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The Influence of the Field of Relative Inertial Forces of a Two-Mass Mechanical System as an Equivalent of Its Internal Interactions on Its Motion

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Abstract

This paper considers a mechanical system consisting of two bodies with masses m_1 and m_2 linked by a kinematic hinged-rod constraint in the form of a rigid straight rod R, allowing them to rotate both relative to each other and to their center of mass. Based on D' Alembert's principle formulated taking into account the constraint axiom for a constrained material point, equations of motion are derived separately for each body (m_1 and m_2) and for their center of mass in the field of relative inertial forces of the diametrically opposite bodies m_2 and m_1 both in vacuum and in a dissipative medium. It is shown that time in the proper reference frames of these bodies flows differently than time in the center-of-mass frame, varying with the kinematic characteristics and mass ratio of the bodies. In the center-of-mass frame and the laboratory frame, time is absolute. In addition, the momentum distribution between the bodies of the closed mechanical system is analyzed, and the relationship between this distribution and its energy balance is established under various initial conditions, demonstrating that its motion in the relative inertial force field satisfies the laws of conservation of momentum and energy, as well as the principle of least action.

Keywords

Two-Mass Mechanical System, Complex Motion, D' Alembert's Principle, Constraint Axiom for Relative Inertial Forces, Equations of Motion, Relativity of Time, Conservation of Momentum and Energy, Minimum Action

1. Introduction

In theoretical mechanics, dynamic problems involving two interacting material bodies m_i and m_j can be reduced to simpler static problems by representing Newton's second law in the form of D' Alembert's principle. Taking into account the constraint theorem, this principle for a constrained material point can be written as follows [1]:

$$\vec{\Phi}_i + \vec{R}_{ji} + \vec{F}_i = 0; i, j = 1,2; i \neq j. \tag{1}$$

Here $\vec{\Phi}_i = -m_i \vec{a}_i$ is the inertial force of body m_i in a fixed laboratory reference frame K ($\vec{a}_i$ is its absolute acceleration in this frame); $\vec{R}_{ji}$ is the reaction of the constraint R directed from body m_i to body m_j, which is counteracted by the inertial force $\vec{\Phi}_i$ of body m_i; and F_i is the external force applied to body m_i in the laboratory frame K.

In D'Alembert's principle (1), the inertial force $\vec{\Phi}_i$ accounts for the non-inertial nature of the proper frame K_i of body m_i, so that this principle is invariant relative to both the frame K_i and the laboratory frame K. In mechanics, this force is considered fictitious [2], introduced solely to reduce dynamic problems to simpler static problems.

The aim of this work is to move from D' Alembert's principle (1) to the dynamic equation for each of the interacting bodies m_i and m_j individually in the field of relative inertial force $\vec{\Phi}_{ji}$ of these bodies, replacing the unknown reactions $\vec{R}_{ji}; i, j = 1,2; i \neq j$ of their constraint R.

We consider a closed mechanical system consisting of two bodies m_i and m_j (MS2) [3] [4] (see **Figure 1(a)**). The bodies m_i and m_j are connected by a rigid straight rod of length $R = r_{12} = r_{21}$ mounted to them via ideal hinges (kinematic hinged-rod constraint R).

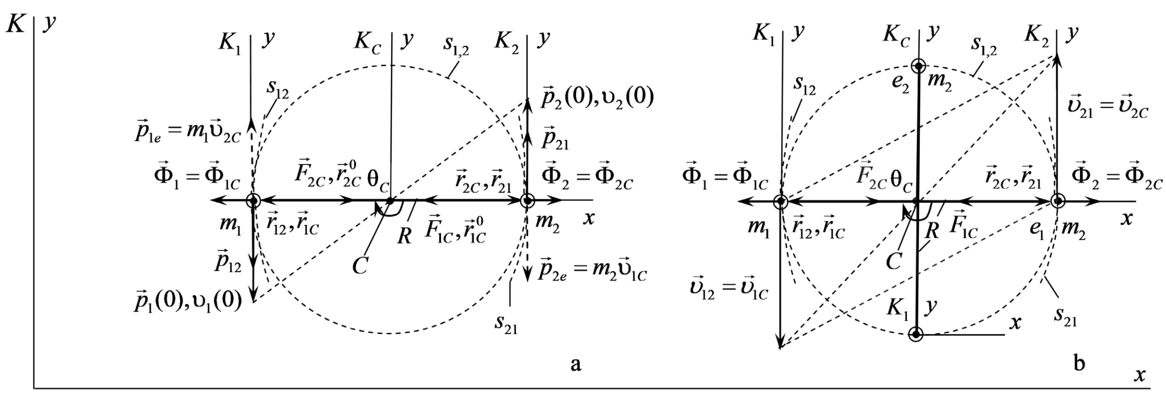

Figure 1. Kinematic diagram of a two-mass mechanical system with a diametric mass distribution $\theta_C = -\pi$ (a) and a map of events e_1 and e_2 showing that for an invariant rotation of body m_2 by an angle $\Delta\varphi = \{0; \pi/2\}$ in frames K_1 and K_C, this body is located at the same point of the laboratory frame K (b).

The MS2 is acted upon by two types of inertial forces:

1) The intrinsic inertial force $\vec{\Phi}_i = -m_i \frac{\mathrm{d}}{\mathrm{d}t} \vec{\upsilon}_i = -m_i \vec{a}_i; i = 1,2$ of body m_i,

included in D'Alembert's principle (1). It is a fundamental property of this body and characterizes its physical ability to counteract the bending of its uniform straight trajectory in the laboratory frame K under the action of any forces (e.g., Hooke's, Coulomb's, or Newton's forces). As applied to the two connected bodies m_i and m_j moving in circular trajectories relative to their center of mass (CM) C, these are their centrifugal inertial forces.

2) The relative inertial force $\vec{\Phi}_{ji} = -\frac{\mathrm{d}}{\mathrm{d}t} m_j \vec{\upsilon}_{ji} = -m_j \vec{a}_{ji}; i, j = 1,2; i \neq j$. It is applied to the working body m_j, which performs relative motion with relative velocity $\vec{\upsilon}_{ij}$ and acceleration $\vec{a}_{ij}; i, j = 1,2; i \neq j$ in the proper frame K_i of the reference body m_i. If the masses of the MS2 bodies are equal ($m_i = m_j$), either of them can be chosen as the reference body. If $m_i \gg m_j$, the body with the greater mass should be chosen as the reference body.

2. Dynamic Equations for the Closed MS2

The dynamic equations for bodies m_1 and m_2 of the closed MS2 are derived based on the kinematic diagram shown in **Figure 1(a)**. The kinematic hinged-rod constraint R between bodies m_1 and m_2 ensures their planar relative translational motion along circular trajectories s_{12} and s_{21} (shown partially in **Figure 1(a)**) in the proper frames K_2 and K_1 of the diametrically opposite bodies m_2 and m_1 and the simultaneous complex planar translational relative motion along trajectory $s_{1,2}$ (which is common for the bodies when their masses $m_1 = m_2$) in the reference frame K_C attached to their CM C. During these motions, the axes of the reference frames K_C, K_1, and K_2 and the fixed laboratory frame K always remain parallel. The kinematic hinged-rod constraint R is massless with ideal hinges, and the bodies m_1 and m_2 are point particles.

Since the kinematic hinged-rod constraint R leads to bending of the trajectories $s_{1,2}$, s_{21}, and s_{12} of bodies m_2 and m_1, it follows that in the reference frames K_C, K_1, and K_2, these bodies are acted upon by the intrinsic $\vec{\Phi}_{iC}$ and relative $\vec{\Phi}_{ji}$ inertial forces of these bodies, which counteract the bending of their trajectories.

The motion of bodies $m_j; j = 1, 2$ together with the proper reference frames K_i of the diametrically opposite bodies $m_i; i = 1, 2; i \neq j$ is translational with respect to the reference frame K and relative with respect to the proper reference frames K_i of bodies m_i. As a result, bodies m_j perform complex translational relative motion along the trajectory $s_{1,2}$ in the reference frame K. This motion generates a cross term U that captures the mutual influence of the translational and relative motions, which will be taken into account when deriving the dynamic equations for the MS2.

