

Collisionless Kinetic Resonances in Open Cosmological Voids

Nikolay N. Fimin 

Federal Research Center, M. V. Keldysh Institute of Applied Mathematics of the Russian Academy of Sciences, Moscow, Russia
Email: oberon@kiam.ru

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Abstract

Cosmological voids are underdense regions whose late-time evolution is controlled by expansion, tidal environment and boundary flows. We treat a void as an open collisionless dark-matter cell, where “collisionless” means that two-body encounters are neglected and “open” means that the phase-space trajectories generated by the transport equation (the kinetic characteristics) may cross the cell boundary, governed by the local Newtonian Vlasov-Poisson system—collisionless phase-space transport coupled self-consistently to Newtonian Poisson gravity—with cosmological constant Λ . The central object is the self-consistent linear Vlasov response resolvent, *i.e.* the inverse of the coupled linear frequency-domain response operator wherever that inverse exists, obtained by combining transport along the stationary-reference characteristics with the Newtonian Poisson response. For complex frequency $\omega \in \mathbb{C}$, it is analytically continued from the upper frequency half-plane $\text{Im } \omega > 0$ across the continuous van Kampen spectrum, *i.e.* the continuum of characteristic transport frequencies. In gravity-dominated cells the response remains associated with reversible phase mixing, understood as dephasing along bounded or mixed characteristics. In the asymptotic void regime, however, the Hessian $\mathcal{H} = \nabla \nabla \Psi$ (the matrix of second derivatives of the effective potential Ψ) has three negative eigenvalues, and the characteristic flow becomes hyperbolic, meaning that nearby trajectories separate exponentially. For perturbations belonging to a Sedláček-type analytic test class, defined by momentum-strip analyticity together with derivative bounds and decay, this geometry permits continuation of the projected resolvent to the lower half-plane, where isolated poles whose generalized modes carry flux away from the cell define Gamow-like kinetic resonances. Projection on the dominant expanding characteristic reduces the spatial response to a Weber, or parabolic-cylinder, equation; finite cosmic-web cell boundaries regularise the outgoing modes by truncating their formal spatial growth, while the leading projected inverted-oscillator model defines a discrete resonance lattice. The exact equal spacing belongs to this

projected Weber normal form—the leading quadratic one-channel representative—rather than to the full six-dimensional self-consistent Fredholm determinant, *i.e.* the determinant of identity minus the response operator. Pole dominance—long-time dominance of pole residues over the continued-spectrum remainder—is conditional on the analytic and decay assumptions stated below. The fundamental mode gives a characteristic void-clearing rate and implies a kinetic outflow scaling proportional to $\rho_\phi \lambda_w^{3/2} R^5$, where ρ_ϕ is a characteristic boundary mass density, $\lambda_w > 0$ is the negative of the most negative Hessian eigenvalue, and R is the characteristic cell radius. The framework is proposed as a continuum theoretical model for resonant relaxation (pole-controlled decay), phase-space leakage (transport across the cell boundary) and kinematic memory (persistent velocity or phase-space signatures) in large cosmological voids, with possible observational diagnostics in void density profiles, cell-frame velocity gradients and boundary anisotropies.

Keywords

Cosmological Voids, Collisionless Dark Matter, Vlasov-Poisson- Λ Model, Kinetic Resonances, Phase-Space Dynamics, Large-Scale Structure

1. Introduction

Cosmological voids occupy most of the volume of the large-scale structure and form the underdense part of the network of clusters, filaments, sheets and voids. Their evolution is usually described in terms of expansion, evacuation of matter, tidal deformation by the surrounding cosmic web and the hierarchy of void merging and embedding. The importance of voids as dynamical and cosmological objects was emphasized by Peebles [1], while the hierarchical picture of void evolution was developed by Sheth and van de Weygaert [2]. Modern web classifiers based on the tidal or velocity-shear tensor, *i.e.* on matrices of spatial derivatives of the gravitational or velocity field, such as those introduced by Hahn *et al.* [3] and Forero-Romero *et al.* [4], distinguish voids, sheets, filaments and knots through the eigenvalue signature of a local Hessian-type tensor. Voids are also used as probes of cosmological expansion and large-scale flows [5]–[7].

The dynamical substrate of this picture is collisionless dark matter. Before shell crossing—the formation of multiple velocity streams at the same position—fluid-like descriptions are often adequate, but the full multi-stream phase-space dynamics are governed by the Vlasov-Poisson system, namely collisionless transport coupled self-consistently to the Newtonian Poisson equation. This distinction is important for voids because their interiors are open regions connected to surrounding walls and filaments by transit orbits and boundary flows. Phase-space approaches to structure formation, including direct Vlasov treatments, phase-space tessellations (piecewise-simplex reconstructions of the phase-space sheet) and topological stream classification (classification by the number and connectiv-

ity of velocity streams), have shown that the six-dimensional distribution function over three positions and three velocities contains information that is not captured by a single-stream fluid approximation [8]-[12]. The present paper therefore treats a void not as an empty geometric domain, but as an open collisionless kinetic cell embedded in the cosmic web.

The main question addressed here is whether such an open cell can support macroscopic resonant response modes. In a bounded gravitational system the linear Vlasov operator possesses the continuous van Kampen spectrum, the continuum generated by transport along characteristics, and relaxation is associated with phase mixing and Landau-type analytic continuation [13] [14]. For open or hyperbolic flows the spectral picture is different. The relevant response is no longer exhausted by reversible mixing on bounded characteristics; poles of analytically continued resolvents may control the long-time behaviour of the projected macroscopic density. Similar ideas appear in kinetic theory through analytic continuation of the Vlasov resolvent [15]-[17] and in the theory of resonant states through rigged Hilbert spaces—test-space/Hilbert-space/dual triples—and outgoing generalized Gamow vectors [18] [19].

We apply this spectral language to void cells in a local Vlasov-Poisson- Λ model. The cosmological constant enters the effective Newtonian potential as a universal quadratic term, while the Newtonian part encodes the density and tidal environment of the cell. The local Hessian of this effective potential classifies the characteristic geometry. In gravity-dominated nodes and filament cores, the relevant characteristic sets are bounded or mixed and the kinetic response remains tied to the continuous spectrum. In a sufficiently underdense void, the effective Hessian may have three expanding directions. The projected flow along the dominant expanding characteristic is then hyperbolic, and the analytic continuation of the projected Vlasov resolvent produces isolated poles in the lower half-plane. These poles are interpreted as macroscopic kinetic resonances of the open void cell.

The formulation used throughout is strictly Newtonian and local. The cell frame is the fixed local Newtonian frame in which the spatial cell and its reference state are time independent. Denote the stationary reference phase-space distribution by $F_0(\mathbf{x}, \mathbf{v})$; “stationary” means $\partial_t F_0 = 0$ in this frame, with $\partial_t \equiv \partial/\partial t$. The independent variables are the physical Euclidean position $\mathbf{x}$, Newtonian time t and the physical cell-frame velocity $\mathbf{v} = d\mathbf{x}/dt$. Here c is the speed of light and Λ is the cosmological constant. No space-time metric, Friedmann-Lemaître-Robertson-Walker (FLRW) line element or relativistic field equation is introduced. The cosmological term is retained explicitly as the homothetic, *i.e.* linear-in-position, acceleration $(c^2\Lambda/3)\mathbf{x}$. A comoving substitution $\mathbf{x} = a(t)\mathbf{q}$, where $a(t)$ is a prescribed scale factor and $\mathbf{q}$ is a comoving coordinate, is itself a time-dependent spatial homothety; it transfers this explicit term into dilation, inertial and velocity-drag terms and, in general, replaces the autonomous (explicitly time-independent) local generator by a non-autonomous (explicitly time-dependent) one. Such a transformation is therefore outside the stationary spectral problem consid-

ered here. In particular, no Hubble-flow/peculiar-velocity decomposition is used.

This $F_0(\mathbf{x}, \mathbf{v})$ is the stationary reference phase-space distribution of the finite open cell, not a homogeneous cosmological background. Its Newtonian potential is generated by the actual finite-cell mass density $\rho_0(\mathbf{x}) = m \int_{\mathbb{R}^3} F_0(\mathbf{x}, \mathbf{v}) d^3v$, where m is the particle mass and d^3v is the Euclidean velocity-volume element. No infinite uniform density is introduced, no spatial mean $\bar{\rho}$ is subtracted, and the replacement $\rho_0 \mapsto \rho_0 - \bar{\rho}$ is never made; hence the Jeans swindle is not used. The Λ term remains a separate force and does not compensate for any removed matter background.

