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Golden Discrepancy and Binary Stability in Quasigeometric Sequences

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Abstract

The golden section φ is the defining irrational constant in golden section quasigeometric integer sequences generated by nearest-integer iteration. Such sequences do not reproduce exact geometric progression; instead, rounding introduces a structured discrepancy whose magnitude decays geometrically while its sign oscillates. The first term h_1 of a quasigeometric sequence H_n is called a parent number p and it satisfies strict computational constraints. Parent numbers can also be generated by perturbing terms of an established quasigeometric sequence H_n by unity, so that $p = h_n + \epsilon$, $\epsilon \in \{-1, +1\}$. The analysis of this perturbation shows that for $2 \le n \le 4$, only the positive or negative branch is admissible, and a transition occurs at $n = 5$ when binary stability is attained. This paper introduces the golden discrepancy decay variable and examines its significance at the critical fifth stage of quasigeometric evolution. Starting from the nearest-integer bound, the analysis establishes the characteristic discrepancy scale $1/(2\varphi^4) \approx 0.072949$ at $n = 5$, the very onset of the stable binary perturbation regime.

Keywords

Quasigeometric Sequence, Golden Ratio, Golden-Section Discrepancy, Discrepancy Decay, Binary Stability, Parent Numbers

1. Introduction

The geometrical constant φ, known as the golden section or golden ratio, has ancient origins. It is defined in The Elements of Euclid [1] as the division of a line into extreme and mean ratio, that is, cutting a line segment into two parts a and b so that the ratio of the total length ($a+b$) to the longer part (a) equals the ratio of the longer part (a) to the shorter part (b), that is

$$\frac{a+b}{a}=\frac{a}{b}$$

The golden section is traditionally linked to the Fibonacci sequence

$$0,1,1,2,3,5,8,\cdots$$

and the Lucas sequence

$$2,1,3,4,7,11,18,\cdots$$

The arithmetic properties of the golden ratio are closely connected with continued fractions and the theory of Diophantine approximation [2]. The theory of the golden section, Fibonacci numbers, and Lucas numbers is extensively studied, see [3]-[8]. Mamombe [8] introduced golden section quasigeometric sequences. Rather than viewing sequences like the traditional Fibonacci and Lucas as isolated phenomena, the framework utilizes specialized algorithms to generate an entire family of recurring Fibonacci-type sequences. A golden section quasigeometric sequence H_n simultaneously satisfies the nearest-integer transformation

$$h_{n+1}=\mathrm{rnd}\left(\varphi h_n\right), n\geq 1 \tag{1.1}$$

and the additive recurrence relation

$$h_{n+2}=h_{n+1}+h_n, n\geq 1 \tag{1.2}$$

where

$$\varphi=\frac{1+\sqrt{5}}{2} \tag{1.3}$$

is the golden section. We state this result and prove it as the fundamental theorem of golden section quasigeometric sequences in the next section.

Throughout this manuscript, $\mathrm{rnd}\left(x\right)$ denotes the standard nearest-integer rounding function mapping $\mathbb{R}\to\mathbb{Z}$, defined such that $\mathrm{rnd}\left(x\right)=\left\lfloor x+\frac{1}{2}\right\rfloor$. In the event of a half-integer tie ($x=k+\frac{1}{2}$ for $k\in\mathbb{Z}$), rounding is resolved upward toward $k+1$. The domain of application is restricted to non-half-integer values because in the present construction ties do not occur since the quantities to which the *rnd* function is applied are irrational.

The term

$$h_1\in\mathbb{Z}$$

is called a parent number

Notice that the ideal golden-scaled value of h_n is φh_n, whereas the actual next term is the selected integer h_{n+1}. The signed difference between these two quantities is here introduced and defined as the golden-section discrepancy, denoted δ_n.

Definition 1.1

Let H_n be a quasigeometric sequence. The golden-section discrepancy δ_n at h_n is defined as the signed difference between the golden-section prediction for h_{n+1} and the actual integer h_{n+1}. Therefore,

$$\delta_n = \varphi h_n - h_{n+1} \tag{1.4}$$

Thus, δ_n measures how far h_{n+1} lies from φh_n. If

$$\delta_n > 0$$

then h_{n+1} lies below the golden-section prediction φh_n. If

$$\delta_n < 0$$

then h_{n+1} lies above the golden-section prediction φh_n.

