 = \left(\frac{2a_p^*}{a(p-2)} \right)^{\frac{2}{p-4}} \quad (1.10)$$

and $A_{\tau_p^2}$ denotes the interior of $A_{\tau_p^2}$.

The threshold constant a^* in Theorem 1.1 is given by

$$a^* = \|w\|_2^2, \quad (1.11)$$

where w is the ground positive solution of the equation

$$-\Delta u + u = |u|^2 u \text{ in } \mathbb{R}^2 \quad (1.12)$$

It is well-known (see [21]-[23]) that w is unique and radially symmetric. Moreover, $|w|$ and $|\nabla w|$ decay exponentially, in the sense that

$$|w|, |\nabla w| = O\left(|x|^{-\frac{1}{2}} e^{-|x|}\right) \text{ as } |x| \rightarrow \infty. \quad (1.13)$$

By the Pohozaev identity, we have

$$\int_{\mathbb{R}^2} |\nabla w|^2 dx = \int_{\mathbb{R}^2} |w|^2 dx = \frac{1}{2} \int_{\mathbb{R}^2} |w|^4 dx. \quad (1.14)$$

Next, in contrast of [24], we investigate the asymptotic behavior of the minimizer u_p of $\mathcal{M}_{p_k, \tau_{p^2}}$ as $p \rightarrow 4^+$. The asymptotic behavior for $p \rightarrow 4^+$

reveals a universal attraction toward the critical state. Indeed, even though the system starts from the supercritical side, its profile near the critical point is captured by the solution at $p = 4$.

Theorem 1.2 For any sequence $\{p_k\}$ with $p_k \rightarrow 4^+$ as $k \rightarrow \infty$, there exists

a corresponding subsequence of minimizers $\{u_{p_k}\}$ of $\mathcal{M}_{p_k, \tau_{p_k}^2}$ (still denoted as $\{u_{p_k}\}$) such that $u_{p_k} \rightarrow u_0$ strongly in $\mathcal{H}$, and u_0 is a global minimizer of the problem $\mathcal{M}_0 := \inf_{u \in S(1)} E_4(u)$, that is, $E_4(u_0) = \mathcal{M}_0$.

Finally, we study the stability of the minimizer u_p .

We first establish the well-posedness of the time-dependent equation

$$\begin{aligned} i\psi_t(x, t) - \Delta \psi(x, t) + |x|^2 \psi(x, t) + \left(\ln |\cdot| * \psi(\cdot, t) \right) \psi(x, t) \\ = a |\psi(x, t)|^{p-2} \psi(x, t) \quad \text{in } \mathbb{R}^2 \times \mathbb{R} \end{aligned} \quad (1.15)$$

with the initial data $\psi_0 \in \mathcal{H}$, that is,

$$\psi(x, 0) = \psi_0(x), \quad x \in \mathbb{R}^2. \quad (1.16)$$

Theorem 1.3 Suppose $\psi_0 \in \mathcal{H}$ and $p \in (4, 4 + \delta)$, $0 < \delta < 1$. Then the Cauchy problem (1.15)-(1.16) admits a global-in-time solution $\psi(x, t)$. Moreover, $\psi(x, t)$ satisfies the conservation laws of energy and mass, namely,

$$E_p(\psi(\cdot, t)) = E_p(\psi_0), \quad M(\psi(\cdot, t)) = M(\psi_0), \quad (1.17)$$

where the corresponding energy and mass are

$$\begin{aligned} E_p(\psi) &= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 |\psi|^2 dx \\ &\quad + \frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x - y| \psi^2(x) \psi^2(y) dy dx - \frac{a}{p} \int_{\mathbb{R}^2} |\psi|^p dx \end{aligned} \quad (1.18)$$

and

$$M(\psi) = \int_{\mathbb{R}^n} |\psi|^2 dx, \quad (1.19)$$

respectively.

Based on Theorem 1.3, we have the following stability result. Let $\mathfrak{M}_p$ denote the set of local minimizers $u_p \in A_{\tau_p^2}$ of the minimization problem $\mathcal{M}_{p, \tau_p^2}$.

Theorem 1.4 Suppose that $0 < a < a^*$ and $p \in (4, 4 + \delta)$ with $\delta \in (0, 1)$. For any $u \in \mathfrak{M}_p$, the normalized standing-wave solutions $\psi(x, t) = e^{-i\mu t} u(x)$ of (1.3) is stable. More precisely, for every $\varepsilon > 0$, there exists $\eta > 0$ such that for any $\varphi \in \mathcal{H}$ satisfying

$$\inf_{u \in \mathfrak{M}} \|\varphi - u\|_{\mathcal{H}} < \eta,$$

we have

$$\sup_{t \geq 0} \inf_{u \in \mathcal{G}} \|\Psi(\cdot, t) - u\|_{\mathcal{H}} < \varepsilon,$$

where Ψ is the unique solution of (1.15) with initial data φ .

The rest of this paper is organized as follows. In Section 2, we collect some preliminary facts that will be used throughout. In Section 3, we prove that $\mathcal{M}_{p, \tau_p^2}$

is attained. The asymptotic behaviour of the minimizer and the stability result are established in Sections 4 and 5, respectively.

2. Preliminaries

In this section, we collect some auxiliary facts that will be used in the sequel.

Let φ_p and w be ground state solutions of (1.12) and (1.7) respectively. By Lemma 2.1 in [25], we have

$$\varphi_p \rightarrow w \text{ in } H^1(\mathbb{R}^2) \quad (2.1)$$

and

$$a_p^* \rightarrow a^* \text{ as } p \rightarrow 4^+. \quad (2.2)$$

It is known from [26] that a_p^* gives the best constant in the Gagliardo-Nirenberg inequality:

$$\int_{\mathbb{R}^2} |u|^p dx \leq \frac{p}{2a_p^*} \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{\frac{p-2}{2}} \int_{\mathbb{R}^2} |u|^2 dx \text{ for } u \in H^1(\mathbb{R}^2). \quad (2.3)$$

We now define on $\mathcal{H}$ the following symmetric bilinear forms

$$B_0(u, u) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u^2(x) u^2(y) dx dy, \quad (2.4)$$

$$B_1(u, u) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(1+|x-y|^2) u^2(x) u^2(y) dx dy \quad (2.5)$$

and

$$B_2(u, u) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \left(1 + \frac{1}{|x-y|^2} \right) u^2(x) u^2(y) dx dy. \quad (2.6)$$

