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Full Discrete Mixed Finite Element Method for the BBM Equation on a Rectangular Grid

Hongying Si, Weifeng Cui

School of Mathematics and Statistics, Shangqiu Normal University, Shangqiu, China
Email: sihongying@126.com

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Abstract

In this paper, the mixed finite element method is applied to two-dimensional nonlinear BBM equations on rectangular meshes. The mixed finite element schemes of semidiscretization and full discretization are established. The space is approximated by a continuous biquadratic finite element and a bilinear finite element, and the time is discretized by the backward Euler difference scheme. The existence and uniqueness of the finite element solution are proved, and the optimal error estimation is obtained. Finally, the numerical results were given, and the stability and convergence of the method were verified.

Keywords

BBM Equation, The Mixed Finite Element, Full Discretization, Error Estimate

1. Introduction

The BBM Equation was proposed by Benjamin, Bona, and Mahony when they studied the unidirectional propagation of nonlinear dispersive long waves in 1972, which is also called the regular long wave equation. The equation is an improvement of the KDV equation in describing the unidirectional propagation of small amplitude and long surface gravity waves in 1 + 1 dimensions. The BBM equation is widely used in linear optics and particle physics. Therefore, it is of great significance to study the numerical solution of the BBM equation. The study of this equation has attracted extensive attention from scholars at home and abroad. Khaled Omrani [1] presented the Galerkin finite element method for the BBM equation: the standard Galerkin method is used for space discretization, and the C-N scheme is used for time discretization. The convergence of the equation is proved. In paper [2], the existence and uniqueness of solutions of the BBM equation in one-dimensional case are discussed. In article [3], Feng Min fu uses the

Crank-Nicolson difference method to discretize the equation. In [4]-[6], the finite difference method and the general method of the equation are studied. In paper [7], Tan Yanmei studied the fully discrete mixed finite element scheme on a triangular element, proved the existence and uniqueness of the finite element solution, and gave an error analysis.

The mixed finite element method was founded by Babuska and Brezzi in the 1970s [8] [9]. In the early 1980s, the mixed finite element method was improved by Falk and Osborn [10] [11]. So far, the mixed finite element method has been widely used to solve partial differential equations, such as bi-harmonic equations [12], parabolic equations [13], Stokes equations [14], and so on. However, the study of a fully discrete mixed finite method for the BBM equation on a rectangular element has not been reported.

On the basis of [8], the mixed finite element method is applied to two-dimensional nonlinear BBM equations on rectangular meshes. The mixed finite element schemes of semidiscretization and full discretization are established. The space is approximated by a continuous biquadratic finite element and a bilinear finite element, and the time is discretized by the backward Euler difference scheme. The existence and uniqueness of the finite element solution are proved, and the optimal error estimation is obtained. In the second part of this paper, the existence and uniqueness of a solution to the BBM equation are established. In the third part, the mixed finite element scheme for semi-discretization on a rectangular grid is established, the existence and uniqueness of the finite element solution are proved, and the error is analyzed. In the fourth part, the convergence of the backward Euler fully discrete mixed finite element scheme is discussed. In the fifth part, a numerical example was given to verify the correctness and effectiveness of this method.

2. Existence and Uniqueness of Generalized Solutions of the BBM Equation

Let $\Omega \subset R^2$ be a smooth, bounded region. In this article, we will consider the following nonlinear BBM equation:

$$\begin{cases} u_t - \Delta u_t = \nabla \cdot f(u), & (x,t) \in \Omega \times [0,T], \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), & x \in \bar{\Omega}. \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < T < \infty$ and $f(u) = -\left(\frac{1}{2}u^2 + u\right)$. Let $H^m(\Omega)$ and $L^2(\Omega) = H^0(\Omega)$ denote the usual Sobolev space of real-valued functions defined on Ω. $(\cdot,\cdot)$ denote the inner product in $L^2(\Omega)$, The norm of this space is the usual Sobolev H^m-norm and it will be denoted by $\|\bullet\|_m$,

$$\|v\|_m = \left[\sum_{|\alpha| \le m} \int_\Omega \left|\partial^\alpha v(x)\right|^2 \mathrm{d}x\right]^{\frac{1}{2}},$$

and Semi-norm

$$\|v\|_m = \left[\sum_{|\alpha|=m} \int_\Omega \left|\partial^\alpha v(x)\right|^2 \mathrm{d}x\right]^{\frac{1}{2}}.$$

Letting $\vec{p} = \nabla u_t$, and using the Green's formula, then we get the mixed variational form of problem (1): find $(u, \vec{p}) : [0,T] \to H_0^1(\Omega) \times \left(L^2(\Omega)\right)^2$, such that

$$\begin{cases} (u_t, v) + (\vec{p}, \nabla v) = (\nabla \cdot f(u), v), & \forall v \in H_0^1(\Omega), \\ (\nabla u_t, \vec{q}) - (\vec{p}, \vec{q}) = 0, & \forall \vec{q} \in \left(L^2(\Omega)\right)^2, \\ u(x,0) = u_0(x), & x \in \bar{\Omega}. \end{cases} \tag{2}$$