The magnitude of δ_n measures the size of the departure from exact golden-section scaling, while its sign specifies the direction, or orientation, of that departure. This discrepancy has an important recursive property. Since

$$\varphi - 1 = \frac{1}{\varphi} \tag{1.5}$$

we have from Equation (1.4) that

$$\delta_{n+1} = \varphi h_{n+1} - h_{n+2}$$

and therefore

$$\delta_{n+1} = \frac{-\delta_n}{\varphi}$$

Repeated application yields

$$\delta_n = \frac{(-1)^{n-1}}{\varphi^{n-1}} \delta_1 \tag{1.6}$$

Equation (1.6) shows immediately that two properties govern the evolution of the discrepancy:

- Its magnitude decays geometrically;
- Its sign reverses at every successive stage.

The term discrepancy also has an established usage in the study of sequences generated from the golden ratio, including low-discrepancy constructions [9]. The discrepancy defined above, however, is a specific signed quantity associated with the nearest-integer quasigeometric transformation.

In this paper we study the generation of parent numbers through perturbation of a quasigeometric sequence by unity. We show that the alternating sign of the golden-section discrepancy determines the transient polarity of admissible branches for $2 \le n \le 4$, while from $n \ge 5$, a threshold condition is satisfied for both perturbation signs, producing a stable two-sided regime.

2. Fundamental Theorem of Quasigeometric Sequences

We shall firstly state and prove the fundamental theorem of quasigeometric sequences.

Theorem 2.1

Let

$$H_n = h_1, h_2, h_3, h_4, h_5, \cdots \tag{2.1}$$

be an integer sequence satisfying the additive recurrence

$$h_{n+2} = h_{n+1} + h_n, n \geq 1$$

If the initial terms $h_1, h_2 \in \mathbb{Z}$ *satisfy the hypothesis that* h_2 *is the unique nearest integer to* φh_1, *equivalently expressed as* $|\varphi h_1 - h_2| < \frac{1}{2}$, *then for every* $n \geq 1$, $h_{n+1} = \mathrm{rnd}(\varphi h_n)$.

Let its golden-section discrepancy be defined by

$$\delta_n = \varphi h_n - h_{n+1}$$

Then

$$\delta_n = \left(-\frac{1}{\varphi}\right)^{n-1} (\varphi h_1 - h_2)$$

Equivalently,

$$\varphi h_n - h_{n+1} = (-1)^{n-1} \varphi^{-(n-1)} (\varphi h_1 - h_2) \tag{2.2}$$

If

$$|\varphi h_1 - h_2| < \frac{1}{2} \tag{2.3}$$

then for every $n \geq 1$,

$$h_{n+1} = \mathrm{rnd}(\varphi h_n).$$

Proof

From the definition of δ_{n+1},

$$\delta_{n+1} = \varphi h_{n+1} - h_{n+2}$$

Using the recurrence relation (1.2),

$$\delta_{n+1} = \varphi h_{n+1} - (h_{n+1} + h_n)$$

Therefore,

$$\delta_{n+1} = (\varphi - 1) h_{n+1} - h_n$$

Since

$$\varphi - 1 = \frac{1}{\varphi},$$

we obtain

$$\delta_{n+1} = \frac{h_{n+1}}{\varphi} - h_n$$

Now,

$$-\frac{\delta_n}{\varphi} = -\frac{\varphi h_n - h_{n+1}}{\varphi} = -h_n + \frac{h_{n+1}}{\varphi}$$

Hence,

$$\delta_{n+1} = -\frac{1}{\varphi}\delta_n$$

Repeated application yields

$$\delta_n = \left(-\frac{1}{\varphi}\right)^{n-1} \delta_1$$

which is Equation (2.2).

Finally, if $|\delta_1| < \frac{1}{2}$, *then*

$$|\delta_n| = \frac{|\delta_1|}{\varphi^{n-1}} < \frac{1}{2}.$$

Since h_{n+1} *is an integer and lies less than one-half from* φh_n, *it is the unique nearest integer to* φh_n. *Therefore,*

$$h_{n+1} = \mathrm{rnd}(\varphi h_n).$$

This completes the proof.

