Journal of Applied Mathematics and Physics, 2026, 14(8), 3063-3088
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

A Reduced Kinetic Stability Model for the Dark-Matter Cosmic Web

Nikolay N. Fimin

Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Moscow, Russia
Email: oberon@kiam.ru

How to cite this paper: Fimin, N.N. (2026) A Reduced Kinetic Stability Model for the Dark-Matter Cosmic Web. *Journal of Applied Mathematics and Physics*, **14**, 3063-3088.
https://doi.org/10.4236/jamp.2026.148150

Received: July 24, 2026
Accepted: August 21, 2026
Published: August 24, 2026

Abstract

We formulate a reduced kinetic stability model for the dark-matter cosmic web in a deliberately restricted finite-time sense. The object is the tangent deformation of prescribed Vlasov characteristics, not spectral or nonlinear stability of a self-consistent Vlasov-Poisson solution. Six-dimensional geodesic-deviation and phase-space-distortion integration is already established; the present addition is a morphology-labeled reporting protocol for an available tangent propagator. A Gaussian-smoothed T-web supplies void, wall, filament, and node labels, while the complete effective Hessian determines the local dynamics. For a frozen conservative physical-coordinate cell, Hessian inertia and quadratic first integrals identify hyperbolic, neutral, and elliptic sectors; comoving behaviour is determined by the full Hubble-containing generator. Finite-time amplification is measured by the standard largest singular-value/FTLE quantity in one fixed full-state metric and on maximal contiguous morphology intervals after exact substep composition. The smoothing length controls the reconstructed Hessian and labels, but its common scalar factor cancels from the induced gain; the position-velocity weighting is set by a declared time scale. The propagator bound is label-blind, so morphology sums are residence-time-labeled bookkeeping rather than causal environmental contributions. Homogeneous expansion is removed with an interaction-picture propagator. Common-anchor factors telescope globally, but their norms depend on the anchor; local two-point factors and a transported-metric interpretation are therefore distinguished. A fully three-dimensional benchmark tests changing eigenframes, classifier-Hessian misalignment, a near-neutral gap, partitioning, anchor dependence, metric choice, and classification tolerances. At a fixed matter Hessian, the cosmological constant gives the usual local downward curvature shift. The paper provides a formal construction and validation protocol, not an empirically validated cosmic-web diagnostic.

Keywords

Kinetic Stability, Cosmic Web, Dark Matter, Characteristic Flow, Symplectic Invariants, Vlasov-Poisson-Λ System

1. Introduction

The cosmic web is organized into voids, walls, filaments, and nodes. This morphology is visible in galaxy surveys and simulations, but it also records information about the density and tidal fields. Collisionless dark matter is described kinetically by the Vlasov equation coupled to Poisson's equation [1]-[4]. Once multistreaming develops, phase-space methods and direct Vlasov approaches retain information that is not contained in a single pressureless-fluid velocity field [5]-[7].

The tangent dynamics used below are not new in themselves. Geodesic-deviation equations and the full six-dimensional phase-space distortion tensor have already been integrated together with dark-matter particle trajectories to resolve fine-grained streams and caustics [8]-[10]. Complementary phase-space-sheet and tessellation methods reconstruct multistream density, caustics, and collisionless-fluid structure [11] [12]. The present paper does not claim to introduce tangent-space integration, caustic tracking, or the generic submultiplicativity of propagator norms.

The induced singular-value gain used below is also standard. For a fixed positive-definite metric W, a propagator $U(t_2,t_1)$ and $\Delta t = t_2 - t_1 > 0$, define the metric Cauchy-Green tensor

$$C_W = \left(W^{1/2}UW^{-1/2}\right)^{\mathrm{T}}\left(W^{1/2}UW^{-1/2}\right).$$

Then, $\|U\|_W = \sqrt{\lambda_{\max}(C_W)}$ and $\Delta t^{-1}\log\|U\|_W$ is the largest finite-time singular-value exponent, *i.e.*, the metric version of the standard finite-time Lyapunov exponent associated with Cauchy-Green strain [13]-[15]. No novelty is claimed for this base measure; the new element is how it is normalized, segmented, background-referenced, and reported with morphology labels.