The trajectories $s_{1,2}$ of bodies m_1 and m_2 in the reference frame K_C are defined by radius vectors $\vec{r}_{1C}$ and $\vec{r}_{2C}$, and the trajectories s_{12} and s_{21} of their relative motion in the proper reference frames K_2 and K_1 of the diametrically opposite bodies m_2 and m_1 are defined by radius vectors $\vec{r}_{12}$ and $\vec{r}_{21}$

with magnitudes $r_{12} = r_{21} = R$. The radius vectors $\vec{r}_{1C}$ and $\vec{r}_{2C}$ are defined by the equations $m_1\vec{r}_{1C} + m_2\vec{r}_{2C} = 0$ and $\vec{r}_{21} = \vec{r}_{2C} - \vec{r}_{1C}$ [1] [5]:

$$\vec{r}_{1C} = -\frac{m_2}{m_1 + m_2}\vec{r}_{21}; \quad \vec{r}_{2C} = \frac{m_1}{m_1 + m_2}\vec{r}_{21}. \tag{2}$$

The kinetic energy of the MS2 in planar motion resulting from the two translational relative motions of its bodies m_1 and m_2 (with an alternate choice of their diametrically opposite bodies m_2 and m_1 as the reference body) is defined by the following generalized coordinates:

- Two polar coordinates r_{1C} and φ_{1C} of the reference body m_1 in the frame K_C of the CM C and one polar coordinate φ_{21} of the working body m_2 in the proper frame K_1 of the reference body m_1.
- Two polar coordinates r_{2C} and φ_{2C} of the reference body m_2 in the reference frame K_C of the CM C and one polar coordinate φ_{12} of the working body m_1 in the proper frame K_2 of the reference body m_2.

The polar coordinates φ_{12} and φ_{21} of the closed MC2 under consideration are kinematically dependent $\varphi_{12} = \varphi_{21} + \pi$ (**Figure 1(a)**). This kinematic dependence satisfies the requirement for dynamic self-consistency $\vec{\Phi}_{12} + \vec{\Phi}_{21} = 0$ of this closed MC2, which is important when the reactions $\vec{R}_{12}$ and $\vec{R}_{21}$ constraints R of its bodies m_1 and m_2 are replaced by their relative inertial forces $\vec{R}_{12} \Rightarrow \vec{\Phi}_{12}$ and $\vec{R}_{21} \Rightarrow \vec{\Phi}_{21}$.

Thus, the total number of independent generalized coordinates describing the planar translational relative motion of the MS2 bodies m_1 and m_2 in the reference frame K is $N = 5$, which equals the number of degrees of freedom of the system $s = 3n - l = 3 \cdot 2 - 1 = 5$. Here $n = 2$ is the number of MS2 bodies m_1 and m_2, each of which have three degrees of freedom in an unconstrained state in the reference frame K; $l = 1$ is one kinematic hinged-rod constraint R.

The hinges themselves do not introduce new independent constraints beyond the condition of constant length $R = \text{const}$ of the rod (they merely provide the kinematic possibility of rotation about their axes). The set of $N = 5$ generalized coordinates determines the total energy E of the MS2 in the reference frame K, which includes the kinetic energy T_{er} of the translational relative motion of bodies m_1 and m_2 and the cross term U due to their two complex translational relative motions [3].

For the chosen generalized coordinates, the motion of the MS2 bodies m_1 and m_2 is characterized by the following momenta:

- Intrinsic momentum $\vec{p}_{iC} = m_i\vec{\upsilon}_{iC}; i = 1,2$ of body m_i, which characterizes the motion of this body together with its proper reference frame K_i in the frame K_C. Here $\vec{\upsilon}_{iC} = \vec{\omega}_{iC} \times \vec{r}_{iC}$ is the velocity of body m_i in the frame K_C and $\vec{\omega}_{iC} = \vec{\omega}_C$ and $\vec{r}_{iC}$ are the angular velocity and radius vector (2) of the translational motion of this body in the frame K_C along a circular trajectory $s_i; i = 1,2$ (**Figure 1(a)**).
- Relative momentum $\vec{p}_{ij} = m_i\vec{\upsilon}_{ij}; i, j = 1,2; i \neq j$ of body m_i, which characterizes the motion of this body along a trajectory s_{ij} in the proper frame K_j

of the diametrically opposite body m_j. Here $\vec{\upsilon}_{ij} = \vec{\omega}_{ij} \times \vec{r}_{ij}$ is the relative velocity of body m_i in the reference frame K_j and $\vec{\omega}_{ij}$ and $\vec{r}_{ij}$ are the angular velocity and radius vector of the translational motion of this body in the frame K_j.

- Translational momentum $\vec{p}_{ie} = m_i \vec{\upsilon}_{ie}; i = 1,2$ body m_i, which characterizes the motion of this body together with the proper reference frame K_j of the diametrically opposite body m_j. Here $\vec{\upsilon}_{ie} = \vec{\upsilon}_{jC} = \vec{\omega}_{jC} \times \vec{r}_{jC}$ is the translational velocity of body m_i together with the proper frame K_j of the diametrically opposite body m_j, expressed in terms of its intrinsic velocity $\vec{\upsilon}_{jC} = \vec{\omega}_{jC} \times \vec{r}_{jC}$ in the frame K_C.

For bodies m_1 and m_2 with initial momenta $\vec{p}_1(0) = -\vec{p}_2(0)$, we establish the relationship between the intrinsic $\vec{p}_{iC} = m_i \vec{\upsilon}_{iC}; i = 1,2$, relative $\vec{p}_{ij} = m_i \vec{\upsilon}_{ij}; i, j = 1,2; i \neq j$, and translational $\vec{p}_{ie} = m_i \vec{\upsilon}_{ie}; i = 1,2$ momenta that satisfies the conservation of momentum ($\vec{p}_C = \overrightarrow{\text{const}}$) of the closed MS2.

Since bodies m_1 and m_2 are linked by the kinematic hinged-rod constraint R, it follows that for their initial momenta $\vec{p}_1(0) = -\vec{p}_2(0)$ (**Figure 1(a)**) under instantaneous transmission of interactions, their distribution in the reference frame K_C is diametric:

$$\theta_C = \arccos\left(p_1(0)/p_2(0)\right) = -\pi. \tag{3}$$

This corresponds to the concepts of optimal periodic motion [6] [7] for a closed MS2 [3] [4], where θ_C is the lag angle of the phase $\varphi_{1C} = \varphi_{21} + \theta_C = \varphi_{21} - \pi$ of the translational motion body m_1 in the reference frame K_C relative to the phase φ_{21} of the translational motion body m_2 in the proper reference frame K_1 of body m_1 (**Figure 1(a)**).

In the MS2 with five ($s = 5$) degrees of freedom under instantaneous transmission of perfectly elastic interaction, the initial momentum $\vec{p}_i(0) = (0,1); i = 1,2$ of each of its bodies m_i is distributed equally between its relative $\vec{p}_{ij}; i, j = 1,2; i \neq j$ and intrinsic $\vec{p}_{iC}; i = 1,2$ momenta. Consequently, when $m_1 = m_2$, the relative and translational velocities are equal $\vec{\upsilon}_{ij} = \vec{\upsilon}_{iC}$, as for body m_2 shown in **Figure 1(b)**. The magnitudes of these velocities $\upsilon_{12} = \upsilon_{1C}$ and $\upsilon_{21} = \upsilon_{2C}$ will be determined based on the kinetic energy conservation law for the MS2 during the analysis of its energy balance. The velocity $\vec{\upsilon}_C$ of the CM C in the reference frame K is $\vec{\upsilon}_C = \left(\vec{p}_1(0) + \vec{p}_2(0)\right)/m = 0$, as shown in **Figure 1(b)**. Acquiring the momenta $\vec{p}_{ij}$ and $\vec{p}_{iC}$, the MS2 bodies m_1 and m_2 begin to perform complex translational relative motion in the reference frame K with absolute velocities

$$\vec{\upsilon}_i = \vec{\upsilon}_{ie} + \vec{\upsilon}_{ij};\ i, j = 1,2;\ i \neq j, \tag{4}$$

where $\vec{\upsilon}_{ie} = \vec{\upsilon}_{jC} = \vec{\omega}_{jC} \times \vec{r}_{jC}$ and $\vec{\upsilon}_{ij} = \vec{\omega}_{ij} \times \vec{r}_{ij}$ are the translational velocity of body m_i together with the proper reference frame K_j of the diametrically opposite body m_j and its relative velocity in this frame.

The definition

$$m\vec{\upsilon}_C = \sum_i m_i \vec{\upsilon}_i = m_1\left(\vec{\upsilon}_{1e} + \vec{\upsilon}_{12}\right) + m_2\left(\vec{\upsilon}_{2e} + \vec{\upsilon}_{21}\right) = \vec{p}_1(0) + \vec{p}_2(0) = 0, \tag{5}$$

obtained taking into account the absolute velocities $\vec{\upsilon}_i$ (4), implies that for the initial momenta $\vec{p}_1(0) = -\vec{p}_2(0)$ of bodies m_1 and m_2, the velocity $\vec{\upsilon}_C$ of their CM C is $\vec{\upsilon}_C = 0$.

