

Prime-Quaternion Addresses for the Free-Group Skeleton of Banach-Tarski Decompositions

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Abstract

We present an arithmetic address code for the free-group action used in the Banach-Tarski construction. Two integer quaternions of norm five induce rational rotations about perpendicular axes. Świerczkowski's criterion proves that these rotations freely generate a group of rank two. For each reduced word, the norm of its unnormalized quaternion product is five raised to the word length, so its 5-adic valuation records Cayley-graph depth. We derive the scalar factor introduced by cancellation in unreduced words and transfer the code to the free part of the sphere after choosing an orbit transversal. The standard four-piece partition of the free group then gives two reconstructions of that free part. The four spherical pieces cannot all be measurable with respect to rotation-invariant surface-area measure. Accurate computation confirms the rotation identities and distinguishes all 4373 reduced words of length at most seven. The contribution is an explicit organization of these algebraic and orbit labels, with their dependence on reduction and on the chosen transversal made precise.

Keywords

Banach-Tarski Theorem, Free Group, Rational Rotation, Quaternion, Paradoxical Decomposition, Non-Measurable Set, Cayley Graph

1. Introduction

The Banach-Tarski theorem concerns finite equidecomposability under Euclid-

ean isometries. For a solid ball, a paradoxical decomposition cannot consist entirely of Lebesgue-measurable pieces, because volume is invariant under those motions and finitely additive on measurable sets. For the sphere, the corresponding obstruction uses rotation-invariant surface-area measure. The acting group explains the difference between dimensions: the isometry groups of the line and plane, regarded as discrete groups, are amenable, whereas the rotation group in three dimensions contains a non-amenable free subgroup. Here amenability means the existence of a left-invariant finitely additive probability measure on all subsets of the group.

The historical development began with Vitali's 1905 construction of a non-measurable subset of the real line [1]. Hausdorff then obtained a paradoxical decomposition of the sphere after removing a countable exceptional set [2]. Banach and Tarski converted this group-theoretic phenomenon into the 1924 theorem that any two bounded subsets of $\mathbb{R}^n$, $n \geq 3$, with nonempty interiors are finitely equidecomposable [3]. Von Neumann introduced the group-theoretic concept now called amenability to explain why the phenomenon depends on the acting group [4]. Tarski subsequently gave an algebraic formulation relating paradoxicality to the nonexistence of invariant finitely additive measures [5]. Later work sharpened both the geometry and the algebra. Robinson reduced ball-doubling to five pieces and proved that five is minimal [6]. De Groot and Dekker developed explicit free subgroups of orthogonal groups [7], while Świerczkowski supplied a particularly economical free rotation construction [8]. Dougherty and Foreman showed that the pieces can be chosen with the property of Baire, first in a short announcement [9] and then in a full treatment [10]. Świerczkowski's later classification proved that two equal-angle rotations about perpendicular axes generate a free subgroup whenever their rational cosine avoids a short exceptional list [11]. Tomkowicz and Wagon give a modern synthesis of the subject, including its links with amenability, set theory, and geometric decomposition [12].

The standard constructions raise a specific question about representation. Once a free rotation pair and an orbit representative are fixed, each point of the free orbit has a unique reduced-word label. We ask how integer quaternions can record both that label and its depth in the Cayley graph, while keeping the arithmetic code separate from the measure of a geometric set.

In this work, we combine a fixed system of unnormalized integer-quaternion lifts, a prime-valuation depth register, an exact cancellation rule, and a transversal-relative orbit address. Quaternion parametrization of rotations, Świerczkowski's freeness criterion, reduced-word normal forms, and transfer of a paradoxical partition through a free action are standard ingredients. The norm identity follows directly from multiplicativity, and the depth register is deliberately redundant. We use these ingredients to distinguish a word, its quaternion lift, an orbit point, and a paradoxical piece within one explicit construction. Sections 3 - 5 carry out this construction for the norm-five generators. We derive the rational rotations, apply the freeness criterion, prove the coding identities, and transfer the standard four-piece partition to sphere orbits.