The term “Gamow” is used here in this limited spectral sense. The resonant modes are not square-integrable eigenfunctions of a closed self-adjoint operator, *i.e.* an operator with closed graph that equals its Hilbert-space adjoint; they are outgoing generalized modes defined after analytic continuation and finite-cell regularisation. Projection on the dominant expanding channel reduces the spatial problem to the Weber equation, the standard parabolic-cylinder differential equation [20] [21], and the finite polyhedral boundary of the cosmic-web cell cuts off the formal spatial growth of outgoing Weber states. This yields a discrete lattice of complex frequencies, whose fundamental mode gives a characteristic clearing rate for the projected density contrast.

The equally spaced lattice derived in Section 5 is the exact resonance set of the one-channel quadratic Weber normal form. It is not assumed to be the exact zero set of the full self-consistent Fredholm determinant. Transverse channels, non-quadratic terms and literal matching at the finite cell boundary enter through a reduced self-energy—the Schur-complement correction generated by the eliminated channels—and can shift or split the ideal poles. Likewise, the analytic test space used for contour deformation is a sufficient mathematical domain, not a claim that every fine-grained nonlinear dark-matter distribution is analytic.

The astrophysical role of the construction is deliberately modest. We do not claim a direct fit to the Hubble tension—the discrepancy between locally and early-Universe inferred expansion rates—or to a particular void catalogue. Instead, the paper develops a continuum kinetic mechanism that may affect void relaxation, phase-space leakage and boundary outflows. The predicted signatures are qualitative but testable in principle: modified void-density relaxation, cell-frame velocity gradients correlated with the local Hessian, anisotropic outflows through void boundaries and kinematic memory in the surrounding filamentary network. In this sense the model is intended as a theoretical complement to numerical and observational studies of void dynamics.

The paper is organized as follows. Section 2 formulates the linear kinetic response and the Fredholm representation. Section 3 describes the gravity-dominated limit and its continuous spectrum. Section 4 discusses mixed saddle topologies. Section 5 derives the hyperbolic void limit, the Weber reduction and the leading projected Gamow-like resonance lattice. The final part of Section 5 interprets the leading mode as a void-clearing law and discusses observable diagnostics.

The appendices collect the contour-deformation argument, the Sedláček functional class and the finite-cell normalization of the resonant amplitudes.

Notation

General Mathematical Conventions

- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$: real, complex, integer, positive-integer and nonnegative-integer sets; $i^2 = -1$, while $\operatorname{Re} z$ and $\operatorname{Im} z$ are the real and imaginary parts of $z \in \mathbb{C}$.
- $|z|$ is the absolute value or complex modulus and $|\mathbf{x}|$ is the Euclidean norm; $|\mathcal{V}|$ denotes spatial volume when $\mathcal{V}$ is a domain. The symbol $\|\cdot\|$ is the norm in the function or operator space specified locally, while d^3x and d^3v are Euclidean volume elements. The operators ∇_x , ∇_v and ∇_p are gradients in position, velocity and momentum; $\nabla \cdot$ and Δ are the spatial divergence and Laplacian. Partial derivatives are denoted by $\partial_j = \partial/\partial x_j$ and analogously by ∂_x , ∂_r , ∂_v or ∂_p for the displayed variable. Finally, $\det$ is the determinant, Tr is the operator trace and diag forms a diagonal matrix.
- A^* denotes the Hilbert-space adjoint of an operator A , whereas z^* or $a^*(x)$ denotes complex conjugation of a scalar or function; $\langle \cdot, \cdot \rangle$ denotes the inner product defined in the relevant space.
- π is the circular constant, e^z is the natural exponential and $\ln z$ is the natural logarithm on the branch specified locally. The notation $A \equiv B$ is a definition, $A \mapsto B$ denotes a stated mapping or replacement, $A \simeq B$ is leading-order equality in the stated approximation, $A \propto B$ is equality up to a factor independent of the displayed scaling variables, and $A \gg B$ denotes the scale separation $|A/B| \gg 1$. Also, $A = \mathcal{O}(g)$ means that $|A/g|$ remains bounded, whereas $A = o(g)$ means that $A/g \rightarrow 0$, in the stated limit.

Constants, Coordinates and Cell Geometry

- G , c and m : Newtonian gravitational constant, speed of light and dark-matter particle mass.
- Λ : Cosmological constant; $\kappa_\Lambda = c^2 \Lambda/3$ is its homothetic Newtonian acceleration coefficient.
- $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, t , $\mathbf{v} = d\mathbf{x}/dt$ and $\mathbf{p} = (p_1, p_2, p_3) = m\mathbf{v}$: physical position, Newtonian time, cell-frame velocity and canonical momentum; x_j and p_j are their Cartesian components, and an overdot denotes differentiation with respect to t .
- $\mathcal{V}_{\text{cell}}$, $V_{\text{cell}} = |\mathcal{V}_{\text{cell}}|$ and $\partial\mathcal{V}_{\text{cell}}$: finite polyhedral spatial domain, its volume and its boundary; $\mathbf{n}_{\text{out}}$ is the outward unit normal.
- $I_{\text{cell}} = [-R_{\text{cell}}, R_{\text{cell}}]$: projected one-dimensional cell interval; R and R_{cell} are the characteristic void radius and effective boundary radius.
- $\rho_0(\mathbf{x}) = m \int F_0(\mathbf{x}, \mathbf{v}) d^3v$ and $\rho_1(\mathbf{x}, t) = m \int f(\mathbf{x}, \mathbf{v}, t) d^3v$: stationary reference and linear perturbation mass densities, where f is the time-dependent linear perturbation of F_0 . The symbol $\rho_\partial(\mathbf{x}, t)$ denotes the mass density on $\partial\mathcal{V}_{\text{cell}}$; without arguments, ρ_∂ is a characteristic boundary value used only in

scaling laws. The positive constant ρ_{norm} is a fixed local normalization, not a subtracted mean; $\bar{\rho}$ denotes a spatial mean only when discussing the subtraction that is not used here.

Potentials and Geometric Rates

- Φ_0 and Φ_1 : stationary reference and perturbed self-consistent Newtonian potentials.
- $\Psi_0 = \Phi_0 - \kappa_\Lambda |\mathbf{x}|^2/2$: effective reference potential per unit mass.
- $\mathcal{H}_{ij} = \partial_i \partial_j \Psi_0$: Hessian matrix of Ψ_0 ; λ_j are its eigenvalues. The point $\mathbf{x}_c$ is a non-degenerate critical point, meaning $\nabla \Psi_0(\mathbf{x}_c) = 0$ and $\det \mathcal{H}(\mathbf{x}_c) \neq 0$, and $\rho_c \equiv \rho_0(\mathbf{x}_c)$ is the actual stationary density there.
- k_M : Morse index, equal to the number of strictly negative eigenvalues of $\mathcal{H}$.
- $\lambda_* = -\lambda_{\text{principal}}(\mathcal{H}) > 0$: dominant kinematic stiffness, where $\lambda_{\text{principal}}$ is the most negative Hessian eigenvalue.
- $\mu_j = \sqrt{-\lambda_j}$ for $\lambda_j < 0$: hyperbolic increments; $\mu = \mu_1$, $\mu_\perp = \max(\mu_2, \mu_3)$ and $\Delta_\mu = \mu - \mu_\perp$ after ordering $\mu_1 \geq \mu_2 \geq \mu_3$.
- $\Omega_j = \sqrt{\lambda_j}$ for $\lambda_j > 0$: local oscillatory frequencies; $\alpha(\mathbf{x}) = 4\pi G \rho_0(\mathbf{x})/\kappa_\Lambda$ is the local density parameter.