Then we have the relation

$$B_0(u, u) = \frac{1}{2} [B_1(u, u) - B_2(u, u)]. \quad (2.7)$$

Using the elementary inequality $0 < \ln(1+r) < r$ for $r > 0$, we obtain

$$\begin{aligned} \frac{1}{2} |B_2(u, u)| &\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \left(1 + \frac{1}{|x-y|^2} \right) u^2(x) u^2(y) dx dy \\ &\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} u^2(x) u^2(y) dx dy. \end{aligned} \quad (2.8)$$

Consequently, by the Gagliardo-Nirenberg inequality (2.3) and the Hardy-Littlewood-Sobolev inequality in dimension two [27]:

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} |u(x)| |v(y)| dx dy \leq C \|u\|_{\frac{4}{3}} \|v\|_{\frac{4}{3}}, \text{ where } u, v \in L^{\frac{4}{3}}(\mathbb{R}^2), \quad (2.9)$$

we have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} u^2(x) u^2(y) dx dy \leq C \left(\int_{\mathbb{R}^2} u^{\frac{8}{3}} dx \right)^{\frac{3}{2}} \leq C \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{\frac{1}{2}}. \quad (2.10)$$

Therefore,

$$\frac{1}{2} |B_2(u, u)| \leq C \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{\frac{1}{2}}. \quad (2.11)$$

Similarly,

$$\begin{aligned} \frac{1}{2}|B_1(u, u)| &\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[\ln(1+|x|^2) + \ln(1+|y|^2) \right] u^2(x) u^2(y) dx dy \\ &\leq 2 \int_{\mathbb{R}^2} \ln(1+|x|^2) u^2(x) dx \int_{\mathbb{R}^2} u^2(y) dy. \\ &\leq 2 \int_{\mathbb{R}^2} |x|^2 u^2(x) dx \int_{\mathbb{R}^2} u^2(y) dy. \end{aligned} \quad (2.12)$$

The following results can be verified as in Lemma 2.2 of [19].

Lemma 2.1 *The following statements hold:*

1. The embedding $\mathcal{H} \hookrightarrow L^q(\mathbb{R}^2)$ is compact, where $q \in [2, +\infty)$.
2. The functionals $B_0(u, u)$, $B_1(u, u)$ and $B_2(u, u)$ are of class C^1 on $\mathcal{H}$.
3. $B_0(u, u)$ is weakly lower semi-continuous on $\mathcal{H}$.

We also have the following compact result.

Lemma 2.2 *Suppose $V \in L_{loc}^\infty(\mathbb{R}^2)$ and $\lim_{|x| \rightarrow \infty} V(x) = \infty$. Then for any $q \in [2, \infty)$, the embedding $\mathcal{H} \hookrightarrow L^q(\mathbb{R}^2)$ is compact.*

3. Existence of Minimizers

In this section, we prove Theorem 1.1. The proof consists of two parts. First, we show $\mathcal{M}_{p,k}$ is attained by some $u \in A_k$, then we prove $u \in A_k$ for some $k > 0$.

Proof of Theorem 1.1 Fix $k > 0$. Let $\{u_n\}$ be a minimizing sequence of $\mathcal{M}_{p,k}$. Since A_k is a bounded closed set in $\mathcal{H}$, $\{u_n\}$ is bounded. Hence, by Lemma 2.2 we may assume that

$$u_n \rightharpoonup u \text{ in } \mathcal{H}; \quad u_n \rightarrow u \text{ in } L^q(\mathbb{R}^2), \quad q \in [2, \infty); \quad u_n \rightarrow u \text{ a.e. } \mathbb{R}^2. \quad (3.1)$$

This yields

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} |\nabla u_n|^2 dx, \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} |u_n|^p dx = \int_{\mathbb{R}^2} |u|^p dx. \quad (3.2)$$

Since $u_n \rightarrow u$ in $L^2(\mathbb{R}^2)$, there exists $v \in L^2(\mathbb{R}^2)$ such that

$$|u_n(x)|, |u(x)| \leq v(x), \quad x \in \mathbb{R}^2$$

for any n [22]. Therefore,

$$\ln(1+|x-y|^2) u_n^2(x) u_n^2(y) \leq \ln(1+|x-y|^2) v^2(x) v^2(y)$$

and

$$\ln\left(1 + \frac{1}{|x-y|^2}\right) u_n^2(x) u_n^2(y) \leq \ln\left(1 + \frac{1}{|x-y|^2}\right) v^2(x) v^2(y).$$

By the Lebesgue dominated convergence theorem, we deduce that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(1+|x-y|^2) u_n^2(x) u_n^2(y) dx dy \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln(1+|x-y|^2) u^2(x) u^2(y) dx dy. \end{aligned}$$

and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \left(1 + \frac{1}{|x-y|^2} \right) u_n^2(x) u_n^2(y) dx dy \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln \left(1 + \frac{1}{|x-y|^2} \right) u^2(x) u^2(y) dx dy. \end{aligned}$$

Consequently,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x-y| u_n^2(x) u_n^2(y) dx dy = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x-y| u^2(x) u^2(y) dx dy. \quad (3.3)$$

Combining (3.2) and (3.3), we obtain

$$\mathcal{M}_{p,k} \leq E_{p,k}(u) \leq \lim_{n \rightarrow \infty} E_{p,k}(u_n) = \mathcal{M}_{p,k}.$$

Thus u is a minimizer of $\mathcal{M}_{p,k}$.

In order to prove that u is a weak solution of (1.3), it is necessary to show $u \in A_k$. To this end, we shall choose $k > 0$ appropriately.