A broad range of web classifiers has been constructed from density, tidal, and velocity-shear tensors, topology, and caustics [16]-[20]. Void and filament catalogs show distinct environment-conditioned density and velocity statistics [21] [22]. Current survey releases provide observational context [23] [24], while cosmological simulations provide the matter field and characteristic trajectories on which a tangent calculation can be performed [25]-[27].

The narrower contribution of this work is a reporting and interpretation layer for such tangent propagators. It combines: 1) a declared T-web morphology; 2) the full local effective Hessian rather than density alone; 3) exact local invariant information in frozen physical cells; 4) one fixed phase-space metric with a declared position-velocity time scale; 5) maximal contiguous reporting intervals, distinct from numerical integration substeps; and 6) an interaction-picture propagator that removes homogeneous Hubble evolution. The resulting interval gains can be grouped by morphology, but the labels do not enter the proof of the norm bound. Therefore, the grouping is not, by itself, evidence that morphology predicts amplification. A predictive cosmic-web diagnostic would require simulation trajectories, finite-difference verification, conditional gain statistics, shuffled-label controls, and an independent validation sample. Those empirical steps are specified but not performed here.

The title uses the term *kinetic stability* in an explicitly reduced and operational sense. A Vlasov distribution is transported by a phase-space characteristic flow, and the derivative of that flow controls deformation of phase-space elements, sensitivity to initial conditions, and amplification of phase-space gradients. Here, "stability" is an umbrella term for invariant stable, unstable, and neutral splitting, Lyapunov boundedness on elliptic subspaces, modal exponential separation on hyperbolic subspaces, and finite-time gain in a declared norm. It is not spectral, nonlinear, or energy-Casimir stability of a self-consistent Vlasov-Poisson equilibrium. The missing self-consistent δf - $\delta\Phi$ feedback is displayed explicitly in Section 2.

Physical separations include the background scale factor, whereas comoving separations measure deformation relative to the Hubble flow. In pure de Sitter space, a physical separation grows exponentially, although the comoving separation is bounded. Raw physical or comoving full-state gains, therefore, contain background effects. The paper separates two objects: the complete propagator, which describes the declared coordinates, and a background-normalized interaction propagator, which isolates the effect of the peculiar Hessian relative to the same $a(t)$ and $H(t)$. Common-anchor interaction factors telescope exactly over a full itinerary, but their norms also contain the background transport from the declared anchor t_0 to the segment start. They are therefore distinguished below from anchor-free local two-point relative factors.

Local invariant information and finite-time gain must also be separated. For a frozen conservative cell in physical coordinates, the inertia of the symmetric Hessian counts hyperbolic, neutral, and elliptic configuration directions. Exact wall-normal or filament-transverse energies exist only when those morphological subspaces are invariant under the complete Hessian. These structural statements do not imply a monotonic decrease of one fixed full-state norm. Global propagation must therefore retain the complete phase-space propagator.

The paper proceeds as follows. Section 2 states the prescribed-field kinetic scope, the physical/comoving relation, the normalization convention, the T-web assignment, and the frozen-cell error control. Section 3 records exact frozen dynamics and local invariants. Section 4 defines maximal morphology intervals, full and background-normalized propagator gains, and the associated finite-time bounds. Section 5 gives a fully three-dimensional controlled benchmark together with partition and sensitivity studies. Sections 6 and 7 discuss the restricted Λ sensitivity, the required simulation validation, and the limitations. The conclusions state only the formal results established in the paper.

The cosmological-constant discussion is correspondingly local. At fixed matter Hessian, Λ gives an isotropic downward shift of the physical curvature spectrum. Background-consistent peculiar evolution must instead be obtained from the comoving propagator and cannot be inferred from that local shift alone [28]-[31].