Kinetic, Operator and Spectral Notation

- F_0 and f : stationary reference distribution and its time-dependent linear perturbation. The normalized one-channel projection is $f_1(x, p, t)$, chosen so that $\delta(x, t) = \int_{\mathbb{R}} f_1(x, p, t) dp = \rho_{1,p}(x, t)/\rho_{\text{norm}}$, where $\rho_{1,p}$ is the projected perturbation density; $f_1(x, p) \equiv f_1(x, p, 0)$ and $\varphi(x) \equiv \delta(x, 0)$.
- For a differentiable test function $g(\mathbf{x}, \mathbf{v})$, $\mathcal{L}_{\text{tr}} g \equiv \mathbf{v} \cdot \nabla_{\mathbf{x}} g + (-\nabla_{\mathbf{x}} \Phi_0 + \kappa_\Lambda \mathbf{x}) \cdot \nabla_{\mathbf{v}} g$ is the stationary-reference characteristic transport generator, and $\mathcal{A} = -i\mathcal{L}_{\text{tr}}$ is its frequency-domain spectral form, so the transport resolvent is $(\omega - \mathcal{A})^{-1}$; self-consistency enters through the Poisson response operator $\hat{K}(\omega)$. In the deep-void quadratic reduction, $\mathcal{L}_{6D} = (\mathbf{p}/m) \cdot \nabla_{\mathbf{x}} - m(\mathcal{H}\mathbf{x}) \cdot \nabla_{\mathbf{p}} = \sum_{j=1}^3 \mathcal{L}_j$, with $\mathcal{L}_j = (\mathbf{p}_j/m) \partial_{x_j} - m\lambda_j x_j \partial_{p_j}$, and $\mathcal{A}_{6D} = -i\mathcal{L}_{6D}$.
- $\omega \in \mathbb{C}$: Fourier-Laplace frequency; $\omega_r \in \mathbb{R}$ is a boundary value and ω_n is an isolated resonance pole.
- $K(\mathbf{x}, \mathbf{x}', \omega)$ and $\hat{K}(\omega)$: response kernel and its integral operator; $S(\mathbf{x}, \omega)$ is the inhomogeneous source and $\mathcal{I}$ is the identity operator.
- $\mathcal{D}(\omega) = \det(\mathcal{I} - \hat{K}(\omega))$ and $\mathcal{D}_2(\omega) = \det_2(\mathcal{I} - \hat{K}(\omega))$: ordinary Fredholm determinant and regularised Hilbert-Schmidt determinant; if ζ_j are the eigenvalues of $\hat{K}$ counted with algebraic multiplicity, then $\det_2(\mathcal{I} - \hat{K}) = \prod_j (1 - \zeta_j) e^{\zeta_j}$.
- $\mathfrak{S}_1$ and $\mathfrak{S}_2$: trace-class and Hilbert-Schmidt ideals; for singular values $s_n(A)$ (the square roots of the eigenvalues of $A^* A$, where A^* is the adjoint of A), $A \in \mathfrak{S}_1$ means $\sum_n s_n(A) < \infty$, while $A \in \mathfrak{S}_2$ means $\sum_n s_n(A)^2 < \infty$.
- $\mathcal{S} \subset L^2 \subset \mathcal{S}'$: Sedláček-type test space, Hilbert space of square-integrable functions and continuous dual; $p_* > 0$ is the independent half-width of the analytic momentum strip.

- For $N \in \mathbb{N}$, P_N is the projection onto the first N Hermite modes; $\mathcal{A}_N = P_N \mathcal{A}_{6D} P_N$ is the corresponding projected spectral generator, while P and $Q = \mathcal{I} - P$ are complementary channel projections and $\mathcal{A}_W$ is the limiting one-channel spectral operator.
- $\Sigma_P(\omega) = P \mathcal{A}_{6D} Q (\omega - Q \mathcal{A}_{6D} Q)^{-1} Q \mathcal{A}_{6D} P$ (when the displayed inverse exists): Schur-complement self-energy of the eliminated channels; $\mathcal{R}(\omega)$ is the projected spatial resolvent.
- Γ_L and Γ_M : Bromwich contour and lower-half-plane deformation arc.
- $\chi > 0$: auxiliary action scale used only to nondimensionalise the Weber normal form; it cancels from the pole frequencies.
- ξ and ν : dimensionless Weber coordinate and parabolic-cylinder index; $D_\nu(\xi)$ is the parabolic-cylinder function.
- For $n \in \mathbb{N}_0$, ψ_n , $\tilde{\psi}_n$, C_n and $\varphi(x)$ are the right resonant mode, adjoint mode, residue amplitude and initial projected density profile.
- $\delta_D^{(d)}$ is the d -dimensional Dirac distribution, characterized by $\int_{\mathbb{R}^d} \delta_D^{(d)}(y - y_0) h(y) d^d y = h(y_0)$ for a test function h ; $\delta_D \equiv \delta_D^{(1)}$. The symbol δ_{nq} is the Kronecker symbol, equal to 1 for $n = q$ and 0 otherwise. The dimensionless projected density contrast is
$$\delta(x, t) = \rho_{1,P}(x, t) / \rho_{\text{norm}} = \int_{\mathbb{R}} f_1(x, p, t) dp.$$

Observable and Asymptotic Quantities

- u , $u(x)$ and $u(r)$: physical cell-frame outflow velocity, its principal-axis component and its radial component, where $r = |x|$ and $e_r = x/r$ is the outward radial unit vector for $r > 0$; no Hubble-flow subtraction is involved.
- $\mathcal{P}_K$: kinetic luminosity, *i.e.* the kinetic-energy flux through the cell boundary.
- $\Delta H_{\text{cell}} = \partial u / \partial x$: projected cell-frame velocity-gradient diagnostic.

2. Integral Formulation and the Fredholm Determinant

Let $\mathcal{V}_{\text{cell}} \subset \mathbb{R}^3$ be the bounded physical spatial domain occupied by the finite cell, let $V_{\text{cell}} = |\mathcal{V}_{\text{cell}}|$ be its volume, and let $\mathbf{n}_{\text{out}}$ denote the outward unit normal on $\partial \mathcal{V}_{\text{cell}}$. Throughout this section, $\mathbf{x} \in \mathbb{R}^3$ is physical Euclidean position, t is Newtonian time, $\mathbf{v} = d\mathbf{x}/dt$ is cell-frame velocity, G is the Newtonian gravitational constant, c is the speed of light and m is the particle mass. Let $F_0(\mathbf{x}, \mathbf{v})$ denote the stationary reference phase-space distribution and let $\Phi_0(\mathbf{x})$ be its self-consistent Newtonian potential. With $\kappa_\Lambda = c^2 \Lambda / 3$, the stationary reference state satisfies

$$\mathbf{v} \cdot \nabla_{\mathbf{x}} F_0 + (-\nabla_{\mathbf{x}} \Phi_0 + \kappa_\Lambda \mathbf{x}) \cdot \nabla_{\mathbf{v}} F_0 = 0, \quad (1)$$

$$\Delta \Phi_0 = 4\pi G \rho_0(\mathbf{x}), \quad (2)$$

$$\rho_0(\mathbf{x}) = m \int_{\mathbb{R}^3} F_0(\mathbf{x}, \mathbf{v}) d^3 v, \quad (3)$$

$$\Phi_0(\mathbf{x}) \rightarrow 0 \quad (|\mathbf{x}| \rightarrow \infty), \quad (4)$$

where $\nabla_{\mathbf{x}}$ and Δ are the spatial gradient and Laplacian, and ρ_0 is the actual stationary mass density of the finite cell. No homogeneous density reservoir and

no subtraction of a spatial mean are introduced, so neither the reference problem nor its linearisation uses the Jeans swindle.

The cell is open for kinetic characteristics. A characteristic may leave $\mathcal{V}_{\text{cell}}$, while a stationary through-flow, when present, is specified by time-independent incoming data on the inflow phase-space boundary

$$\Gamma_- = \left\{ (\mathbf{x}, \mathbf{v}) \in \partial\mathcal{V}_{\text{cell}} \times \mathbb{R}^3 : \mathbf{v} \cdot \mathbf{n}_{\text{out}} < 0 \right\}. \quad (5)$$

Zero inflow is included as a special case. The explicit force $\kappa_\Lambda \mathbf{x}$ is homothetic. Replacing $\mathbf{x}$ by a comoving variable $a(t)\mathbf{q}$ would introduce dilation, inertial and velocity-drag terms and would change the autonomous generator used in the Fourier-Laplace and Fredholm analysis. No comoving coordinates, scale factor or peculiar-velocity convention is employed below; no relativistic metric is required for this generalized Newtonian formulation.