Corollary 2.1: Quasigeometric Fibonacci Sequence

For

$$H_n = F_n = 1, 2, 3, 5, 8, \cdots \tag{2.4}$$

we have

$$h_1 = 1,\ h_2 = 2,$$

and therefore

$$\varphi h_1 - h_2 = \varphi - 2 = -\varphi^{-2}$$

By Theorem 2.1,

$$\varphi h_n - h_{n+1} = (-1)^n \varphi^{-(n+1)} \tag{2.5}$$

Consequently, Relation (1.1) *holds.*

The additive recurrence and the golden-section transformation are therefore not unrelated descriptions accidentally producing similar numbers. In this construction, they are complementary descriptions of the same sequence. Theorem 2.1 proves that every Fibonacci-type integer sequence whose initial discrepancy satisfies $|\varphi h_1 - h_2| < \frac{1}{2}$ obeys the golden-section rounding law. For the initial value $h_1 = 1$, this rounding law uniquely produces $h_2 = 2$; hence the quasigeometric construction (1.1) and the additive recurrence (1.2) describe the same sequence. This represents unification of a nonlinear geometric operation with a linear arithmetic relation.

The ordinary Fibonacci recurrence tells us how to calculate the next number once two preceding numbers are known. The quasigeometric construction adds a different kind of information: it explains how one term may be geometrically extrapolated from the preceding term. The sequence is consequently a discrete, integer-valued expression of golden-section growth.

3. Parent Numbers and the Generation of Quasigeometric Sequences

3.1. The Parent Number

An integer sequence H_n is called a golden section quasigeometric sequence when it satisfies simultaneously the quasigeometric generation rule and the addi-

tive recurrence thus:

$$\begin{cases} h_{n+1} = \mathrm{rnd}\left(\varphi h_n\right), n \geq 1 \\ h_{n+2} = h_{n+1} + h_n, n \geq 1 \end{cases} \tag{3.1}$$

The important consequence is that the additive recurrence alone is not sufficient to characterize this class of integer sequences [8]. The quasigeometric rule supplies the principal structure from which the recurrence follows. The converse need not hold. This distinction is central to the definition of a quasigeometric sequence. The construction links multiplicative growth by the irrational factor φ with an integral additive recurrence. This gives the sequence its quasigeometric character: its local generation is multiplicative, whereas its exact integral structure is additive. The two descriptions are therefore not independent rules but complementary aspects of the same construction.

The seed value from which a quasigeometric Fibonacci sequence is generated is called a parent number. In this framework, a parent number is an integer satisfying the defining quasigeometric conditions.

Definition 3.1

Let $p \in \mathbb{Z}$ *be an integer seed. A parent number is defined through the predicate*:

$$\begin{cases} q(p) = rnd(\varphi p) : \text{generated successor} \\ u(p) = q(p) - p : \text{successor difference} \\ \mathrm{rnd}\left(\varphi u(p)\right) \neq p : \text{non-degeneracy condition} \end{cases} \tag{3.2}$$

Thus, a parent number is not an arbitrary seed of the additive recurrence; it is an integer seed that satisfies the golden section generation rule together with the required non-degeneracy condition. This point explicitly distinguishes quasigeometric sequences from the usual elementary presentation of a second-order recurrence, in which two starting values are specified arbitrarily. Here, the second value is constrained by the golden-section quasigeometric rule, while the seed value itself is strictly defined by Relation (3.2).

While classical linear recurrences and matrix exponentiation such as $\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$ compute sequence terms efficiently in $O(\log n)$ time, they treat initial conditions as unconstrained arbitrary integers. The quasigeometric framework introduces a constrained generative classification: it restricts initial seeds to those satisfying the golden-section non-degeneracy criterion. This provides a direct structural mapping between non-linear golden scaling and linear integer sequences, allowing the identification of exact stability thresholds that standard linear matrix methods cannot detect.

3.2. Parent Number Detection

The definition of a parent number provides a direct detection procedure. Given an integer p, one first generates the candidate successor using the golden-section

transformation and then tests the defining conditions. A candidate is accepted only when the generated terms satisfy the required non-degeneracy and recurrence conditions.