2. Scope of Reduced Kinetic Stability and Local Equations

2.1. Characteristic Transport and the Meaning of Reduced Kinetic Stability

In physical coordinates, the collisionless distribution function satisfies

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\Psi\cdot\nabla_v f = 0, \tag{1}$$

where

$$\Psi(x,t) = \Phi(x,t) - \frac{\Lambda c^2}{6}|x|^2,\ \Delta_x\Phi = 4\pi G\rho,\ \rho = m\int f\,\mathrm{d}^3v. \tag{2}$$

Let $\mathcal{Z} = (X, V)$ and let $\mathcal{F}_{t,t_0}$ denote the phase-space characteristic flow generated by a sufficiently regular background field Ψ_0:

$$\dot{X} = V,\ \dot{V} = -\nabla\Psi_0(X,t),\ \mathcal{Z}(t) = \mathcal{F}_{t,t_0}\mathcal{Z}(t_0). \tag{3}$$

The Vlasov density is constant along characteristics,

$$f\left(\mathcal{F}_{t,t_0}z_0, t\right) = f(z_0, t_0). \tag{4}$$

Equivalently,

$$\nabla_z f(z,t) = D\mathcal{F}_{t_0,t}(z)^{\mathrm{T}}\,\nabla_z f\left(\mathcal{F}_{t_0,t}z, t_0\right), \tag{5}$$

so deformation of the tangent map controls phase-space gradient amplification and filamentation. Thus, $D\mathcal{F}_{t,t_0}$ is part of the kinetic transport itself. For a spatial deviation ξ and velocity deviation $\nu = \dot{\xi}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \nu\end{pmatrix} = \begin{pmatrix}0 & I\\ -K_{\text{phys}}(t) & 0\end{pmatrix}\begin{pmatrix}\xi\\ \nu\end{pmatrix},\ K_{\text{phys}}(t) = D_x^2\Psi_0\left(X_0(t), t\right), \tag{6}$$

whose configuration block is

$$\ddot{\xi} = -K_{\text{phys}}(t)\,\xi. \tag{7}$$

Definition 1 (Reduced kinetic stability). *On a finite interval, reduced kinetic stability means the invariant splitting, Lyapunov boundedness, exponential separation, and norm-dependent finite-time gain properties of the tangent map* $D\mathcal{F}_{t,t_0}$ *associated with the Vlasov characteristic transport in a prescribed background. The term includes stable, unstable, and neutral local sectors in the standard dynamical-systems sense* [32]. *It does not mean spectral, nonlinear, or energy-Casimir stability of the self-consistent distribution function.*

For comparison, the self-consistent linearization about f_0 contains

$$\partial_t\delta f + v\cdot\nabla_x\delta f - \nabla\Psi_0\cdot\nabla_v\delta f = \nabla\delta\Phi\cdot\nabla_v f_0, \tag{8}$$

$$\Delta_x \delta\Phi = 4\pi G m \int \delta f \, \mathrm{d}^3 v. \tag{9}$$

The characteristic variation then contains the additional force $-\nabla\delta\Phi\left(X_0,t\right)$. Equations (8) and (9) describe collective feedback, Jeans-type modes, and phase mixing. They are not solved in the present model. The adjective "kinetic" therefore identifies the phase-space Vlasov transport from which the characteristic map is obtained, while the qualifier "reduced" records the omission of self-consistent perturbation feedback.

2.2. Physical and Comoving Separations

Write $x = a(t)r$, with comoving position r and $H = \dot{a}/a$. In the usual peculiar-potential formulation, a reference trajectory and a neighboring comoving separation η obey

$$\ddot{r} + 2H\dot{r} = -\frac{1}{a^2}\nabla_r \phi, \tag{10}$$

$$\ddot{\eta} + 2H\dot{\eta} = -K_{\mathrm{pec}}(t)\eta,\ \ K_{\mathrm{pec}}(t) = \frac{1}{a^2}D_r^2\phi, \tag{11}$$

where

$$\Delta_r \phi = 4\pi G a^2 \left(\rho - \bar{\rho}\right). \tag{12}$$

The physical separation is $\xi = a\eta$. Direct differentiation gives

$$\ddot{\xi} = -K_{\mathrm{phys}}\xi,\ \ K_{\mathrm{phys}} = K_{\mathrm{pec}} - \frac{\ddot{a}}{a}I. \tag{13}$$

For pressureless matter plus Λ,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\bar{\rho}}{3} + \frac{\Lambda c^2}{3}, \tag{14}$$

so

$$K_{\mathrm{phys}} = K_{\mathrm{pec}} + \frac{4\pi G\bar{\rho}}{3}I - \frac{\Lambda c^2}{3}I. \tag{15}$$

Using Equation (12) gives $\operatorname{tr} K_{\mathrm{phys}} = 4\pi G\rho - \Lambda c^2$, as in the total physical-potential formulation.