For any $u \in S(1)$, from (2.3) and (2.11) we deduce

$$\begin{aligned} E_p(u) &\geq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{8} [B_1(u, u) - B_2(u, u)] \\ &\quad - \frac{a}{2a_p^*} \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{\frac{p-2}{2}} \int_{\mathbb{R}^2} |u|^2 dx \\ &\geq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{a}{2a_p^*} \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{\frac{p-2}{2}} - C \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Consider the function

$$h(t) = \frac{1}{2} t^2 - \frac{a}{2a_p^*} t^{p-2}.$$

It attains its maximum at

$$\tau_p := \left(\frac{2a_p^*}{a(p-2)} \right)^{\frac{1}{p-4}}. \quad (3.4)$$

Let

$$I_p(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |x-y| u^2(x) u^2(y) dx dy - \frac{a}{p} \int_{\mathbb{R}^2} |u|^p dx. \quad (3.5)$$

Then

$$I_p(u) \geq \frac{1}{2} \tau_p^2 - \frac{a}{2a_p^*} \tau_p^{p-2} - C \tau_p = \frac{p-4}{2(p-2)} \tau_p^2 - C \tau_p := g(\tau_p).$$

Choose $\theta = \frac{p-4}{4(p-2)}$, then

$$g(\tau_p) \geq \theta \tau_p^2 \rightarrow +\infty \text{ as } p \rightarrow 4^+.$$

Since $E_p(u) \geq I_p(u)$,

$$\inf_{u \in \partial A_{\tau_p}^2} E_p(u) \geq \inf_{u \in \partial A_{\tau_p}^2} I_p(u) \geq \frac{p-4}{2(p-2)} \tau_p^2 - C \tau_p. \quad (3.6)$$

We now show that $u \in A_{t_p^2}$.

Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ be such that $\varphi \geq 0$ and $\|\varphi\|_2^2 = 1$. Choose $t_p > 0$ such that

$$\varphi_{t_p}(x) = t_p \varphi(t_p x)$$

satisfying

$$\int_{\mathbb{R}^2} |\nabla \varphi_{t_p}|^2 dx = t_p^2 \int_{\mathbb{R}^2} |\nabla \varphi|^2 dx = \frac{1}{4} \theta \tau_p^2,$$

which implies $\varphi_{t_p} \in A_{t_p^2}$. Thus,

$$t_p = \frac{1}{2} \left(\theta \int_{\mathbb{R}^2} |\nabla \varphi|^2 dx \right)^{-\frac{1}{2}} \tau_p.$$

This implies that $t_p \rightarrow \infty$ as $p \rightarrow 4^+$, since $\tau_p \rightarrow +\infty$.

Set $M := \sup_{x \in \text{supp } \varphi} |x|^2$. By Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}^2} |x|^2 |\varphi_{t_p}|^2 dx = \int_{\mathbb{R}^2} \left| \frac{x}{t_p} \right|^2 |\varphi|^2 dx \leq \left| \frac{M}{t_p} \right|^2 \int_{\mathbb{R}^2} |\varphi|^2 dx \rightarrow 0 \quad (3.7)$$

as $p \rightarrow 4^+$. Moreover, since

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u^2(x) u^2(y) dx dy = \frac{1}{2} (B_1(u, u) - B_2(u, u)),$$

by (2.12) and (3.7) we obtain

$$\left| B_1(\varphi_{t_p}, \varphi_{t_p}) \right| \leq \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 |\varphi_{t_p}|^2 dx \rightarrow 0 \text{ as } p \rightarrow 4^+. \quad (3.8)$$

In view of (3.7),

$$\left| B_2(\varphi_{t_p}, \varphi_{t_p}) \right| \leq C \left(\int_{\mathbb{R}^2} |\nabla \varphi_{t_p}|^2 dx \right)^{\frac{1}{2}} \leq \frac{1}{2} C \theta^{1/2} \tau_p.$$

Consequently,

$$\begin{aligned} E_p(\varphi_{t_p}) &\leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \varphi_{t_p}|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 |\varphi_{t_p}|^2 dx \\ &\quad + \frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| \varphi_{t_p}^2(x) \varphi_{t_p}^2(y) dx dy \\ &\leq \frac{1}{4} \theta \tau_p^2 + C \left(\frac{1}{4} \theta \tau_p^2 \right)^{1/2} + o(1) \\ &\leq \frac{1}{2} \theta \tau_p^2 < g(\tau_p) \end{aligned} \quad (3.9)$$

for $p > 4$ sufficiently close to 4. Hence we obtain

$$\mathcal{M}_{p, \tau_p^2} \leq E_p(\varphi_{t_p}) < \inf_{u \in \partial A_{\tau_p^2}} E_p(u). \quad (3.10)$$

The proof is complete. $\square$

4. Asymptotic Behavior

In this section, we investigate the asymptotic behavior of the minimizers u_p of

$$\mathcal{M}_{p, \tau_p^2} \text{ as } p \rightarrow 4^+.$$

By Proposition 2.3 in [19], critical points $u \in \mathcal{H}$ of E_p on the L^2 -sphere are strong solutions of (1.3) in $W^{2,q}(\mathbb{R}^2)$ for all $q \geq 1$, and, in the present setting, they are classical solutions in $C^2(\mathbb{R}^2)$. Moreover, it is shown in the same proposition that for every $\alpha > 0$,

$$|u(x)|e^{\alpha|x|} \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

Consequently, u satisfies the Pohozaev identity:

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx = \int_{\mathbb{R}^2} |x|^2 |u|^2 dx + \frac{1}{4} \left(\int_{\mathbb{R}^2} |u|^2 dx \right)^2 + \frac{(p-2)a}{p} \int_{\mathbb{R}^2} |u|^p dx \quad (4.1)$$

Lemma 4.1 Any sequence of minimizers $\{u_p\}$ of $\mathcal{M}_{p,\tau_p^2}$ is uniformly bounded in $\mathcal{H}$ for $p > 4$ sufficiently close to 4.

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ be such that

$$0 \leq \varphi \leq 1 \text{ and } \|\varphi\|_2 = 1.$$

Since τ_p^2 given in (1.10) tends to infinity as $p \rightarrow 4^+$, we have $\varphi \in A_{\tau_p^2}$ for $p > 4$ sufficiently close to 4. Therefore,

$$E_p(u_p) = \inf_{u \in A_{\tau_p^2}} E_p(u) \leq E_p(\varphi). \quad (4.2)$$

By the Gagliardo-Nirenberg inequality (2.3),

$$\int_{\mathbb{R}^2} |\varphi|^p dx \leq \frac{p}{2a_p^*} \left(\int_{\mathbb{R}^2} |\nabla \varphi|^2 dx \right)^{\frac{p-2}{2}} \int_{\mathbb{R}^2} |\varphi|^2 dx, \quad (4.3)$$

which implies, since $a_p^* \rightarrow a^*$ as $p \rightarrow 4^+$, that

$$\int_{\mathbb{R}^2} |\varphi|^p dx \leq C \quad (4.4)$$

for some constant $C > 0$ independent of p .

Moreover, the estimate

$$\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| \varphi^2(x) \varphi^2(y) dx dy \right| \leq C \quad (4.5)$$

combined with (4.4) yields

$$E_p(\varphi) \leq C \quad (4.6)$$

with $C > 0$ independent of p .