Step 1: $p \to \mathrm{rnd}(\varphi p)$

Step 2: test the parent number condition using Relation (3.2)

Step 3: generate subsequent terms

Step 4: verify $h_{n+2} = h_{n+1} + h_n, n \geq 1$

This procedure may be implemented as an algorithm over the natural numbers. It is important, however, to distinguish detection from generation. The algorithm does not assume in advance that every integer is a parent number; rather, it tests whether a candidate integer can function as a valid seed.

A second detection technique arises from transformations of already established quasigeometric sequences. Such results provide structural detection rules in addition to the direct computational test. This leads to the Law of Polarity discussed below.

3.3. The Law of Polarity

Given the quasigeometric sequence

$$F_n = 1, 2, 3, 5, 8, \cdots \tag{3.3}$$

We state and prove the following lemma.

Lemma 3.1

Let F_n *be the quasigeometric sequence* (3.3). *For all* $n \geq 5$, $f_n \pm 1 = h_1$, *where* H_n *is a quasigeometric sequence.*

Proof

Here we are called upon to prove that for every $n \geq 5$, *both integers* $f_n - 1$ *and* $f_n + 1$ *are parent numbers.*

Let

$$\epsilon \in \{-1, +1\}$$

and define

$$p = f_n + \epsilon \tag{3.4}$$

We shall prove that p *is a parent number.*

The golden ratio

$$\varphi = \frac{1+\sqrt{5}}{2}$$

satisfies

$$\varphi^2 = \varphi + 1$$

For the sequence (3.3), *Binet's formula gives*

$$f_n = \frac{\varphi^{n+1} - (-\varphi)^{-(n+1)}}{\sqrt{5}} \tag{3.5}$$

such that

$$\varphi f_n = f_{n+1} - (-\varphi)^{-(n+1)} \tag{3.6}$$

Therefore,

$$\varphi p = \varphi(f_n + \epsilon)$$

Substituting the Identity (3.6) *gives*

$$\varphi p = f_{n+1} - (-\varphi)^{-(n+1)} + \epsilon\varphi \tag{3.7}$$

Now write $\varphi = 2 + (\varphi - 2)$

Hence

$$\epsilon\varphi = 2\epsilon + \epsilon(\varphi - 2)$$

Consequently,

$$\varphi p = f_{n+1} + 2\epsilon + \epsilon(\varphi - 2) - (-\varphi)^{-(n+1)} \tag{3.8}$$

Define

$$q = f_{n+1} + 2\epsilon \tag{3.9}$$

It follows that

$$\varphi p - q = \epsilon(\varphi - 2) - (-\varphi)^{-(n+1)} \tag{3.10}$$

Taking absolute values,

$$|\varphi p - q| \le |\varphi - 2| + \varphi^{-(n+1)} \tag{3.11}$$

Since $n \ge 5$,

$$\varphi^{-(n+1)} \le \varphi^{-6}$$

Numerically,

$$|\varphi - 2| = 0.381966\cdots$$

and

$$\varphi^{-6} = 0.055728\cdots$$

Therefore,

$|\varphi p - q| \le 0.381966\cdots + 0.055728\cdots$; equality occurs at n = 5 for $\varepsilon = +1$.

Thus,

$$|\varphi p - q| \le 0.437694 < \frac{1}{2}$$

Since q *is an integer, this proves that* q *is the nearest integer to* φp. *Hence*,

$$q = \mathrm{rnd}(\varphi p)$$

We have therefore established the first condition required for p *to be a parent number. Now consider the difference between* q *and* p. *We have that*

$$q - p = (f_{n+1} + 2\epsilon) - (f_n + \epsilon)$$

Therefore,

$$q - p = f_{n+1} - f_n + \epsilon$$

Hence,

$$q - p = f_{n-1} + \epsilon \tag{3.12}$$

Let

$$u = q - p$$

Then

$$u = f_{n-1} + \epsilon$$

We must finally verify the non-degeneracy condition in the definition of a parent number.