Definition (5) yields the following relationship between the translational $\vec{p}_{ie} = m_i\vec{\upsilon}_{ie} = m_i\vec{\upsilon}_{jC}$ and relative $\vec{p}_{ij} = m_i\vec{\upsilon}_{ij} = m_i\vec{\omega}_{ij} \times \vec{r}_{ij}; i, j = 1, 2; i \neq j$ momenta:

$$\vec{p}_{1e} + \vec{p}_{12} = 0; \vec{p}_{2e} + \vec{p}_{21} = 0. \tag{6}$$

In view of the definition $m_1\vec{r}_{1C} + m_2\vec{r}_{2C} = 0$ for the CM C, the translational momenta $\vec{p}_{ie} = m_i\vec{\upsilon}_{ie} = m_i\vec{\upsilon}_{jC}; i, j = 1, 2; i \neq j$ are linked by a similar relation:

$$\vec{p}_{1e} + \vec{p}_{2e} = 0. \tag{7}$$

Then, it follows from definition (5) that

$$\vec{p}_{12} + \vec{p}_{21} = 0 \tag{8}$$

According to (6)-(8), the momentum distribution of the closed MS2 satisfies the equalities

$$\vec{p}_{1C} = -\vec{p}_{2C}; \quad \vec{p}_{1C} = -\vec{p}_{21}; \quad \vec{p}_{2C} = -\vec{p}_{12}; \quad \vec{p}_{12} = -\vec{p}_{21}, \tag{9}$$

which ensure the conservation of its momentum

$$\vec{p}_C = \vec{p}_{1C} + \vec{p}_{2C} + \vec{p}_{12} + \vec{p}_{21} = \overrightarrow{\text{const}} = 0, \tag{10}$$

for $\vec{\upsilon}_C = 0$ (5). Equalities (9) will be used to analyze the energy balance of the MS2 for two sets of initial momenta $\vec{p}_1(0) = -\vec{p}_2(0)$ and $\vec{p}_1(0) = 0$, $\vec{p}_2(0) = (0,1)$ of its bodies m_1 and m_2.

When $m_1 = m_2$, equalities (6) reduce the sums of the translational $\vec{\upsilon}_{ie}$ and relative $\vec{\upsilon}_{ij}$ velocities to zero ($\vec{\upsilon}_{1e} + \vec{\upsilon}_{12} = 0$ and $\vec{\upsilon}_{2e} + \vec{\upsilon}_{21} = 0$), which is consistent with (5), resulting in zero absolute velocity $\vec{\upsilon}_i = 0$ (4) for each of the bodies m_i in the reference frame K. However, since the translational $\vec{\upsilon}_{ie}$ and relative $\vec{\upsilon}_{ij}$ velocities of the bodies m_i are not individually zero, they uniquely determine the kinetic T_{er} and total E energies of the MS2 in the reference frame K, which will be used when defining them in terms of the zero absolute velocity $\vec{\upsilon}_i$ (4).

We temporarily reduce the number of degrees of freedom of the closed MS2 to two ($s = 2$): along the Cartesian coordinate x_i of the reference body $m_i; i = 1, 2$ in the reference frame K and along the polar coordinate $\varphi_{ji}; i, j = 1, 2; i \neq j$ of the working body m_j in the proper frame K_i of the reference body m_i. Then, the reaction $\vec{R}_{ji}$ of the kinematic hinged-rod constraint R applied to any of the selected reference bodies m_i can be replaced by the relative inertial force $\vec{R}_{ji} = \vec{\Phi}_{ji} = m_j\omega_{ji}^2\vec{r}_{ji}$ of the conditionally discarded working body m_j (this will be subsequently mathematically proven using the example of MS2 with $s = 2$). As a result, from (1) we obtain the dynamic equation for any of the selected reference bodies m_i of the closed MS2 with $s = 2$ and $\vec{F}_i = 0$ in the form of D'Alembert's principle or Newton's second law in the field of the relative inertial force of the discarded working body m_j:

$$\vec{\Phi}_i + \vec{\Phi}_{ji} = 0; m_i\vec{a}_i = \vec{\Phi}_{ji}; \ i, j = 1, 2; \ i \neq j, \tag{11}$$

which does not contain the unknown reactions $\vec{R}_{ji}$ (1) of the kinematic hinged-rod constraint R and where $\vec{a}_i$ is the absolute acceleration of the reference body m_i in the laboratory frame K.

We now determine the form of the relative inertial forces $\vec{\Phi}_{ji}$ (11) for the closed MS2 with $N = s = 5$ shown in **Figure 1**, a when its bodies m_1 and m_2 perform complex planar translational relative motion in the reference frame K. In view of the absolute velocities $\vec{\upsilon}_i$ (4), the total kinetic energy of this simultaneous complex planar motion of the two bodies m_1 and m_2 can be represented as [3]

$$T_{er} = \frac{1}{2}\sum_i m_i \vec{\upsilon}_i^2 = \frac{1}{2}m_1\vec{\upsilon}_{1e}^2 + \frac{1}{2}m_2\vec{\upsilon}_{2e}^2 + \frac{1}{2}m_1\vec{\upsilon}_{12}^2 + \frac{1}{2}m_2\vec{\upsilon}_{21}^2 + U, \tag{12}$$

where U is the cross term of the translational relative motion, defined in [5] as the generalized kinetic potential

$$U = m_1\upsilon_{1e}\upsilon_{12}\cos\theta_C + m_2\upsilon_{2e}\upsilon_{21}\cos\theta_C = -\left(m_1\upsilon_{2C}\upsilon_{12} + m_2\upsilon_{1C}\upsilon_{21}\right), \tag{13}$$

which characterizes the contribution of the relative motion ($\vec{\upsilon}_{12r}, \vec{\upsilon}_{21r}$) of bodies m_1 and m_2 to their translational motion ($\vec{\upsilon}_{1e} = \vec{\upsilon}_{2C}, \vec{\upsilon}_{2e} = \vec{\upsilon}_{1C}$); here $\theta_C = \widehat{\vec{\upsilon}_{2e}\vec{\upsilon}_{21}} = -\pi$ is the angle between the vectors of the translational $\vec{\upsilon}_{2e}$ and relative $\vec{\upsilon}_{21}$ velocities (**Figure 1(a)**).

The sum of the first and second terms included in T_{er} (12) is the kinetic energy of the translational motion T_{Ce} of bodies m_1 and m_2 in the reference frame since K, taking into account the absolute velocities $\vec{\upsilon}_i$ (4), it characterizes the translational momentum $\vec{p}_{Ce} = \left(m_1 + m_2\right)\vec{\upsilon}_C = \sum_i m_i\vec{\upsilon}_i = m_1\vec{\upsilon}_{1e} + m_1\vec{\upsilon}_{12} + m_2\vec{\upsilon}_{2e} + m_2\vec{\upsilon}_{21} = m_1\vec{\upsilon}_{1e} + m_2\vec{\upsilon}_{2e}$ of MS2 as a whole.

The sum of the first and second terms in T_{er} (12) is the kinetic energy of the translational motion of bodies m_1 and m_2. Taking into account the permutation $m_1 \Leftrightarrow m_2$, valid for equal masses $m_1 = m_2$, this sum satisfies the equality

$$T_{Ce} = \frac{1}{2}m_1\vec{\upsilon}_{1e}^2 + \frac{1}{2}m_2\vec{\upsilon}_{2e}^2 = \frac{1}{2}m_1\vec{\upsilon}_{2C}^2 + \frac{1}{2}m_2\vec{\upsilon}_{1C}^2 = \frac{1}{2}m_1\vec{\upsilon}_{1C}^2 + \frac{1}{2}m_2\vec{\upsilon}_{2C}^2, \tag{14}$$

expressed in terms of the translational momenta $\vec{p}_{1e} = m_1\vec{\upsilon}_{1e} = m_1\vec{\upsilon}_{2C}$ and $\vec{p}_{2e} = m_2\vec{\upsilon}_{2e} = m_2\vec{\upsilon}_{1C}$ of bodies m_1 and m_2 together with the proper frames K_2 and K_1 of the diametrically opposite bodies m_2 and m_1; this defines the physical meaning of the sum T_{Ce}.

The sum of the third and fourth terms of the kinetic energy T_{er} (12) is the kinetic energy $T_r = \frac{1}{2}m_1\upsilon_{12}^2 + \frac{1}{2}m_2\upsilon_{21}^2$ of the relative motion of body m_1 in the frame K_2 and body m_2 in the frame K_1.