2. Equidecomposability and the Free Part of the Sphere

We set a group G act on a set X . Two subsets $E, F \subseteq X$ are called G -equidecomposable if there are finite partitions $E = \sqcup_{r=1}^m E_r$ and $F = \sqcup_{r=1}^m F_r$, together with elements $g_1, \dots, g_m \in G$, such that

$$F_r = g_r E_r \quad (1 \leq r \leq m). \quad (1)$$

Equation (1) means that corresponding pieces differ only by allowed group motions. The symbol $\sqcup$ denotes disjoint union.

A subset $E \subseteq X$ is G -paradoxical if it has a finite partition

$$E = \left(\sqcup_{r=1}^m A_r \right) \sqcup \left(\sqcup_{s=1}^n B_s \right) \quad (2)$$

and group elements $g_r, h_s \in G$ for which both $\sqcup_r g_r A_r = E$ and $\sqcup_s h_s B_s = E$. Thus the same original pieces can be rearranged into two disjoint copies of E .

For a ball $B \subset \mathbb{R}^3$, the classical theorem asserts paradoxicality under the Euclidean isometry group. If every piece in such a decomposition were Lebesgue measurable, invariance and finite additivity would give $\mu(B) = 2\mu(B)$, which contradicts $0 < \mu(B) < \infty$. At least one piece must therefore lie outside the domain of Lebesgue measure. The conclusion is non-measurability, not failure of finite additivity on measurable sets.

We choose $F_2 = \langle a, b \rangle$ be the free group on two generators. Its elements are reduced words in the alphabet $\Sigma = \{a, a^{-1}, b, b^{-1}\}$. A word $w = s_1 \cdots s_n$ is reduced when $s_{r+1} \neq s_r^{-1}$ for every r . Its reduced length is denoted by $\ell(w) = n$, and the identity element e has length zero.

The Cayley graph $\Gamma(F_2, \Sigma)$ is a 4-regular tree. Its vertices are reduced words, and an edge joins w to the reduction of ws for each $s \in \Sigma$. Word length is the graph distance from e .

Suppose that a faithful representation $\rho: F_2 \rightarrow SO(3)$ has been fixed. The explicit representation used in this paper is constructed in Section 3. Every nonidentity rotation of the unit sphere S^2 fixes exactly two antipodal points. So by defining the exceptional set D and its complement X by

$$D = \bigcup_{\substack{g \in F_2 \\ g \neq e}} \text{Fix}(\rho(g)), \quad X = S^2 - D. \quad (3)$$

Since F_2 is countable and each set $\text{Fix}(\rho(g))$ contains two points, D is countable. It is also invariant under the action: if x is fixed by $\rho(g)$, then $\rho(h)x$ is fixed by $\rho(hgh^{-1})$. Thus X is invariant. Its action is free, meaning that $\rho(g)x = x$ with $x \in X$ forces $g = e$.

Throughout the spherical construction, σ denotes completed rotation-invariant surface-area measure on the unit sphere, normalised by $\sigma(S^2) = 4\pi$. Since D is countable, $\sigma(D) = 0$ and $\sigma(X) = 4\pi$. Measurability of subsets of X will always refer to this measure.

The orbits of F_2 partition X . The Axiom of Choice selects a transversal $M \subset X$, meaning that M contains exactly one point from each orbit. The orbit

map

$$\Theta: F_2 \times M \rightarrow X, \quad \Theta(g, m) = \rho(g)m, \quad (4)$$

is then bijective. Surjectivity follows from the definition of a transversal. For injectivity, suppose $\rho(g)m = \rho(h)n$. The points $m, n \in M$ belong to the same orbit, hence $m = n$. Freeness then gives $h^{-1}g = e$, so $g = h$.

Equation (4) is the precise location of word addresses in the Banach-Tarski argument. A point of X is not intrinsically a word. It becomes a pair (g, m) only after the representation ρ , the generators a, b , and a transversal M have been chosen.