We now claim that

$$\int_{\mathbb{R}^2} |\nabla u_p|^2 dx \leq C, \quad (4.7)$$

with $C > 0$ independent of p . Indeed, by the Pohozaev identity (4.1) we deduce, for $p > 4$ close to 4, that

$$\begin{aligned} 2 \int_{\mathbb{R}^2} |\nabla u_p|^2 dx &= 2E_p(u_p) - \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u_p^2(x) u_p^2(y) dx dy \\ &\quad + \frac{1}{4} + \frac{a(4-p)}{p} \int_{\mathbb{R}^2} |u_p|^p dx \\ &\leq 2E_p(u_p) - \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u_p^2(x) u_p^2(y) dx dy + \frac{1}{4}. \end{aligned} \quad (4.8)$$

Note that $B_1(u, u) \geq 0$. From (2.7) and (2.11), we have

$$\begin{aligned} & -\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u_p^2(x) u_p^2(y) dx dy \\ & \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln\left(1 + \frac{1}{|x-y|^2}\right) u_p^2(x) u_p^2(y) dx dy \leq C \left(\int_{\mathbb{R}^2} |\nabla u_p|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

From (4.2) and (4.6), we have

$$E_p(u_p) \leq C.$$

Therefore,

$$\int_{\mathbb{R}^2} |\nabla u_p|^2 dx \leq C + C \left(\int_{\mathbb{R}^2} |\nabla u_p|^2 dx \right)^{\frac{1}{2}}, \quad (4.9)$$

which proves the claim (4.7). Combining (4.7) with the Pohozaev identity (4.1), we obtain

$$\int_{\mathbb{R}^2} |x|^2 |u_p|^2 dx \leq \int_{\mathbb{R}^2} |\nabla u_p|^2 dx \leq C.$$

Hence,

$$\|u_p\|_{\mathcal{H}} \leq C.$$

The proof of Lemma 4.1 is complete. $\square$

Proof of Theorem 1.4. Let $\{u_p\}$ be a sequence of minimizers of $\mathcal{M}_{p, \tau_p^2}$. By Lemma 4.1, $\|u_p\|_{\mathcal{H}} \leq C$. In view of Lemma 2.2, there exists a sequence p_k , $p_k \rightarrow 4^+$ as $k \rightarrow \infty$ such that the corresponding sequence of minimizers $\{u_{p_k}\}$ of $\mathcal{M}_{p_k, \tau_{p_k}^2}$ satisfies

$$u_{p_k} \rightharpoonup u_0 \text{ in } \mathcal{H} \text{ and } u_{p_k} \rightarrow u_0 \text{ in } L^q(\mathbb{R}^2)$$

for $q \in [2, +\infty)$ as $k \rightarrow \infty$. The Lagrange multiplier λ_k satisfies

$$\begin{aligned} \lambda_k &= \int_{\mathbb{R}^2} \left(|\nabla u_{p_k}|^2 + |x|^2 |u_{p_k}|^2 \right) dx \\ &+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u_{p_k}^2(x) u_{p_k}^2(y) dx dy - \int_{\mathbb{R}^2} |u_{p_k}|^{p_k} dx, \end{aligned}$$

which implies that $\{\lambda_k\}$ is bounded. Thus, up to a subsequence, we may assume $\lambda_k \rightarrow \lambda_0$.

Furthermore, for any $\varphi \in C_0^\infty(\mathbb{R}^2)$, we write

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} \left(|u_{p_k}|^{p_k-2} u_{p_k} - |u_0|^2 u_0 \right) \varphi dx \right| \\ & \leq \left| \int_{\mathbb{R}^2} \left(|u_{p_k}|^{p_k-2} u_{p_k} - |u_{p_k}|^2 u_{p_k} \right) \varphi dx \right| + \left| \int_{\mathbb{R}^2} \left(|u_{p_k}|^2 u_{p_k} - |u_0|^2 u_0 \right) \varphi dx \right|. \end{aligned}$$

In view of the weak convergence,

$$\left| \int_{\mathbb{R}^2} \left(|u_{p_k}|^2 u_{p_k} - |u_0|^2 u_0 \right) \varphi dx \right| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

The the mean value theorem and the L^p interpolation inequality imply

$$\left| \int_{\mathbb{R}^2} \left(|u_{p_k}|^{p_k-2} u_{p_k} - |u_{p_k}|^2 u_{p_k} \right) \varphi dx \right| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence, applying the Lebesgue dominated convergence theorem and passing to the limit $k \rightarrow \infty$ in the weak formulation

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(\nabla u_{p_k} \nabla \varphi + |x|^2 u_{p_k} \varphi \right) dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u_{p_k}^2(y) dy u_{p_k}(x) \varphi(x) dx \\ &= \int_{\mathbb{R}^2} |u_{p_k}|^{p_k-2} u_{p_k} \varphi dx + \lambda_k \int_{\mathbb{R}^2} u_{p_k} \varphi dx, \end{aligned} \quad (4.10)$$

we obtain that u_0 satisfies

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(\nabla u_0 \nabla \varphi + |x|^2 u_0 \varphi \right) dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u_0^2(y) dy u_0(x) \varphi(x) dx \\ &= \int_{\mathbb{R}^2} |u_0|^2 u_0 \varphi dx + \lambda_0 \int_{\mathbb{R}^2} u_0 \varphi dx. \end{aligned} \quad (4.11)$$

Similarly, the Lebesgue dominated convergence theorem, we have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u_{p_k}^2(x) u_{p_k}^2(y) dx dy \rightarrow \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u_0^2(x) u_0^2(y) dx dy$$

as $k \rightarrow \infty$. Moreover,

$$\int_{\mathbb{R}^2} |u_{p_k}|^{p_k} dx = \int_{\mathbb{R}^2} |u_{p_k}|^{p_k} dx - \int_{\mathbb{R}^2} |u_{p_k}|^4 dx + \int_{\mathbb{R}^2} |u_{p_k}|^4 dx.$$