Since $\left|\varphi u - \left(f_n + 2\epsilon\right)\right| = \varphi^{-2} + \varphi^{-n} \le \varphi^{-2} + \varphi^{-5} < \frac{1}{2}$, we obtain:

$$\mathrm{rnd}\left(\varphi u\right) = f_n + 2\epsilon \tag{3.13}$$

On the other hand, from Equation (3.4),

$$p = f_n + \epsilon$$

Therefore

$$\mathrm{rnd}\left(\varphi u\right) = f_n + 2\epsilon \neq p \tag{3.14}$$

Thus the non-degeneracy condition is satisfied, completing the proof.

The proof to Lemma 3.1 leads to another important result. Equations (3.4) and (3.9) indicate that

$$p = f_n + \epsilon$$

and

$$q = f_{n+1} + 2\epsilon$$

Notice that

$$p + q = f_{n+2} + 3\epsilon$$

Therefore:

Corollary 3.1

Let F_n *be the quasigeometric sequence* (3.3) *and* $\epsilon \in \{-1, +1\}$ *as defined above. For fixed* $n \ge 5$, *there exists a quasigeometric sequence* H_r *defined by*

$$h_r = f_{n+r-1} + \epsilon f_r,\ n \ge 5,\ r \ge 1 \tag{3.15}$$

Lemma 3.1 allows us to predict parent numbers from the quasigeometric sequence F_n, and Corollary 3.1 deduces that these parent numbers generate quasigeometric sequences derived from F_n, see also [7]. Here enters the notion of polarity. Notice that

$$\epsilon \in \{-1, +1\} \tag{3.16}$$

The sequence H_r derived from F_n when $\epsilon = -1$ shall be designated $H_r^{(-)}$, and when $\epsilon = +1$ the sequence shall be designated $H_r^{(+)}$. This designation shall apply to any sequence $H_n^{(\epsilon)}$ derived from another quasigeometric sequence. Definition 3.1 and Relation (3.2) characterize whether a candidate seed is a parent number. Therefore, parent numbers:

- Are generated computationally;
- Are derived from existing quasigeometric sequences.

Lemma 3.2 and theorems 3.1 and 3.2 deal with the derivation of parent numbers from existing quasigeometric sequences. These theorems are also proved in [8], but here we provide a direct method of proof.

Lemma 3.2

Let $H_n^{(+)} = \{h_n\}$ *be a golden-section quasigeometric sequence whose initial discrepancy satisfies*

$$-1/2 < \delta_1 < -1/(2\varphi).$$

For

$$p_n(\varepsilon) = h_n + \varepsilon, \varepsilon \in \{-1, +1\},$$

and $2 \le n \le 4$, *exactly one perturbation is a parent number. Its sign is*

$$\varepsilon_n = (-1)^{n+1}.$$

Consequently,

$$n = 2 : h_2 - 1, \ n = 3 : h_3 + 1, \ n = 4 : h_4 - 1$$

are the unique admissible perturbations. Thus the transient polarity is

$$- \to + \to -.$$

Proof to Lemma 3.2 is provided in the Appendix.

Theorem 3.1

In the quasigeometric sequence $H_n^{(+)}$, *for all* $n \ge 5$, *both perturbation branches become admissible, meaning* $h_n + \varepsilon$, $\varepsilon \in \{-1, +1\}$, *is a parent number.*

Proof

Let a perturbed candidate be defined as

$$p_n = h_n + \varepsilon, \varepsilon \in \{-1, +1\} \tag{3.17}$$

Applying the golden section transformation yields:

$$\varphi p_n = \varphi(h_n + \varepsilon) = h_{n+1} + 2\varepsilon + \delta_n + \varepsilon(\varphi - 2).$$