Theorem 2.1 Equation (2) has a unique solution $(u, \vec{p}) \in H_0^1(\Omega) \times \left(L^2(\Omega)\right)^2$, and satisfy

$$\|u(t)\|_1 = \|u(0)\|_1, \forall t > 0.$$

Proof Since there be a unique solution u in Equation (1), let $(u, \vec{p}) = (u, \nabla u_t)$, be the solution of Equation (2), take $\vec{q} = \nabla u, v = u$, then

$$(u_t, u) + (\nabla u_t, \nabla u) = (\nabla \cdot f(u), u).$$

Because

$$(\nabla \cdot f(u), u) = \nabla(f(u), u) - \nabla(\varphi(u)),$$

where $\varphi'(u) = f(u)$, according to the boundary value condition $u|_{\partial\Omega} = 0$ and $\varphi(0) = 0$, then

$$(\nabla \cdot f(u), u) = \int_\Omega \nabla \cdot (f(u)u) \mathrm{d}x = 0. \tag{3}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_0^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_0^2 = 0,$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_1^2 = 0. \tag{4}$$

Integrating Equation (4) from 0 to *t*, we can get

$$\|u(t)\|_1 = \|u(0)\|_1, \forall t > 0. \tag{5}$$

Now prove uniqueness, let $(u^*, \vec{p}^*)$ be another solution of Equation (2), according to Equation (5), we get

$$\|u^*(t)\|_1 = \|u^*(0)\|_1, \forall t > 0.$$

Substitute u into (2), then subtracting (2), we get

$$\begin{cases} \left(u_t - u_t^*, v\right) + \left(\vec{p} - \vec{p}^*, \nabla v\right) = \left(\nabla \cdot \left(f(u) - f(u^*)\right), v\right), & \forall v \in H_0^1(\Omega), \\ \left(\nabla\left(u_t - u_t^*\right), \vec{q}\right) - \left(\vec{p} - \vec{p}^*, \vec{q}\right) = 0, & \forall \vec{q} \in \left(L^2(\Omega)\right)^2. \end{cases} \tag{6}$$

Taking $\vec{q}=\nabla\left(u-u^*\right), v=u_t-u_t^*$ in (6), according to (3), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|u-u^*\right\|_0^2+\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\left(u-u^*\right)\right\|_0^2=0,$$

Then

$$\left\|u-u^*\right\|_1^2=\left\|u\left(0\right)-u^*\left(0\right)\right\|_1^2=\left\|u_0\left(x\right)-u_0\left(x\right)\right\|_1^2=0,$$

such that $u=u^*$, let $\vec{q}=\vec{p}-\vec{p}^*$, we have $\vec{p}=\vec{p}^*$.

This completes the proof.

3. Existence and Error Analysis of Semi-Discrete Mixed Finite Element Solutions

Let T_{2h} is quasi-uniform rectangular partition of Ω, T_{1h} is a sub-rectangle obtained by connecting the midpoints of the four sides of $T_{2h}, T_1, T_2, T_3, T_4$, as shown in Figure 1:

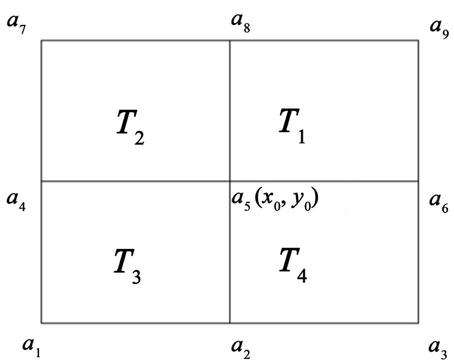

Figure 1. Rectangular partition.

$V_i\left(i=1,2\right)$ is a bi i degree polynomial space defined on T_{ih}, obviously, $V_i\subset H^1\left(\Omega\right)$, Let $X_h=V_2\cap H_0^1\left(\Omega\right)$, $\vec{M}_h=V_1$, the finite element space is defined as follows

$$X_h=\left\{v_h\in H_0^1\left(\Omega\right)\cap C^0\left(\bar{\Omega}\right): v_h\big|_K\in Q_2\left(K\right), K\in T_{2h}\right\},$$

$$\vec{M}_h=\left\{\vec{q}=\left(q_1,q_2\right)\in\left(L^2\left(\Omega\right)\right)^2:\vec{q}\big|_{K'}\in\left(Q_1\left(K'\right)\right)^2, K'\in T_{1h}\right\},$$

where $Q_2\left(K\right)$ is a biquadratic polynomial space on a large element K, $Q_1\left(K'\right)$ is a bilinear polynomial space on a sub-element $K^{'}$.

Definition

$$\left\|u\right\|_K=\sqrt{\int_K\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)\mathrm{d}x\mathrm{d}y},$$

$$u\left(a_i\right)=u_i.$$

Let I_2u be a biquadratic Lagrange interpolation polynomial, which is defined on the nodes of T, I_1u be a piecewise bilinear Lagrange interpolation polynomial, which is defined on four sub-rectangles T_1, T_2, T_3, T_4, and there hold

Lemma 3.1 [12] $\forall u \in C(\Omega)$, then

$$\|I_2 u - I_1 u\|_a \le \sqrt{\frac{5}{6}} \|I_1 u\|_a,$$

where $\|w\|_a^2 = (\nabla w, \nabla w) = |w|_1$ and $I_i : C^0 \to V_i \, (i = 1, 2)$ is a be i degree Lagrange interpolation operator defined on T_{ih}.