By Brézis-Nirenberg lemma and the convergence of u_{p_k} in $L^q(\mathbb{R}^2)$ for any $q \geq 2$, we obtain

$$\int_{\mathbb{R}^2} |u_{p_k}|^4 dx = \int_{\mathbb{R}^2} |u_{p_k} - u_0|^4 dx + \int_{\mathbb{R}^2} |u_0|^4 dx + o(1) = \int_{\mathbb{R}^2} |u_0|^4 dx + o(1)$$

as $k \rightarrow \infty$. Using the mean value theorem, we have

$$|u_{p_k}|^{p_k} - |u_{p_k}|^4 = (p_k - 4) |u_{p_k}|^{4+\theta_k(p_k-4)} \ln|u_{p_k}|, \quad \theta_k \in (0,1).$$

We estimate

$$\begin{aligned} |u_{p_k}|^{4+\theta_k(p_k-4)} |\ln|u_{p_k}|| &\leq C |u_{p_k}|^{4+\theta_k(p_k-4)} \left(|u_{p_k}|^{-1} + |u_{p_k}| \right) \\ &\leq C \left(|u_{p_k}|^{3+\theta_k(p_k-4)} + |u_{p_k}|^{5+\theta_k(p_k-4)} \right). \end{aligned}$$

By the L^p interpolation inequality, we deduce that

$$\|u_{p_k}\|_{3+\theta_k(p_k-4)} \leq \|u_{p_k}\|_2^{\alpha_1} \|u_{p_k}\|_4^{1-\alpha_1} \leq C, \quad \|u_{p_k}\|_{5+\theta_k(p_k-4)} \leq \|u_{p_k}\|_2^{\alpha_2} \|u_{p_k}\|_6^{1-\alpha_2} \leq C,$$

where

$$\frac{1}{3+\theta_k(p_k-4)} = \frac{\alpha_1}{2} + \frac{1-\alpha_1}{4}, \quad \frac{1}{5+\theta_k(p_k-4)} = \frac{\alpha_2}{2} + \frac{1-\alpha_2}{6}.$$

Therefore,

$$\left| \int_{\mathbb{R}^2} \left(|u_{p_k}|^{p_k} - |u_{p_k}|^4 \right) dx \right| \leq (p_k - 4) C \rightarrow 0$$

as $k \rightarrow \infty$.

Thus, from the equality

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(|\nabla u_{p_k}|^2 + |x|^2 |u_{p_k}|^2 \right) dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u_{p_k}^2(x) u_{p_k}^2(y) dx dy \\ &= \int_{\mathbb{R}^2} |u_{p_k}|^{p_k} dx + \lambda_k \int_{\mathbb{R}^2} |u_{p_k}|^2 dx, \end{aligned}$$

we deduce that

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(|\nabla (u_{p_k} - u_0)|^2 + |x|^2 |u_{p_k} - u_0|^2 \right) dx + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| u_0^2(x) u_0^2(y) dx dy \\ &= \lambda_0 - \int_{\mathbb{R}^2} \left(|\nabla u_0|^2 + |x|^2 |u_0|^2 \right) dx + \int_{\mathbb{R}^2} |u_0|^4 dx + o(1) \end{aligned}$$

as $k \rightarrow \infty$. This together with (4.11) yields

$$\int_{\mathbb{R}^2} \left(|\nabla (u_{p_k} - u_0)|^2 + |x|^2 |u_{p_k} - u_0|^2 \right) dx = o(1).$$

That is, $u_{p_k} \rightarrow u_0$ in $\mathcal{H}$ as $k \rightarrow \infty$. In the same way, we can deduce that

$$E_{p_k}(u_{p_k}) \rightarrow E_4(u_0).$$

Thus, for any $\varepsilon > 0$ there exists $k_0 > 0$ such that $E_{p_k}(u_{p_k}) > E_4(u_0) - \varepsilon$ whenever $k \geq k_0$.

We now claim that $E_4(u_0) = \mathcal{M}_0 := \inf_{u \in S(1)} E_4(u)$. Indeed, if it is not the case, we would have $c_0 := E_4(u_0) - \mathcal{M}_0 > 0$. Moreover, by the definition of $\mathcal{M}_0$, there exists a $w \in S(1)$ such that $\mathcal{M}_0 > E_4(w) - \varepsilon$. Therefore,

$$E_{p_k}(u_{p_k}) > E_4(u_0) - \varepsilon = c_0 + \mathcal{M}_0 - \varepsilon > c_0 + E_4(w) - 2\varepsilon.$$

Note that for k sufficiently large, $w \in A_{\tau_{p_k}^2}$ since $\tau_{p_k}^2 \rightarrow \infty$ if $k \rightarrow \infty$. Hence, choosing $\varepsilon < \frac{1}{2}c_0$, we obtain

$$E_{p_k}(u_{p_k}) > c_0 + \mathcal{M}_0 - \varepsilon \geq c_0 + E_4(w) - 2\varepsilon > \mathcal{M}_{p_k, \tau_{p_k}^2} + \frac{1}{2}c_0 > \mathcal{M}_{p_k, \tau_{p_k}^2},$$

which is a contradiction to $E_{p_k}(u_{p_k}) = \mathcal{M}_{p_k, \tau_{p_k}^2}$. The assertion follows. $\square$

5. Stability

In this section, we prove the stability for any minimizer $u \in \mathfrak{M}_p$. More precisely, we shall prove Theorem 1.4.

Let $\psi_0 \in \mathcal{H}$ be the initial data of (1.15). It was established in [28] [29] [30] the local well-posedness of the Cauchy problem of (1.15) with initial data (1.16).

Lemma 5.1 *Let $\psi_0 \in \mathcal{H}$. Then there exists a unique solution $\psi \in C([0, T]; \mathcal{H})$ of the Cauchy problem (1.15)-(1.16) for some maximal existence time $T \in (0, \infty]$, and $\psi(\cdot, t)$ satisfies the energy and mass conservation laws (1.17).*

Proof of Theorem 1.3. By [26] [30], in order to prove global existence, it suffices to obtain a time-independent bound on $\|\psi(\cdot, t)\|_{\mathcal{H}}$. We proceed as follows.

$$\begin{aligned}
\|\psi(\cdot, t)\|_{\mathcal{H}}^2 &= 2E_p(\psi(\cdot, t)) - \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| \psi(x, t)^2 \psi(y, t)^2 dx dy \\
&\quad + \frac{2a}{p} \int_{\mathbb{R}^2} |\psi(x, t)|^p dx \\
&\leq 2E_p(\psi(\cdot, t)) + B_2(\psi(\cdot, t), \psi(\cdot, t)) + \frac{2a}{p} \int_{\mathbb{R}^2} |\psi(x, t)|^p dx.
\end{aligned} \tag{5.1}$$