The potential U (13) defines the relative inertial forces of bodies m_1 and m_2 during their complex translational relative motion:

$$\begin{aligned} \vec{F}_{1C} &= -\frac{\partial U}{\partial r_{2C}} \cdot \vec{r}_{2C}^{\,0} = m_1\omega_{2C}\omega_{12}r_{12} \cdot \vec{r}_{1C}^{\,0}; \\ \vec{F}_{2C} &= -\frac{\partial U}{\partial r_{1C}} \cdot \vec{r}_{1C}^{\,0} = m_2\omega_{1C}\omega_{21}r_{21} \cdot \vec{r}_{2C}^{\,0}, \end{aligned} \tag{15}$$

which, in view of equality (14), act on the diametrically opposite bodies m_2 and m_1 in the reference frame K [3] [4]. Here $\vec{r}_{1C}^{\,0} = -\vec{r}_{2C}/r_{2C}$ and $\vec{r}_{2C}^{\,0} = -\vec{r}_{1C}/r_{1C}$ are the unit vectors opposing the radius vectors $\vec{r}_{2C}$ and $\vec{r}_{1C}$ (2) and defining the direction of the relative inertial forces $\vec{F}_{1C}$ and $\vec{F}_{2C}$ toward the CM C (see **Figure 1(a)**).

Internal momenta, which take into account the thermal motion of molecules, and the rotation of atoms in the crystal lattice of bodies MC2 and their core are not considered in the kinetic energy of MC2, as their small contribution to the motion of its CM C makes them extremely insignificant.

Thus, in the mathematical decomposition of kinetic energy T_{er} (12) into transportive T_{Ce} and relative T_r components, with their cross term in the form of kinetic potential U (13), they have different physical meanings, since they are expressed in terms of the transportive $\vec{\upsilon}_{1e} = \vec{\upsilon}_{2C}$, $\vec{\upsilon}_{2e} = \vec{\upsilon}_{1C}$ and relative $\vec{\upsilon}_{12}$, $\vec{\upsilon}_{21}$ velocities of bodies m_1 and m_2 MC2, so that when summed, the same momentum of these bodies is not considered more than once.

Using the equalities $p_{1C} = p_{12}$ and $p_{2C} = p_{21}$ (6) and the radii r_{1C} and r_{2C} (2), the angular velocities $\omega_{1C} = \omega_{2C} = \omega_C$ of bodies m_1 and m_2 in the reference frame K_C and their angular velocities ω_{12} and ω_{21} in the proper frames K_2 and K_1 of the diametrically opposite bodies m_2 and m_1 (see (15)), can be expressed as follows [1] [3] [4]:

$$\begin{gathered} \omega_C = \omega_{1C} = \omega_{2C} = \omega_{12} + \omega_{21}; \\ \omega_{12} = \frac{m_2 \omega_C}{m_1 + m_2};\ \omega_{21} = \frac{m_1 \omega_C}{m_1 + m_2}. \end{gathered} \tag{16}$$

Based on the kinematic feasibility of the motion of bodies m_1 and m_2 of the MS2 without its destruction, it is necessary that the angular displacements of these bodies in the reference frames K, K_C, K_1, and K_2 be invariant:

$$\Delta\varphi = \omega_C \Delta t_C = \omega_{12} \Delta t_2 = \omega_{21} \Delta t_1, \tag{17}$$

where $\Delta t_1 = t_1 - t_0$, $\Delta t_2 = t_2 - t_0$, and $\Delta t_C = t_C - t_0$ are the time increments and t_0 is the unified initial time in all reference frames; t_1 and t_2 is the proper time in the reference systems K_1 and K_2; t_C is the proper time in the reference frame K_C, which is absolute $t_C = t$ with respect to the time t in the reference frame K.

Physically, invariant (17) characterizes the fact that for any angular displacement $\Delta\varphi = \{0; \pi/2\}$ of body m_2 in the reference frames K_1 and K_C (which is also true for body m_1 in the reference frames K_2 and K_C), this body will be located at the same point of the trajectory $s_{1,2}$ of the reference frame K, as shown, e.g., for this body by events e_1 and e_2 in **Figure 1(b)**. In other words, it determines that for angular velocities ω_{21} and ω_{12} (16) in reference frames K_1 and K_2, different from the angular velocity ω_C in reference frame K_C, the time increments Δt_C, Δt_1 and Δt_2 in these frames must be different. That is, in reference frame K_1 or K_2, where the angular velocity ω_{21} or ω_{12} (16) is higher ω_C, the proper time t_1 or t_2 flows more slowly (the intervals between events are longer), and vice versa. Therefore, the clock rates in reference frames K_1 and K_2, due to different angular velocities ω_{21} and ω_{12}, are determined through the proper time t_1 and t_2. Since the proper times t_1 and t_2 are different in reference frames K_1 and K_2, the observed clock rates of this process in these systems will also differ. It is important to note that for the considered MS2 model, relativistic time dilation is not introduced (as in Einstein's special theory of relativity [8]). It operates within the framework of classical mechanics, but takes into account that during the relative motion of parts of the MC2 (rotation), the very concept of locality of time for reference systems K_1 and K_2 in relation to time $t_C = t$ in the reference system K_C is derived on the basis of angular coordinates $\varphi_C = \omega_C t_C$, $\varphi_{21} = \omega_{21} t_1$ and $\varphi_{12} = \omega_{12} t_2$, associated with time t_C, t_1 and t_2 in reference systems K_C, K_1 and K_2 through the kinematics of the MC2—angular velocities ω_C, ω_{21} and ω_{12}, and is not postulated on the current model of the clock.

From the invariant $\Delta\varphi$ (17) one can determine the proper time t_1 and t_2 in the reference systems K_1 and K_2:

$$t_1 = \frac{m_1 + m_2}{m_1} t_C; \;\; t_2 = \frac{m_1 + m_2}{m_2} t_C. \tag{18}$$

where time t_C in the reference frame K_C is absolute $t_C = t$.

The relativity of times t_1 and t_2 (18) is not associated with the relativity of time in Einstein's general relativity theory [8], which is based on the constancy of the speed of light $c = \text{const}$ in all observer systems moving uniformly relative to a light source. A general property of the relativity of time (18) in comparison with Einstein's special relativity theory [8] is that as the mass of the reference body m_i increases (which is equivalent to the strengthening of its gravitational field), the rate of time flow in the proper frame K_i of this body slows down and approaches the rate of time t_C in the reference frame K_C of the CM C, which is absolute ($t_C = t$).

Taking into account the relative inertial forces $\vec{F}_{1C}$ and $\vec{F}_{2C}$ (15), D'Alembert's principle and Newton's second law (11) for bodies m_1 and m_2 of the closed MS2 under consideration take the form

$$\vec{\Phi}_i + \vec{F}_{jC} = 0; \; m_i \vec{a}_i = \vec{F}_{jC}; \; i, j = 1, 2; \; i \neq j. \tag{19}$$

The first Equation in (19) characterizes the equilibrium of body m_i in the laboratory reference frame K during uniform motion along a geodesic trajectory s_i, while the second characterizes its dynamics in the same frame K.

The relative inertial forces $F_{jC} = m_j \omega_{iC} \omega_{ji} r_{ji}; i, j = 1, 2; i \neq j$ in Equation (19) have the same dimensions as the relative inertial forces

$\vec{\Phi}_{ji} = -m_j \omega_j^2 r_{ji}; i, j = 1, 2; i \neq j$ in Equation (11). They act through the constraint R on the diametrically opposite body m_i, forming an inertial domain (ID) [3] [4] that excludes the unknown reactions $\vec{R}_{ji}$ (1) of this constraint R from D'Alembert's principle. Therefore, D' Alembert's principle (1) and the dynamic Equations (11) and (19) become invariant in the fixed laboratory reference frame

K, as the interaction of bodies m_i and m_j via the reactions $\vec{R}_{ji}$ of the constraint R is equivalent to the action of the relative inertial forces $\vec{\Phi}_{ji}$ and $\vec{F}_{ji}$. Moreover, due to the equivalence of gravitational and inertial masses, the ID field can be viewed as a central dynamic attractive force field whose action on the MS2 is equivalent to that of the external force field.

The MS2 model, represented by Equation (19), explicitly excludes constraint R reactions $\vec{R}_{ji}$ and Newton's third law. It is constructed not using the geometry of a rigid rod R per se, but through additional dynamic assumptions about the nature of the total momentum distribution and phase relationships. In this MS2 model, momentum transfer between its bodies is accomplished through the action of relative inertial forces $\vec{F}_{jC}$ (19). That is, the redistribution mechanism consists of the action of the relative inertial force $\vec{F}_{jC}$ of the working body m_j through the constraint R on the supporting body m_i.