Lemma 3.2 [15] The discrete BB condition holds, that is, there is a constant $\beta > 0$ independent of h, such that

$$\sup_{\vec{q}_h \in \vec{M}_h} \frac{(\vec{q}_h, \nabla v_h)}{\|\vec{q}_h\|_0} \ge \beta \|v_h\|_1, \forall v_h \in X_h. \tag{7}$$

Proof From $v_h \in X_h$, then $v_h \in H_1^0$, and $v_h|_K \in Q_2(K)$, according to Poincaré inequality, $\nabla v_h \in (L^2(\Omega))^2$ and $\nabla v_h|_{K'} \in (Q_1(K'))^2$, we get

$$\sup_{\vec{q}_h \in \vec{M}_h} \frac{(\vec{q}_h, \nabla v_h)}{\|\vec{q}_h\|_0} \ge \frac{(\nabla v_h, \nabla v_h)}{\|\nabla v_h\|_0} = |v_h|_1 \ge \beta \|v_h\|_1, \forall v_h \in X_h.$$

The proof is completed.

For all $v \in H^1(\Omega), \vec{q} = (q_1, q_2) \in (L^2(\Omega))^2$, we define the interpolation operator Π_h^2 and Π_h^1 as follows:

$$\Pi_h^2 : H^1(\Omega) \to X_h, \Pi_h^1 : (L^1(\Omega))^2 \to \vec{M}_h.$$

According to the projection property and duality principle, we get **Lemma 3.3** and **Lemma 3.4**.

Lemma 3.3 [16] Existence of interpolation operator
$\Pi_h^2 : H^1(\Omega) \to X_h$, such that

$$(v - \Pi_h^2(v_h), v_h) = 0, \forall v_h \in X_h,$$

$$\|\Pi_h^2(v)\|_0 \le \|v\|_0, \forall v \in H^1(\Omega),$$

When $\forall v \in H^r(\Omega)$, then

$$\|v - \Pi_h^2(v)\|_s \le Ch^{r-s} |v|_r, s = 0, 1, s \le r \le m + 1. \tag{8}$$

Lemma 3.4 [17] Existence of an interpolation operator $\Pi_h^1 : (L^2(\Omega))^2 \to \vec{M}_h$, such that

$$(p - \Pi_h^1(\vec{p}), \vec{q}_h) = 0, \forall \vec{q}_h \in \vec{M}_h,$$

$$\|\Pi_h^1(\vec{p})\|_0 \le \|\vec{p}\|_0, \forall \vec{p} \in (L^2(\Omega))^2,$$

When $\vec{p} \in \vec{M}_h$, then

$$\|\Pi_h^1(\vec{p})\|_s \le Ch^{r-s} |\vec{p}|_r, s = 0, 1, s \le r \le m. \tag{9}$$

Lemma 3.5 [18] (Gronwall Lemma) Suppose $g(t)$ be integrable on $[0, t_1]$ and positive everywhere, $c \ge 0$ is a constant. If $\psi(t) \in C^0([0, t_1])$, and there is the following inequality

$$0 \le \psi(t) \le c + \int_0^t g(s)\psi(s)\,\mathrm{d}s, \forall t \in [0, t_1],$$

then

$$0 \le \psi(t) \le c \cdot \exp\left(\int_0^t g(s)\,\mathrm{d}s\right), \forall t \in [0, t_1].$$

Lemma 3.6 [19] Let $(a_n),(b_n),(c_n)$ be positive columns, (c_n) monotonically increasing and satisfy

$$a_n + b_n \le c_n + \lambda \sum a_j, n \ge 1, \lambda \ge 0,$$

and

$$a_0 + b_0 \le c_0,$$

then

$$a_n + b_n \le c_n \cdot \exp(n\lambda), n \ge 0.$$

Now, we consider the semi-discrete form of (2), find $(u_h, \vec{p}_h):[0,T] \to X_h \times M_h$, such that

$$\begin{cases} (u_{ht}, v_h) + (\vec{p}_h, \nabla v_h) = (\nabla \cdot f(u_h), v_h) & \forall v_h \in X_h, \\ (\nabla u_{ht}, \vec{q}_h) - (\vec{p}_h, \vec{q}_h) = 0, & \forall \vec{q}_h \in M_h, \\ u(x,0) = \Pi_h^2 u_0(x), & x \in \bar{\Omega}. \end{cases} \tag{10}$$

Theorem 3.1 [7] Equation (10) has a unique solution

$$(u_h, \vec{p}_h):[0,T] \to X_h \times M_h.$$

Theorem 3.2 The solution $(u, \vec{p}),(u_h, \vec{p}_h)$ of (2) and (10), respectively, suppose $u, u_t \in H^3(\Omega)$, $\vec{p} \in \left(H^2(\Omega)\right)^2$, then there holds

$$\|u - u_h\|_1 \le ch^2 \left[\|u\|_3^2 + \int_0^t \left(\|u\|_3^2 + \|u_t\|_3^2\right)\mathrm{d}\tau\right]^{\frac{1}{2}}, \tag{11}$$

and

$$\|\vec{p} - \vec{p}_h\|_0 \le ch^2 \left[\left(\|\vec{p}\|_2^2 + \|u\|_3^2\right) + \int_0^t \left(\|u\|_3^2 + \|u_t\|_3^2\right)\mathrm{d}\tau\right]^{\frac{1}{2}}. \tag{12}$$