The coordinate interpretation is exposed by a de Sitter check. If $\phi = 0$, $a = \mathrm{e}^{Ht}$ and $H^2 = \Lambda c^2/3$, then

$$\eta(t) = C_1 + C_2\mathrm{e}^{-2Ht} \tag{16}$$

remains bounded, whereas

$$\xi(t) = C_1\mathrm{e}^{Ht} + C_2\mathrm{e}^{-Ht},\ \ K_{\mathrm{phys}} = -H^2 I. \tag{17}$$

Thus the physical exponent H is background expansion, not a cosmic-web instability. The representation used for a finite-time estimate must be fixed in advance. Comoving variables are appropriate for peculiar deformation relative to the Hubble flow; physical variables measure actual physical separation over a stated finite interval. Asymptotic Lyapunov exponents need not be invariant under the unbounded transformation $\xi = a\eta$.

2.3. Fixed State Scaling and Phase-Space Metric

A numerical propagator norm is meaningless until configuration and velocity components have been placed on one declared physical scale. For a cosmological analysis window starting at t_0, the primary convention of this paper is

$$\ell_*^{\mathrm{com}} = R_s, \ \ell_*^{\mathrm{phys}} = a(t_0) R_s, \ \tau_* = H(t_0)^{-1}. \tag{18}$$

The dimensionless full states are

$$z_{\mathrm{com}} = \begin{pmatrix} \eta/\ell_*^{\mathrm{com}} \\ \tau_* \dot{\eta}/\ell_*^{\mathrm{com}} \end{pmatrix}, \ z_{\mathrm{phys}} = \begin{pmatrix} \xi/\ell_*^{\mathrm{phys}} \\ \tau_* \dot{\xi}/\ell_*^{\mathrm{phys}} \end{pmatrix}. \tag{19}$$

The default reporting metric is the Euclidean norm of these dimensionless states, *i.e.*, $W_* = I_6$. In dimensional variables $Y = (q, \dot{q})^{\mathrm{T}}$, the same convention is

$$\|Y\|_{W_*}^2 = \frac{|q|^2}{\ell_*^2} + \frac{\tau_*^2 |\dot{q}|^2}{\ell_*^2}, \ W_* = \ell_*^{-2} \begin{pmatrix} I & 0 \\ 0 & \tau_*^2 I \end{pmatrix}. \tag{20}$$

The common factor ℓ_*^{-2} is a unit conversion rather than an independent regulator of induced gain. For every $c > 0$,

$$\|U\|_{cW} = \|U\|_W, \ \frac{\|UY\|_{cW}}{\|Y\|_{cW}} = \frac{\|UY\|_W}{\|Y\|_W}.$$

Consequently, changing ℓ_* alone does not change an operator gain or a normalized selected-state ratio. The smoothing length R_s remains physically important because it determines the smoothed classifier tensor and Hessian, and hence the dynamics and morphology intervals. The relative weighting of the position and velocity blocks in the full-state metric is set by τ_* (and by any additional nonscalar choice of W).

The scales and metric are fixed for the complete itinerary; they are not reset at a web transition. Cross-redshift or cross-cosmology comparisons must report R_s as a dynamical/smoothing input and must report t_0, τ_* and W as the phase-space metric convention. A sensitivity analysis in τ_* is mandatory whenever an alternative dynamical time is used. In a dimensionless algebraic benchmark with $H = 0$, $\tau_* = 1$ is simply part of the stated test units and carries no claim of cross-redshift comparability.