Let $f(\mathbf{x}, \mathbf{v}, t)$ be a small perturbation of F_0 and let $\Phi_1(\mathbf{x}, t)$ be its perturbed self-consistent potential. The linearized Vlasov equation is

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} \Phi_1 \cdot \nabla_{\mathbf{v}} F_0 + (-\nabla_{\mathbf{x}} \Phi_0 + \kappa_\Lambda \mathbf{x}) \cdot \nabla_{\mathbf{v}} f = 0. \quad (6)$$

The acceleration $\kappa_\Lambda \mathbf{x}$ is the generalized Newtonian cosmological term discussed in Refs. [22]-[24]. The unperturbed phase-space characteristics $(\mathbf{x}'(\tau), \mathbf{v}'(\tau))$, with backward time $\tau \leq 0$, satisfy

$$\frac{d\mathbf{x}'}{d\tau} = \mathbf{v}', \quad \frac{d\mathbf{v}'}{d\tau} = -\nabla_{\mathbf{x}'} \Phi_0(\mathbf{x}') + \kappa_\Lambda \mathbf{x}'. \quad (7)$$

For the complex Fourier-Laplace frequency $\omega \in \mathbb{C}$, apply the one-sided transform

$$f(\mathbf{x}, \mathbf{v}, \omega) = \int_0^\infty f(\mathbf{x}, \mathbf{v}, t) e^{i\omega t} dt, \quad \text{Im } \omega > 0. \quad (8)$$

Integration along the unperturbed characteristics gives

$$f(\mathbf{x}, \mathbf{v}, \omega) = \int_{-\infty}^0 e^{-i\omega\tau} \left[\nabla_{\mathbf{x}'} \Phi_1(\mathbf{x}', \omega) \cdot \nabla_{\mathbf{v}'} F_0(\mathbf{x}', \mathbf{v}') + f(\mathbf{x}', \mathbf{v}', 0) \right] d\tau. \quad (9)$$

Define the physical perturbation density by $\rho_1(\mathbf{x}, t) = m \int_{\mathbb{R}^3} f(\mathbf{x}, \mathbf{v}, t) d^3v$. The perturbed Poisson equation is $\Delta \Phi_1 = 4\pi G \rho_1$ with $\Phi_1(\mathbf{x}, t) \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$. Denote the free-space Green function by

$$\mathcal{G}(\mathbf{x}, \mathbf{x}'') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}''|}, \quad \Delta_{\mathbf{x}} \mathcal{G}(\mathbf{x}, \mathbf{x}'') = \delta_D^{(3)}(\mathbf{x} - \mathbf{x}''), \quad (10)$$

where $\delta_D^{(3)}$ is the three-dimensional Dirac distribution, defined by $\int_{\mathbb{R}^3} \delta_D^{(3)}(\mathbf{y} - \mathbf{y}_0) h(\mathbf{y}) d^3y = h(\mathbf{y}_0)$ for every test function h . The whole-space Poisson condition is distinct from the open kinetic boundary at $\partial\mathcal{V}_{\text{cell}}$ and from the outgoing condition used later in the projected Weber problem.

Let $\hat{K}(\omega)$ be the potential-response integral operator with kernel $K(\mathbf{x}, \mathbf{x}', \omega)$,

$$(\hat{K}(\omega)\Phi_1)(\mathbf{x}) = \int_{\mathcal{V}_{\text{cell}}} K(\mathbf{x}, \mathbf{x}', \omega) \Phi_1(\mathbf{x}', \omega) d^3\mathbf{x}'. \quad (11)$$

Here $\mathbf{x}''$ and $\mathbf{v}$ are the phase-space variables at $\tau = 0$, while

$(\mathbf{x}'(\tau; \mathbf{x}'', \mathbf{v}), \mathbf{v}'(\tau; \mathbf{x}'', \mathbf{v}))$ is the corresponding backward characteristic; below, $\mathbf{v}'$ in $\nabla_{\mathbf{v}'} F_0$ denotes that characteristic velocity. Then the perturbed potential obeys the inhomogeneous Fredholm equation

$$\Phi_1(\mathbf{x}, \omega) = (\hat{K}(\omega)\Phi_1)(\mathbf{x}) + S(\mathbf{x}, \omega), \quad (12)$$

where S is the source generated by the initial perturbation. The kernel is

$$K(\mathbf{x}, \mathbf{x}', \omega) = 4\pi Gm \int_{\mathbb{R}^3} d^3x'' \int_{\mathbb{R}^3} d^3v \int_{-\infty}^0 d\tau e^{-i\omega\tau} \nabla_{\mathbf{x}''} \mathcal{G}(\mathbf{x}, \mathbf{x}'') \cdot \nabla_{\mathbf{v}'} F_0(\mathbf{x}', \mathbf{v}') \delta_D^{(3)}(\mathbf{x}' - \mathbf{x}'(\tau; \mathbf{x}'', \mathbf{v})). \quad (13)$$

Sufficient compactness hypothesis. Assume that $\mathcal{V}_{\text{cell}}$ is bounded and Lipschitz, meaning that its boundary is locally the graph of a Lipschitz function; finite polyhedral cells are included. Assume that the reference characteristic flow is C^1 (continuously differentiable) away from a measure-zero grazing set of characteristics tangent to the boundary and that, for an integer $k \geq 1$, $F_0 \in C^k$ (it has continuous derivatives through order k) has spatial support in $\overline{\mathcal{V}_{\text{cell}}}$, where the overline denotes closure. For multi-indices $\alpha, \beta \in \mathbb{N}_0^3$ (triples of nonnegative integers) entering the kernel, assume

$$|\partial_{\mathbf{x}}^{\alpha} \partial_{\mathbf{v}}^{\beta} F_0(\mathbf{x}, \mathbf{v})| \leq C_{\alpha\beta} (1 + |\mathbf{v}|)^{-s-|\beta|}, \quad s > 4, \quad (14)$$

where $\partial_{\mathbf{x}}^{\alpha} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ and analogously for $\partial_{\mathbf{v}}^{\beta}$, $C_{\alpha\beta}$ are finite constants, and s is a velocity-decay exponent, not the particle mass. Assume also that the retarded susceptibility, i.e. the causal linear response map from the perturbing potential to the induced density, is bounded in a specified velocity-weighted Hilbert space $H_w = L^2(\mathcal{V}_{\text{cell}} \times \mathbb{R}^3, w(\mathbf{v}) d^3x d^3v)$ with positive weight w . After velocity integration, require the effective potential-response kernel to belong to $L^2(\mathcal{V}_{\text{cell}} \times \mathcal{V}_{\text{cell}})$. Then $\hat{K}(\omega)$ belongs to the Hilbert-Schmidt ideal $\mathfrak{S}_2$ and is therefore compact—it maps bounded sets to relatively compact sets—for $\text{Im } \omega > 0$. These conditions are sufficient, not necessary. Smoothness of F_0 alone does not make the raw three-dimensional response trace class. The trace-class condition $\hat{K}(\omega) \in \mathfrak{S}_1$ is used only when the singular values defined in the Notation section are summable, for example after additional smoothing or a genuinely finite-dimensional Hermite-spatial Galerkin reduction, i.e. a finite modal projection. Otherwise the determinant is the regularised Hilbert-Schmidt determinant $\det_2(\mathcal{I} - \hat{K})$ [25].

The map $\omega \mapsto \hat{K}(\omega)$ is analytic in the upper half-plane in the corresponding operator norm. For $\text{Im } \omega > 0$, the factor $e^{-i\omega\tau}$ ensures uniform convergence of the time integral on compact subsets and permits differentiation under the integral sign.

In the trace-class reduction, isolated resonances are represented by zeros of

$$\mathcal{D}(\omega) \equiv \det(\mathcal{I} - \hat{K}(\omega)) = 0, \quad (15)$$

where $\mathcal{I}$ is the identity operator. With only Hilbert-Schmidt regularity, use instead $\mathcal{D}_2(\omega) = \det_2(\mathcal{I} - \hat{K}(\omega))$; for eigenvalues ζ_j of $\hat{K}$, counted with alge-

braic multiplicity, the regularised determinant is $\det_2(\mathcal{I} - \hat{K}) = \prod_j (1 - \zeta_j) e^{\zeta_j}$. Below, $\mathcal{D}$ denotes the appropriate ordinary or regularised determinant. The trace-class formula is understood through the Plemelj-Smithies expansion, the trace expansion of the Fredholm determinant.

For analytic continuation, use the rigged Hilbert-space triple $\mathcal{S} \subset L^2 \subset \mathcal{S}'$, where $\mathcal{S}$ is the Sedláček-type test space defined by strip analyticity, derivative bounds and momentum decay in **Appendix B**, and $\mathcal{S}'$ is its continuous dual. The isolated complex frequencies ω_n are poles of the continued response; their outgoing Gamow vectors belong to $\mathcal{S}'$ rather than to L^2 . Thus the continued response is interpreted as a map $\hat{K}(\omega): \mathcal{S} \rightarrow \mathcal{S}'$.

Interpretation. The transition from filamentary or sheet-like cells to void cells is represented spectrally by deformation of the characteristic zeros of $\mathcal{D}(\omega)$ as the Hessian signature changes. The Morse index is therefore a control parameter for the passage from mixed phase-mixing behaviour to hyperbolic resonant response.

3. The Gravity-Dominated Limit: Bounded Orbits and the Continuous Spectrum

To illustrate the Fredholm determinant $\mathcal{D}(\omega)$, let $\mathbf{x}_c$ be a non-degenerate critical point of Ψ_0 , i.e. $\nabla \Psi_0(\mathbf{x}_c) = 0$ and $\det \mathcal{H}(\mathbf{x}_c) \neq 0$, and let $\rho_c = \rho_0(\mathbf{x}_c)$ be the actual stationary mass density there. Consider dense nodes and filament cores satisfying $4\pi G \rho_c \gg \kappa_\Lambda$. The effective reference potential per unit mass is

$$\Psi_0(\mathbf{x}) = \Phi_0(\mathbf{x}) - \frac{\kappa_\Lambda}{2} |\mathbf{x}|^2. \quad (16)$$

At this critical point the Hessian field is defined by

$$\mathcal{H}_{ij} = \partial_i \partial_j \Psi_0. \quad (17)$$

The Morse index k_M is the number of strictly negative eigenvalues of $\mathcal{H}$.