Proof Let

$$u - u_h = \left(u - \Pi_h^2 u\right) + \left(\Pi_h^2 u - u_h\right) = \eta + \xi,$$

$$\vec{p} - \vec{p}_h = \left(\vec{p} - \Pi_h^1 \vec{p}\right) + \left(\Pi_h^1 \vec{p} - \vec{p}_h\right) = \vec{\rho} + \vec{\theta}.$$

For all $v_h \in X_h, \vec{q}_h \in M_h$, the error equation is

$$\begin{cases} (\xi_t, v_h) + (\vec{\theta}, \nabla v_h) = -(\eta_t, v_h) - (\vec{\rho}, \nabla v_h) + (\nabla \cdot (f(u) - f(u_h)), v_h), \\ (\nabla \xi_t, \vec{q}_h) - (\vec{\theta}, \vec{q}_h) = (\vec{\rho}, \vec{q}_h) - (\nabla \eta_t, \vec{q}_h). \end{cases} \tag{13}$$

Taking $v_h \in \xi, \vec{q}_h \in \nabla \xi$, and adding the two formulas of (13), we have

$$(\xi_t, \xi) + (\nabla \xi_t, \nabla \xi) = -(\eta_t, \xi) - (\nabla \eta_t, \nabla \xi) + ((f(u) - f(u_h)), \nabla \xi)$$

According to the Young inequality, the Cauchy-Schwarz inequality and the Lipschitz condition $\left\|f(u)-f(u_h)\right\|_0 \le c\|u-u_h\|_0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|_0^2+\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\xi\|_0^2 \le c\left[\|\eta_t\|_0^2+\|\nabla\eta_t\|_0^2+\left(\|\xi\|_0^2+\|\eta\|_0^2\right)\right] +\varepsilon\|\xi\|_0^2+2\varepsilon\|\nabla\xi\|_0^2. \tag{14}$$

Since $\xi(0)=0$, integrate (14) from 0 to t, and then according to **Lemma 3.5** and (8), we get

$$\|\xi\|_0^2+\|\nabla\xi\|_0^2 \le ch^4\int_0^t\left(\|u\|_3^2+\|u_t\|_3^2\right)\mathrm{d}\tau. \tag{15}$$

Using the triangle inequality and the interpolation theory, we get

$$\|u-u_h\|_1^2 \le \|\eta\|_1^2+\|\xi\|_1^2 \le ch^4\left[\|u\|_3^2+\int_0^t\left(\|u\|_3^2+\|u_t\|_3^2\right)\mathrm{d}\tau\right].$$

The proof of (11) is completed. Taking $v_h=\xi_t, \vec{q}_h=\nabla\xi_t$, and adding the two formulas of (13), we have

$$(\xi_t,\xi_t)+(\nabla\xi_t,\nabla\xi_t)=-(\eta_t,\xi_t)-(\nabla\eta_t,\nabla\xi_t)+\left(\left(f(u)-f(u_h)\right),\nabla\xi_t\right).$$

Using the same method as (15), then there hold

$$\|\xi_t\|_0^2+\|\nabla\xi_t\|_0^2 \le ch^4\int_0^t\left(\|u\|_3^2+\|u_t\|_3^2\right)\mathrm{d}\tau. \tag{16}$$

Taking $\vec{q}_h=\vec{\theta}$, then

$$\left\|\vec{\theta}\right\|_0^2=\left(\vec{\theta},\vec{\theta}\right)=\left(\nabla\xi_t,\vec{\theta}\right)+\left(\nabla\eta_t,\vec{\theta}\right)-\left(\vec{\rho},\vec{\theta}\right).$$

According to the Young inequality, the Cauchy-Schwarz inequality, we get

$$\left\|\vec{\theta}\right\|_0^2 \le c\left(\|\nabla\xi_t\|_0^2+\|\vec{\rho}\|_0^2+\|\nabla\eta_t\|_0^2\right)+\varepsilon\left\|\vec{\theta}\right\|_0^2.$$

Then, according to **Lemma 3.3**, **Lemma 3.4**, and (16), we get

$$\left\|\vec{\theta}\right\|_0^2 \le ch^4\int_0^t\left(\|u\|_3^2+\|u_t\|_3^2\right)\mathrm{d}\tau+ch^4\left(\|\vec{p}\|_2^2+\|u_t\|_3^2\right).$$

Thus, using trigonometric inequality and **Lemma 3.4**, we obtain

$$\|\vec{p}-\vec{p}_h\|_0^2 \le \|\vec{\rho}\|_0^2+\left\|\vec{\theta}\right\|_0^2 \le ch^4\left[\left(\|\vec{p}\|_2^2+\|u_t\|_3^2\right)+\int_0^t\left(\|u\|_3^2+\|u_t\|_3^2\right)\mathrm{d}\tau\right].$$

The proof of (12) is completed.

4. Backward Euler, Fully Discrete Mixed Finite Element Scheme

For any given positive integer N, let $\Delta t=\frac{T}{N}$, denote the size of the time discretization and $0=t_0<t_1<\cdots<t_N=T, \Delta t=\frac{T}{N}, t_n=n\Delta t, n=0,1,\cdots,N$.