2.4. Newtonian Domain and Boundary Convention

The local Newtonian description requires, in addition to sub-horizon scale,

$$\frac{HL}{c} \ll 1, \ \frac{|\phi|}{c^2} \ll 1, \ \frac{|u_{\mathrm{pec}}|}{c} \ll 1, \tag{21}$$

where L is the analysed scale and $u_{\mathrm{pec}} = a\dot{r}$. For a frozen background over one cell, one also requires

$$H \Delta t_{\mathrm{cell}} \ll 1, \ |\dot{H}| \Delta t_{\mathrm{cell}}^2 \ll 1. \tag{22}$$

The peculiar Poisson equation is understood with the boundary convention of the simulation or reconstruction, typically a periodic box or a local patch with prescribed boundary data. The formally divergent potential of an infinite homogeneous Newtonian medium is not used. The physical Hessian is assembled from the background-subtracted peculiar Hessian and the Friedmann background as in Equation (15); this also prevents double counting of the Λ contribution.

2.5. Trace-Tidal Decomposition and Morphology Assignment

At a physical point, the matter Hessian can be decomposed as

$$D_x^2\Phi_0 = \frac{4\pi G\rho_0}{3} I + \mathcal{E}_0, \quad \operatorname{tr}\mathcal{E}_0 = 0, \tag{23}$$

so

$$K_{\text{phys}} = \frac{4\pi G\rho_0 - \Lambda c^2}{3} I + \mathcal{E}_0. \tag{24}$$

Density fixes only the trace. Individual principal-curvature signs depend on the traceless tidal field and on the boundary data. A density-only isotropic formula is therefore conditional on $\mathcal{E}_0 \simeq 0$.

Morphology is assigned independently of the subsequent stability calculation. Smooth the comoving density contrast with a Gaussian kernel,

$$\delta_R(r) = \int W_{R_s}(r - r')\delta(r')\mathrm{d}^3r', \quad W_{R_s}(r) = \frac{\mathrm{e}^{-|r|^2/(2R_s^2)}}{(2\pi R_s^2)^{3/2}}, \tag{25}$$

using the fiducial scale

$$R_s = 2h^{-1}\,\text{Mpc}. \tag{26}$$

Define

$$\Delta_r\varphi_R = \delta_R, \quad T_R = D_r^2\varphi_R. \tag{27}$$

Let $\tau_1 \ge \tau_2 \ge \tau_3$ be the eigenvalues of T_R. With threshold $\tau_{\text{th}} = 0$, the number

$$N_+ = \#\{i : \tau_i > 0\} \tag{28}$$

assigns void, wall, filament and node for $N_+ = 0,1,2,3$, respectively [17] [18]. The positive eigenvector is the wall normal; the two positive eigenvectors span the filament transverse plane. Numerical classification also requires a declared eigenvalue tolerance. Define the classifier margin

$$d_{\text{class}}(t) = \min_i \left|\tau_i(t) - \tau_{\text{th}}\right|. \tag{29}$$

If $d_{\text{class}} < \delta_{\text{class}}$, the label is flagged as unresolved rather than silently assigned. Such transition intervals remain in the propagator composition but are not used as evidence for a morphology-specific effect.

For consistent smoothing,

$$K_{\text{pec}}^{(R)} = 4\pi G\bar{\rho}T_R, \tag{30}$$

and the physical effective curvatures are

$$\kappa_i^{\text{phys}} = 4\pi G\bar{\rho}\left(\tau_i + \frac{1}{3}\right) - \frac{\Lambda c^2}{3}. \tag{31}$$

The morphology label retains only the count in Equation (28); it does not determine the values or signs in Equation (31). All local growth or confinement statements, therefore, use the complete Hessian.