Moreover, the MS2 model is constructed under the following key assumptions:

- The transfer of momentum from the working body m_j to the supporting body m_i with a larger mass is carried out by means of the inertial force $\vec{F}_{jC}$ of the working body m_j. This replaces "instantaneity" in the classical sense with a controlled redistribution, which can be described as instantaneous within the framework of the proposed MS2 model with a specific distribution of its total momentum $\vec{p}_C$ (10).
- To conserve the total momentum $\vec{p}_C = \overrightarrow{\text{const}}$ of a closed MS2, the condition is introduced that the phases of the interacting bodies in the reference frame of their center of mass are shifted by angle $\theta_C = -\pi$ (3). This is not simply a kinematic condition, but a dynamic condition that effectively defines the following rule: for any initial momentum distribution, a closed MS2 instantaneously tends to ensure the phase condition $\theta_C = -\pi$, which is equivalent to an instantaneous redistribution within the framework of its adopted model.

From the perspective of classical mechanics, the introduction of instantaneous momentum redistribution through phase conditions and relative inertial forces are additional postulates, not consequences of the standard equations of classical mechanics. It is these postulates that make the proposed MS2 model internally consistent within its own logic, but simultaneously take it beyond the traditional approach.

Replacing $T_{Ce} = \frac{1}{2}m_1\vec{\upsilon}_{1e}^2 + \frac{1}{2}m_2\vec{\upsilon}_{2e}^2$ in (12) by $T_{Ce} = \frac{1}{2}m_1\vec{\upsilon}_{1C}^2 + \frac{1}{2}m_2\vec{\upsilon}_{2C}^2$ in accordance with equality to (14) and taking into account that the external potential energy of the closed MS2 is zero ($\Pi = 0$) yields its Lagrangian

$$\begin{aligned} L = T_{er} &= \frac{1}{2}m_1\vec{\upsilon}_{1C}^2 + \frac{1}{2}m_2\vec{\upsilon}_{2C}^2 + \frac{1}{2}m_1\vec{\upsilon}_{12}^2 + \frac{1}{2}m_2\vec{\upsilon}_{21}^2 + U \\ &= \frac{1}{2}m_1\omega_{1C}^2 r_{1C}^2 + \frac{1}{2}m_2\omega_{2C}^2 r_{2C}^2 + \frac{1}{2}m_1\omega_{12}^2 r_{12}^2 + \frac{1}{2}m_2\omega_{21}^2 r_{21}^2 \\ &\quad -\left(m_1\omega_{2C}r_{2C}\omega_{12}r_{12} + m_2\omega_{1C}r_{1C}\omega_{21}r_{21}\right) \end{aligned} \tag{20}$$

in the reference frame K for $m_1 = m_2$.

The partial derivatives of the Lagrangian function L (20) of the form $-\frac{\partial L}{r_{1C}}=0$ and $-\frac{\partial L}{r_{2C}}=0$ yield the generalized forces

$$\begin{aligned}&-m_1\omega_{1C}^2 r_{1C}+m_2\omega_{1C}\omega_{21}r_{21}=0;\\&-m_2\omega_{2C}^2 r_{2C}+m_1\omega_{2C}\omega_{12}r_{12}=0\end{aligned}\tag{21}$$

in the form of the D' Alembert principle (19) in curvilinear coordinates, in the field of relative inertial forces $\vec{F}_{jC}$, which establishes their equivalence to reactions $\vec{R}_{ij}$.

Indeed, the first terms of (21) define the intrinsic inertial forces $\vec{\Phi}_1=\vec{\Phi}_{1C}=-m_1\omega_{1C}^2\vec{r}_{1C}$ and $\vec{\Phi}_2=\vec{\Phi}_{2C}=-m_2\omega_{1C}^2\vec{r}_{2C}$ of bodies m_1 and m_2 (**Figure 1(a)**), which are directed away from CM C.

The second terms define the relative inertial forces $\vec{F}_{2C}=m_2\omega_{1C}\omega_{21}r_{21}\cdot\vec{r}_{2C}^{\,0}$ and $\vec{F}_{1C}=m_1\omega_{2C}\omega_{12}r_{12}\cdot\vec{r}_{1C}^{\,0}$ of bodies m_2 and m_1 during their complex translational relative motion, where the choice of direction for the unit vectors $\vec{r}_{1C}^{\,0}=-\vec{r}_{2C}/r_{2C}$ and $\vec{r}_{2C}^{\,0}=-\vec{r}_{1C}/r_{1C}$ toward the CM C (**Figure 1(a)**) is due to the fact that the forces $\vec{F}_{2C}$ and $\vec{F}_{1C}$ are counteracted by the intrinsic inertial forces $\vec{\Phi}_1$ and $\vec{\Phi}_2$ of bodies m_1 and m_2 (21).

Equation (19), expressed in terms of the relative inertial forces $\vec{F}_{1C}$ and $\vec{F}_{2C}$ (15), allow an independent analysis of the dynamics of each of MS2 bodies m_1 and m_2 without taking into account the unknown reactions $\vec{R}_{ji}$ (1) of their constraint R. Since these forces form a balanced system of forces $\vec{F}_{1C}+\vec{F}_{2C}=0$, their action on the closed MS2 system satisfies the conservation of its momentum ($\vec{p}_C=\overrightarrow{\text{const}}$).

We rederive the dynamic Equation (19) for the MS2 for the case where the initial momenta of its bodies m_1 and m_2 are $\vec{p}_1(0)=0$ and $\vec{p}_2(0)=(0,1)$, respectively, as shown in **Figure 2(a)**.

In the MS2 with five ($s=5$) degrees of freedom, the initial momentum $\vec{p}_2(0)=(0,1)$ at time $t=0$ is instantaneously redistributed between the bodies m_1 and m_2 of the MS2 and its CM C so that the translational velocities $\vec{\upsilon}_{1e},\vec{\upsilon}_{2e}$ ($\vec{\upsilon}_{1e}=\vec{\upsilon}_{2C}$, $\vec{\upsilon}_{2e}=\vec{\upsilon}_{1C}$) and the relative velocities of its bodies m_1 and m_2 $\vec{\upsilon}_{21},\vec{\upsilon}_{12}$ become equal to each other: $\vec{\upsilon}_{12}=\vec{\upsilon}_{1C}$ and $\vec{\upsilon}_{21}=\vec{\upsilon}_{2C}$ (**Figure 2(b)**). Their magnitudes $\upsilon_{12}=\upsilon_{1C}$ and $\upsilon_{21}=\upsilon_{2C}$ will be determined from the energy conservation law of the closed MS2 in the subsequent analysis of its energy balance. The velocity $\vec{\upsilon}_C$ of the CM C in the reference frame K is $\vec{\upsilon}_C=p_2(0)/m$, as shown in **Figure 2(a)**. Acquiring the momenta $\vec{p}_{ij}$ and $\vec{p}_{iC}$, bodies m_1 and m_2 begin to perform complex translational relative motion along the circle $s_{1,2C}$ relative to the frame K_C (**Figure 2(b)**). Due to the simultaneous rectilinear and uniform motion of the CM C of these bodies in the reference frame K, this motion follows spiral trajectories (not shown in **Figure 2**).

Based on the definition

$$\vec{\upsilon}_C=\frac{\sum_i m_i\vec{\upsilon}_i}{m_1+m_2}=\frac{m_1\vec{\upsilon}_{ie}+m_1\vec{\upsilon}_{12}+m_2\vec{\upsilon}_{2e}+m_2\vec{\upsilon}_{21}}{m_1+m_2}=\frac{m_1\vec{\upsilon}_{ie}+m_2\vec{\upsilon}_{2e}}{m_1+m_2},\tag{22}$$

obtained taking into account the absolute velocities $\vec{\upsilon}_i$ (4), the motion of the CM C can be considered translational $\vec{\upsilon}_C \equiv \vec{\upsilon}_{Ce}$.