For smooth function $\varphi^n=\varphi(t_n)$, let

$$\partial_t\varphi^n=\frac{1}{\Delta t}\left(\varphi^n-\varphi^{n-1}\right), \partial_t\nabla\varphi^n=\frac{1}{\Delta t}\left(\nabla\varphi^n-\nabla\varphi^{n-1}\right).$$

Now we consider the backward Euler full discretization scheme of (2), find $\left(u_h^n, \vec{p}_h^n\right):[0,T]\to X_h\times\vec{M}_h\,(n=1,2,\cdots,N)$, such that

$$\begin{cases}\left(\partial_t u_h^n, v_h\right)+\left(\vec{p}_h^n,\nabla v_h\right)=\left(\nabla\cdot f\left(u_h^n\right),v_h\right), & \forall v_h\in X_h,\\ \left(\partial_t u_h^n,\vec{q}_h\right)-\left(\vec{p}_h^n,\vec{q}_h\right)=0, & \forall\vec{q}_h\in M_h,\\ u(x,0)=\Pi_h^2 u_0(x), & x\in\bar{\Omega}.\end{cases}\tag{17}$$

Theorem 4.1 The solution $u^n,\vec{p}^n,\left(u_h^n,\vec{p}_h^n\right)$ of (2) and (17), respectively, suppose $u,u_t,u_{tt}\in H^3(\Omega)$, $\vec{p},\vec{p}_t\in\left(H^2(\Omega)\right)^2$, then there holds

$$\left\|u_n-u_h^n\right\|_1^2\le C\left[h^4+(\Delta t)^2\right],\tag{18}$$

and

$$\left\|\vec{p}^n-\vec{p}_h^n\right\|_0^2\le C\left[h^4+(\Delta t)^2\right].\tag{19}$$

Proof Taking $t=t_n, v=v_h,\vec{q}=\vec{q}_h$ in (2), then

$$\begin{cases}\left(u_t^n,v_h\right)+\left(\vec{p}^n,\nabla v_h\right)=\left(\nabla\cdot f\left(u^n\right),v_h\right), & \forall v_h\in X_h,\\ \left(\nabla u_t^n,\vec{q}_h\right)-\left(\vec{p}^n,\vec{q}_h\right)=0, & \forall\vec{q}_h\in M_h,\\ u(x,0)=u_0(x), & x\in\bar{\Omega}.\end{cases}\tag{20}$$

Subtracting (17) from (20), we have

$$\begin{cases}\left(\partial_t u^n-\partial_t u_h^n,v_h\right)+\left(\vec{p}^n-\vec{p}_h^n,\nabla v_h\right)=\left(\nabla\cdot\left(f\left(u^n\right)-f\left(u_h^n\right)\right),v_h\right)\\ \qquad\qquad+\left(R_1^n,v_h\right), \qquad \forall v_h\in X_h,\\ \left(\partial_t\nabla u^n-\partial_t\nabla u_h^n,\vec{q}_h\right)-\left(\vec{p}^n-\vec{p}_h^n,\vec{q}_h\right)=\left(R_2^n,\vec{q}_h\right), \qquad \forall\vec{q}_h\in M_h.\end{cases}\tag{21}$$

where $R_1^n=\partial_t\left(u^n-u_t^n\right), R_2^n=\partial_t\left(\nabla u^n-\nabla u_t^n\right)$.

For simplicity, we introduce the following notation:

$$u^n-u_h^n=\left(u^n-\Pi_h^2u^n\right)+\left(\Pi_h^2u^n-u_h^n\right)=\eta^n+\xi^n,$$
$$\vec{p}^n-\vec{p}_h^n=\left(\vec{p}^n-\Pi_h^1\vec{p}^n\right)+\left(\Pi_h^1\vec{p}^n-\vec{p}_h^n\right)=\vec{\rho}^n+\vec{\theta}^n.$$

So (21) becomes

$$\begin{cases}\left(\partial_t\xi^n,v_h\right)+\left(\vec{\theta}^n,\nabla v_h\right)=-\left(\partial_t\eta^n,v_h\right)-\left(\vec{\rho}^n,\nabla v_h\right)+\left(R_1^n,v_h\right)\\ \qquad\qquad+\left(\nabla\cdot\left(f\left(u^n\right)-f\left(u_h^n\right)\right),v_h\right), \quad \forall v_h\in X_h,\\ \left(\partial_t\nabla\xi^n,\vec{q}_h\right)-\left(\vec{\theta}^n,\vec{q}_h\right)=-\left(\partial_t\nabla\eta^n,\vec{q}_h\right)+\left(\vec{\rho}^n,\vec{q}_h\right)\\ \qquad\qquad+\left(R_2^n,\vec{q}_h\right), \qquad \forall\vec{q}_h\in M_h.\end{cases}\tag{22}$$

Taking $v_h=\xi^n,\vec{q}_h^n=\nabla\xi^n$ in (22), we get

$$\begin{aligned}\left(\partial_t\xi^n,\xi^n\right)+\left(\partial_t\nabla\xi^n,\nabla\xi^n\right)=&-\left(\partial_t\eta^n,\xi^n\right)-\left(\partial_t\nabla\eta^n,\nabla\xi^n\right)\\ &+\left(\nabla\cdot\left(f\left(u^n\right)-f\left(u_h^n\right)\right),\xi^n\right)\\ &+\left(R_1^n,\xi^n\right)+\left(R_2^n,\nabla\xi^n\right).\end{aligned}\tag{23}$$

