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Carrier-Resolved Burnside Data for CSS Lattice Codes: Incidence Complexes, Ground-Space Reduction, and X-Cube Sewing

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Abstract

We attach carrier data to CSS stabilizer calculations by using the effective Burnside category of a finite support meet-semilattice. A cellular CSS Hamiltonian determines a literal support lattice and a Pauli coefficient; chosen generator-relation presentations determine carrier-labelled Burnside chain complexes. Their binary evaluations retain point, line, plane, tube, and leaf labels while reproducing the usual plaquette, Bacon-Shor, and periodic X-cube ranks. The microscopic coefficient detects a pinned local qubit. Ground-space compression removes this ancilla dependence and produces a monotone diagram of operator systems whose support changes by bounded thickening under finite-depth local circuits. For the X-cube code, the affine-leaf syndrome complexes recover the eight quotient sectors. We lift coordinate leaf insertion to an isomorphism of global CSS complexes and give its block action on directional logical sectors: the two toric-code classes enter the two transverse summands. This yields a $\mathbb{Z}/2$ sewing class for direction-framed presentations. Its scalar shadow $\log_2 \mathrm{GSD} \bmod 2$ is invariant when the free layers are two-dimensional topological Pauli-stabilizer codes, and a semion layer shows why the statement does not extend to unrestricted foliated resources. Thus the paper separates presentation-level carrier data from the two restricted equivalence invariants proved here.

Keywords

CSS Stabilizer Codes, Effective Burnside Category, X-Cube Model, Fracton Order, Foliated Equivalence

1. Introduction

Subsystem-symmetric and fracton lattice models retain information that is lost when one forgets the carrier of an operator or excitation. A logical operator may have a rigid line representative, and a charge may be point-, line-, or plane-mobile. The purpose of this paper is to organize that information in a finite, calculable language.

The construction has two algebraic levels. First, a cellular CSS Hamiltonian has a canonical literal support lattice and a Pauli coefficient. Second, a chosen presentation lifts its generator-relation matrices to complexes of formal carriers and spans. The first records microscopic Pauli degrees of freedom; the second records the carriers used in a calculation. Dual syndrome carriers and mobility labels are therefore stated as certified presentation data rather than inferred from an abstract Hamiltonian.

The main results are as follows. We construct the support-Burnside complexes for the plaquette, Bacon-Shor, and X-cube examples and compute their evaluated homology. For the X-cube code we give explicit affine-line and affine-leaf presentations of logical and syndrome data. We then lift the standard CNOT leaf-sewing circuit to a map of global CSS complexes, including its exact binary support action and its block action on the directional logical summands. Finally, we separate the microscopic Pauli coefficient from its ground-space compression and prove controlled stability of the latter under finite-depth circuits and specified resource layers.

Two scope choices are used throughout. The direction-resolved sewing class belongs to presentations with a chosen coordinate frame. The unframed parity statement uses only two-dimensional topological Pauli-stabilizer layers as free resources; Proposition 11.2 gives the obstruction for arbitrary two-dimensional phases. These hypotheses keep the carrier and phase statements distinct without repeating them after each calculation.

Section 12 places the construction alongside categorical CSS codes, local logical algebras, and the established X-cube foliation literature. In particular, code-space compression and X-cube sewing are used as prior ingredients; the contributions here are their carrier-resolved Burnside organization and the explicit directional block calculation.

2. Finite Support Systems and Burnside Spans

Definition 2.1 (Support system)**.** A finite support system *is a finite meet-semilattice* $(P,\le,\wedge)$ *with a greatest element* 1. *We allow a distinguished null label* $\perp$ *with* $\perp\wedge p=\perp$. *The null label means an empty carrier, not a point. A rank or dimension function, when used, is extra structure.*

Let $\mathrm{Fam}(P)$ be the category of finite sets A equipped with a label $\lambda_A : A\to P$. A morphism $f : A\to B$ satisfies $\lambda_A(a)\le\lambda_B(f(a))$. We write p for the singleton labelled by p.

A support-Burnside chain complex may contain the singleton labelled by $\perp$ as a formal summand. Here "empty carrier" describes its geometric label, not its underlying one-element set: this singleton is not the empty family $\varnothing$ in

$\mathrm{Fam}(P)$. Consequently, the constant span coefficient used below is not null-reduced and gives $\underline{\mathbb{F}}_2(\perp) \cong \mathbb{F}_2$, whereas $\underline{\mathbb{F}}_2(\varnothing) = 0$. Such a summand is a formal generator or relation certified to have empty support, not a point carrier or a physical Pauli degree of freedom. When empty carriers are to be annihilated, one must instead use a null-reduced coefficient satisfying $M(\perp) = 0$, such as the Pauli coefficient of Definition 4.2. No null-labelled generator occurs in the concrete code complexes below.

Lemma 2.2. *The category* $\mathrm{Fam}(P)$ *has finite coproducts and pullbacks. For* $A \to C \leftarrow B$, *the pullback has underlying set* $A \times_C B$ *and label* $\lambda_A(a) \wedge \lambda_B(b)$ *at* (a,b).

Proof. Disjoint union gives coproducts. If a labelled family maps to both A and B over C, its label lies below both pullback labels and hence below their meet, which proves the labelled universal property. □

Declare every morphism ingressive and egressive and write

$$\mathrm{A}_P^{\mathrm{eff}} := \mathrm{A}^{\mathrm{eff}}\left(\mathrm{Fam}(P)\right).$$

Its morphisms are spans, composed by pullback. For $p \le q$, the spans

$$p \xleftarrow{\mathrm{id}} p \to q, \quad q \leftarrow p \xrightarrow{\mathrm{id}} p$$

give transfer and restriction, respectively. Thus all base-change identities are built into span composition. This is the effective-Burnside construction specialized to finite labelled sets; its general categorical form is standard [1]. Only its additive span category is used below.

3. Support-Burnside Chain Complexes

The relation complexes used for CSS codes are covariant: a large carrier can impose a relation on smaller carriers.

Let $\mathcal{B}_P = \pi_0 \mathrm{A}_P^{\mathrm{eff}}$ and let $\mathcal{B}_P^{\oplus}$ be its additive completion. A morphism in $\mathcal{B}_P^{\oplus}$ is a finite integer matrix of spans.

Definition 3.1 (Support-Burnside chain complex)**.** *A support-Burnside chain complex is a bounded chain complex in* $\mathcal{B}_P^{\oplus}$ *whose terms are finite sums of singleton carriers. An ordinary Mackey functor* $M : \mathcal{B}_P^{\oplus} \to \mathrm{Vect}_{\mathbb{F}_2}$ *sends it covariantly to an ordinary binary chain complex. We write*

$$H_n^M(C_\bullet) := H_n\left(M(C_\bullet)\right).$$

We use the standard span-linearization coefficient throughout the code calculations.

Definition 3.2 (Constant span coefficient)**.** *For a labelled finite set* A, *set*

$$\underline{\mathbb{F}}_2(A) = \mathbb{F}_2\langle a \in A \rangle.$$

For a span $A \xleftarrow{\ell} U \xrightarrow{r} B$, *send the basis vector* a *to the sum of all* $u \in U$ *with* $\ell(u) = a$, *placed at* $r(u)$. *Extend linearly over* $\mathbb{F}_2$.