In the gravity-dominated regime the effective potential has a local minimum, so $k_M = 0$. The Hessian is locally diagonalizable, and the cosmological contribution is perturbative:

$$\Omega_j = \Omega_j^{(N)} + \mathcal{O}(\Lambda), \quad (18)$$

where Ω_j is the j th local oscillatory frequency and $\Omega_j^{(N)}$ is its purely Newtonian value. Assume local Liouville integrability: there are three functionally independent integrals I_a , $a = 1, 2, 3$, whose differentials are linearly independent on the regular set and which satisfy $\{I_a, I_b\} = 0$, where, for differentiable phase-space functions A and B , the canonical Poisson bracket is $\{A, B\} = \sum_{j=1}^3 (\partial_{x_j} A \partial_{p_j} B - \partial_{p_j} A \partial_{x_j} B)$. This supplies action-angle variables $(\mathbf{J}, \boldsymbol{\theta})$, where $\mathbf{J}$ are the actions and $\boldsymbol{\theta}$ their conjugate angles, for the reference Hamiltonian

$$H_0(\mathbf{x}, \mathbf{v}) = \frac{1}{2} m |\mathbf{v}|^2 + m \Psi_0(\mathbf{x}). \quad (19)$$

In these coordinates $H_0 = H_0(\mathbf{J})$, $\dot{\mathbf{J}} = 0$ and $\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega}(\mathbf{J})$, where a dot de-

notes d/dt . The unperturbed characteristics are bounded quasi-periodic orbits, meaning that their angle variables advance linearly on invariant tori, the level sets $\mathbf{J} = \text{const}$. In the continuum limit, the absolutely continuous action measure—a measure possessing a density with respect to Lebesgue measure—generates the continuous spectrum and hence a branch cut of the analytically continued resolvent, *i.e.* a curve across which its two boundary values differ. With $\ell \in \mathbb{Z}^3$ denoting the harmonic multi-index and $\boldsymbol{\Omega}(\mathbf{J})$ the orbital-frequency vector, the discrete harmonic sum is replaced by integration over actions.

Let $\mathcal{F}(\ell, \mathbf{J})$ denote the resulting spectral weight, let $\omega_r \in \mathbb{R}$ be a real boundary frequency and let $\varepsilon_\omega > 0$ approach zero. If $\mathcal{F} \in L^1 \cap C^1$, meaning integrable and continuously differentiable in the variables used below, the Sokhotski-Plemelj boundary value is

$$\begin{aligned} & \lim_{\varepsilon_\omega \rightarrow 0^+} \text{Tr} \hat{K}(\omega_r + i\varepsilon_\omega) \\ &= \mathcal{P} \int \frac{\mathcal{F}(\ell, \mathbf{J})}{\omega_r - \ell \cdot \boldsymbol{\Omega}(\mathbf{J})} d^3 J - i\pi \int \mathcal{F}(\ell, \mathbf{J}) \delta_D(\omega_r - \ell \cdot \boldsymbol{\Omega}(\mathbf{J})) d^3 J, \end{aligned} \quad (20)$$

where Tr denotes the operator trace, $\mathcal{P}$ is the Cauchy principal value and δ_D is the Dirac distribution.

The real-axis branch cut is the van Kampen continuous spectrum generated by the continuum of characteristic frequencies. Collective discrete modes, including an Antonov-type self-gravitating collective instability or stable oscillatory modes, are isolated roots

$$\mathcal{D}(\omega) = 0, \quad \text{Im} \omega > 0. \quad (21)$$

Thus, in the $k_M = 0$ domain, Λ perturbs the Newtonian orbital frequencies $\boldsymbol{\Omega}(\mathbf{J})$ without changing the qualitative continuous-spectrum structure of the local harmonic approximation.

4. Intermediate Topologies: Saddle-Point Dynamics and Mixed Spectra

Between dense nodes and voids, filaments and sheets are described locally by saddle points of Ψ_0 , *i.e.* critical points whose Hessian has both positive and negative eigenvalues. The density parameter evaluated at the saddle point, where

$\rho_c = \rho_0(\mathbf{x}_c)$, is

$$\alpha \equiv \alpha(\mathbf{x}_c) = \frac{4\pi G \rho_c}{\kappa_\Lambda}, \quad (22)$$

and is of order unity. Let λ_j , $j \in \{1, 2, 3\}$, be the eigenvalues of the symmetric Hessian $\mathcal{H}$ in an orthonormal principal-axis frame. The Morse index k_M equals the number of negative λ_j : $k_M = 1$ gives one expanding and two bounded directions, while $k_M = 2$ gives two expanding and one bounded direction.

Let $\mathcal{B} = \{j : \lambda_j > 0\}$ be the set of bounded axes and $\mathcal{U} = \{j : \lambda_j < 0\}$ the set of expanding axes. Define $\Omega_j = \sqrt{\lambda_j}$ for $j \in \mathcal{B}$ and $\mu_j = \sqrt{-\lambda_j}$ for $j \in \mathcal{U}$. Along an expanding direction the local characteristic is

$$x'_j(\tau) = B_j e^{\mu_j \tau} + C_j e^{-\mu_j \tau}, \quad (23)$$

where B_j and C_j are integration constants. Restriction to the stable invariant manifold, the set of trajectories bounded as $\tau \rightarrow -\infty$, sets $C_j = 0$. The separated characteristics are therefore

$$x'_j(\tau) = \begin{cases} A_j \cos(\Omega_j \tau + \phi_j), & j \in \mathcal{B}, \\ B_j e^{\mu_j \tau}, & j \in \mathcal{U}, \end{cases} \quad (24)$$

where A_j is an oscillation amplitude and ϕ_j its phase.

Let $\ell \in \mathbb{Z}$ be the harmonic number of a bounded direction j , and let $h \in \mathcal{U}$ label an expanding direction. The time integral in the Fredholm kernel contains

$$\int_{-\infty}^0 e^{-i\omega\tau} e^{i\ell\Omega_j\tau} e^{\mu_h\tau} d\tau = \frac{i}{\omega - \ell\Omega_j + i\mu_h}. \quad (25)$$

The hyperbolic increment μ_h therefore gives the resonant denominator a negative imaginary part. Under the continuation and resolvent bounds stated in **Appendix A**, deformation of the contour used in the inverse Fourier-Laplace transform shows that, for data in the Sedláček class, pole terms dominate the long-time low-frequency asymptotics of the projected observable; without those hypotheses, the branch-cut contribution need not be subdominant.

A structural threshold occurs when a bounded frequency vanishes, $\Omega_j \rightarrow 0$, because the action-angle representation then develops secular transit-time terms, *i.e.* terms that grow with the transit time. Write $\hat{K}(\omega, \alpha)$ for the response operator at density parameter α and $\mathcal{D}(\omega, \alpha)$ for its ordinary or regularised determinant. The roots of

$$\mathcal{D}(\omega, \alpha) = 0 \quad (26)$$

can bifurcate near $\omega = 0$ as a Hessian eigenvalue changes sign. For each fixed ω in the analytic domain, we use Kato's Type-(A) analytic-family hypothesis in α , meaning that the operator domain is independent of α and $\hat{K}(\omega, \alpha)g$ is analytic in α for every domain vector g [26]. Under this hypothesis a simple, multiplicity-one resonance branch $\omega(\alpha)$ varies analytically until it meets another root or the continuous spectrum.

5. The Asymptotic Void Limit: Hyperbolic Divergence and Gamow Resonances

Use physical position $\mathbf{x} = (x_1, x_2, x_3)$ and canonical cell-frame momentum $\mathbf{p} = m\mathbf{v}$. In the quadratic neighbourhood of a deep-void critical point ($k_M = 3$), meaning that terms beyond second order in the local Taylor expansion of the effective potential are neglected, the six-dimensional linearized transport generator is

$$\mathcal{L}_{6D} = \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}} - m(\mathcal{H}\mathbf{x}) \cdot \nabla_{\mathbf{p}}. \quad (27)$$

After an orthogonal transformation to the eigenvectors of the symmetric Hes-

sian, $\mathcal{H} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_j < 0$, and

$$\mathcal{L}_{6D} = \sum_{j=1}^3 \mathcal{L}_j, \quad \mathcal{L}_j = \frac{p_j}{m} \partial_{x_j} - m \lambda_j x_j \partial_{p_j}. \quad (28)$$

Thus $\mu_j = \sqrt{-\lambda_j}$ is the exponential increment along axis j . The associated frequency generator is $\mathcal{A}_{6D} = -i\mathcal{L}_{6D}$.