The free action gives a paradoxical decomposition of X , the complement of D in S^2 . The standard Banach-Tarski proof then absorbs the countable set D , extends the sphere decomposition radially to the punctured ball, and absorbs the centre. These steps are classical and are treated in detail in [6] [12]. They are logically separate from the arithmetic code developed below. The code applies directly to the free orbit structure on X ; the auxiliary absorption pieces do not acquire unique reduced-word labels from Equation (4).

3. An Explicit Rational Free Rotation Pair

3.1. Quaternionic Rotations

A real quaternion can be written as $q = \alpha + \beta i + \gamma j + \delta k$, where $i^2 = j^2 = k^2 = ijk = -1$. Its conjugate is $\bar{q} = \alpha - \beta i - \gamma j - \delta k$, and its norm is

$$N(q) = q\bar{q} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2. \quad (5)$$

The norm in Equation (5) is multiplicative: $N(qr) = N(q)N(r)$. A nonzero quaternion acts on the imaginary subspace $\text{Im } \mathbb{H} \cong \mathbb{R}^3$ by $v \mapsto qvq^{-1}$. Multiplying q by a nonzero real scalar does not change this action.

In coordinates, the induced rotation is

$$R(q) = \frac{1}{N(q)} \begin{pmatrix} \alpha^2 + \beta^2 - \gamma^2 - \delta^2 & 2(\beta\gamma - \alpha\delta) & 2(\beta\delta + \alpha\gamma) \\ 2(\beta\gamma + \alpha\delta) & \alpha^2 - \beta^2 + \gamma^2 - \delta^2 & 2(\gamma\delta - \alpha\beta) \\ 2(\beta\delta - \alpha\gamma) & 2(\gamma\delta + \alpha\beta) & \alpha^2 - \beta^2 - \gamma^2 + \delta^2 \end{pmatrix}. \quad (6)$$

Equation (6) gives $R(q) \in SO(3)$, satisfies $R(qr) = R(q)R(r)$, and identifies q and $-q$.

3.2. The Prime-Norm Generators

For the integer quaternions, we choose

$$q_a = 1 + 2i, \quad q_b = 1 + 2j, \quad N(q_a) = N(q_b) = 5. \quad (7)$$

Substitution into Equation (6) gives the rotations

$$A_5 = R(q_a) = \frac{1}{5} \begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & -4 \\ 0 & 4 & -3 \end{pmatrix}, \quad B_5 = R(q_b) = \frac{1}{5} \begin{pmatrix} -3 & 0 & 4 \\ 0 & 5 & 0 \\ -4 & 0 & -3 \end{pmatrix}. \quad (8)$$

The axis coordinates are fixed: the $(1, 1)$ entry of A_5 and the $(2, 2)$ entry of

B_5 are both one after division by five. The remaining two-dimensional blocks describe the rotations in the perpendicular coordinate planes.

Direct multiplication yields

$$A_5^T A_5 = B_5^T B_5 = I_3, \quad \det A_5 = \det B_5 = 1. \quad (9)$$

Consequently $A_5, B_5 \in SO(3, \mathbb{Q})$. The first is a rotation about the x -axis and the second about the y -axis. They have the same angle θ , determined by

$$\cos \theta = -\frac{3}{5}, \quad \sin \theta = \frac{4}{5}. \quad (10)$$

The axes are perpendicular, and the cosine in Equation (10) is rational.

3.3. Freeness

Let us set $U, V \in SO(3, \mathbb{R})$ be rotations about two perpendicular axes through the origin, with a common rotation angle φ . Assume that $\cos \varphi \in \mathbb{Q}$. Then U and V freely generate a free group of rank two if and only if $\cos \varphi \notin \{0, \pm 1/2, \pm 1\}$. Here “freely generate” means that no nonempty reduced word in U, U^{-1}, V, V^{-1} equals the identity.