2.6. Frozen Cells, Rotating Eigenframes, and an Error Bound

Choose one representation $\sigma \in \{\text{phys}, \text{com}\}$ for the entire itinerary. With a fixed time scale $\tau_* > 0$, define

$$\begin{aligned} z_{\text{phys}} &= \begin{pmatrix} \xi/\ell_*^{\text{phys}} \\ \tau_*\dot{\xi}/\ell_*^{\text{phys}} \end{pmatrix}, \quad \dot{z}_{\text{phys}} = L_{\text{phys}}(t)\, z_{\text{phys}}, \\ L_{\text{phys}}(t) &= \begin{pmatrix} 0 & \tau_*^{-1} I \\ -\tau_* K_{\text{phys}}(t) & 0 \end{pmatrix}. \end{aligned} \tag{32}$$

$$\begin{aligned} z_{\text{com}} &= \begin{pmatrix} \eta/\ell_*^{\text{com}} \\ \tau_*\dot{\eta}/\ell_*^{\text{com}} \end{pmatrix}, \quad \dot{z}_{\text{com}} = L_{\text{com}}(t)\, z_{\text{com}}, \\ L_{\text{com}}(t) &= \begin{pmatrix} 0 & \tau_*^{-1} I \\ -\tau_* K_{pec}(t) & -2H(t) I \end{pmatrix}. \end{aligned} \tag{33}$$

Let $U(t, t_0)$ be the exact propagator of $\dot{z} = L(t) z$. In a cell $[t_0, t_0 + \Delta]$, freeze $L_0 = L(t_0)$ and use $U_0(\Delta) = \mathrm{e}^{L_0 \Delta}$. For any submultiplicative matrix norm, set

$$M = \sup_{t_0 \le s \le t_0 + \Delta} \|L(s)\|, \quad \varepsilon = \sup_{t_0 \le s \le t_0 + \Delta} \|L(s) - L_0\|. \tag{34}$$

Variation of constants gives the explicit estimate

$$\left\|U(t_0 + \Delta, t_0) - \mathrm{e}^{L_0 \Delta}\right\| \le \varepsilon \Delta \exp\left[\max\left\{M, \|L_0\|\right\} \Delta\right]. \tag{35}$$

Thus, the frozen-cell approximation is controlled directly by the variation of the full first-order generator, not merely by instantaneous eigenvalue ratios.

If $E(t)$ is the orthogonal matrix of eigenvectors of $K(t)$, $D = E^{\mathrm{T}} K E = \operatorname{diag}(\kappa_i)$ and $\Omega = E^{\mathrm{T}} \dot{E}$, then $\xi = Eq$ gives

$$\ddot{q} + 2\Omega\dot{q} + \left[D + \dot{\Omega} + \Omega^2\right] q = 0. \tag{36}$$

The terms $2\Omega\dot{q}$, $\dot{\Omega} q$ and $\Omega^2 q$ show why slow principal-frame rotation, its derivative, and spectral gaps must be checked. Quantities such as $|\dot{\kappa}_i| / |\kappa_i|^{3/2}$ and $\|\Omega\| / \sqrt{|\kappa_i|}$ are useful diagnostics, but without an error estimate such as Equation (35), they are not sufficient theorems. Near eigenvalue crossings, the invariant subspace, rather than an individual eigenvector, must be propagated. More quantitatively, suppose a spectral cluster of $K(t_0)$ is separated from the complementary spectrum by a gap $\delta_{\text{gap}} > 0$ and $\|K(t) - K(t_0)\|_2 < \delta_{\text{gap}}/2$. A Davis-Kahan projector estimate gives

$$\|P(t) - P(t_0)\|_2 \le \frac{2\|K(t) - K(t_0)\|_2}{\delta_{\text{gap}}}, \tag{37}$$

for the corresponding invariant-subspace projectors [33]. This supplies a basis-independent localization check in addition to the full propagator error bound.

2.7. Basis-Independent Local Signatures and Exact Invariants

The stability information carried by a frozen symmetric Hessian is not tied to a particular eigenvector convention. Let P_-, P_0 and P_+ be the spectral projectors of K onto its negative, zero, and positive spectral subspaces, and define

$$\mathfrak{I}(K)=(n_-,n_0,n_+),\ n_\alpha=\operatorname{rank}P_\alpha. \tag{38}$$

The triple $\mathfrak{I}(K)$ is the inertia of the quadratic form and n_- is its Morse index. It is invariant under orthogonal frame changes and, more generally, under nonsingular congruences by Sylvester's law of inertia [34]. In numerical work, the zero subspace is replaced by a declared tolerance $|\kappa_i|\le\kappa_{\text{tol}}$. The T-web count N_+ and the effective-Hessian inertia $\mathfrak{I}(K)$ are intentionally kept separate: the former is a morphology label, while the latter is the local stability signature. This inertia classification refers specifically to the frozen conservative physical-coordinate equation $\ddot{q}=-K_{\text{phys}}q$. In the comoving representation, the Hubble block in L_{com} changes the spectral type; consequently, comoving stability is classified by the full first-order generator or its finite-time propagator, not by the sign of K_{pec} alone.