The parent-number condition requires $q_n = \mathrm{rnd}(\varphi p_n)$ *and*

$$\mathrm{rnd}[\varphi(q_n - p_n)] \ne p_n.$$

1. ***The transient regime*** $2 \le n \le 4$ ***is proved in Lemma*** **3.2**

2. ***The stable regime*** $n \ge 5$

$$p_n = h_n + \varepsilon, \varepsilon \in \{-1, +1\}$$

$$\varphi p_n = h_{n+1} + 2\varepsilon + \delta_n + \varepsilon(\varphi - 2)$$

$$|\delta_n| < 1/(2\varphi^4)$$

$$|\delta_n + \varepsilon(\varphi - 2)| < 1/(2\varphi^4) + (2 - \varphi) < 1/2$$

This strict inequality forces the rounded term to safely evaluate to

$$q_n = \mathrm{rnd}(\varphi p_n) = h_{n+1} + 2\varepsilon$$

$$q_n - p_n = h_{n-1} + \varepsilon$$

$$|\delta_{n-1}| < 1/(2\varphi^3)$$

$$|\delta_{n-1} + \varepsilon(\varphi - 2)| < 1/(2\varphi^3) + (2 - \varphi) = 1/2$$

$$\mathrm{rnd}[\varphi(q_n - p_n)] = h_n + 2\varepsilon \neq h_n + \varepsilon = p_n$$

Thus the non-degeneracy condition holds for both $\varepsilon = -1$ *and* $\varepsilon = +1$. *Hence* $h_n - 1$ *and* $h_n + 1$ *are both parent numbers for every* $n \geq 5$. *Here, geometric decay places the perturbed values safely on the required side of the rounding threshold, so both branches become admissible.*

The transient regime is genuinely one-sided: the admissible polarity alternates as $- \to + \to -$, while the opposite perturbation fails the parent-number condition at $n = 2,3,4$. At $n = 5$, both perturbations become admissible and remain so for all $n \geq 5$. Thus $n = 5$ is the first stage of binary stability.

Theorem 3.2 is the sign-reversed counterpart of Theorem 3.1. The two classes of sequences, that is, $H_n^{(+)}$ and $H_n^{(-)}$, are mirror images in terms of the golden-section discrepancy:

$$H_n^{(+)} : - \to + \to - \to (-,+)$$

$$H_n^{(-)} : + \to - \to + \to (-,+)$$

Theorem 3.2

In the quasigeometric sequence $H_n^{(-)}$, $h_2 + 1, h_3 - 1, h_4 + 1$ *are parent numbers. From* $n \geq 5$, *both branches become admissible, that is,* $h_n \pm 1$ *is a parent number.*

Proof

The initial discrepancy of $H_n^{(-)}$ *has the opposite sign and satisfies the reflected bounds of* $H_n^{(+)}$, *therefore the admissible polarity at* $n = 2,3,4$ *is reversed. For* $n \geq 5$, *the absolute error bound* $|\delta_n| < \frac{1}{2\varphi^4}$ *ensures that* δ_n *remains strictly within the central open interval* $\left(-\frac{1}{2}, \frac{1}{2}\right)$ *relative to the rounding boundaries for both perturbation branches.* ■

The initial discrepancy determines the orientation of the transient. The recurrence reverses the sign of the discrepancy at each stage, while geometric decay eventually places the system in the stable two-sided regime.

Theorems 3.1 and 3.2 show that for the two polarity classes of quasigeometric sequences, $n = 5$ is the point at which the parent-number construction enters the binary stability regime. We now need to examine the golden-section discrepancy to understand what is happening to the sequence at this point.

4. Golden Discrepancy Decay Variable

From the Introduction, the discrepancy satisfies the recurrence

$$\delta_{n+1} = -\delta_n / \varphi$$

Repeated application gives

$$\delta_n = \left((-1)^{n-1} / \varphi^{n-1}\right)\delta_1$$

Thus two features govern the discrepancy: its sign reverses at each successive stage, while its magnitude contracts geometrically. These properties are established directly in the Introduction.

It is useful to separate the sequence-dependent initial discrepancy from the universal part of its evolution. Define the golden discrepancy decay variable by

$$D_n = (-1)^{n-1} / \varphi^{n-1} \tag{4.1}$$

Then the discrepancy can be written as

$$\delta_n = D_n \delta_1 \tag{4.2}$$

The distinction is conceptually important. The value δ_1 depends on the particular initial pair of the sequence, whereas D_n depends only on the golden-section transformation at stage n. Consequently, different quasigeometric sequences can possess different initial discrepancy magnitudes while sharing the same decay law.