By the Gagliardo-Nirenberg inequality (2.3),

$$\int_{\mathbb{R}^2} |\psi(\cdot, t)|^p dx \leq \frac{p}{2a_p^*} \left(\int_{\mathbb{R}^2} |\nabla \psi(\cdot, t)|^2 dx \right)^{\frac{p-2}{2}} \int_{\mathbb{R}^2} |\psi(\cdot, t)|^2 dx. \tag{5.2}$$

Similarly, from (2.11) we have

$$-\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| \psi^2(x, t) \psi^2(y, t) dx dy \leq B_2(\psi(\cdot, t), \psi(\cdot, t)).$$

By (2.8) and (2.10),

$$B_2(\psi(\cdot, t), \psi(\cdot, t)) \leq C_0 \|\psi(\cdot, t)\|_{\frac{8}{3}}^2, \tag{5.3}$$

where

$$C_0 = \pi^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \left\{ \frac{\Gamma(1)}{\Gamma(2)} \right\}^{\frac{1}{2}} = 2\pi^{\frac{1}{2}}.$$

By (5.2),

$$\int_{\mathbb{R}^2} |\psi(\cdot, t)|^{\frac{8}{3}} dx \leq \frac{4C_0}{3a_{\frac{8}{3}}^*} \left(\int_{\mathbb{R}^2} |\nabla \psi(\cdot, t)|^2 dx \right)^{\frac{1}{3}} \int_{\mathbb{R}^2} |\psi(\cdot, t)|^2 dx. \tag{5.4}$$

Using the conservation laws of energy and mass (1.17) together with (5.1)-(5.4), we obtain

$$\begin{aligned}
\|\psi(\cdot, t)\|_{\mathcal{H}}^2 &\leq 2E_p(\psi_0) + \left(\frac{4C_0 \|\psi_0\|_2^2}{3a_{\frac{8}{3}}^*} \right)^{\frac{3}{4}} \left(\int_{\mathbb{R}^2} |\nabla \psi(\cdot, t)|^2 dx \right)^{\frac{1}{4}} \\
&\quad + \frac{a \|\psi_0\|_2^2}{a_p^*} \left(\int_{\mathbb{R}^2} |\nabla \psi(\cdot, t)|^2 dx \right)^{\frac{p-2}{2}}.
\end{aligned} \tag{5.5}$$

Since $p \in (4, 4 + \delta)$ with $\delta \in (0, 1)$, the desired global existence assertion follows from (5.5). Indeed, if there existed $T_0 > 0$ such that

$$\lim_{t \rightarrow T_0} \|\nabla \psi(\cdot, t)\|_2 = +\infty,$$

we would obtain a contradiction to (5.5). This completes the proof of Theorem 1.3.

We now turn to the proof of stability.

Proof of Theorem 1.4.

We argue by contradiction. Suppose that for each $n \in \mathbb{N}$, there exist $\varepsilon_0 > 0$, functions φ_n and times t_n such that

$$\inf_{u \in \mathfrak{M}_p} \|\varphi_n - u\|_{\mathcal{H}} < \frac{1}{n} \quad (5.6)$$

and

$$\inf_{u \in \mathfrak{M}_p} \|\Psi_n(\cdot, t_n) - u\|_{\mathcal{H}} \geq \varepsilon_0, \quad (5.7)$$

where Ψ_n is the solution of (1.15) with the initial data φ_n . By (1.17) and (5.6), we have

$$E_p(\Psi_n(\cdot, t_n)) = E_p(\varphi_n) \rightarrow \mathcal{M}_{p, \tau_p^2} \quad (5.8)$$

and

$$\|\Psi_n(\cdot, t_n)\|_2^2 = \|\varphi_n\|_2^2 \rightarrow 1, \quad (5.9)$$

as $n \rightarrow \infty$.

Since $\tau_p \rightarrow +\infty$ as $p \rightarrow 4^+$, using (5.6), (5.8) and (5.9), for $p \in (4, 4 + \delta)$ as in Theorem 1.1 we may choose $N_0 \in \mathbb{N}$ such that for all $n > N_0$,

$$\|\varphi_n\|_2^2 \leq 2 \text{ and } \|\nabla \varphi_n\|_2^2 < \|\varphi_n\|_2^2 \tau_p^2. \quad (5.10)$$

We now claim that for any $n > N_0$,

$$\|\nabla \Psi_n(\cdot, t_n)\|_2^2 < \|\varphi_n\|_2^2 \tau_p^2. \quad (5.11)$$

If not, there would exist $n > N_0$ such that

$$\|\nabla \Psi_n(\cdot, t_n)\|_2^2 \geq \|\varphi_n\|_2^2 \tau_p^2.$$

By (5.10) and the continuity of solutions of (1.15) with respect to t , there exists $\tau_n \leq t_n$ such that

$$\|\nabla \Psi_n(\cdot, \tau_n)\|_2^2 = \|\varphi_n\|_2^2 \tau_p^2 \leq 2\tau_p^2. \quad (5.12)$$

Set

$$w_n = \frac{1}{\|\Psi_n(\cdot, \tau_n)\|_2} \Psi_n(x, \tau_n) = \frac{1}{\|\varphi_n\|_2} \Psi_n(x, \tau_n).$$

Then,

$$\|w_n\|_2^2 = 1, \quad \|\nabla w_n\|_2^2 = \tau_p^2. \quad (5.13)$$

Consequently,

$$E_p(w_n) \geq \inf_{u \in \partial A_{\tau_p^2}} E_p(u). \quad (5.14)$$

On the other hand,

$$\begin{aligned} \|\Psi_n(\cdot, \tau_n)\|_{\mathcal{H}}^2 &= 2E_p(\Psi_n(\cdot, \tau_n)) - \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| \Psi_n(x, \tau_n)^2 \Psi_n(y, \tau_n)^2 dx dy \\ &\quad + \frac{2a}{p} \int_{\mathbb{R}^2} |\Psi_n(x, \tau_n)|^p dx \\ &\leq 2E_p(\Psi_n(\cdot, \tau_n)) + B_2(\Psi_n(\cdot, \tau_n), \Psi_n(\cdot, \tau_n)) + \frac{2a}{p} \int_{\mathbb{R}^2} |\Psi_n(x, \tau_n)|^p dx. \end{aligned} \quad (5.15)$$