Theorem 1 (Świerczkowski’s criterion applied to $p = 5$). The subgroup $G_5 = \langle A_5, B_5 \rangle$ is a free group of rank two.

Proof. The matrices in Equation (8) act on Euclidean three-space by rotations through the same angle θ about the perpendicular coordinate axes. Equation (10) gives the rational value $\cos \theta = -3/5$, which lies outside the exceptional set in Świerczkowski’s criterion. Theorem 1.1, therefore implies that A_5 and B_5 freely generate G_5 .

Theorem 1 gives the faithful representation

$$\rho_5 : F_2 \hookrightarrow SO(3, \mathbb{Q}), \quad \rho_5(a) = A_5, \quad \rho_5(b) = B_5. \quad (11)$$

Faithfulness means that $\rho_5(w) = I_3$ only for $w = e$, and that distinct reduced words have distinct rotation matrices.

4. The Prime-Quaternion Address

We define the lift of each letter in Σ by the four-entry map

$$a \mapsto q_a = 1 + 2i, \quad a^{-1} \mapsto q_{a^{-1}} = 1 - 2i,$$

$$b \mapsto q_b = 1 + 2j, \quad b^{-1} \mapsto q_{b^{-1}} = 1 - 2j.$$

Each lift has norm five. For $s \in \Sigma$, the inverse-letter lift is the conjugate, $q_{s^{-1}} = \bar{q}_s = 5q_s^{-1}$. Thus conjugate lifts represent inverse rotations only projectively: $R(\bar{q}_s) = R(q_s)^{-1}$ since a nonzero real scalar does not change the induced rotation. They are not multiplicative inverses in the quaternion algebra.

For a reduced word $w = s_1 s_2 \cdots s_n$, define its unnormalised quaternion lift by

$$Q_5(w) = q_{s_1} q_{s_2} \cdots q_{s_n}, \quad Q_5(e) = 1. \quad (12)$$

The lift uses integer arithmetic. Since $R(qr) = R(q)R(r)$, Equations (6), (11),

and (12) imply

$$R(Q_5(w)) = \rho_5(w). \quad (13)$$

Thus the projective quaternion in Equation (12) and the rational rotation matrix in Equation (11) carry the same group element.

We set $v_5(m)$ denote the 5-adic valuation of a positive integer m , namely the largest $r \geq 0$ for which 5^r divides m .

Theorem 2 (prime-depth identity). For every reduced word $w \in F_2$, put $n = \ell(w)$. Then

$$N(Q_5(w)) = 5^n \text{ and } v_5(N(Q_5(w))) = n. \quad (14)$$

Proof. For $w = s_1 \cdots s_n$, where $n = \ell(w)$, by Equation (7) and the definition of the inverse lifts, $N(q_{s_r}) = 5$ for every r . Multiplicativity of the quaternion norm gives $N(Q_5(w)) = \prod_{r=1}^n N(q_{s_r}) = \prod_{r=1}^n 5 = 5^n$. Taking v_5 gives $v_5(N(Q_5(w))) = n = \ell(w)$.

Theorem 2 grades the address by depth. It does not assign volume, mass, or probability to a subset of the sphere.

Prime-quaternion code can be defined as:

$$C_5(w) = (Q_5(w), v_5(N(Q_5(w)))) \quad (15)$$

Proposition 1 (injectivity). The map C_5 is injective on reduced words.

Proof. If $C_5(u) = C_5(v)$, then $Q_5(u) = Q_5(v)$. Equation (13) gives $\rho_5(u) = \rho_5(v)$. The representation ρ_5 is faithful by Theorem 1, so $u = v$. $\square$

The second component of Equation (15) is intentionally redundant: Theorem 2 shows that it is exactly the word length. Its purpose is to expose the depth as a prime valuation, not to create a stronger invariant. In particular, word length depends on the chosen generating set and is not invariant under arbitrary changes of generators or conjugation.