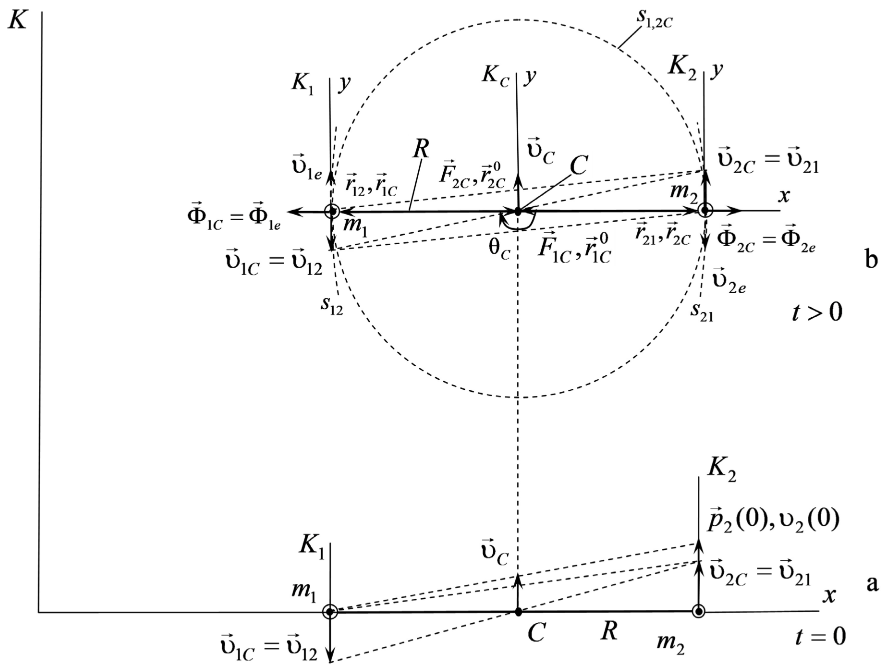

Figure 2. Kinematics of the MS2 system whose bodies m_1 and m_2 have initial momenta $\vec{p}_1(0) = 0$ and $\vec{p}_2(0) = (0,1)$.

Then, the translational velocities $\vec{\upsilon}_{ie}$ of bodies m_1 and m_2 are the sum

$$\vec{\upsilon}_{ie,\text{abs}} = \vec{\upsilon}_C + \vec{\upsilon}_{ie}, i = 1,2, \tag{23}$$

whose cross terms vanish when it is squared:

$$m_1\vec{\upsilon}_C \cdot (\vec{\omega}_C \times \vec{r}_{1C}) + m_2\vec{\upsilon}_C \cdot (\vec{\omega}_C \times \vec{r}_{1C}) = \vec{\upsilon}_C \cdot (\vec{\omega}_C \times (m_1\vec{r}_{1C} + m_2\vec{r}_{2C})) = 0. \tag{24}$$

Replacing the translational velocity $\vec{\upsilon}_{ie}$ in (12) by $\vec{\upsilon}_{ie,\text{abs}}$ (23) yields the kinetic energy T_{er} of the MS2 in the form

$$T_{er} = \frac{1}{2}m\vec{\upsilon}_C^2 + \frac{1}{2}m_1\vec{\upsilon}_{1e}^2 + \frac{1}{2}m_2\vec{\upsilon}_{2e}^2 + \frac{1}{2}m_1\vec{\upsilon}_{12}^2 + \frac{1}{2}m_2\vec{\upsilon}_{21}^2 + U, \tag{25}$$

where $T_C = \frac{1}{2}m\vec{\upsilon}_C^2$ is the kinetic energy of the rectilinear uniform motion of the CM C with velocity $\vec{\upsilon}_C$ in the reference frame K (**Figure 2(b)**).

The kinetic energy (25) differs from Koenig's kinetic energy by containing the translational kinetic energy term $T_e = \frac{1}{2}m_1\vec{\upsilon}_{1e}^2 + \frac{1}{2}m_2\vec{\upsilon}_{2e}^2$ and the generalized kinetic potential $U = -(m_1\upsilon_{2C}\upsilon_{12} + m_2\upsilon_{1C}\upsilon_{21})$, which defines the relative inertial forces $\vec{F}_{1C}$ and $\vec{F}_{2C}$ (15).

Based on Galileo's principle of relativity, when $\vec{p}_C = \overrightarrow{\text{const}}$, the term T_C in (25) can be omitted when constructing the Lagrangian of the MS2 under consideration. Then, taking into account the permutation of masses $m_1 \Leftrightarrow m_2$ (14), allowable when $m_1 = m_2$, its Lagrangian can be written in the form

$$\begin{aligned} L = T_{er} &= \frac{1}{2} m_1 \vec{\upsilon}_{2C}^2 + \frac{1}{2} m_2 \vec{\upsilon}_{1C}^2 + \frac{1}{2} m_1 \vec{\upsilon}_{12}^2 + \frac{1}{2} m_2 \vec{\upsilon}_{21}^2 + U \\ &= \frac{1}{2} m_1 \omega_{2C}^2 r_{2C}^2 + \frac{1}{2} m_2 \omega_{1C}^2 r_{1C}^2 + \frac{1}{2} m_1 \omega_{12}^2 r_{12}^2 + \frac{1}{2} m_2 \omega_{21}^2 r_{21}^2 \\ &\quad - \left(m_1 \omega_{2C} r_{2C} \omega_{12} r_{12} + m_2 \omega_{1C} r_{1C} \omega_{21} r_{21} \right), \end{aligned} \tag{26}$$

similar to (20).

From the Lagrangian L (26), one can determine the relative inertial forces $\vec{F}_{1C}$ and $\vec{F}_{2C}$ (15) and the Equation of motion (19) for each of the MS2 bodies (m_1 and m_2).

3. Dynamic Equations for the MS2 System with $s = 2$ Degrees of Freedom in a Dissipative Medium

The absolute momenta of the reference and working bodies m_1 and m_2 ($m_1 > m_2$) of the MS2 in the reference frame K can be expressed in terms of their Cartesian coordinates

$$x_1 = x_1 \text{ and } x_2 = x_1 + R\cos\varphi_{21} \tag{27}$$

as follows [3] [4]:

$$p_{1x} = m_1 \frac{\mathrm{d}}{\mathrm{d}t} x_1; \quad p_{2x} = m_2 \frac{\mathrm{d}}{\mathrm{d}t} x_2 = m_2 \frac{\mathrm{d}}{\mathrm{d}t} x_1 + m_2 \frac{\mathrm{d}}{\mathrm{d}t} R\cos\varphi_{21}. \tag{28}$$

The generalized independent coordinates of the MS2 can be chosen to be the Cartesian coordinate x_1 of the reference body m_1 moving along the x axis of the reference frame K and the polar coordinate $\varphi_{21} = \omega_{21} t_1$ of the working body m_2 rotating relative to the reference body m_1 in its proper frame K_1, in which time t_1 is absolute ($t_1 = t_C = t$), according to (16) and (17) with $\omega_{12} = 0$ and the relative velocity $\omega_{21} = \text{const}$, which is maintained constant by the internal active pair of forces [3] [4].

Then, the absolute momentum MS2 can be represented as

$$p_x = \sum_i p_{ix} = p_{Cx} + p_{21x} = \left(m_1 + m_2 \right) \frac{\mathrm{d}}{\mathrm{d}t} x_1 + m_2 \frac{\mathrm{d}}{\mathrm{d}t} R\cos\varphi_{21}. \tag{29}$$

It is composed of the intrinsic momentum of the reference body m_1 with the reduced mass $m = m_1 + m_2$, which, together with proper reference frame K_1 performs absolute motion in the reference frame K, and the relative momentum of the working body m_2 in the reference frame K_1:

$$p_{Cx} = \left(m_1 + m_2 \right) \frac{\mathrm{d}}{\mathrm{d}t} x_1, \quad p_{21x} = m_2 \frac{\mathrm{d}}{\mathrm{d}t} R\cos\varphi_{21}. \tag{30}$$

Using the momenta p_{1x} and p_{2x} (28) and accounting for the constraint axiom for the constrained material point and the dissipative force

$$F_{1x} = -\mu_1 \dot{x}_1 \tag{31}$$

acting on the reference body m_1 moving with velocity $\dot{x}_1$ along the x axis of the reference frame K, Newton's second law for each of the interacting bodies

m_1 and m_2 individually can be expressed as

$$\mathrm{d}p_{1x}/\mathrm{d}t = X_{12} + F_{1x};\ \mathrm{d}p_{2x}/\mathrm{d}t = X_{21}, \tag{32}$$

where $X_{12} = -X_{21}$ are the projections of the reactions $\vec{R}_{ij}$ of the constraint R of bodies m_1 and m_2 onto the x axis of the reference frame K, which satisfy Newton's third law [1].