Note that p is fixed, using (2.3), (5.12) and (5.13), we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |\Psi_n(x, \tau_n)|^p dx &\leq \frac{p}{2a_p^*} \left(\int_{\mathbb{R}^2} |\nabla \Psi_n(x, \tau_n)|^2 dx \right)^{\frac{p-2}{2}} \left(\int_{\mathbb{R}^2} |\Psi_n(x, \tau_n)|^2 dx \right) \\ &\leq \frac{2^{\frac{p-4}{2}} C}{a_p^*} \tau_p^{p-2} \leq C. \end{aligned} \quad (5.16)$$

Similarly, from (2.12), we have

$$-\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| \Psi_n(x, \tau_n)^2 \Psi_n(y, \tau_n)^2 dx dy \leq C \|\Psi_n(x, \tau_n)\|_{\mathcal{H}}^{\frac{1}{2}}. \quad (5.17)$$

Inserting (5.16) and (5.17) into (5.15), and using (5.8) we deduce that

$$\|\Psi_n(\cdot, \tau_n)\|_{\mathcal{H}}^2 \leq C. \quad (5.18)$$

Let $\lambda_n = \|\varphi_n\|_2^{-1}$. Then, by the definition of E_p ,

$$\begin{aligned} E_p(w_n) &\leq E_p(\Psi_n(\cdot, \tau_n)) + \frac{1}{2} |1 - \lambda_n^2| \|\Psi_n(\cdot, \tau_n)\|_{\mathcal{H}}^2 \\ &\quad + \frac{1}{4} |1 - \lambda_n^4| \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln|x-y| \Psi_n(x, \tau_n)^2 \Psi_n(y, \tau_n)^2 dx dy \right| \\ &\quad + \frac{a}{p} |1 - \lambda_n^p| \int_{\mathbb{R}^2} |\Psi_n(x, \tau_n)|^p dx. \end{aligned}$$

From (5.9), (5.16)-(5.18), we obtain

$$E_p(w_n) \leq E_p(\varphi_n) + C \left(\frac{1}{2} |1 - \lambda_n^2| + |1 - \lambda_n^4| + |1 - \lambda_n^p| \right). \quad (5.19)$$

On the other hand, from (3.10) and (5.8), we have

$$\mathcal{M}_{p, \tau_p^2} < \inf_{u \in \partial A_{\tau_p^2}} E_p(u),$$

and

$$E_p(\varphi_n) \rightarrow \mathcal{M}_{p, \tau_p^2}$$

as $n \rightarrow \infty$. Hence, for all sufficiently large $n \geq N_0$,

$$E_p(\varphi_n) + C \left(\frac{1}{2} |1 - \lambda_n^2| + |1 - \lambda_n^4| + |1 - \lambda_n^p| \right) < \inf_{u \in \partial A_{\tau_p^2}} E_p(u).$$

Combining this with (5.19), we arrive at

$$E_p(w_n) < \inf_{u \in \partial A_{\tau_p^2}} E_p(u),$$

which contradicts to (5.14). Therefore, the claim (5.11) holds.

Next, following the same argument as in (5.18), we can prove that

$$\|\Psi_n(\cdot, t_n)\|_{\mathcal{H}}^2 \leq C. \quad (5.20)$$

Let $v_n = \lambda_n \Psi_n(\cdot, t_n)$, where $\lambda_n = \|\varphi_n\|_2^{-1}$. Then,

$$\int_{\mathbb{R}^2} |v_n|^2 dx = \lambda_n^2 \int_{\mathbb{R}^2} |\Psi_n(x, t_n)|^2 dx = \lambda_n^2 \int_{\mathbb{R}^2} |\varphi_n|^2 dx = 1,$$

and by (5.11),

$$\int_{\mathbb{R}^2} |\nabla v_n|^2 dx = \lambda_n^2 \int_{\mathbb{R}^2} |\nabla \Psi_n(x, t_n)|^2 dx < \lambda_n^2 \|\varphi_n\|_2^2 \tau_p^2 = \tau_p^2$$

for n sufficiently large. Hence, $v_n \in A_{\tau_p^2}$. Moreover, since $\lambda_n \rightarrow 1$ as $n \rightarrow \infty$, it follows from (5.20) that

$$E_p(\lambda_n \Psi_n(x, t_n)) - E_p(\Psi_n(x, t_n)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, by (5.8),

$$\begin{aligned} E_p(v_n) &= E_p(\Psi_n(\cdot, t_n)) + (E_p(\lambda_n \Psi_n(x, t_n)) - E_p(\Psi_n(x, t_n))) \\ &= E_p(\varphi_n) + o(1) \rightarrow \mathcal{M}_{p, \tau_p^2} \end{aligned}$$

as $n \rightarrow \infty$.

Therefore, $\{v_n\}$ is a minimizing sequence of $\mathcal{M}_{p, \tau_p^2}$. As in the proof of Theorem 1.1, we can show that v_n converges strongly in $\mathcal{H}$ to a local minimizer u of $\mathcal{M}_{p, \tau_p^2}$. Consequently, using (5.9) and (5.20), we obtain

$$\begin{aligned} \|\Psi_n(\cdot, t_n) - u\|_{\mathcal{H}} &\leq \|v_n - u\|_{\mathcal{H}} + \|\Psi_n(\cdot, t_n) - v_n\|_{\mathcal{H}} \\ &= \|v_n - u\|_{\mathcal{H}} + (\Psi_n(\cdot, t_n)(1 - \lambda_n))_{\mathcal{H}} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, which contradicts (5.7). The proof is complete.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- [1] Pekar, S.I. (1954) Untersuchungen über die Elektronentheorie der Kristalle. Akademie Verlag Berlin.
- [2] Lieb, E.H. (1977) Existence and Uniqueness of the Minimizing Solution of Choquard's Nonlinear Equation. *Studies in Applied Mathematics*, **57**, 93-105. <https://doi.org/10.1002/sapm197757293>
- [3] Penrose, R. (1996) On Gravity's Role in Quantum State Reduction. *General Relativity and Gravitation*, **28**, 581-600. <https://doi.org/10.1007/bf02105068>
- [4] Ambrosetti, A. and Ruiz, D. (2008) Multiple Bound States for the Schrödinger-Poisson Problem. *Communications in Contemporary Mathematics*, **10**, 391-404. <https://doi.org/10.1142/s021919970800282x>
- [5] Azzollini, A. and Pomponio, A. (2008) Ground State Solutions for the Nonlinear Schrödinger-Maxwell Equations. *Journal of Mathematical Analysis and Applications*, **345**, 90-108. <https://doi.org/10.1016/j.jmaa.2008.03.057>
- [6] Cingolani, S., Clapp, M. and Secchi, S. (2012) Multiple Solutions to a Magnetic Nonlinear Choquard Equation. *Zeitschrift für angewandte Mathematik und Physik*, **63**, 233-248. <https://doi.org/10.1007/s00033-011-0166-8>
- [7] Moroz, V. and Van Schaftingen, J. (2013) Groundstates of Nonlinear Choquard Equations: Existence, Qualitative Properties and Decay Asymptotics. *Journal of Functional Analysis*, **265**, 153-184. <https://doi.org/10.1016/j.jfa.2013.04.007>
- [8] Bellazzini, J., Jeanjean, L. and Luo, T. (2013) Existence and Instability of Standing