The first values are

$$D_n = 1, \frac{-1}{\varphi}, \frac{1}{\varphi^2}, \frac{-1}{\varphi^3}, \frac{1}{\varphi^4}, \cdots \tag{4.3}$$

The alternating sign gives the discrepancy an oscillatory character, while the factor $1/\varphi^{n-1}$ gives geometric decay. The golden ratio therefore enters the construction not merely as an approximate ratio between consecutive terms, but as the parameter governing the attenuation of the departure from exact golden scaling. The nearest-integer bound applies initially, so

$$|\delta_1| < 1/2$$

At $n = 5$, the decay variable gives

$$D_5 = 1/\varphi^4$$

Hence

$$|\delta_5| < 1/\left(2\varphi^4\right) \tag{4.4}$$

Since

$$\varphi = \frac{1+\sqrt{5}}{2}$$

we obtain

$$\frac{1}{2\varphi^4} = 0.0729490\cdots \tag{4.5}$$

This number is therefore not introduced as an adjustable parameter. It follows from the initial nearest-integer bound together with four stages of golden discrepancy decay. The resulting chain is half-unit bound → golden discrepancy decay → $n = 5$ → $1/(2\varphi^4) \approx 0.072949$.

5. Conclusions

The investigation establishes that the golden-section discrepancy serves as a natural dynamical framework for nearest-integer quasigeometric sequences, where its magnitude decays geometrically while its sign oscillates according to the universal decay variable

$$D_n = (-1)^{n-1} / \varphi^{n-1}$$

Through the analysis of perturbations of h_n by unity, the research reveals a strict structural evolution across early stages, demonstrating that the transient regime for $2 \le n \le 4$ is strictly one-sided and governed by alternating branch polarities.

The critical transition at the fifth stage ($n = 5$) marks the emergence of binary stability, where the geometric attenuation of the discrepancy reduces its upper bound to $1/2\varphi^4 \approx 0.072949$, allowing both positive and negative perturbation branches to satisfy the non-degeneracy condition.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix: Proof of Lemma 3.2

Put

$$a = -\delta_1.$$

From the hypothesis,

$$1/(2\varphi) < a < 1/2. \tag{3.18}$$

Since

$$\delta_n = \left((-1)^{n-1}/\varphi^{n-1}\right)\delta_1,$$

we obtain

$$\delta_n = (-1)^n a/\varphi^{n-1}. \tag{3.19}$$

Also, using $\varphi^2 = \varphi + 1$,

$$\varphi - 2 = -1/\varphi^2. \tag{3.20}$$

For a perturbation

$$p_n(\varepsilon) = h_n + \varepsilon,$$

the golden-section transformation gives

$$\begin{aligned}\varphi p_n(\varepsilon) &= \varphi(h_n + \varepsilon)\\ &= h_{n+1} + \delta_n + \varphi\varepsilon\\ &= h_{n+1} + 2\varepsilon + \delta_n + \varepsilon(\varphi - 2).\end{aligned}$$

Define the rounding error

$$E_n(\varepsilon) = \delta_n - \varepsilon/\varphi^2. \tag{3.21}$$

Hence

$$\varphi p_n(\varepsilon) = h_{n+1} + 2\varepsilon + E_n(\varepsilon). \tag{3.22}$$

The branch having the same sign as δ_n. *From* (3.19), *the sign of* δ_n *is* $(-1)^n$. *Let therefore*

$$\varepsilon = (-1)^n. \tag{3.23}$$

Then

$$E_n(\varepsilon) = (-1)^n\left(a/\varphi^{n-1} - 1/\varphi^2\right).$$

For $2 \le n \le 4$,

$$\left|a/\varphi^{n-1} - 1/\varphi^2\right| < 1/\varphi^2 < 1/2.$$

Therefore

$$q_n = \mathrm{rnd}\left(\varphi p_n(\varepsilon)\right) = h_{n+1} + 2\varepsilon. \tag{3.24}$$

It follows that

$$q_n - p_n(\varepsilon) = h_{n-1} + \varepsilon.$$

Applying the golden-section transformation again,

$$\varphi\left(q_n - p_n(\varepsilon)\right) = \varphi\left(h_{n-1} + \varepsilon\right)$$
$$= h_n + 2\varepsilon + \delta_{n-1} - \varepsilon/\varphi^2.$$