From the left of the above formula, which can be rewritten as

$$\left(\partial_t\xi^n,\xi^n\right)=\frac{1}{2\Delta t}\left(\left\|\xi^n\right\|_0^2-\left\|\xi^{n-1}\right\|_0^2+\left\|\xi^n-\xi^{n-1}\right\|_0^2\right),$$

$$\left(\partial_t\nabla\xi^n,\nabla\xi^n\right)=\frac{1}{2\Delta t}\left(\left\|\nabla\xi^n\right\|_0^2-\left\|\nabla\xi^{n-1}\right\|_0^2+\left\|\nabla\xi^n-\nabla\xi^{n-1}\right\|_0^2\right).$$

Using the Cauchy-Schwarz inequality and the Young Inequality, we get

$$\left(\partial_t\eta^n,\xi^n\right)+\left(\partial_t\nabla\eta^n,\nabla\xi^n\right)\le C\left(\left\|\partial_t\eta^n\right\|_0^2+\left\|\partial_t\nabla\eta^n\right\|_0^2\right)+\varepsilon\left(\left\|\xi^n\right\|_0^2+\left\|\nabla\xi^n\right\|_0^2\right),\quad (24)$$

And

$$\left(R_1^n,\xi^n\right)+\left(R_2^n,\nabla\xi^n\right)\le C\left(\left\|R_1^n\right\|_0^2+\left\|R_2^n\right\|_0^2\right)+\varepsilon\left(\left\|\xi^n\right\|_0^2+\left\|\nabla\xi^n\right\|_0^2\right).\quad (25)$$

further, using the Cauchy-Schwartz inequality, the Young Inequality, and the Lipschitz condition, we get

$$\begin{aligned}\left(\nabla\cdot\left(f\left(u^n\right)-f\left(u_h^n\right)\right),\xi^n\right)&\le C\left\|u^n-u_h^n\right\|_0^2+\varepsilon\left\|\nabla\xi^n\right\|_0^2\\&\le C\left(\left\|\xi^n\right\|_0^2+\left\|\eta^n\right\|_0^2\right)+\varepsilon\left\|\nabla\xi^n\right\|_0^2.\end{aligned}\quad (26)$$

According to the Taylor formula, we get

$$\left\|R_1^n\right\|_0^2+\left\|R_2^n\right\|_0^2\le C\left(\Delta t\right)^2\left(\left\|u_{tt}\left(t_{n+\theta_1}\right)\right\|_0^2+\left\|\nabla u_{tt}\left(t_{n+\theta_2}\right)\right\|_0^2\right).\quad (27)$$

substitute (24), (25), (26) in (23), then

$$\begin{aligned}&\frac{1}{2\Delta t}\left[\left\|\xi^n\right\|_0^2+\left\|\nabla\xi^n\right\|_0^2-\left(\left\|\xi^{n-1}\right\|_0^2+\left\|\nabla\xi^{n-1}\right\|_0^2\right)\right]\\&\le C\left(\left\|\partial_t\eta^n\right\|_0^2+\left\|\partial_t\nabla\eta^n\right\|_0^2+\left\|\eta^n\right\|_0^2\right)\\&\quad+C\left(\Delta t\right)^2\left(\left\|u_{tt}\left(t_{n+\theta_1}\right)\right\|_0^2+\left\|\nabla u_{tt}\left(t_{n+\theta_2}\right)\right\|_0^2\right)+3\varepsilon\left(\left\|\xi^n\right\|_0^2+\left\|\nabla\xi^n\right\|_0^2\right).\end{aligned}$$

Multiply both ends of the above formula by $4\Delta t$, taking $\varepsilon=\frac{1}{6\Delta t}$, and add from 0 to N, we get

$$\begin{aligned}\left\|\xi^n\right\|_0^2+\left\|\nabla\xi^n\right\|_0^2\le C\Delta t\sum_{k=0}^{N}&\left[\left\|\partial_t\eta^k\right\|_0^2+\left\|\partial_t\nabla\eta^k\right\|_0^2+\left\|\eta^k\right\|_0^2\right]\\&+\left(\Delta t\right)^2\left(\left\|u_{tt}\left(t_{k+\theta_1}\right)\right\|_0^2+\left\|\nabla u_{tt}\left(t_{k+\theta_2}\right)\right\|_0^2\right).\end{aligned}$$

From **Lemma 3.3** and **Lemma 3.6**, we have

$$\left\|\xi^n\right\|_0^2+\left\|\nabla\xi^n\right\|_0^2\le C\left[h^{2m}\int_0^{t_n}\left(\left\|u\right\|_3^2+\left\|u_t\right\|_3^2\right)\mathrm{d}\tau+\left(\Delta t\right)^2\int_0^{t_n}\left\|u_{tt}\right\|_1^2\mathrm{d}\tau\right].\quad (28)$$

According to the regularity of u, we have

$$\left\|\xi^n\right\|_1^2=\left\|\xi^n\right\|_0^2+\left\|\nabla\xi^n\right\|_0^2\le C\left[h^4+\left(\Delta t\right)^2\right].\quad (29)$$