- Waves with Prescribed Norm for a Class of Schrödinger-Poisson Equations. *Proceedings of the London Mathematical Society*, **107**, 303-339.
<https://doi.org/10.1112/plms/pds072>
- [9] Jeanjean, L. and Luo, T. (2013) Sharp Nonexistence Results of Prescribed L^2 -Norm Solutions for Some Class of Schrödinger-Poisson and Quasi-Linear Equations. *Zeitschrift für angewandte Mathematik und Physik*, **64**, 937-954.
<https://doi.org/10.1007/s00033-012-0272-2>
- [10] Lions, P.L. (1984) Solutions of Hartree-Fock Equations for Coulomb Systems. *Communications in Mathematical Physics*, **109**, 33-97.
<https://doi.org/10.1007/bf01205672>
- [11] Luo, T. (2014) Multiplicity of Normalized Solutions for a Class of Nonlinear Schrödinger-Poisson-Slater Equations. *Journal of Mathematical Analysis and Applications*, **416**, 195-204. <https://doi.org/10.1016/j.jmaa.2014.02.038>
- [12] Stubbe, J. (2008) Bound States of Two-Dimensional Schrödinger-Newton Equations. arXiv: 0807.4059v1.
- [13] Azzollini, A. (2021) The Planar Schrödinger-Poisson System with a Positive Potential. *Nonlinearity*, **34**, 5799-5820. <https://doi.org/10.1088/1361-6544/ac0230>
- [14] Cingolani, S. and Jeanjean, L. (2019) Stationary Waves with Prescribed L^2 -Norm for the Planar Schrödinger-Poisson System. *SIAM Journal on Mathematical Analysis*, **51**, 3533-3568. <https://doi.org/10.1137/19m1243907>
- [15] Du, M. and Weth, T. (2017) Ground States and High Energy Solutions of the Planar Schrödinger-Poisson System. *Nonlinearity*, **30**, 3492-3515.
<https://doi.org/10.1088/1361-6544/aa7eac>
- [16] Guo, Y., Liang, W. and Li, Y. (2023) Existence and Uniqueness of Constraint Minimizers for the Planar Schrödinger-Poisson System with Logarithmic Potentials. *Journal of Differential Equations*, **369**, 299-352.
<https://doi.org/10.1016/j.jde.2023.06.007>
- [17] Wang, W., Zhang, W. and Li, Y. (2022) Minimizers of the Planar Schrödinger-Newton Equations. *Complex Variables and Elliptic Equations*, **67**, 151-161.
<https://doi.org/10.1080/17476933.2020.1816987>
- [18] Wang, C. and Zhang, S. (2025) Uniqueness of Minimizers for the Mass Subcritical Planar Schrödinger-Poisson System with Logarithmic Convolution Potential. *Discrete and Continuous Dynamical Systems*, **45**, 794-820.
<https://doi.org/10.3934/dcds.2024113>
- [19] Cingolani, S. and Weth, T. (2016) On the Planar Schrödinger-Poisson System. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, **33**, 169-197.
<https://doi.org/10.1016/j.anihpc.2014.09.008>
- [20] Guo, Y. and Seiringer, R. (2014) On the Mass Concentration for Bose-Einstein Condensates with Attractive Interactions. *Letters in Mathematical Physics*, **104**, 141-156.
<https://doi.org/10.1007/s11005-013-0667-9>
- [21] Guo, Y., Zeng, X. and Zhou, H. (2014) Concentration Behavior of Standing Waves for Almost Mass Critical Nonlinear Schrödinger Equations. *Journal of Differential Equations*, **256**, 2079-2100. <https://doi.org/10.1016/j.jde.2013.12.012>
- [22] Berestycki, H. and Lions, P.L. (1983) Nonlinear Scalar Field Equations, II Existence of Infinitely Many Solutions. *Archive for Rational Mechanics and Analysis*, **82**, 347-375. <https://doi.org/10.1007/bf00250556>
- [23] Kwong, M.K. (1989) Uniqueness of Positive Solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$. *Archive for Rational Mechanics and Analysis*, **105**, 243-266.

- <https://doi.org/10.1007/bf00251502>
- [24] Willem, M. (1996) Minimax Theorems. Birkhauser.
 - [25] Yang, J. and Yang, J. (2022) Normalized Solutions and Mass Concentration for Supercritical Nonlinear Schrödinger Equations. *Science China Mathematics*, **65**, 1383-1412. <https://doi.org/10.1007/s11425-020-1793-9>
 - [26] Weinstein, M. (1983) Nonlinear Schrödinger Equations and Sharp Interpolation Estimates. *Communications in Mathematical Physics*, **87**, 567-576. <https://doi.org/10.1007/bf01208265>
 - [27] Lieb, E.H. and Loss, M. (2001) Analysis. In: *Graduate Studies in Mathematics* (2nd Edition, Vol. 14), American Mathematical Society.
 - [28] Cazenave, T. (1989) An Introduction to Nonlinear Schrödinger Equations Vol. 22. Textos de Metodos Matematicos.
 - [29] Oh, Y.G. (1989) Cauchy Problem and Ehrenfest's Law of Nonlinear Schrödinger Equations with Potentials. *Journal of Differential Equations*, **81**, 255-274. [https://doi.org/10.1016/0022-0396\(89\)90123-x](https://doi.org/10.1016/0022-0396(89)90123-x)
 - [30] Zhang, J. (2000) Stability of Attractive Bose-Einstein Condensates. *Journal of Statistical Physics*, **101**, 731-746. <https://doi.org/10.1023/a:1026437923987>