Since $\varepsilon = (-1)^n$,

$$\delta_{n-1} = (-1)^{n-1} a/\varphi^{n-2} = -\varepsilon a/\varphi^{n-2}.$$

Thus

$$\varphi\left(q_n - p_n(\varepsilon)\right) = h_n + 2\varepsilon - \varepsilon\left(a/\varphi^{n-2} + 1/\varphi^2\right). \tag{3.25}$$

Define

$$T_n = 1/\varphi^2 + a/\varphi^{n-2}. \tag{3.26}$$

Because $2 \le n \le 4$, *the smallest possible value occurs at* $n = 4$. *Using* (3.18),

$$T_n \ge T_4 > 1/\varphi^2 + 1/\left(2\varphi^3\right).$$

But $\varphi^3 = 2\varphi + 1$, *and hence*

$$1/\varphi^2 + 1/\left(2\varphi^3\right) = 1/2. \tag{3.27}$$

Therefore

$$T_n > 1/2,\ 2 \le n \le 4. \tag{3.28}$$

Moreover, from $a < 1/2$,

$$T_n < 1/\varphi^2 + 1/2 < 1.$$

Consequently the correction in (3.25) *crosses exactly one nearest-integer boundary in the direction* $-\varepsilon$. *Therefore,*

$$\mathrm{rnd}\left[\varphi\left(q_n - p_n(\varepsilon)\right)\right] = h_n + \varepsilon = p_n(\varepsilon). \tag{3.29}$$

The non-degeneracy condition therefore fails. Hence the branch

$$\varepsilon = (-1)^n$$

is inadmissible throughout $2 \le n \le 4$.

The opposite branch. *There remains*

$$\varepsilon = (-1)^{n+1}. \tag{3.30}$$

For this sign,

$$E_n(\varepsilon) = (-1)^n\left(a/\varphi^{n-1} + 1/\varphi^2\right). \tag{3.31}$$

Set

$$R_n = 1/\varphi^2 + a/\varphi^{n-1}. \tag{3.32}$$

Using (3.18),

$$R_3 > 1/\varphi^2 + 1/\left(2\varphi^3\right) = 1/2,$$

whereas

$$R_4 < 1/\varphi^2 + 1/\left(2\varphi^3\right) = 1/2. \tag{3.33}$$

Thus the unique rounding-boundary transition occurs between stages 3 *and* 4.

For $n=2,3$, $R_n>1/2$. *Equation* (3.22) *therefore rounds by one unit in the direction* $(-1)^n=-\varepsilon$, *giving*

$$q_n=h_{n+1}+\varepsilon.$$

Hence

$$q_n-p_n(\varepsilon)=h_{n-1},$$

and therefore

$$\mathrm{rnd}\left[\varphi\left(q_n-p_n(\varepsilon)\right)\right]=\mathrm{rnd}\left(\varphi h_{n-1}\right)=h_n.$$

Since

$$h_n\neq h_n+\varepsilon=p_n(\varepsilon),$$

the non-degeneracy condition is satisfied.

For $n=4$, *Equation* (3.33) *gives* $R_4<1/2$, *so no additional rounding displacement occurs and*

$$q_4=h_5+2\varepsilon.$$

Here $\varepsilon=-1$, *and hence*

$$q_4-p_4=h_3-1.$$

Furthermore,

$$\varphi\left(h_3-1\right)=h_4-2+\left(\delta_3+1/\varphi^2\right).$$

From (3.18),

$$-1/\left(2\varphi^2\right)<\delta_3<-1/\left(2\varphi^3\right),$$

so

$$\left|\delta_3+1/\varphi^2\right|<1/2.$$

Therefore

$$\mathrm{rnd}\left[\varphi\left(h_3-1\right)\right]=h_4-2\neq h_4-1=p_4.$$

Thus the non-degeneracy condition also holds at $n=4$.

We conclude that for every $2\le n\le 4$, *the unique admissible sign is*

$$\varepsilon_n=(-1)^{n+1}.$$

Hence

$$h_2-1,\ h_3+1,\ h_4-1$$

are the transient parent numbers, with polarity

$$-\to+\to-.$$

□