Summing the left and right sides of Equation (32) and taking into account that $\sum_{i,j=1,2;\, i\neq j} X_{ij} = 0$ yields D' Alembert's principle and Newton's second law

$$\Phi_{1x} + \Phi_{21x} + F_{1x} = 0;\ m\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = \Phi_{21x} + F_{1x}, \tag{33}$$

in the form (11) in the reference frame K for the dissipative external medium, where $\Phi_{1x} = -\frac{\mathrm{d}p_{Cx}}{\mathrm{d}t} = -(m_1 + m_2)\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2}$ is the intrinsic inertial force of the reference body m_1 with the reduced mass $m = m_1 + m_2$ and $\Phi_{21x} = -\frac{\mathrm{d}p_{21x}}{\mathrm{d}t} = -m_2\frac{\mathrm{d}^2 R\cos\varphi_{21}}{\mathrm{d}t^2} = m_2\omega_{21}^2 R\cos\varphi_{21}$ is the intrinsic inertial force of the working body m_2 in the reference frame K_1.

Equation (33) are expressed in terms of the relative inertial force $\vec{\Phi}_{21}$ of the working fluid m_2 and can be obtained directly from the Lagrangian MC2, written relative to the Cartesian coordinate $x_1 = f(\varphi_{21})$, which does not contain constraint R reactions $\vec{R}_{ij}$ [3], which establishes the equivalence of constraint R reactions $\vec{R}_{ij}$ and the relative inertial force $\vec{\Phi}_{21}$ of the working fluid m_2. Equation (33) also show that under $m_1 \neq m_2$ MC2 it is reduced to a single body with mass $m = m_1 + m_2$ with $s \leq 3$ degrees of freedom, moving in a dissipative medium in a field of relative inertial force $\vec{\Phi}_{21}$ along a rectilinear trajectory.

The solution of the inhomogeneous differential equation in the form of Newton's second law (33) was obtained in [3] [4] for the initial momenta $\vec{p}_1 = 0$ and $\vec{p}_2 = (0,1)$ of the MS2 bodies m_1 and m_2. It is the sum of the general solution of the homogeneous Equation (33) (with its right-hand side set to zero) and the particular solution of this inhomogeneous equation.

The general solution characterizes the transient process occurring in the MS2 at the beginning of its motion and determines the position x_O of the force center O of the MS2 ID on the x axis of the reference frame K after the completion of the transient process.

The particular solution describes the steady-state linear harmonic oscillation of the reference body m_1 with reduced mass $m = m_1 + m_2$ along the x axis of the reference frame K in the vicinity of the force center O of the ID:

$$x_{1O} = A_1\cos\varphi_{1O} = -A_1\cos(\varphi_{21} + \varepsilon), \tag{34}$$

where

$$A_1 = a\Big/\sqrt{1+\xi^2};\ \varphi = \theta_C + \varepsilon = -\pi + \varepsilon;\ \varepsilon = \mathrm{arctg}\,\xi; \tag{35}$$

$\varphi_{1O} = \varphi_{21} + \varphi$ and A_1 are the phase and amplitude of the linear harmonic oscillation of the reference body m_1; $\varphi = \varphi_{1O} - \varphi_{21}$ is the lag angle of the phase

φ_{1O} of the linear harmonic oscillation of the reference body m_1 relative to the phase φ_{21} of the rotational motion of the working body m_2, including the dissipative angle ε that takes into account the viscosity of the external medium; $a = m_2 R/(m_1 + m_2)$ is the amplitude coefficient; $\xi = 2\gamma_1/\omega_{21}$ is the dissipative parameter; $\gamma_1 = \mu_1/[2(m_1 + m_2)]$ is the motion damping coefficient of the reference body m_1 with reduced mass $m = m_1 + m_2$; and μ_1 is the linear viscous drag coefficient of the external medium to the motion of the reference body m_1.

The known coordinate x_{1O} (34) of the reference body m_1 and the coordinate transformations (27) ensure the determination of the coordinate x_{2O} of the working body m_2 in the reference frame K_O attached to the force center O. Then, the coordinate x_{CO} of the CM C of the MS2 in this reference frame can be determined by the formula

$$x_{CO} = (m_1 x_{10} + m_2 x_{20})/m, \tag{36}$$

where m_1 is the actual mass of the reference body m_1.

4. Numerical Analysis of the Energy Balance and Least Action for the MS2

The closed MS2 shown in **Figure 1** and **Figure 2** has the following intrinsic parameters: $m_1 = m_2 = 1$ kg; $R = 1$ m; $r_{1C} = m_2 R/(m_1 + m_2) = 0.5$ m and $r_{2C} = m_1 R/(m_1 + m_2) = 0.5$ m.

1) Initial conditions and kinematic characteristics of the MS2 (**Figure 1(a)**)

The magnitudes of the initial momenta $\vec{p}_1(0) = -\vec{p}_2(0)$ of bodies m_1 and m_2 are $p_1(0) = m_1 \upsilon_1(0) = 1\ \mathrm{kg \cdot m/s}$ and $p_2(0) = m_2 \upsilon_2(0) = 1\ \mathrm{kg \cdot m/s}$, where $\upsilon_1(0) = 1\ \mathrm{m/s}$ and $\upsilon_2(0) = 1\ \mathrm{m/s}$ are the magnitudes of the velocities of bodies m_1 and m_2 in the reference frame K before their interaction.

The kinetic energy T_{er} (12) of the MS2 under perfectly elastic interactions must satisfy the conservation relation

$$\frac{1}{2} m_1 \vec{\upsilon}_1^2(0) + \frac{1}{2} m_2 \vec{\upsilon}_2^2(0) = \frac{1}{2} m_1 \vec{\upsilon}_{2C}^2 + \frac{1}{2} m_2 \vec{\upsilon}_{1C}^2 + \frac{1}{2} m_1 \vec{\upsilon}_{12}^2 + \frac{1}{2} m_2 \vec{\upsilon}_{21}^2. \tag{37}$$

Taking into account the MS2 momentum distribution (9), for $m_1 = m_2$ we can write

$$\upsilon_{1C} = \upsilon_{2C} = \upsilon_{12} = \upsilon_{21}. \tag{38}$$

Then, for the initial velocities $\upsilon_1(0) = \upsilon_2(0) = 1\ \mathrm{m/s}$, relation (37) yields

$$\upsilon_{1C} = \upsilon_{12} = \upsilon_{2C} = \upsilon_{21} = \frac{\sqrt{2}}{2} = 0.707\ \mathrm{m/s}. \tag{39}$$

After bodies m_1 and m_2 begin to interact (at $t > 0$), the kinematic characteristics of the MS2 system take the following values: $\omega_{12} = \frac{\upsilon_{21}}{R} = 0.707\ \mathrm{s^{-1}}$; $\omega_{21} = \frac{\upsilon_{21}}{R} = 0.707\ \mathrm{s^{-1}}$ and $\omega_C = \omega_{1C} = \omega_{2C} = \omega_{12} + \omega_{21} = 1.414\ \mathrm{s^{-1}}$ (**Figure 1(a)**).

In the reference frame K, components of the total energy of the MS2 are

$$T_e = \frac{1}{2}m_1\vec{\upsilon}_{1e}^2 + \frac{1}{2}m_2\vec{\upsilon}_{2e}^2 = \frac{1}{2}m_1\omega_C^2 r_{2C}^2 + \frac{1}{2}m_2\omega_C^2 r_{1C}^2 = 0.5\ \text{kg}\cdot\text{m}^2/\text{s}; \quad (40)$$

$$T_r = \frac{1}{2}m_1\upsilon_{12}^2 + \frac{1}{2}m_2\upsilon_{21}^2 = \frac{1}{2}m_1\omega_{12}^2 R^2 + \frac{1}{2}m_2\omega_{21}^2 R^2 = 0.5\ \text{kg}\cdot\text{m}^2/\text{s}; \quad (41)$$

$$\begin{aligned} U &= -\left(m_1\upsilon_{2C}\upsilon_{12} + m_2\upsilon_{1C}\upsilon_{21}\right) \\ &= -\left(m_1\omega_C r_{2C}\omega_{12}R + m_2\omega_C r_{1C}\omega_{21}R\right) = -1\ \text{kg}\cdot\text{m}^2/\text{s}. \end{aligned} \quad (42)$$

The energy balance of the MS2 for initial momenta $\vec{p}_1(0) = -\vec{p}_2(0)$ is shown in **Table 1**.

Table 1. MS2 energy balance.