Lemma 3.3. *Definition 3.2 defines an additive Mackey functor* $\underline{\mathbb{F}}_2 : \mathcal{B}_P^{\oplus} \to \mathrm{Vect}_{\mathbb{F}_2}$.

Proof. The matrix coefficient of a composite span counts pairs in the pullback of the two middle legs, exactly the coefficient obtained by composing the two linear maps. Disjoint unions give direct sums, and null-labelled generators are treated like all other labelled generators. □

Thus a restriction span from a tube to one of its boundary lines is a well-defined boundary map, and applying $\mathbb{F}_2$ gives the familiar binary coefficient 1.

4. Canonical CSS Support Data and Certified Carriers

We first isolate the part of the construction that is canonical for a cellular CSS Hamiltonian, before making any choice of a relation basis or a mobility carrier.

Definition 4.1 (Cellular CSS Hamiltonian). *A cellular CSS Hamiltonian is a finite cell complex* Λ, *a finite set* Q *of qubits carried by closed cell subcomplexes* $\overline{q} \subset \Lambda$, *and two orthogonal subspaces*

$$S_X, S_Z \subset \mathbb{F}_2^Q, \quad S_X \perp S_Z.$$

For $a \in \mathbb{F}_2^Q$, *put* $\operatorname{supp}(a) = \bigcup_{a_q = 1} \overline{q}$. *Its canonical support system is the finite lattice*

$$P_\Lambda^{\text{can}} = \operatorname{Sub}(\Lambda)$$

of all cell subcomplexes of Λ, *ordered by inclusion.*

This basis-free choice contains every literal Pauli and stabilizer support and is unchanged by a change of stabilizer generators. It is finite because Λ is finite. A calculation may replace it by the finite sublattice generated by the carriers it actually uses; that is a stated reduction, not the canonical construction.

Definition 4.2 (Pauli support coefficient). *For* $p \in P_\Lambda^{\text{can}}$, *let*

$$Q(p) = \{q \in Q : \overline{q} \subset p\}, \quad \mathcal{P}_\Lambda(p) = \mathbb{F}_2^{Q(p)}.$$

If $p \le q$, *the transfer is the coordinate inclusion* $\mathcal{P}_\Lambda(p) \hookrightarrow \mathcal{P}_\Lambda(q)$ *and the restriction is coordinate deletion* $\mathcal{P}_\Lambda(q) \twoheadrightarrow \mathcal{P}_\Lambda(p)$. *On a labelled finite family take the direct sum over its elements, and on a span use restriction along its left leg followed by transfer along its right leg.*

Proposition 4.3 (Canonical Pauli Mackey coefficient). *Definition 4.2 defines an additive null-reduced Mackey functor*

$$\mathcal{P}_\Lambda : \mathcal{B}^{\oplus}_{P_\Lambda^{\text{can}}} \to \mathrm{Vect}_{\mathbb{F}_2}.$$

The assignment $(\Lambda, Q, S_X, S_Z) \mapsto (P_\Lambda^{\text{can}}, \mathcal{P}_\Lambda)$ *is functorial for cellular CSS isomorphisms.*

Proof. For $p, q \le r$, coordinate inclusion and coordinate deletion obey the Beck-Chevalley identity

$$\mathrm{res}_p^r \mathrm{tr}_q^r = \mathrm{tr}_{p \wedge q}^p \mathrm{res}_{p \wedge q}^q.$$

This is the identity on precisely the coordinates carried by $p \wedge q$ and zero on all others. Hence span composition is respected. A cellular CSS isomorphism bijects qubits and their closed carriers, and carries the two stabilizer subspaces to the corresponding subspaces, so it induces the stated isomorphism of support systems and coefficients. ☐

Restricting a global stabilizer to a smaller carrier need not produce another stabilizer. Accordingly, global relation complexes require the certified carriers introduced below.

The canonical coefficient is strictly finer than the usual global data in a limited but useful sense.

Proposition 4.4 (Not determined by global logical and quotient-charge data). *The Pauli support coefficient is not determined by the global CSS logical spaces together with any quotient of the syndrome group in which syndromes created inside a contractible ball are trivial.*

Proof. On an otherwise unused cell in a small contractible ball, adjoin a decoupled qubit q with the single Z_q stabilizer. The two systems live on the same cell complex and have the same canonical support lattice. The new X-error coordinate has nonzero syndrome and therefore contributes no X logical class; the new Z_q is itself a stabilizer and contributes no Z logical class. Its syndrome is created by the operator X_q in that ball, hence is zero in every stated quotient-charge group. In contrast, on the qubit carrier $b=\overline{q}$ one has $\mathcal{P}_{\Lambda'}(b)\cong\mathbb{F}_2$, whereas the unextended system has $\mathcal{P}_{\Lambda}(b)=0$. Thus these global invariants cannot recover the coefficient. ☐

Thus $\mathcal{P}_{\Lambda}$ detects presentation-local information, including a phase-trivial pinned qubit. Section 10 instead compresses local operators to the ground space.

Definition 4.5 (Certified cellular CSS presentation). *A certified cellular CSS presentation is a cellular CSS Hamiltonian together with chosen bases of the displayed relations, local Pauli processes, and syndrome generators, and a carrier assignment by primal or barycentric-dual subcomplexes. Every component of a displayed relation must be contained in the carrier of that relation. The enlarged support system is the finite lattice generated by the canonical carriers and these certified carriers.*

For literal Pauli supports the carrier is the smallest cell subcomplex containing the qubits on which the operator acts. For the standard X-cube cellular realization, the syndrome of a Z operator on an edge is carried by the transverse dual plaquette containing its four dual-cube vertices. This dual statement depends on that cellular realization; it is not available for an arbitrary abstract CSS matrix.

Proposition 4.6 (Certified presentation functoriality). *A certified cellular CSS presentation determines support-Burnside chain complexes for every displayed generator-relation or local-syndrome matrix. Carrier-preserving maps of certified presentations induce maps of these complexes and commute with evaluation by* $\mathbb{F}_2$.

Proof. Every component of a certified relation is a containment of carriers and hence a restriction or transfer span in the enlarged support-Burnside category. Matrices of these spans compose as the corresponding matrices of relations. Applying the functor of Definition 3.2 is functorial. ☐

5. Plaquette and Bacon-Shor Calculations

Let $\Lambda = \mathbb{Z}/L_x\mathbb{Z} \times \mathbb{Z}/L_y\mathbb{Z}$ and let $V = \mathbb{F}_2^{\Lambda}$. The plaquette constraint is

$$(Dq)_{ij} = q_{ij} + q_{i+1,j} + q_{i,j+1} + q_{i+1,j+1}.$$

Theorem 5.1 (Plaquette symmetry space). *There is a natural presentation*

$$\ker D \cong \frac{\mathbb{F}_2^{L_y}[x] \oplus \mathbb{F}_2^{L_x}[y]}{\left\langle \left(1_{L_y}, 1_{L_x}\right) \right\rangle}, \quad \dim_{\mathbb{F}_2} \ker D = L_x + L_y - 1.$$

Proof. Every $q_{ij} = r_j + c_i$ lies in the kernel. Conversely, the plaquette equation propagates a chosen row and column to all sites. The sole redundancy is the simultaneous constant shift of r and c. ☐

The support statement has the following chain-level refinement. In $\mathcal{B}_{P_2}^{\oplus}$, take

$$K_1 = [xy] \xrightarrow{\partial} K_0 = [x]^{\oplus L_y} \oplus [y]^{\oplus L_x}, \quad \partial(1) = \left(1_{L_y}, 1_{L_x}\right),$$

where the components are restriction spans from xy to x and y. Then $H_0^{\mathbb{F}_2}(K_\bullet) \cong \ker D$. The generators have line carriers and the single relation has full two-dimensional carrier.