Taking $v_h=\partial_t\xi^n,\vec{q}_h=\partial_t\nabla\xi^n$ in (22), using the Lipschitz condition, we get

$$
\begin{aligned}
\left\|\partial_t\xi^n\right\|_0^2+\left\|\partial_t\nabla\xi^n\right\|_0^2 &= -\left(\partial_t\eta^n,\partial_t\xi^n\right)-\left(\partial_t\nabla\eta^n,\partial_t\nabla\xi^n\right)\\
&\quad+\left(\nabla\cdot\left(f\left(u^n\right)-f\left(u_h^n\right)\right),\partial_t\xi^n\right)\\
&\quad+\left(R_1^n,\partial_t\xi^n\right)+\left(R_2^n,\partial_t\nabla\xi^n\right)\\
&\le C\left(\left\|\partial_t\eta^n\right\|_0^2+\left\|\partial_t\nabla\eta^n\right\|_0^2+\left\|R_1^n\right\|_0^2+\left\|R_2^n\right\|_0^2\right)\\
&\quad+\varepsilon\left(\left\|\partial_t\xi^n\right\|_0^2+\left\|\partial_t\nabla\xi^n\right\|_0^2\right).
\end{aligned}
$$

Multiply both ends of the above formula by 2, taking $\varepsilon=\frac{1}{2}$, then according to **Lemma 3.3** and the regularity of u, we get

$$\left\|\partial_t\xi^n\right\|_0^2+\left\|\partial_t\nabla\xi^n\right\|_0^2\le C\left[h^4+(\Delta t)^2\right]. \tag{30}$$

Then, using **Lemma 3.3** and the triangle inequality, the proof of (18) is completed. Letting $\vec{q}_h=\vec{\theta}^n$ the second formula of (22), we get

$$
\begin{aligned}
\left\|\vec{\theta}^n\right\|_0^2 &= \left(\partial_t\nabla\xi^n,\vec{\theta}^n\right)+\left(\partial_t\nabla\eta^n,\vec{\theta}^n\right)+\left(\vec{\rho}^n,\vec{\theta}^n\right)+\left(R_2^n,\vec{\theta}^n\right)\\
&\le 4\varepsilon\left\|\vec{\theta}^n\right\|_0^2+C\left(\left\|\partial_t\nabla\xi^n\right\|_0^2+\left\|\partial_t\nabla\eta^n\right\|_0^2+\left\|\vec{\rho}^n\right\|_0^2+\left\|R_2^n\right\|_0^2\right).
\end{aligned}
$$

Letting $\varepsilon=\frac{1}{8}$, from (29), **Lemma 3.3**, and **Lemma 3.4**, then there holds

$$\left\|\vec{\theta}^n\right\|_0^2\le C\left[h^4+(\Delta t)^2\right]. \tag{31}$$

then using the triangle inequality and the interpolation theorem, the proof of (19) is completed.

5. Numerical Example

Positioning Consider the following BBM Equation:

$$
\begin{aligned}
&u_t-u_{xxt}-u_{yyt}+u_x+u_y+uu_x+uu_y=0, && (x,y,t)\in\Omega\times(0,T],\\
&u(x,y,0)=u_0(x,y)=x^2(1-x)^2y^2(1-y)^2, && (x,y)\in\partial\Omega.
\end{aligned}
$$

where $\Omega=(0,1)\times(0,1)$. The region Ω is divided into uniform rectangles along the X and Y directions, Q_2 element approximation is adopted for the velocity u space, and Q_1 element approximation is adopted for the P space.

Since the exact solution of problem (2) is unknown, we use the numerical solution on the fine grid as the exact solution to estimate the error. We take some different values of Δt and h to verify the format of this paper, as shown in the following table.

Table 1. $\Delta t=0.1, h=1/10, e=u^*-u_h, \eta=p^*-p_h$.

t	$\|e\|_{L^2(\Omega)}$	$\|e\|_{H^1(\Omega)}$	$\|\eta\|_{L^2(\Omega)}$
0.2	6.31×10^{-7}	3.71×10^{-4}	1.31×10^{-6}
0.4	6.80×10^{-7}	4.75×10^{-4}	8.20×10^{-7}

Continued

0.6	7.19×10^{-7}	4.91×10^{-4}	6.27×10^{-7}
0.8	7.20×10^{-7}	5.01×10^{-4}	4.83×10^{-7}
1.0	7.01×10^{-7}	5.44×10^{-4}	3.04×10^{-7}

Table 2. $\Delta t=0.1, h=1/20, e=u^*-u_h, \eta=p^*-p_h$.

t	$\|e\|_{L^2(\Omega)}$	$\|e\|_{H^1(\Omega)}$	$\|\eta\|_{L^2(\Omega)}$
0.2	4.00×10^{-8}	8.71×10^{-5}	9.28×10^{-8}
0.4	5.00×10^{-8}	1.35×10^{-4}	5.49×10^{-8}
0.6	6.00×10^{-8}	1.52×10^{-4}	4.23×10^{-8}
0.8	6.00×10^{-8}	1.73×10^{-4}	3.58×10^{-8}
1.0	5.00×10^{-8}	1.86×10^{-4}	2.67×10^{-8}

Table 3. $\Delta t=0.1, h=1/40, e=u^*-u_h, \eta=p^*-p_h$.