The Weber reduction is an asymptotic one-channel projection of this full six-dimensional, *i.e.* three-position plus three-momentum, phase space. Choose axis 1 along the most negative Hessian eigenvalue and expand the conjugate momentum dependence in Hermite functions, the standard orthonormal Gaussian-polynomial basis. For an integer $N \geq 1$, let P_N project onto the first N Hermite functions and define $\mathcal{A}_N = P_N \mathcal{A}_{6D} P_N$ after restriction to the principal spatial channel. Assume strong-resolvent convergence to a limiting one-channel operator $\mathcal{A}_W$: for one, hence every, complex z in a common resolvent set (a set on which both $\mathcal{A}_N - z$ and $\mathcal{A}_W - z$ have bounded inverses) and every test vector g , $\|(\mathcal{A}_N - z)^{-1}g - (\mathcal{A}_W - z)^{-1}g\| \rightarrow 0$ as $N \rightarrow \infty$. At the macroscopic-density level, the omitted transverse and high-Hermite part is denoted by $\mathcal{E}_N(t) = o(1)$ as $t \rightarrow \infty$. A bound $\mathcal{E}_N(t) = \mathcal{O}(e^{-\gamma t})$ would require a positive spectral-gap rate γ .

Order the increments so that $\mu_1 \geq \mu_2 \geq \mu_3$ and set

$$\mu = \mu_1, \quad \mu_{\perp} = \max(\mu_2, \mu_3), \quad \Delta_{\mu} = \mu - \mu_{\perp}. \quad (29)$$

The scalar one-channel estimate requires

$$\Delta_{\mu} > 0, \quad \Delta_{\mu} t \gg 1, \quad (30)$$

together with transverse and high-Hermite couplings small relative to the isolated-channel separation. If $\Delta_{\mu} t$ is of order unity or smaller, the nearly degenerate two- or three-dimensional hyperbolic block must be retained; its resonances are multi-indexed and may split under coupling, so it is not represented by the scalar lattice below.

Let $\text{Flux}_{\parallel}$ and $\text{Flux}_{\perp}$ denote the magnitudes of the macroscopic flux along the dominant axis and across it. Under the gap hypothesis,

$$\frac{\text{Flux}_{\perp}}{\text{Flux}_{\parallel}} = \mathcal{O}(e^{-\Delta_{\mu} t}) \rightarrow 0, \quad (31)$$

and the transverse contribution to the density response is $\mathcal{O}(e^{-\Delta_{\mu} t})$. The principal spectral generator $\mathcal{A}_1 = -i\mathcal{L}_1$ therefore controls the long-time one-channel resolvent.

When $4\pi G\rho_c < \kappa_{\Lambda}$ at the deep-void critical point, the dominant stiffness

$$\lambda_{*} = -\lambda_1 = \mu^2 > 0 \quad (32)$$

produces the local inverted-oscillator effective potential per unit mass

$$\Psi(x) = -\frac{1}{2} \lambda_{*} x^2, \quad (33)$$

where $x = x_1$ is the principal physical coordinate. The equation $m\ddot{x} = m\lambda_{*}x$

gives $x(t) \propto e^{\mu t}$.

The velocity-space Hilbert space $L^2(\mathbb{R}^3, d^3v)$ of square-integrable functions of v contains the continuous Vlasov spectrum but not the outgoing resonant states as square-integrable eigenfunctions. Restricting the initial perturbation to the Sedláček-type space $\mathcal{S}$ defined in **Appendix B** permits continuation of the projected spatial resolvent into the lower ω half-plane. Pole dominance still requires the continuation and contour bounds of **Appendix A** and nonzero excitation of the relevant residues; for nonanalytic fine-grained data, branch-cut or algebraic phase-mixing terms may remain leading.

After a normal-form similarity reduction—an invertible change of variables that conjugates the dominant quadratic spectral generator $\mathcal{A}_1$ to its Weber representative—introduce a positive auxiliary action scale χ solely to keep the Weber normal form dimensionally explicit. The principal spatial factor is represented by

$$\hat{\mathcal{W}} = -\frac{\chi^2}{2m} \frac{d^2}{dx^2} - \frac{m}{2} \mu^2 x^2, \quad \hat{\mathcal{W}}\psi = \chi\omega\psi, \quad (34)$$

where $\psi(x)$ is the spatial normal-form mode. Here $\hat{\mathcal{W}}$ is a classical spectral normal-form operator, not a quantum Hamiltonian; χ only nondimensionalises the reduction and cancels from the pole frequencies. Its eigenvalue equation is

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\chi^2} \left(\chi\omega + \frac{m}{2} \mu^2 x^2 \right) \psi = 0. \quad (35)$$

With the dimensionless complex coordinate

$$\xi = x \sqrt{\frac{2m\mu}{\chi}} e^{-i\pi/4}, \quad (36)$$

the equation becomes the canonical Weber equation for the parabolic-cylinder function $D_\nu(\xi)$,

$$\frac{d^2\psi}{d\xi^2} + \left(\frac{i\omega}{\mu} - \frac{\xi^2}{4} \right) \psi = 0. \quad (37)$$

Comparison with $\psi'' + (\nu + 1/2 - \xi^2/4)\psi = 0$, where primes denote differentiation with respect to ξ , gives

$$\omega = -i\mu \left(\nu + \frac{1}{2} \right). \quad (38)$$

The outgoing Gamow continuation, meaning the analytic branch whose generalized mode carries flux away from the cell along the unstable characteristic, selects $\nu = n$ with $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. The corresponding ideal projected poles are

$$\omega_n = -i\mu \left(n + \frac{1}{2} \right), \quad n \in \mathbb{N}_0. \quad (39)$$

The outgoing analytic condition is not an arbitrary reflecting wall (zero normal

flux) or Dirichlet wall ($\psi = 0$ at the boundary) at the finite cell boundary. The associated modes grow formally as $|x| \rightarrow \infty$, while their finite-cell amplitudes are regularised on I_{cell} as described in **Appendix C**.

Equation (39) is exact for the projected quadratic Weber model, not for the full six-dimensional self-consistent determinant. Let P be the projection onto the dominant spatial-Hermite channel and let $Q = \mathcal{I} - P$ be its complement. Whenever ω belongs to the resolvent set of $QA_{6D}Q$, i.e. whenever $\omega - QA_{6D}Q$ has a bounded inverse on the Q channel, the projected frequency resolvent has the Schur-complement form

$$P(\omega - A_{6D})^{-1}P = [\omega - PA_{6D}P - \Sigma_P(\omega)]^{-1}, \quad (40)$$

$$\Sigma_P(\omega) = PA_{6D}Q(\omega - QA_{6D}Q)^{-1}QA_{6D}P, \quad (41)$$

where $\Sigma_P(\omega)$ is the self-energy generated by the eliminated channels. Neglecting Σ_P and the non-quadratic remainder gives the equally spaced lattice. If these corrections are analytic and small compared with the separation of simple projected poles, analytic Fredholm perturbation theory—the perturbation theory of isolated zeros and poles for analytic compact-operator families—moves them to nearby resonances of the reduced self-consistent response; otherwise no exact lattice is claimed for the full determinant.

Under the hypotheses of **Appendix A**, the time-domain projected density contrast has the residue expansion

$$\delta(x, t) = \sum_{n=0}^{\infty} C_n \psi_n(x) e^{-i\omega_n t}. \quad (42)$$

Let $\varphi(x)$ be the initial projected density profile on I_{cell} , let ψ_n be the right resonant mode and let $\tilde{\psi}_n$ be its adjoint mode. With $*$ denoting complex conjugation,

$$C_n = \frac{\int_{I_{\text{cell}}} \tilde{\psi}_n^*(x) \varphi(x) dx}{\int_{I_{\text{cell}}} \tilde{\psi}_n^*(x) \psi_n(x) dx}. \quad (43)$$

Define the finite-cell inner product by $\langle a, b \rangle_{I_{\text{cell}}} = \int_{I_{\text{cell}}} a^*(x) b(x) dx$. For mode indices $n, q \in \mathbb{N}_0$, if the biorthogonal normalization

$$\langle \tilde{\psi}_n, \psi_q \rangle_{I_{\text{cell}}} = \delta_{nq} \quad (44)$$

is imposed, where $\delta_{nq} = 1$ for $n = q$ and 0 otherwise is the Kronecker symbol, the denominator in (43) is unity.