For the rectangular Bacon-Shor presentation with m columns and n rows, vertical XX and horizontal ZZ gauge generators give

$$\left[\left[mn, 1, (m-1)(n-1), \min\{m,n\}\right]\right] \ [2].$$

The protected X and Z logical operators have row and column carriers, respectively. This is a statement about the specified rectangular presentation; no canonical carrier is assigned to an abstract logical class without its representative.

6. Periodic X-Cube Logical Relations

Let $\Lambda = \mathbb{Z}_{L_x} \times \mathbb{Z}_{L_y} \times \mathbb{Z}_{L_z}$, $V = L_x L_y L_z$, and $S = L_x + L_y + L_z$. Put one qubit on every oriented edge. For a cube c, let B_c be the product of X on its twelve edges; for a vertex v, let A_v^{μ} be the product of Z on the four edges in the plane normal to μ. We use the standard periodic X-cube presentation of Ref. [3].

Lemma 6.1 (Stabilizer ranks). *For* $L_x, L_y, L_z \geq 3$,

$$\operatorname{rank}\langle B_c \rangle = V - S + 2, \quad \operatorname{rank}\langle A_v^{\mu} \rangle = 2V - S + 1, \quad k = 2S - 3.$$

Proof. Writing $\Delta_\mu = 1 + T_\mu$ over $\mathbb{F}_2$, cube relations obey $\Delta_y \Delta_z f = \Delta_x \Delta_z f = \Delta_x \Delta_y f = 0$. The first equation gives the rectangle identity

$$f_{ijk} = f_{ij0} + f_{i0k} + f_{i00}.$$

The other two give $f_{ij0} = f_{i00} + f_{0j0} + f_{000}$ and $f_{i0k} = f_{i00} + f_{00k} + f_{000}$. Hence every relation has the form $f_{ijk} = a_i + b_j + c_k$. The kernel of $(a,b,c) \mapsto f$ consists of constant triples whose sum is zero, so the cube-relation space has dimension $S-2$ and the cube-stabilizer rank is $V - S + 2$.

For a cross relation, let $g_\mu(v)$ be the coefficient of A_v^{μ}. Cancellation on the three edge orientations is equivalent to

$$\Delta_x\left(g_y + g_z\right) = 0, \quad \Delta_y\left(g_x + g_z\right) = 0, \quad \Delta_z\left(g_x + g_y\right) = 0.$$

The V pointwise triples (h,h,h) are the local relations $A_v^x A_v^y A_v^z = 1$. Modulo them, set $g_z = 0$ and put $p = g_x + g_z$, $q = g_y + g_z$. The preceding system becomes $\Delta_y p = 0$, $\Delta_x q = 0$, and $\Delta_z(p+q) = 0$. Its solutions are

$$p(i,j,k) = a_i + c_k, \quad q(i,j,k) = b_j + c_k.$$

The parameters have the one-dimensional redundancy obtained by adding the same constant to a,b,c. The quotient relation space therefore has dimension $S-1$, and the full cross-relation space has dimension $V+S-1$. Thus the cross rank is $3V-(V+S-1) = 2V-S+1$. Finally, $k = 3V-(V-S+2)-(2V-S+1) = 2S-3$. □

For fixed transverse coordinates, let W_{jk}^x be the rigid x-directed X loop and define the other directions cyclically. Put

$$\mathcal{W}(A,B) = \mathbb{F}_2^{A\times B}/\mathrm{im}(\Delta_A \otimes \Delta_B).$$

Theorem 6.2 (Affine-line calculation). *The X-logical sector has the directional presentation*

$$L_X \cong W_x \oplus W_y \oplus W_z, \quad W_x = \mathcal{W}\left(\mathbb{Z}_{L_y}, \mathbb{Z}_{L_z}\right),$$

and cyclically. Hence

$$\dim W_x = L_y + L_z - 1, \quad \dim W_y = L_x + L_z - 1, \quad \dim W_z = L_x + L_y - 1.$$

Proof. Multiplication of the cube stabilizers along an x-directed tube with transverse corner (j,k) gives

$$W_{jk}^x + W_{j+1,k}^x + W_{j,k+1}^x + W_{j+1,k+1}^x = 0,$$

which is the image of $\Delta_y \otimes \Delta_z$. Conversely, the usual plane-by-plane reduction by cube stabilizers removes contractible X-support segments and leaves a sum of rigid loops; repeating the reduction on adjacent planes produces exactly these four-loop relations (compare Ref. [4]). Thus the x sector is $\mathcal{W}\left(\mathbb{Z}_{L_y}, \mathbb{Z}_{L_z}\right)$.

On a cyclic set A, $\ker \Delta_A$ is the constant subspace, so $\operatorname{rank} \Delta_A = |A|-1$. Therefore

$$\operatorname{rank}(\Delta_A \otimes \Delta_B) = (|A|-1)(|B|-1), \quad \dim \mathcal{W}(A,B) = |A|+|B|-1.$$

The cyclic sectors have total dimension $2S-3$, equal to the independently computed logical dimension in the preceding lemma. Hence the spanning set has neither additional relations nor a missing sector, proving the direct-sum presentation. □

Let $P_{\mathrm{XC}}(\Lambda)$ be the lattice generated by literal primal tubes and loops (and, below, their dual carriers). The logical calculation is the support-Burnside chain complex

$$K_1^X = \bigoplus_{\mu,a,b}\left[\tau_{ab}^{\mu}\right] \xrightarrow{d_X} K_0^X = \bigoplus_{\mu,a,b}\left[\ell_{ab}^{\mu}\right],$$

where, for example,

$$d_X\left(\tau_{jk}^x\right) = \ell_{jk}^x + \ell_{j+1,k}^x + \ell_{j,k+1}^x + \ell_{j+1,k+1}^x.$$

Each component is a restriction span because the literal tube contains the corresponding boundary loop. Therefore

$$H_0^{\mathbb{F}_2}\left(K_\bullet^X\right) \cong L_X .$$

This is the carrier-resolved span presentation of the binary relation matrix.