t	$\|e\|_{L^2(\Omega)}$	$\|e\|_{H^1(\Omega)}$	$\|\eta\|_{L^2(\Omega)}$
0.2	2.00×10^{-10}	2.31×10^{-5}	9.28×10^{-8}
0.4	1.00×10^{-8}	3.23×10^{-5}	5.49×10^{-8}
0.6	1.00×10^{-8}	3.78×10^{-5}	4.23×10^{-8}
0.8	1.00×10^{-8}	4.24×10^{-5}	3.58×10^{-8}
1.0	2.00×10^{-10}	4.75×10^{-5}	2.70×10^{-8}

Table 4. $t=1, \Delta t=h^2, e=u^*-u_h, \eta=p^*-p_h$.

h	$\|e\|_{H^1(\Omega)}$	*rate*	$\|\eta\|_{L^2(\Omega)}$	*rate*
1/4	7.92×10^{-4}	-	5.78×10^{-7}	-
1/8	1.96×10^{-4}	2.11	1.39×10^{-7}	2.01
1/16	4.93×10^{-5}	1.97	3.08×10^{-8}	2.05
1/32	1.34×10^{-5}	1.87	7.96×10^{-9}	1.96

The numerical results (**Tables 1-4**) show that the scheme is feasible, and further verify that the hybrid stabilization method presented in this paper is stable and convergent.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Omrani, K. (2006) The Convergence of Fully Discrete Galerkin Approximations for the Benjamin-Bona-Mahony Equation. *Applied Mathematics and Computation*, **180**, 614-621. https://doi.org/10.1016/j.amc.2005.12.046

[2] Benjamin, T.B., Bona, J.L. and Mahony, J.J. (1972) Model Equations for Long Waves in Nonlinear Dispersive Systems. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, **272**, 47-78. https://doi.org/10.1098/rsta.1972.0032

[3] Li, S.K., Feng, M.F. and Li, S. (2003) Finite Difference Approximate Solutions for the Benjamin-Bona-Mahony Equation. *Journal of Sichuan Normal University (Natural Science)*, **26**, 363-365. (In Chinese)

[4] Luo, Z.D. and Liu, R.X. (1999) Mixed Finite Element Analysis and Nonnumerical Simulation for Burgers Equation. *Mathematica Numerica Sinica*, **21**, 257-268.

[5] Medeiros, L.A. and Menzala, G.P. (1977) Existence and Uniqueness for Periodic Solutions of the Benjamin-Bona-Mahony Equation. *SIAM Journal on Mathematical Analysis*, **8**, 792-799. https://doi.org/10.1137/0508062

[6] Qin, Y., Kong, H., Luo, D. and Feng, M. (2015) A Fully Discrete Mixed Finite Element Method for the Benjamin-Bona-Mahony (BBM) Equation. *Acta Mathematicae Applicatae Sinica*, **38**, 597-609. (In Chinese)

[7] Tan, Y.M., Kong, H., Luo, D. and Feng, M.F. (2015) A Fully Discrete Mixed Finite Method for the Benjamin-Bona-Mahony Equation. *Journal d'Analyse Mathématique*, **38**, 597-609.

[8] Babuška, I. (1971) Error-Bounds for Finite Element Method. *Numerische Mathematik*, **16**, 322-333. https://doi.org/10.1007/bf02165003

[9] Brezzi, F. (1974) On the Existence, Uniqueness and Approximation of Saddle-Point Problems Arising from Lagrangian Multipliers. *SIAM Journal on Numerical Analysis*, **13**, 185-197.

[10] Wan, L.H. and Xue, X.J. (2004) Mathematical Basis of Finite Element Method. Science Press. (In Chinese)

[11] Luo, Z.D. (2006) Foundation and Application of Mixed Finite Element Method. Science Press. (In Chinese)

[12] Si, H.Y. and Chen, S.C. (2012) A New Mixed Finite Element Scheme for the Biharmonic Equation. *Mathematica Numerica Sinica*, **34**, 173-182. (In Chinese)

[13] Zhu, W.J. and Zhang, H.C. (2016) Analysis of a Mixed Finite Element Method for a Fourth-Order Parabolic Equation. *Journal of Hebei University (Natural Science Edition)*, **40**, 189-194. (In Chinese)

[14] Si, H.Y. (2016) New Mixed Finite Element Schemes for Steady Stokes Problem. *Journal of Anhui University (Natural Sciences)*, **40**, 15-18. (In Chinese)

[15] Chen, S.C. and Chen, H.R. (2010) New Mixed Finite Schemes for Second Order Elliptic Problem. *Numerical Mathematics Sinica*, **32**, 213-218. (In Chinese)

[16] Luo, Z.D. (1996) Foundation and Application of Mixed Finite Element Method. Shandong Education Press. (In Chinese)

[17] Luo, Z.D. (1992) An Improved Mixed Finite Element Scheme for Stokes Equation.

Acta Mathematica Scientia, **12**, 85-86. (In Chinese)

[18] Girault, V. and Raviart, P.A. (1979) An Analysis of a Mixed Finite Element Method for the Navier-Stokes Equations. *Numerische Mathematik*, **33**, 235-271. https://doi.org/10.1007/bf01398643

[19] Temam, R. (1984) Navier-Stokes Equation: Theory and Numerical Analysis. 3rd Edition, North-Holland Publishing Company.