Let $\rho_{1,P}(x, t)$ be the projected physical perturbation density and define the dimensionless contrast by $\delta(x, t) = \rho_{1,P}(x, t) / \rho_{\text{norm}}$, where the positive normalization ρ_{norm} is fixed in time and is not a spatial-mean subtraction. In the leading local one-channel closure, reference-state gradients, reference through-flow and transverse fluxes are assigned to the remainder, and $\rho_{1,P}$ obeys $\partial_t \rho_{1,P} + \partial_x(u \rho_{1,P}) \simeq 0$, where $u = u_x$ is the principal-axis component of the physical cell-frame outflow velocity $\mathbf{u}$. Along $dx/dt = u$, the material derivative $d/dt \equiv \partial_t + u \partial_x$ gives

$d(\ln|\delta|)/dt + \partial_x u \approx 0$. Together with the fundamental decay rate derived below, this defines the projected cell-frame rate increment

$$\Delta H_{\text{cell}} \equiv \frac{\partial u}{\partial x} \approx \frac{1}{2} \sqrt{\lambda_*}. \quad (45)$$

This is an intrinsic velocity-gradient diagnostic, not a decomposition into a Hubble flow and a peculiar component.

Astrophysical Implications: Void Clearing and Kinematic Diagnostics

The transition from a continuous van Kampen spectrum to the leading projected Gamow-like lattice in a void ($k_M = 3$) suggests a possible kinetic contribution to void evolution. In particle-based N -body descriptions, where N is the number of interacting simulation particles, evacuation is usually attributed to expansion, gravitational instability and the tidal environment. Here the same process is described through the analytic structure of the projected collisionless response operator.

For the fundamental mode $n = 0$, with $\omega_0 = -i\mu/2$, write $\delta_0(x)$ for its initial spatial amplitude and $\delta_{\text{void}}(x, t)$ for the corresponding leading projected void contrast. Then

$$\delta_{\text{void}}(x, t) \approx \delta_0(x) e^{-\mu t/2}. \quad (46)$$

This is a leading-order prediction of the local hyperbolic and one-channel approximation, not a universal empirical law for all voids.

Along the projected principal axis the same mode gives $\partial_x u \approx \mu/2$ and hence $u(x) \approx \mu x/2$ after choosing $u(0) = 0$. A separate isotropic spherical extrapolation writes $\mathbf{u} = u(r)\mathbf{e}_r$, where $r = |\mathbf{x}|$ and $\mathbf{e}_r = \mathbf{x}/r$ for $r > 0$, so $\nabla \cdot \mathbf{u} = r^{-2} \partial_r [r^2 u(r)]$. Imposing $\nabla \cdot \mathbf{u} \approx \mu/2$ and regularity at $r = 0$ then gives $u(r) \approx \mu r/6$. The kinetic-energy flux through the physical boundary is

$$\mathcal{P}_K = \oint_{\partial V_{\text{cell}}} \frac{1}{2} \rho_\delta(\mathbf{x}, t) u^2 \mathbf{u} \cdot d\mathbf{S}, \quad (47)$$

where $\oint$ denotes integration over the closed boundary, $\rho_\delta(\mathbf{x}, t)$ is the local boundary mass density and $d\mathbf{S}$ is the outward-oriented surface element. Replacing it by a characteristic boundary value ρ_δ for the scaling estimate gives

$$\mathcal{P}_K \propto \rho_\delta (\mu R)^3 R^2 \propto \rho_\delta \lambda_*^{3/2} R^5. \quad (48)$$

The R^5 dependence is a scaling law conditional on dominance of the leading hyperbolic resonance, not a calibrated luminosity formula.

Possible diagnostics are correlations of density-profile relaxation with Hessian eigenvalues, anisotropic cell-frame outflows and kinematic memory in surrounding walls and filaments. The directly predicted local quantity is

$$\Delta H_{\text{cell}} = \frac{\partial u}{\partial x} \approx \frac{1}{2} \sqrt{\lambda_*}. \quad (49)$$

Comparison with an independently inferred large-scale expansion rate is an ob-

servational post-processing step, not a background/peculiar-velocity decomposition and not an explanation of the Hubble tension (the discrepancy between local and early-Universe expansion-rate inferences). The signal will depend on observer position, survey window, line-of-sight averaging and the void finder used to define the cell boundary.

6. Conclusions

This analysis develops a spectral description of collisionless dark-matter response in open cosmological void cells. By mapping the geometric characteristics of the effective collisionless flow, we argue that the kinematic stiffness parameter, $\lambda_* = -\lambda_{\text{principal}}(\mathcal{H})$, dictates a topological transition in the local kinetic response.

In gravity-dominated structures, such as overdense nodes and filaments ($k_M \leq 2$), the local macroscopic potential supports bounded invariant sets. The kinetic response is asymptotically governed by the continuous van Kampen spectrum on the real axis, which encodes the classical phase-mixing mechanism within the standard Hilbert space L^2 . In this regime, macroscopic structural relaxation is consistent with reversible dynamics.

The transition into a cosmological void ($k_M = 3$), where vacuum repulsion dominates ($\lambda_* > 0$), modifies this behavior. The continuous exponential separation transforms the stationary-reference macroscopic dynamics into a purely hyperbolic flow. By restricting the initial phase-space perturbations to the Sedláček functional class and projecting the integral operator onto the stable manifold, we analytically continue the projected macroscopic resolvent. In this hyperbolic regime, for analytic data and the lower-half-plane resolvent estimates stated in **Appendix A**, the contribution of the continuous spectrum becomes asymptotically subdominant as $t \rightarrow \infty$.

Consequently, the macroscopic phase-space evolution is, under the stated assumptions, asymptotically governed by the residues of the analytically continued resolvent at the Gamow poles. The reduction of the divergent flow along the dominant characteristic axis limits the spatial dynamics to the canonical Weber differential equation. The outgoing continuation of the dominant quadratic Weber normal form quantizes its isolated roots into the leading projected lattice of complex frequencies:

$$\omega_n = -i\mu \left(n + \frac{1}{2} \right), \quad n \in \mathbb{N}_0. \quad (50)$$

These discrete poles define the leading projected macroscopic kinetic resonances; transverse self-consistency, non-quadratic terms and literal finite-boundary matching can shift or split them in the full problem. The analysis suggests that the Λ -dominated hyperbolic geometry can act as a condition for effective macroscopic irreversibility. Within the stated analytic and one-channel approximation, the hyperbolic flow can replace the leading projected phase-mixing contribution by quantized resonant decay; this is a conditional mechanism rather than a universal law governing every void.

Quantitatively, this spectral transition implies that cosmological voids may act as dynamically active components of the cosmic web rather than passively expanding underdense regions. When it is excited and dominant, the fundamental projected Gamow mode generates a persistent collisionless matter outflow toward surrounding structures. Governed by the continuity equation, the associated kinetic luminosity scales as:

$$\mathcal{P}_K \propto \rho_{\phi} \lambda_*^{3/2} R^5. \quad (51)$$

This mechanism provides a kinetic pathway through which hyperbolic void expansion may contribute to the redistribution of collisionless matter between void interiors and the surrounding cosmic web.

Declaration of Generative AI and AI-Assisted Technologies in the Writing Process

The author used OpenAI ChatGPT for language editing, structural organisation of the manuscript, LaTeX and bibliography-formatting assistance, and preparation of submission-support materials. No figures, observational data sets or numerical results were generated by AI tools. The author reviewed the manuscript and remains fully responsible for the scientific content, accuracy and originality of the work.

Data Availability

No new observational or numerical data were generated or analysed in this study.

Conflicts of Interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Asymptotic Pole Dominance and Contour Deformation

The projected spatial resolvent is non-Hermitian, meaning that it need not equal its adjoint, and its long-time response is determined by analytic continuation in ω .

Theorem A.1 (Asymptotic Pole Dominance). *Let $\delta(x, t)$ be the macroscopic density response on $I_{\text{cell}} = [-R_{\text{cell}}, R_{\text{cell}}]$, and let $f_1 \in \mathcal{S}$ be the projected initial phase-space perturbation. Assume that the projected resolvent $\mathcal{R}(\omega)$ has a meromorphic continuation—analytic except for isolated poles—through the real axis to $\text{Im } \omega > -\sigma_0$, where $\sigma_0 > 0$. On every pole-free line $\text{Im } \omega = -\sigma$, with $0 < \sigma < \sigma_0$, assume $\|\mathcal{R}(\omega)\| \leq C_\sigma (1 + |\omega|)^{-1-\varepsilon_R}$ for constants $C_\sigma > 0$ and $\varepsilon_R > 0$. If a pole is excited by the initial state (its residue coefficient is nonzero) and lies above that line, the long-time response is represented by the residues above the shifted contour. Formally, the full set of poles gives the Mittag-Leffler, i.e. pole-residue, expansion*

$$\delta(x, t) = \sum_{n=0}^{\infty} C_n \psi_n(x) e^{-i\omega_n t}. \quad (52)$$

More precisely, let j label the excited poles above the shifted line, with ω_j , ψ_j and C_j their pole frequencies, right resonant modes and residue coefficients:

$$\delta(x, t) = \sum_{\text{Im } \omega_j > -\sigma} C_j \psi_j(x) e^{-i\omega_j t} + \mathcal{E}_\sigma(x, t), \quad \|\mathcal{E}_\sigma(\cdot, t)\| \leq M_\sigma e^{-\sigma t}, \quad (53)$$

where $\mathcal{E}_\sigma$ is the shifted-contour remainder, $\|\cdot\|$ is the chosen projected-response norm and M_σ is a time-independent constant.