For reference, Table 1 collects every generator and relation family displayed in the X-cube calculations. "Primal" and "dual" refer to the standard cubic cellulation and its barycentric dual. For $p \subseteq q$, write

$$r_{q,p} := \left(q \xleftarrow{\iota} p \xrightarrow{\mathrm{id}} p\right), \quad t_{p,q} := \left(p \xleftarrow{\mathrm{id}} p \xrightarrow{\iota} q\right)$$

for the restriction and transfer spans. Put $\mathrm{b}_c = \bigcup_{e \subset \partial c} e$ for the primal edge frame of a cube and $\mathrm{a}_v^\mu = \bigcup_{e \ni v, e \perp \hat{\mu}} e$ for a primal cross. For a binary relation $R = \sum_a G_a$, set $\mathrm{c}(R) = \bigcup_a \mathrm{c}(G_a)$. To fix independent stabilizer relation bases, take the reference leaves $j=0$ and $k=0$ and use the R^B and R^A families

$$R_{x,i}^B = \sum_{j,k} B_{c_{ijk}} \quad (\text{all } i), \qquad R_{y,j}^B = \sum_{i,k} B_{c_{ijk}} \quad (j \neq 0),$$

$$R_{z,k}^B = \sum_{i,j} B_{c_{ijk}} \quad (k \neq 0), \qquad R_v^{A,0} = A_v^x + A_v^y + A_v^z \quad (\text{all } v),$$

$$R_i^{A,x} = \sum_{j,k} A_{v_{ijk}}^x \quad (\text{all } i), \qquad R_j^{A,y} = \sum_{i,k} A_{v_{ijk}}^y \quad (\text{all } j),$$

$$R_k^{A,z} = \sum_{i,j}\left(A_{v_{ijk}}^x + A_{v_{ijk}}^y\right) \quad (k \neq 0).$$

They contain $S-2$ and $V+S-1$ relations, respectively. All unlisted matrix components are zero, and $\mathbb{F}_2$ sends every listed span to the binary coefficient 1.

Table 1. Carrier certificate for the displayed periodic X-cube presentation.

Family	Role	Certified carrier	Span components or placement
B_c	X-stabilizer generator	primal cube frame b_c	the twelve restrictions $r_{\mathrm{b}_c,e}$ to boundary-edge qubit carriers
A_v^μ	Z-stabilizer generator	primal four-edge cross a_v^μ	the four restrictions $r_{\mathrm{a}_v^\mu,e}$ to its edge qubit carriers
$R_{\mu,s}^B$	cube-stabilizer relation basis	primal union $\mathrm{c}\left(R_{\mu,s}^B\right)$ of the participating cube frames	one restriction $r_{\mathrm{c}(R),\mathrm{b}_c}$ for every B_c occurring in R
$R_v^{A,0} = A_v^x + A_v^y + A_v^z$	local cross relation	primal six-edge star $\bigcup_\mu \mathrm{a}_v^\mu$	three restrictions $r_{\mathrm{c}(R),\mathrm{a}_v^\mu}$
$R_s^{A,\mu}$	nonlocal cross-relation basis	primal union $\mathrm{c}\left(R_s^{A,\mu}\right)$ of the participating cross carriers	one restriction $r_{\mathrm{c}(R),\mathrm{a}_v^\nu}$ for every A_v^ν occurring in R

Continued

ℓ_{ab}^{μ}	rigid-loop logical generator	primal rigid μ-loop at transverse coordinates (a,b)	corresponding summand of K_0^X
τ_{ab}^{μ}	four-loop relation	union of cube frames in the one-cell-thick primal μ-tube	four restrictions to the displayed boundary loops in K_0^X
p_{ijk}	fracton-syndrome generator	barycentric-dual vertex	corresponding summand of K_0^f
σ_e	fracton edge process	transverse barycentric-dual plaquette of the primal edge e	four restrictions $r_{\sigma_e,p}$ to its dual corner vertices
$\ell^{\mu}(r)$	lineon-syndrome generator	primal rigid μ-line through r	corresponding summand of K_0^{ℓ}
$e_{\mu}(r)$	lineon pair process	the same primal rigid μ-line	two identity spans to $\ell^{\mu}(r)$ and $\ell^{\mu}(r+\hat{\mu})$
p_r	local lineon relation	primal point r	transfers $t_{p_r,L_r^x}, t_{p_r,L_r^y}, t_{p_r,L_r^z}$, imposing $\ell^x+\ell^y+\ell^z=0$

7. A Coordinate X-Cube Leaf-Sewing Map

We now construct the microscopic map for inserting a coordinate leaf in the periodic cubic X-cube presentation.

Insert a new xy leaf α immediately before a z leaf β, and write Λ^+ for the lattice with L_z replaced by L_z+1. Every old z edge e piercing α is split into edges e^- and e^+. Starting with the old X-cube code, a toric code on α, and one $|0\rangle$ qubit on each e^+, define

$$E_z=\prod_{e\pitchfork\alpha}\mathrm{CNOT}_{e^-\to e^+}.$$

We take e^+ to be the half-edge in the slab from α to β; e^- is the other half. Below, the superscripts on z_a^- and z_a^+ instead label the two endpoints of a, so both of those edges lie in the α-β slab. For an x- or y-edge a of α, let a_β be the corresponding edge of β. If a is x-oriented, let z_a^- and z_a^+ be the two adjacent z-edges joining the endpoints of a to β. The sewing layer is the translation-invariant CNOT circuit

$$S_z=\prod_{a\in E_x(\alpha)\sqcup E_y(\alpha)}\mathrm{CNOT}_{a\to a_\beta}\prod_{a\in E_x(\alpha)}\mathrm{CNOT}_{a\to z_a^-}\mathrm{CNOT}_{a\to z_a^+},\quad U_z=S_zE_z.$$

All gates in S_z commute: their controls are on α and none of their targets is a control. This is the $x \leftrightarrow y$ exchanged coordinate specialization of the even-faced-prism circuit of Ref. [5], whose written convention uses the other in-plane orientation for the extra controls. The cubic lattice symmetry makes the two conventions equivalent, and cyclic formulas give the other leaf insertions.

For a CSS Hamiltonian $H = \left(S_X, S_Z \subset \mathbb{F}_2^Q\right)$, write its global X-CSS complex as

$$\mathrm{C}_X\left(H\right): S_X \hookrightarrow \mathbb{F}_2^Q \xrightarrow{\sigma_Z} S_Z^\vee, \quad \sigma_Z\left(v\right)\left(z\right) = \left\langle z, v\right\rangle.$$

Its middle homology is the X logical space. The product-state ancillas in the input to U_z have zero middle homology.

Theorem 7.1 (Framed coordinate leaf sewing). *Assume* $L_x, L_y, L_z \geq 3$; *the enlarged lattice has sizes* $\left(L_x, L_y, L_z + 1\right)$. *Conjugation by* U_z *induces an isomorphism of the global CSS complexes*

$$\mathrm{C}_X\left(H_{\mathrm{XC}}\left(\Lambda\right)\right) \oplus \mathrm{C}_X\left(H_{\mathrm{TC}}\left(\alpha\right)\right) \oplus \mathrm{C}_X\left(H_{|0\rangle}\right) \xrightarrow{\cong} \mathrm{C}_X\left(H_{\mathrm{XC}}\left(\Lambda^+\right)\right).$$

Consequently, on X *logical spaces it induces*

$$\Phi_{z,X}: L_X\left(\Lambda\right) \oplus H_1\left(\alpha; \mathbb{F}_2\right) \xrightarrow{\cong} L_X\left(\Lambda^+\right).$$

If γ_x, γ_y *are the two toric-code* X *loops on* α, *then*

$$\Phi_{z,X}\left(W_x \oplus \mathbb{F}_2\left[\gamma_x\right]\right) = W_x^+,$$
$$\Phi_{z,X}\left(W_y \oplus \mathbb{F}_2\left[\gamma_y\right]\right) = W_y^+,$$
$$\Phi_{z,X}\left(W_z\right) = W_z^+.$$

Here the superscript $+$ *denotes the directional summand on* Λ^+. *Thus a* z-*leaf insertion changes the directional rank inventory by* $\rho_z = u_x + u_y$; *cyclically, a* μ-*leaf insertion changes it by* ρ_μ.