Initial conditions	T_C	T_e	T_r	U	E_D	E
$\vec{p}_1(0) = -\vec{p}_2(0)$	0	0.5	0.5	−1	0	0
$\vec{p}_1(0) = 0; \vec{p}_2(0) = (0,1)$	0.25	0.125	0.125	−0.25	0	0.25

Since the kinetic energy $T_{er} = T_e + T_r = 1$ of the MS2 (**Table 1**) is equal to the initial kinetic energy $T_{er} = T(0) = m_1\upsilon_1^2(0)/2 + m_2\upsilon_2^2(0)/2 = 1$ of its bodies m_1 and m_2, its kinetic energy is conserved (37). The zero total energy of the MS2 $E = E_D = 0$, where $E_D = T_{er} + U$, implies that the kinetic energy T_{er} of bodies m_1 and m_2 is periodically converted into their generalized kinetic potential U and vice versa. For example, relative to the x axis (**Figure 1(a)**), the kinetic potential of bodies m_1 and m_2 is at a maximum ($U = \max$) and their kinetic energy is at a minimum ($T_{er} = \min$). When the bodies rotate by an angle of $\pi/2$, the situation is reversed. The global minimum of the total energy $E = E_D = 0$ MS2 corresponds to the least action ($L = \min$) (20).

2) Initial conditions and kinematic characteristics of the MS2 (**Figure 2(a)**)

The magnitudes of the initial momenta $\vec{p}_1(0) = 0$; $\vec{p}_2(0) = (0,1)$ of bodies m_1 and m_2 are $p_1(0) = 0$ and $p_2(0) = m_2\upsilon_2(0) = 1\ \text{kg}\cdot\text{m/s}$, where $\upsilon_2(0) = 1\ \text{m/s}$ is the magnitude of the velocity of body m_2 in the reference frame K before its interaction with body m_1.

The kinetic energy T_{er} (12) of the MS2 under perfectly elastic interaction must satisfy the conservation relation

$$\frac{1}{2}m_2\vec{\upsilon}_2^2(0) - \frac{1}{2}m_2\vec{\upsilon}_C^2 = \frac{1}{2}m_1\vec{\upsilon}_{2C}^2 + \frac{1}{2}m_2\vec{\upsilon}_{1C}^2 + \frac{1}{2}m_1\vec{\upsilon}_{12}^2 + \frac{1}{2}m_2\vec{\upsilon}_{21}^2. \quad (43)$$

Taking into account the momentum distribution of the MS2 (9) for $m_1 = m_2$ we can write $\upsilon_{1C} = \upsilon_{2C} = \upsilon_{12} = \upsilon_{21}$ (30). Then, for an initial velocity $\upsilon_2(0) = 1\ \text{m/s}$ of body m_2 and a velocity $\upsilon_C = \dfrac{m_2\upsilon_2(0)}{m_1 + m_2} = 0.5\ \text{m/s}$ of the CM C, relation (43) yields

$$\upsilon_{1C} = \upsilon_{12} = \upsilon_{2C} = \upsilon_{21} = \frac{1}{\sqrt{8}} = 0.354\ \text{m/s}. \quad (44)$$

After bodies m_1 and m_2 begin to interact (at $t > 0$), the kinematic characteristics of the MS2 are as follows: $\omega_{12}=\frac{\upsilon_{12}}{2R}=0.354\ \text{s}^{-1}$; $\omega_{21}=\frac{\upsilon_{21}}{2R}=0.354\ \text{s}^{-1}$ and $\omega_C=\omega_{1C}=\omega_{2C}=\omega_{12}+\omega_{21}=0.708\ \text{s}^{-1}$ (**Figure 2(a)** and **Figure 2(b)**).

In the reference frame K, the components of the total energy of the MS2 are

$$T_C=\frac{1}{2}(m_1+m_2)\vec{\upsilon}_C^2=0.25\ \text{kg}\cdot\text{m}^2/\text{s}; \tag{45}$$

$$T_e=\frac{1}{2}m_1\vec{\upsilon}_{1e}^2+\frac{1}{2}m_2\vec{\upsilon}_{2e}^2=\frac{1}{2}m_1\omega_C^2r_{2C}^2+\frac{1}{2}m_2\omega_C^2r_{1C}^2=0.125\ \text{kg}\cdot\text{m}^2/\text{s}; \tag{46}$$

$$T_r=\frac{1}{2}m_1\upsilon_{12}^2+\frac{1}{2}m_2\upsilon_{21}^2=\frac{1}{2}m_1\omega_{12}^2R^2+\frac{1}{2}m_2\omega_{21}^2R^2=0.125\ \text{kg}\cdot\text{m}^2/\text{s}; \tag{47}$$

$$\begin{aligned} U &= -\left(m_1\upsilon_{2C}\upsilon_{12}+m_2\upsilon_{1C}\upsilon_{21}\right) \\ &= -\left(m_1\omega_C r_{2C}\omega_{12}R+m_2\omega_C r_{1C}\omega_{21}R\right)=-0.25\ \text{kg}\cdot\text{m}^2/\text{s}. \end{aligned} \tag{48}$$

Table 1 shows the energy balance of the MS2 system when its bodies m_1 and m_2 have initial momenta $\vec{p}_1(0)=0$ and $\vec{p}_2(0)=(0,1)$ kg·m²/s. For these initial momenta, the kinetic energy of the MS2 contains an additional term in the form of the kinetic energy $T_C=(m_1+m_2)\upsilon_C^2/2=0.25$ of its CM C. This term depends on the initial conditions $\vec{p}_1(0)=0$; $\vec{p}_2(0)=(0,1)$ and can be omitted since $\vec{p}_C=\overrightarrow{\text{const}}$. Then, the kinetic energy $T_{er}=T_e+T_r=0.25$ of the MS2 (**Table 1**) is equal to the initial kinetic energy $T_{er}=T(0)=m_1\upsilon^2(0)/2+m_2\upsilon^2(0)/2-m_1\upsilon_C^2/2$ of its bodies m_1 and m_2 minus the omitted term T_C, so that the kinetic energy of the MS2 in the form (43) is conserved. The total energy of the MS2 for the initial conditions $\vec{p}_1(0)=0$ and $\vec{p}_2(0)=(0,1)$ is not zero: $E=T_C+E_D=0.25$, where $E_D=T_{er}+U=0$ is a term characterizing the periodic conversion of the kinetic energy T_{er} of bodies m_1 and m_2 into their generalized kinetic potential U and vice versa. For example, relative to the x axis (**Figure 2(b)**), the kinetic potential of bodies m_1 and m_2 is at a maximum ($U=\max$) and their kinetic energy is at a minimum ($T_{er}=\min$). When the bodies rotate by an angle of $\pi/2$, the situation is reversed. The local minimum of the total energy $E=T_C+E_D=0.25$ of the MS2 for $E_D=T_{er}+U=0$ corresponds to the least action ($L=\min$) (26).

5. Conclusions

Thus, based on D' Alembert's principle (1) formulated taking into account the constraint axiom for a constrained material point, equations of motion for each of bodies m_1 and m_2 of the closed MS2 system were derived describing their dynamics in a fixed laboratory reference frame K in the field of the relative inertial forces of the diametrically opposite bodies m_2 and m_1. The obtained results are extended to the case of their complex translational relative translational motion for the closed MS2 system with five ($s=5$) degree of freedom, as well as to their motion for the MS2 with two ($s=2$) degrees of freedom in a dissipative medium.

It is shown that for the MS2 with five ($s=5$) degrees of freedom, time

($t_1 = \frac{m_1 + m_2}{m_1} t_C$ and $t_2 = \frac{m_1 + m_2}{m_2} t_C$) in the proper reference frames K_1 and K_2 of the interacting bodies m_1 and m_2 flows differently than time in the reference frame K_C of their CM C. In the frame K_C and in the laboratory frame K, time is absolute ($t_C = t$).

By constructing the Lagrangian of the closed MS2 with five ($s = 5$) degrees of freedom, it is shown that its motion in the field of relative inertial forces satisfies the laws of conservation of momentum and energy, as well as the principle of least action under the specified initial conditions.

The relevance of this study lies in the fact that it establishes the relationship between the relativity of kinematic variables (including time) and the dynamics of each of the interacting bodies m_i and m_j of the MS2 in the stationary laboratory reference frame K through the structure of the system Lagrangian that defines the intrinsic ($\vec{\Phi}_i$) and relative ($\vec{\Phi}_{ij}$ and $\vec{F}_{iC}$) inertial forces of its bodies m_i and m_j. The relative inertial forces $\vec{\Phi}_{ij}$ and $\vec{F}_{iC}$ of the interacting bodies m_i and m_j act are equivalent to the reaction $\vec{R}_{ij}$ of their constraint R in D' Alembert's principle (1), which defines the dynamic Equations (11) and (19), thereby simplifying the dynamic analysis of complex mechanical systems.

The results of this work can find useful in applications for the synchronization and dynamic analysis of complex motions in robotics, the design of propulsion devices in low-dissipation media [3] [4] [9], as well as for the synchronization of complex orbital motions of artificial and natural celestial bodies interacting via a gravitational field.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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