The reduction step must precede depth assignment. Let $u = t_1 \cdots t_m$ be an arbitrary word, and let $\text{red}(u) = w$ have length n . We define the raw quaternion product $Q_5^{\text{raw}}(u) = q_{t_1} \cdots q_{t_m}$.

Proposition 2 (exact cancellation factor). If $c = (m - n)/2$ is the number of inverse-pair cancellations required to reduce u , then

$$Q_5^{\text{raw}}(u) = 5^c Q_5(w). \quad (16)$$

Proof. Each elementary cancellation removes an adjacent pair ss^{-1} or $s^{-1}s$. Its quaternion product is $q_s \bar{q}_s = N(q_s) = 5$, a real scalar that commutes with every quaternion. Removing c such pairs therefore removes the scalar factor 5^c , leaving the product associated with the reduced word.

Equation (16) explains why the valuation of a raw product records raw symbolic history rather than reduced group depth. For example, aa^{-1} has raw product 5 and raw norm 25, although its reduced word is e and its reduced depth is zero.

We take $w = aba^{-1}$. Quaternion multiplication gives

$$Q_5(w) = (1+2i)(1+2j)(1-2i) = 5-6j+8k, \quad N(Q_5(w)) = 125 = 5^3. \quad (17)$$

Equation (17) agrees with $\ell(w) = 3$. The associated rotation, obtained either from Equation (6) or by multiplying $A_5 B_5 A_5^{-1}$, is

$$\rho_5(w) = \begin{pmatrix} \frac{3}{5} & -\frac{16}{25} & -\frac{12}{25} \\ \frac{16}{25} & \frac{3}{125} & -\frac{96}{125} \\ \frac{12}{25} & -\frac{96}{125} & \frac{53}{125} \end{pmatrix}. \quad (18)$$

Exact calculation gives $\rho_5(w)^T \rho_5(w) = I_3$ and $\det \rho_5(w) = 1$.

Address Growth

We set $L_n = \{w \in F_2 : \ell(w) = n\}$. There are four choices for the first letter of a nonempty reduced word and three choices thereafter, because the inverse of the preceding letter is forbidden. Hence

$$|L_0| = 1, \quad |L_n| = 4 \cdot 3^{n-1} \quad (n \geq 1). \quad (19)$$

The number of addresses of depth at most n is consequently

$$|\{w : \ell(w) \leq n\}| = 1 + \sum_{r=1}^n 4 \cdot 3^{r-1} = 2 \cdot 3^n - 1. \quad (20)$$

At exact depth $n \geq 1$, every quaternion code has norm 5^n , while there are $4 \cdot 3^{n-1}$ distinct addresses. We define the information count I_n as the base-two logarithm of this number. A fixed-length binary encoding requires $\lceil I_n \rceil$ bits. The information count and its asymptotic ratio to the logarithm of the common norm are

$$I_n = \log_2 |L_n| = 2 + (n-1) \log_2 3, \quad \lim_{n \rightarrow \infty} \frac{I_n}{n \log_2 5} = \frac{\log 3}{\log 5} \approx 0.682606 \quad (21)$$

Figure 1 compares these two growth rates. The norm records depth but does not distinguish the different words at that depth.

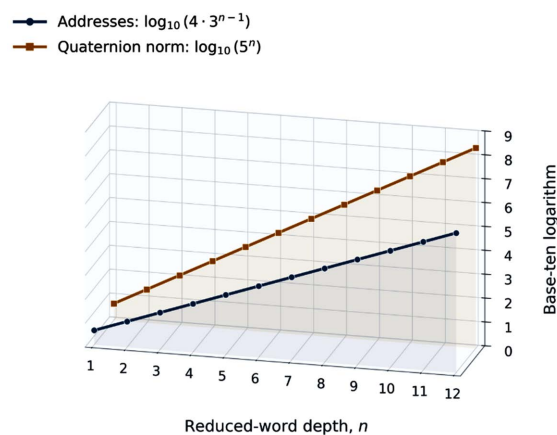


Figure 1. Number of reduced words and common quaternion norm at depths one through twelve, shown on a base-ten logarithmic scale.