Proof. The edge-splitting gates implement $Z_e \mapsto Z_{e^-}$, $X_e \mapsto X_{e^-} X_{e^+}$ and add the stabilizer $Z_{e^-} Z_{e^+}$. For each gate $\mathrm{CNOT}_{c \to t}$ in S_z,

$$X_c \mapsto X_c X_t, \quad X_t \mapsto X_t, \quad Z_c \mapsto Z_c, \quad Z_t \mapsto Z_c Z_t.$$

We record the local substitution explicitly. Write $r = \left(i, j\right) \in \mathbb{Z}_{L_x} \times \mathbb{Z}_{L_y}$, and let x_r^λ and y_r^λ denote the positively oriented in-plane edges based at r in the leaf $\lambda \in \left\{\alpha, \beta\right\}$. Let p_r be the z-edge ending at $r \in \alpha$, and let q_r be the z-edge from $r \in \alpha$ to the corresponding vertex of β. Thus p_r and q_r are the two pieces e^- and e^+ of the old piercing edge. The gates in S_z give

$$\mathrm{Ad}_{S_z}\left(X_{x_r^\alpha}\right) = X_{x_r^\alpha} X_{x_r^\beta} X_{q_r} X_{q_{r+\hat{x}}}, \quad \mathrm{Ad}_{S_z}\left(X_{y_r^\alpha}\right) = X_{y_r^\alpha} X_{y_r^\beta},$$
$$\mathrm{Ad}_{S_z}\left(Z_{q_r}\right) = Z_{x_{r-\hat{x}}^\alpha} Z_{x_r^\alpha} Z_{q_r}, \qquad \mathrm{Ad}_{S_z}\left(Z_{v_r^\beta}\right) = Z_{v_r^\alpha} Z_{v_r^\beta} \quad \left(v = x, y\right),$$

and fix the other single-edge factors appearing below.

Use the toric-code convention in which

$$T_r^X = X_{x_r^\alpha} X_{y_{r+\hat{x}}^\alpha} X_{x_{r+\hat{y}}^\alpha} X_{y_r^\alpha},$$
$$T_r^Z = Z_{x_{r-\hat{x}}^\alpha} Z_{x_r^\alpha} Z_{y_{r-\hat{y}}^\alpha} Z_{y_r^\alpha}$$

are respectively the plaquette and star stabilizers on α. Let $\bar{B}_r$ be the old cube bisected by α, let $B_{r,-}$ and $B_{r,+}$ be the output cubes immediately below and above α, and let $\bar{A}^{\mu}_{r,\beta}$ denote an old cross based on β. Direct multiplication of the displayed single-edge images gives

$$\begin{aligned} &\mathrm{Ad}_{U_z}\left(\bar{B}_r\right) = B_{r,-}B_{r,+}, && \mathrm{Ad}_{U_z}\left(T_r^X\right) = B_{r,+},\\ &\mathrm{Ad}_{U_z}\left(\bar{A}^{\mu}_{r,\beta}\right) = A^{\mu}_{r,\alpha}A^{\mu}_{r,\beta} \quad \left(\mu = x,y,z\right), && \mathrm{Ad}_{U_z}\left(T_r^Z\right) = A^{z}_{r,\alpha},\\ &\mathrm{Ad}_{U_z}\left(Z^{\mathrm{anc}}_{q_r}\right) = Z_{p_r}Z_{q_r}Z_{x^{\alpha}_{r-\hat{x}}}Z_{x^{\alpha}_{r}} = A^{y}_{r,\alpha}. \end{aligned}$$

Here $Z^{\mathrm{anc}}_{q_r}$ is the input $\left|0\right\rangle$ stabilizer: E_z first sends it to $Z_{p_r}Z_{q_r}$, and the two x-edge controls incident on r supply the remaining factors. For $\mu = x, y$, E_z sends the lower z-edge in the old β-cross to p_r; the common q_r factor in $A^{\mu}_{r,\alpha}A^{\mu}_{r,\beta}$ cancels. For $\mu = z$, the four in-plane target factors acquire precisely the four corresponding controls on α. Cubes and crosses outside this local slab map to their corresponding output generators.

These identities also exhibit preimages for every new generator. Indeed,

$$B_{r,+} = \mathrm{Ad}_{U_z}\left(T_r^X\right),\quad B_{r,-} = \mathrm{Ad}_{U_z}\left(\bar{B}_r T_r^X\right).$$

The toric star and the ancilla give $A^{z}_{r,\alpha}$ and $A^{y}_{r,\alpha}$, while the pointwise relation $A^{x}_{r,\alpha}A^{y}_{r,\alpha}A^{z}_{r,\alpha} = 1$ gives $A^{x}_{r,\alpha}$. Multiplying these by the images of the old β-crosses then gives each $A^{\mu}_{r,\beta}$. Hence conjugation identifies the input and output X- and Z-stabilizer subspaces, with a local preimage for every output generator. These formulas are the algebraic version, in the $x \leftrightarrow y$-exchanged convention of the displayed circuit S_z, of Figure 7 in Ref. [5].

As a global consistency check, put $N = L_xL_y$. The output has $V^+ = V + N$ vertices and $S^+ = S + 1$, so the preceding rank lemma gives

$$\operatorname{rank} S_X^+ = V^+ - S^+ + 2 = V + N - S + 1.$$

On the input side the old X-cube and toric-code X-stabilizer ranks are $V - S + 2$ and $N - 1$, while the $\left|0\right\rangle$ ancillas contribute no X-stabilizers. Their sum is the same. Similarly,

$$\operatorname{rank} S_Z^+ = 2V^+ - S^+ + 1 = 2V + 2N - S,$$

whereas the input ranks are $\left(2V - S + 1\right) + \left(N - 1\right) + N$; the last term is the N independent ancilla Z-stabilizers. Thus the input and output rank totals agree on both CSS sides.

The circuit is CSS-preserving. If F_z is its binary map on X supports, then its map on Z supports is F_z^{-T}. The two stabilizer equalities and preservation of the binary pairing make F_z commute with both arrows in the displayed global X-CSS complex in Section 7. Hence it is the asserted isomorphism of CSS complexes.

The carrier map is also explicit. Let d_zR replace every piercing edge in an X support R by $e^- \cup e^+$, and let Γ_z be the ordered pairs $\left(c,t\right)$ occurring in S_z. Because Pauli supports add over $\mathbb{F}_2$, the exact support is

$$\kappa_z^X\left(R\right) = d_zR \,\Delta\, \Delta_{\substack{(c,t)\in\Gamma_z\\ c\in d_zR}}\left\{t\right\} = \operatorname{supp}\left(U_zX_RU_z^{\dagger}\right).$$