Proof. The argument has four steps.

Step 1: Kinetic dispersion integral. Let v_x be the principal-axis component of the cell-frame velocity and let $p = mv_x$ be its canonical momentum. By the normalization specified in the Notation section, the projected density contrast is

$$\delta(x, t) = \int_{\mathbb{R}} f_1(x, p, t) dp. \quad (54)$$

For $\text{Im } \omega > 0$, the transform convention of Section 2 gives the free-streaming response

$$\delta(x, \omega) = i \int_{\mathbb{R}} \left(\omega + i \frac{p}{m} \partial_x \right)^{-1} f_1(x, p) dp. \quad (55)$$

The inverse operator in parentheses is the frequency resolvent of $-i(p/m)\partial_x$; the prefactor i follows from the one-sided transform, and the real ω axis carries the continuous-spectrum branch cut.

Step 2: Sedláček continuation. For $f_1 \in \mathcal{S}$, the momentum dependence is holomorphic in a horizontal strip with the decay specified in **Appendix B**. The p contour can therefore be deformed as ω crosses the real axis, up to the singularities or strip boundary. The continued response defines

$$\delta(x, \omega) = \mathcal{R}(\omega) \varphi(x), \quad (56)$$

where φ is the initial projected density profile.

Step 3: Temporal contour deformation. Let $\Gamma_L = \{\omega \in \mathbb{C} : \text{Im } \omega = \gamma_L\}$, with $\gamma_L > 0$, be the horizontal Bromwich contour used for inverse Fourier-Laplace transformation. Then

$$\delta(x, t) = \frac{1}{2\pi} \int_{\Gamma_L} \mathcal{R}(\omega) \varphi(x) e^{-i\omega t} d\omega. \quad (57)$$

Append the lower-half-plane semicircular deformation arc Γ_M of radius $M > 0$ and deform the contour across the isolated poles ω_j . The residue theorem gives the pole sum plus the shifted-contour and arc contributions.

Step 4: Vanishing of the arc. On the bounded interval I_{cell} , second-order differential resolvents have the high-frequency estimate $\|\mathcal{R}(\omega)\| = \mathcal{O}(|\omega|^{-1/2})$ before continuation. For the theorem, require on the deformed contour the stronger integrable bound

$$\|\mathcal{R}(\omega)\| = \mathcal{O}(|\omega|^{-1-\varepsilon_R}). \quad (58)$$

Because $e^{-i\omega t}$ decays in the lower half-plane for $t > 0$, Jordan's lemma—the large-semicircle estimate for such decaying contour integrals—gives

$$\lim_{M \rightarrow \infty} \int_{\Gamma_M} \mathcal{R}(\omega) \varphi(x) e^{-i\omega t} d\omega = 0. \quad (59)$$

The shifted line gives the remainder bound in (53). Thus the excited poles above that line dominate the asymptotics under the stated continuation and decay hypotheses. For nonanalytic data or an obstructing branch singularity, algebraic phase-mixing terms can remain comparable to, or larger than, the pole terms. ■

Appendix B. The Sedláček Class and Kinetic-Macroscopic Projection

The restriction on initial data is a sufficient domain for contour deformation; it is not asserted to describe every exact nonlinear dark-matter distribution.

Sedláček-type functional class $\mathcal{S}$. Let $x \in I_{\text{cell}}$ be the projected physical coordinate and let p be its conjugate canonical momentum. Fix a strip half-width $p_* > 0$, a decay parameter $0 < \varepsilon_p < 1$ and an integer differentiability order $k \geq 1$. A function $f_1(x, p)$ belongs to $\mathcal{S}$ if: 1) for each x , it extends holomorphically, *i.e.* complex-analytically, to $|\text{Im } p| < p_*$; 2) for each substrip parameter $0 \leq \eta < p_*$, it is C^k in x , with all mixed derivatives through total order k continuous on the closed substrip $|\text{Im } p| \leq \eta$; and 3) for integers $a, b \geq 0$ with $a + b \leq k$, the seminorms

$$q_{\eta, a, b}(f_1) = \sup_{\substack{x \in I_{\text{cell}} \\ |\text{Im } p| \leq \eta}} (1 + |\text{Re } p|)^{1+\varepsilon_p+b} \left| \partial_x^a \partial_p^b f_1(x, p) \right| < \infty \quad (60)$$

are finite for every $0 \leq \eta < p_*$, where $\sup$ denotes the supremum over the displayed set. The family $q_{\eta, a, b}$ defines the topology used here. It supplies momentum integrability, derivative control and decay sufficient to move the contour and eliminate the vertical pieces at large $|\text{Re } p|$.

The width p_* is an independent regularity parameter of the prepared or coarse-grained perturbation. No proportionality $p_* \propto \mu^{-1}$ is assumed: without a specified nondimensionalisation and smoothing model, such a relation is neither required nor justified. The increment μ fixes the ideal Weber pole spacing, whereas p_* limits the admissible momentum-contour deformation.

An example is the Cauchy-Lorentz profile

$$f_1(x, p) = \varphi(x) \frac{p_*}{\pi(p^2 + p_*^2)}, \quad (61)$$

where $\varphi(x)$ is the projected spatial amplitude. It is analytic in the open strip $|\operatorname{Im} p| < p_*$ and has poles on the strip boundary at $p = \pm ip_*$.

For $\operatorname{Im} \omega > 0$, let $f_1(x, p, \omega) = \int_0^\infty f_1(x, p, t) e^{i\omega t} dt$ be the one-sided Fourier-Laplace transform. The projected density is

$$\mathcal{S}(x, \omega) = \int_{\mathbb{R}} f_1(x, p, \omega) dp, \quad (62)$$

and the frequency resolvent $(\omega + i(p/m)\partial_x)^{-1}$ has a moving singularity in the complex p plane. Strip analyticity allows the contour to be deformed until a singularity of f_1 or the strip boundary is reached, thereby defining $\mathcal{R}(\omega)$ and isolating its outgoing poles from the kinetic continuum.

Membership in $\mathcal{S}$ is not claimed for an exact fine-grained post-shell-crossing distribution. Caustics—singular projections of folded phase-space sheets—and filamentary sheets may violate strip analyticity, in which case the branch-cut term need not be subdominant. Finite velocity dispersion, numerical filtering or observational coarse graining, *i.e.* smoothing over finite resolution, can motivate regularised test data, but none proves that the exact distribution belongs to $\mathcal{S}$. Pole excitation and dominance therefore depend on the initial data and on the projected observable.

Appendix C. Finite Polyhedral Cell Decomposition and Amplitude Coefficients C_n

For the projected pole expansion

$$\mathcal{S}(x, t) = \sum_{n=0}^{\infty} C_n \psi_n(x) e^{-i\omega_n t}, \quad (63)$$

the ideal frequencies are $\omega_n = -i\mu(n + 1/2)$ and the right modes ψ_n are Weber parabolic-cylinder functions. Let $\tilde{\psi}_n$ be the adjoint modes and $\varphi(x)$ the initial projected density profile. Then

$$C_n = \frac{\int_{I_{\text{cell}}} \tilde{\psi}_n^*(x) \varphi(x) dx}{\int_{I_{\text{cell}}} \tilde{\psi}_n^*(x) \psi_n(x) dx}. \quad (64)$$

For $n, q \in \mathbb{N}_0$, if $\langle \tilde{\psi}_n, \psi_q \rangle_{I_{\text{cell}}} = \delta_{nq}$, the denominator is unity.

The physical cell is the finite domain $\mathcal{V}_{\text{cell}}$ of volume V_{cell} , and the projected interval is $I_{\text{cell}} = [-R_{\text{cell}}, R_{\text{cell}}]$. This finite interval truncates the formal growth of the outgoing Weber functions and makes the displayed biorthogonal integrals fi-

nite. It regularises amplitudes but does not prove that the full cell has the ideal equally spaced spectrum. Literal boundary matching or non-quadratic exterior fields can shift or split the Weber poles; Equation (39) remains the leading projected normal-form lattice.