5. Transfer to the Banach-Tarski Orbit Decomposition

For $s \in \Sigma$, we let H_s be the set of nonempty reduced words whose first letter is s . Let $L = \{a^{-r} : r \geq 1\}$ be the negative a -ray. By defining

$$\begin{aligned} P_1 &= H_a \cup \{e\} \cup L, \\ P_2 &= H_{a^{-1}} - L, \\ P_3 &= H_b, \\ P_4 &= H_{b^{-1}}. \end{aligned} \quad (22)$$

the four sets in Equation (22) are disjoint and cover F_2 . Left multiplication and free reduction give

$$aP_2 = P_2 \sqcup P_3 \sqcup P_4, \quad bP_4 = P_1 \sqcup P_2 \sqcup P_4. \quad (23)$$

To verify the first identity, write a word of P_2 as $a^{-r}v$, where $r \geq 1$ and v begins with b or b^{-1} . Multiplication by a either decreases r by one or, when $r = 1$, exposes the first letter of v . This produces exactly $P_2 \sqcup P_3 \sqcup P_4$. The second identity follows because $bH_{b^{-1}}$ is the complement of H_b in F_2 .

Equations (22) and (23) imply two disjoint reconstructions:

$$F_2 = P_1 \sqcup aP_2, \quad F_2 = P_3 \sqcup bP_4. \quad (24)$$

Thus the four original pieces P_1, P_2, P_3, P_4 give two reconstructions of F_2 . **Figure 2** shows the partition through depth three. Colours indicate membership in Equation (22), and black outlines mark the negative a -ray assigned to P_1 .

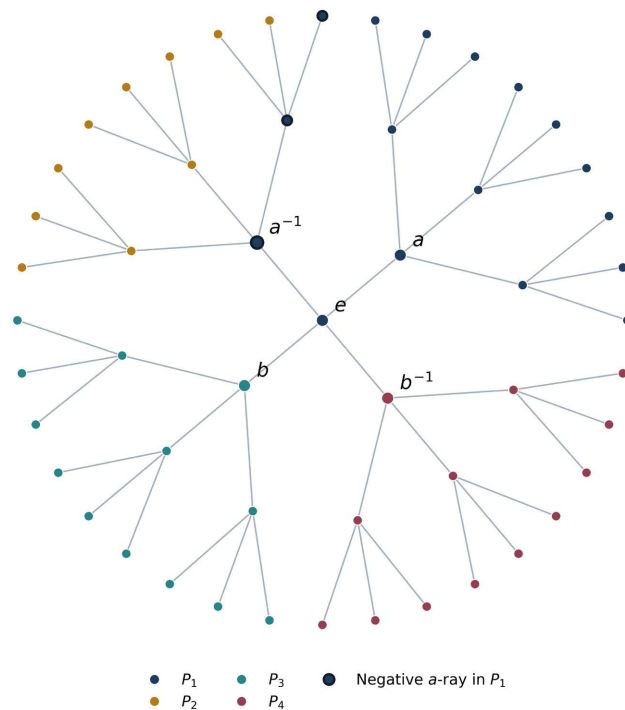


Figure 2. The prefix partition on the Cayley tree through depth three. The identity and the negative a -ray belong to P_1 ; all other vertices are coloured by their first letter.

By retaining the free action of F_2 on X , the complement of D in S^2 , and the transversal M from Section 2, we define four subsets of X by

$$E_r = P_r M = \{\rho_5(g)m : g \in P_r, m \in M\}, \quad 1 \leq r \leq 4. \quad (25)$$

The bijection in Equation (4) shows that the E_r are disjoint and cover X . Transferring Equation (24) through the action gives

$$X = E_1 \sqcup A_5 E_2, \quad X = E_3 \sqcup B_5 E_4. \quad (26)$$

Equation (26) is the four-piece paradoxical decomposition of the free part of the sphere. It uses the same rotations that define the quaternion code.