Here $\triangle$ denotes symmetric difference; it records cancellations when two controls hit the same target or when a target was already occupied. The monotone union envelope

$$\bar{\kappa}_z^X(R) = d_z R \cup \bigcup_{\substack{(c,t)\in\Gamma_z \\ c\in d_z R}} \{t\}$$

contains the exact support and is the certified output carrier. In particular, conjugating an old rigid x, y, or z loop preserves its direction, while conjugating the toric x and y loops gives, modulo output cube stabilizers, respectively the product of the new and the adjacent x- and y-directed rigid loops. The induced maps can be written without a circuit diagram. For finite cyclic sets A, B, the map

$$\vartheta_{A,B} : \mathcal{W}(A,B) \to \left\{(a,b) \in \mathbb{F}_2^A \oplus \mathbb{F}_2^B : \epsilon_A(a) = \epsilon_B(b)\right\},\quad [e_{ab}] \mapsto (e_a, e_b)$$

is an isomorphism: the four-term rectangle relations lie in its kernel, and the source and target have the same dimension. Let $i : \mathbb{Z}_{L_z} \hookrightarrow \mathbb{Z}_{L_z+1}$ include the old leaves. In the ϑ coordinates, the two nontrivial blocks of $\Phi_{z,X}$ are

$$\left((a,b),t\right) \mapsto \left(a, i(b) + t\left(e_\alpha + e_\beta\right)\right) \quad \text{from } W_x \oplus \mathbb{F}_2[\gamma_x] \text{ to } W_x^+,$$
$$\left((a,b),t\right) \mapsto \left(a, i(b) + t\left(e_\alpha + e_\beta\right)\right) \quad \text{from } W_y \oplus \mathbb{F}_2[\gamma_y] \text{ to } W_y^+.$$

The first formula is exactly the old-loop inclusion together with the product of the loops at α and β, and the second is its cyclic analogue. Both preserve the equal-parity condition. To check injectivity, suppose an image vanishes. Its e_α coefficient is t, because the new leaf α is omitted from $\operatorname{im} i$; hence $t = 0$. Then $i(b) = 0$ and $a = 0$, so the input vanishes. The W_z block is the identity on the analogous equal-parity presentation. Therefore the three images lie in the summands displayed in (7.1). Their dimensions are

$$\dim W_x + 1 = L_y + L_z = \dim W_x^+,\ \dim W_y + 1 = L_x + L_z = \dim W_y^+,\ \dim W_z = \dim W_z^+,$$

and the complex isomorphism is injective on logical homology. This proves (7.1) and the rank statement. ☐

The union envelope $\bar{\kappa}_z^X$ records the carrier enlargement of the chosen representatives under the sewing circuit.

8. Affine-Leaf X-Cube Syndrome Homology

Use the barycentric dual lattice for fracton syndromes. A cube excitation is labelled by its dual vertex p_{ijk}; the syndrome of a Z operator on a primal edge is labelled by the transverse dual plaquette σ_e, which contains the four relevant dual vertices. Thus the support-Burnside complex is

$$K_1^f = \bigoplus_e [\sigma_e] \to K_0^f = \bigoplus_{i,j,k} \left[p_{ijk}\right],$$

with the four restriction components determined by the boundary vertices of each σ_e. It evaluates under $\mathbb{F}_2$ to

$$D_f : \mathbb{F}_2\left[E(\Lambda)\right] \xrightarrow{\partial_f} \mathbb{F}_2\left\{f(i,j,k)\right\},$$

where

$$\partial_f e_x = \Delta_y \Delta_z f,\quad \partial_f e_y = \Delta_x \Delta_z f,\quad \partial_f e_z = \Delta_x \Delta_y f.$$

For cross excitations, label $\ell^\mu(i,j,k)$ by the rigid μ-line through that vertex. An edge process in the μ direction and its two output lineons have the same rigid-line carrier. More explicitly, take

$$K_1^\ell = \bigoplus_{\mu,r}\left[L_r^\mu\right] \oplus \bigoplus_r \left[p_r\right] \to K_0^\ell = \bigoplus_{\mu,r}\left[L_r^\mu\right].$$

The first summand maps by the two restriction components along the corresponding rigid line. The point carrier p_r maps by three transfer components to L_r^x, L_r^y, L_r^z. Consequently, the second summand imposes the local relation $\ell^x + \ell^y + \ell^z = 0$. After evaluation this is the ordinary complex

$$D_\ell : \mathbb{F}_2\left[E(\Lambda)\right] \to \bigoplus_{r\in\Lambda} \mathbb{F}_2\left\{\ell^x(r), \ell^y(r), \ell^z(r)\right\} \Big/ \left\langle \ell^x(r) + \ell^y(r) + \ell^z(r)\right\rangle,$$

with $\partial_\ell e_\mu(r) = \ell^\mu(r) + \ell^\mu(r+\hat{\mu})$.

For affine leaf spaces

$$\Pi^x = \mathbb{F}_2\left\{p_i^x : i \in \mathbb{Z}_{L_x}\right\},\quad \Pi^y = \mathbb{F}_2\left\{p_j^y : j \in \mathbb{Z}_{L_y}\right\},\quad \Pi^z = \mathbb{F}_2\left\{p_k^z : k \in \mathbb{Z}_{L_z}\right\},$$

let $\Pi = \Pi^x \oplus \Pi^y \oplus \Pi^z$ and let ϵ_μ sum the coefficients in Π^μ. Define the familiar affine-leaf charge maps by

$$\pi_f\left(f(i,j,k)\right) = p_i^x + p_j^y + p_k^z,$$

and

$$\pi_\ell\left(\ell^x(i,j,k)\right) = p_j^y + p_k^z,$$

with cyclic analogues.

Theorem 8.1 (Affine-leaf syndrome calculation)**.** *Assume* $L_x, L_y, L_z \ge 3$. *The induced maps give isomorphisms*

$$H_0(D_f) \cong \Pi_f := \left\{(a,b,c) : \epsilon_x(a) = \epsilon_y(b) = \epsilon_z(c)\right\},$$

$$H_0(D_\ell) \cong \Pi_\ell := \left\{(a,b,c) : \epsilon_x(a) + \epsilon_y(b) + \epsilon_z(c) = 0\right\}.$$

Thus their dimensions are $S-2$ *and* $S-1$, *respectively.*

Proof. For example, $\partial_f e_x$ consists of four points at fixed i and adjacent j,k. Under π_f, the x-leaf basis vector occurs four times and every relevant y- or z-leaf basis vector occurs twice, so the image is zero; the other edge orientations are cyclic. The image of $f(0,0,0)$ has parity triple $(1,1,1)$, while

$$\pi_f\left(f(i,0,0) + f(0,0,0)\right) = \left(p_i^x + p_0^x, 0, 0\right),$$

and the cyclic differences generate the three even-parity leaf subspaces. These elements generate every triple of equal parity, proving that the induced fracton map is onto Π_f.

For lineons, the two endpoints of $\partial_\ell e_\mu$ have the same affine-leaf charge, hence cancel. Moreover, $\pi_\ell\left(\ell^x+\ell^y+\ell^z\right)=0$, so π_ℓ descends through the local relation. Differences of parallel lineons generate each even-parity leaf subspace. One lineon of each orientation has parity vector, respectively, $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$; these span the parity triples of total sum zero. Thus the lineon map is onto Π_ℓ.

It remains to rule out additional classes in either cokernel. The matrices of ∂_f and ∂_ℓ are the cube and cross check matrices, so the rank lemma gives ranks $V-S+2$ and $2V-S+1$. Therefore

$$\dim\operatorname{coker}\partial_f=V-(V-S+2)=S-2,$$

while the lineon codomain has dimension $2V$ after the V independent local relations, and

$$\dim\operatorname{coker}\partial_\ell=2V-(2V-S+1)=S-1.$$

These equal $\dim\Pi_f$ and $\dim\Pi_\ell$. The two surjections are therefore isomorphisms. ☐

Corollary 8.2 (Quotient syndrome sectors). *Assume* $L_x,L_y,L_z\geq 3$. *Let* $\left(\Pi^\mu\right)^{\mathrm{ev}}=\ker\epsilon_\mu$. *In the two syndrome sectors define the planon subgroups being quotiented by*

$$\mathcal{P}_f=\left(\Pi^x\right)^{\mathrm{ev}}\oplus\left(\Pi^y\right)^{\mathrm{ev}}\oplus\left(\Pi^z\right)^{\mathrm{ev}}\subset\Pi_f,\quad \mathcal{P}_\ell=\left(\Pi^x\right)^{\mathrm{ev}}\oplus\left(\Pi^y\right)^{\mathrm{ev}}\oplus\left(\Pi^z\right)^{\mathrm{ev}}\subset\Pi_\ell.$$

Each even-leaf space is generated by adjacent pairs $p_i^\mu+p_{i+1}^\mu$. *Under the preceding syndrome isomorphisms these pairs are precisely the elementary planon attachments in the coordinate plane normal to* μ. *Thus* $\mathcal{P}_f$ *and* $\mathcal{P}_\ell$ *contain every class generated by the declared planon attachments, and quotient superselection removes exactly all of that intended planon data. There are three even-leaf summands in each syndrome sector, and the resulting quotients are*

$$Q^{(0)}:=\Pi_f/\mathcal{P}_f\cong\mathbb{F}_2,\quad Q^{(1)}:=\Pi_\ell/\mathcal{P}_\ell\cong\left\{\left(q_x,q_y,q_z\right)\in\mathbb{F}_2^3:q_x+q_y+q_z=0\right\}\cong\mathbb{F}_2^2,$$

and therefore $Q_{\mathrm{XC}}\cong\mathbb{F}_2^3$. *This recovers the eight known quotient-superselection sectors while retaining the prequotient planon data; compare Refs.* [6] [7].

9. Framed Sewing Parity

A *direction-split polarized presentation* includes a specified coordinate frame, a decomposition

$$L_X=L_X^x\oplus L_X^y\oplus L_X^z$$

and carrier certificates for the three summands. A framed admissible move is a direction-preserving isomorphism of certified presentations, or one of the three CNOT sewing maps of Theorem 7.1 and its inverse. Its toric layer inventory is

$$\rho_x=u_y+u_z,\quad \rho_y=u_x+u_z,\quad \rho_z=u_x+u_y.$$

Proposition 9.1 (Framed sewing parity). *For framed admissible moves, the class of*

$$m_X=\left(\dim L_X^x\right)u_x+\left(\dim L_X^y\right)u_y+\left(\dim L_X^z\right)u_z$$

is preserved in

$$K_{\text{fol}}^{\text{dir}} = \mathbb{Z}\{u_x, u_y, u_z\} / \langle \rho_x, \rho_y, \rho_z \rangle \cong \mathbb{Z}/2.$$

The isomorphism is total parity.

Proof. A direction-preserving isomorphism fixes m_X. Theorem 7.1 changes it by the displayed ρ_μ for a μ-leaf sewing or its inverse. The three resource vectors have determinant 2, and total parity vanishes on each and is surjective.□

For the periodic X-cube presentation,

$$m_X = (L_y + L_z - 1)u_x + (L_x + L_z - 1)u_y + (L_x + L_y - 1)u_z,$$

whose total coefficient is odd. Thus this *framed presentation* has nonzero sewing parity. The directional lift still depends on the frame. Its total parity, however, is simply the parity of the total logical dimension; Section 11 identifies the precise phase category in which that numerical shadow no longer needs a frame.

10. Ground-Space Compression and Exact-Resource Stability

We now remove the pinned-qubit obstruction of Proposition 4.4. Let a local Hamiltonian H act on $\mathcal{H}_\Lambda = \otimes_{q\in Q} \mathcal{H}_q$, let G_H be its ground space, and let P_H be the orthogonal projector onto G_H. For a carrier p, write $\mathcal{B}(p)$ for the linear space of operators supported on the tensor factors whose closed carriers lie in p.

Definition 10.1 (Phase-reduced support diagram). *The* phase-reduced support diagram *of* H *is the monotone assignment*

$$\mathcal{O}_H(p) = \left\{ (P_H A P_H)\big|_{G_H} : A \in \mathcal{B}(p) \right\} \subseteq \operatorname{End}(G_H).$$

Each value is a concrete operator system: a unital, adjoint-closed complex linear subspace of $\operatorname{End}(G_H)$. *Inclusions of carriers induce inclusions of operator systems.*

Choose the adjacency metric on cells and let $N_R p$ denote the closed R-neighborhood of a carrier. Two support diagrams are R*-controlled equivalent* if there is a unitary W between their ambient ground spaces such that

$$\operatorname{Ad}_W \mathcal{O}_H(p) \subseteq \mathcal{O}_{H'}(N_R p), \quad \operatorname{Ad}_{W^{-1}} \mathcal{O}_{H'}(p) \subseteq \mathcal{O}_H(N_R p)$$

for every carrier, after the evident identification of the two cellulations. Allowing a common larger R makes this an equivalence relation on uniformly local families. It forgets bounded microscopic displacements but retains the large-scale support at which an operator acts nontrivially on the ground space.

Proposition 10.2 (Relation to the Pauli coefficient). *For a CSS stabilizer Hamiltonian,* $\mathcal{O}_H(p)$ *is spanned by*

$$(P_H X^a Z^b P_H)\big|_{G_H}, \quad a, b \in \mathcal{P}_\Lambda(p).$$

Thus $\mathcal{O}_H$ *is obtained from two copies of the canonical Pauli support coefficient by Pauli representation, linearization over* $\mathbb{C}$, *and ground-space compression. If a qubit in a pure product state is adjoined, the reduced diagram is canonically unchanged after forgetting that qubit's carrier.*

Proof. Local Pauli operators form a basis of $\mathcal{B}(p)$, proving the spanning statement. For a pinned ancilla with state $|0\rangle$ and projector $P_0 = |0\rangle\langle 0|$, expand a local operator as $\sum_i A_i \otimes B_i$. Compression gives

$$(P_H \otimes P_0)\left(\sum_i A_i \otimes B_i\right)(P_H \otimes P_0) = \sum_i \langle 0|B_i|0\rangle P_H A_i P_H \otimes P_0.$$

The ancilla therefore adds no ground-space operator beyond scalar multiples of the old ones. In particular, the counterexample of Proposition 4.4 vanishes after phase reduction. □

Theorem 10.3 (Finite-depth and resource stability). *Suppose a depth-d circuit U with gate diameter at most r maps G_H unitarily onto $G_{H'}$. Then $\mathcal{O}_H$ and $\mathcal{O}_{H'}$ are R-controlled equivalent for $R = dr$.*

More generally, suppose the two ground spaces become related by such a circuit after adjoining pure product-state ancillas and arbitrary gapped two-dimensional resource-layer ground spaces. Then their phase-reduced diagrams become controlled equivalent after tensoring with the corresponding layer diagrams. Consequently the controlled-equivalence class of $\mathcal{O}_H$, modulo tensor factors from two-dimensional resource layers, is an invariant of the exact operational resource equivalence relation just stated.