Suppose all four E_r were measurable with respect to σ . Rotation invariance and finite additivity applied to Equation (26) would give

$$\sigma(E_1) + \sigma(E_2) = \sigma(X), \quad \sigma(E_3) + \sigma(E_4) = \sigma(X).$$

Since the four pieces also partition X , adding these equalities would imply $\sigma(X) = 2\sigma(X)$, contrary to $\sigma(X) = 4\pi$. Hence the pieces cannot all be surface-measurable. This argument alone does not determine the measurability of any specified E_r .

For each $x \in X$, Equation (4) gives a unique pair $(g, m) \in F_2 \times M$ with $x = \rho_5(g)m$. We can define the M -relative address of x by

$$\alpha_M(x) = (m, C_5(g)). \quad (27)$$

The representative m identifies the orbit and $C_5(g)$ identifies the point within that orbit. Proposition 1 and the bijectivity of Θ make Equation (27) injective. The dependence on M is unavoidable. Selecting M uses choice, and a different transversal changes the address of every point in the affected orbits. The code is therefore exact after the choices have been fixed, but it is not canonical. Equation (25) distinguishes the finite partition from the word labels. Each piece E_r is indexed by the entire language P_r and all representatives in M . A single word g instead specifies one point $\rho_5(g)m$ in each orbit. Thus $C_5(g)$ is a code within an orbit, P_r is a set of words, and $E_r = P_r M$ is a subset of the sphere. The equality $v_5(N(Q_5(g))) = \ell(g)$ concerns a word and its chosen lift. It does not describe the surface-area measure of E_r . The argument above shows that the four pieces cannot all be surface-measurable; it does not establish non-measurability of each individual piece.

Equation (26) treats X , not the complete sphere or solid ball. The countable set D can be absorbed by a separate equidecomposition, and the resulting sphere decomposition can be extended along radial half-open segments. A further absorption handles the centre. Robinson's refinement gives five pieces for the ball and proves that four cannot suffice [6]. These auxiliary steps do not alter Theorems 1 and 2, but they do show why the present address should be described as a code for the free-group skeleton rather than as a complete constructive listing of the five ball pieces.

6. Conclusion

The integer quaternions $1+2i$ and $1+2j$ give an exact arithmetic representation of the free-group action considered here. Their rational rotations have perpendicular axes and common-angle cosine $-3/5$, so Świerczkowski's criterion proves freeness for this pair. For a reduced word w , the unnormalised quaternion product $Q_5(w)$ projects to the same rotation as the matrix word and satisfies $N(Q_5(w)) = 5^{\ell(w)}$. The 5-adic valuation of the norm is exactly the Cayley-graph depth. For unreduced words, every inverse-pair cancellation contributes an explicit scalar factor 5. These identities give a reproducible, depth-graded code and explain its algebraic content without attaching physical meaning to the quaternion norm. After removing the countable fixed-point set and choosing a transversal, each point of the remaining sphere has a unique orbit representative and reduced-word label. The four word sets then give four spherical subsets satisfying Equation (26). These subsets cannot all be measurable with respect to rotation-invariant surface-area measure. Each piece is a union indexed by a set of words and the transversal, rather than a single word. The resulting distinction is exact. Addressability concerns unique labels inside a chosen free orbit structure. Measurability concerns membership in the domain of an additive measure. The prime-quaternion code supplies the former and leaves the latter unchanged. Future work could compare other prime-norm quaternion pairs, study the dependence of depth statistics on the generating set, or formalise coding on Baire-property decompositions. Any physical or tensorial extension would require separate axioms and cannot be inferred from the set-theoretic decomposition alone.

Author Contributions

Payam Danesh and Raoul Bianchetti jointly developed the mathematical framework, carried out the analysis, and checked the proofs. Both authors contributed to drafting and revising the manuscript and approved the final version.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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