Proof. If A is supported in p, the light-cone bound puts $UAU^\dagger$ in $\mathcal{B}(N_R p)$. Since $UP_H U^\dagger = P_{H'}$, conjugation by $W = U|_{G_H}$ gives the first inclusion in the definition of R-controlled equivalence in Section 10. Applying the same argument to $U^\dagger$ gives the second. Product-state factors disappear by Proposition 10.2. For a two-dimensional layer T, local operators are spanned by tensor products, so

$$\mathcal{O}_{H\otimes T}(p) = \mathcal{O}_H(p \cap \Lambda) \otimes \mathcal{O}_T(p \cap T).$$

The same light-cone argument applied after adjoining all resource layers proves the second statement, and passage to the quotient by those declared free tensor factors proves the last. □

Remark 10.4 Theorem 10.3 applies to arbitrary finite-depth local unitaries, without a CSS or frame-preservation hypothesis. It concerns the exact operational relation in the theorem; extending it to gapped paths would require approximate operator systems and quasi-local tails.

11. Unframed Stabilizer Parity and Its Obstruction

For a binary stabilizer Hamiltonian let

$$k(H) = \log_2 \dim G_H, \quad \nu_2(H) = k(H) (\operatorname{mod} 2).$$

Definition (Periodic stabilizer-foliated equivalence). *Periodic stabilizer-foliated equivalence is the relation generated by the following moves and their inverses:*

1) *bounded periodic reblocking or unblocking of a size-independent number of adjacent cells to reach a common cellulation, without adding or deleting phys-*

ical degrees of freedom;

2) *conjugation by a finite-depth local unitary of bounded range on that cellulation*;

3) *adjoining or removing decoupled pure product-state stabilizer ancillas*;

4) *stacking or unstacking a two-dimensional translation-invariant topological Pauli-stabilizer layer satisfying the no-frustration, translation-invariance, and topological-order hypotheses of Ref.* [8], *on a compatible closed coordinate torus.*

A compatible closed coordinate torus is a periodic cell subcomplex normal to one coordinate direction, closed by the ambient periodic identifications, whose two periods and induced cellulation are inherited from the ambient system after the permitted bounded reblocking. The torus and its periods are required to be sufficiently large for the cited classification theorem. A change of L_x, L_y, *or* L_z *is allowed only when it is realized by a finite sequence of the listed moves, as in the leaf insertion of Theorem* 7.1; *bare lattice resizing is not an additional move.*

An allowed resource layer is locally Clifford equivalent to finitely many toric codes together with trivial stabilizer factors and therefore encodes an even number of qubits on its torus.

Theorem 11.1 (Unframed stabilizer-foliated parity). *The number* ν_2 *is invariant under the periodic stabilizer-foliated equivalence relation just defined. For the periodic X-cube code with* $L_x, L_y, L_z \geq 3$,

$$\nu_2\left(H_{\mathrm{XC}}\right) = 2\left(L_x + L_y + L_z\right) - 3 \equiv 1 \pmod 2,$$

whereas a product-state Hamiltonian has $\nu_2 = 0$. *Hence periodic X-cube is not stabilizer-foliated equivalent to a product state within this resource category. No coordinate splitting of the logical space is required.*

Proof. A unitary equivalence preserves $\dim G_H$, and a pure product-state ancilla has a one-dimensional ground space. Stacking a resource layer adds its number of encoded qubits to $k(H)$. By Ref. [8], every allowed layer is a finite product of toric-code and trivial stabilizer factors; on a torus each toric-code factor encodes two qubits and each trivial factor encodes none. Periodic reblocking changes only the cell labels, not the ground space. Every generator of the equivalence relation therefore changes k by an even number. A lattice-period change has no independent case because it is permitted only when implemented by those generators. The X-cube and product-state values follow from the stabilizer rank calculation of Section 6. □

Under the total-parity isomorphism of Proposition 9.1, the framed class m_X maps to

$$\dim L_X^x + \dim L_X^y + \dim L_X^z = \dim L_X = k(H) \pmod 2.$$

Thus Proposition 9.1 is a direction-resolved lift of Theorem 11.1; only its total-parity shadow survives after the frame is forgotten.

Proposition 11.2 (No extension to unrestricted resource layers). *The invariant* ν_2 *does not extend to a general foliated equivalence in which arbitrary gapped two-dimensional phases are free resources.*

Proof. The bosonic $\nu = 1/2$ Laughlin topological order (the chiral semion theory) has two ground states on a torus [9]. Stacking one such layer multiplies the three-dimensional ground-state degeneracy by 2 and flips $\log_2 \mathrm{GSD} \bmod 2$. Since the layer is free in the unrestricted foliated relation, ν_2 cannot be constant on its equivalence classes. For resource layers whose torus degeneracy is not a power of two, ν_2 is not even defined. □

12. Related Work and Contribution

The effective Burnside category used here is a specialization of the standard span construction underlying Mackey functors [1]. Categorical structures for CSS and stabilizer codes are already developed by Cowtan and Burton [10] and by Comfort [11]. The contribution of Sections 2-4 is the carrier-resolved organization of a specified cellular CSS presentation: support labels are retained on the spans, and evaluation recovers the ordinary binary relation matrices. The Pauli coefficient of Definition 4.2 is part of that presentation-level organization.

Prior work develops both ground-space compression and supported logical algebras. Quantum error correction of observable algebras appears in Ref. [12]. Haah studies local logical algebras and their behavior under shallow circuits [13], while Kanno and Shimada use compressed code algebras [14]. Section 10 uses these ideas in an all-carrier diagram and proves the corresponding controlled-support statement for the exact resource relation specified in Theorem 10.3.

The X-cube model, its subextensive logical rank, and its foliated organization originate in Refs. [3] [5]. In particular, the CNOT sewing of a toric-code leaf and its two added logical qubits are prior results [5]; related circuit and gluing constructions appear in Ref. [15]. Quotient superselection sectors [7], foliated entanglement quantities [16], compactification data [4], and recoverable information [17] provide distinct established diagnostics, while cage-net models delimit the original foliation scheme [18]. The calculation here supplies an explicit map of global CSS complexes, the carrier envelope, and the directional block formula in Section 7 for the framed cubic presentation.

The results can therefore be summarized as follows:

1) support-Burnside complexes retain carrier data in familiar CSS relation matrices;

2) ground-space compression produces a controlled all-carrier diagram for the exact operational equivalence relation stated here; and

3) coordinate X-cube sewing yields the directional block formula and its framed-groupoid parity, whose scalar shadow is invariant for the restricted stabilizer resource category of Theorem 11.1.

13. Conclusions

Carrier-resolved Burnside complexes provide compact bookkeeping for CSS relations without discarding the supports on which generators and relations are realized. The plaquette, Bacon-Shor, and periodic X-cube examples recover the standard binary ranks, while the affine-leaf calculation retains the planon data before quotienting. For coordinate X-cube sewing, the CNOT circuit acts as an isomorphism of global CSS complexes; its exact support map, carrier envelope, and directional blocks make the two added toric logical classes explicit.

The three resulting levels have different invariance properties. The Pauli coefficient is presentation-local. Ground-space compression gives a controlled diagram for the exact operational resource relation of Theorem 10.3. Directional parity belongs to the framed sewing groupoid, while $k \bmod 2$ survives after forgetting the frame only for the restricted stabilizer-layer category of Theorem 11.1; Proposition 11.2 gives the obstruction for unrestricted two-dimensional resources.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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