For the conservative physical frozen cell, write $q=\xi$, $p=\dot{\xi}$ and $q_\alpha=P_\alpha q$, $p_\alpha=P_\alpha p$. On the negative subspace set $K_-:=K|_{\operatorname{Ran}P_-}<0$ and $M=(-K_-)^{1/2}$. The hyperbolic modal vectors

$$a_+=p_-+Mq_-,\ a_-=p_--Mq_- \tag{39}$$

obey

$$\dot{a}_+=Ma_+,\ \dot{a}_-=-Ma_-. \tag{40}$$

Their bilinear product

$$I_-=a_+^{\mathrm{T}}a_-=p_-^{\mathrm{T}}p_-+q_-^{\mathrm{T}}K_-q_- \tag{41}$$

is an exact, generally indefinite invariant. On the positive subspace, $K_+:=K|_{\operatorname{Ran}P_+}>0$ and

$$I_+=p_+^{\mathrm{T}}p_++q_+^{\mathrm{T}}K_+q_+ \tag{42}$$

is exact and positive definite. On the neutral subspace, p_0 is constant and q_0 may drift linearly. These quantities combine into the frozen-cell Hamiltonian

$$\mathcal{H}_K(q,p)=\frac{1}{2}p^{\mathrm{T}}p+\frac{1}{2}q^{\mathrm{T}}Kq,\ \frac{\mathrm{d}\mathcal{H}_K}{\mathrm{d}t}=0. \tag{43}$$

Table 1 lists the morphology-local specializations. The curvature condition, rather than the morphology label alone, is decisive.

There is also a structural invariant that survives arbitrary time dependence of a symmetric physical Hessian. For the unscaled canonical state $Y=(q,p)^{\mathrm{T}}$, set

$$J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix},\ A(t)=\begin{pmatrix}0 & I\\ -K_{\text{phys}}(t) & 0\end{pmatrix}. \tag{44}$$

Because $A^{\mathrm{T}}J + JA = 0$, the physical tangent propagator satisfies

$$U_Y(t,t_0)^{\mathrm{T}} JU_Y(t,t_0) = J,\ \det U_Y(t,t_0) = 1. \tag{45}$$

Thus, the symplectic form and phase volume are exact invariants even when the cell energy is not conserved [35] [36].

The comoving equation has an analogous canonical statement. For any two solutions η_1, η_2 of Equation (11),

$$\mathcal{W}_{12}(t) = a(t)^2\left[\eta_1^{\mathrm{T}}\dot{\eta}_2 - \dot{\eta}_1^{\mathrm{T}}\eta_2\right] \tag{46}$$

is constant. Equivalently, with canonical momentum $\pi = a^2\dot{\eta}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\eta\\ \pi\end{pmatrix} = \begin{pmatrix}0 & a^{-2}I\\ -a^2K_{\mathrm{pec}} & 0\end{pmatrix}\begin{pmatrix}\eta\\ \pi\end{pmatrix}, \tag{47}$$

which is symplectic. The contraction of phase volume in the noncanonical variables $(\eta, \dot{\eta})$, therefore, reflects the time-dependent coordinate scaling, not the destruction of the canonical invariant.

The quadratic energy is only a local invariant when K is frozen. For a smoothly varying physical Hessian,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{K(t)} = \frac{1}{2}q^{\mathrm{T}}\dot{K}(t)q. \tag{48}$$

At an idealized cell boundary t_j, where q and p are continuous but the frozen Hessian changes from K_j to K_{j+1},

$$\mathcal{H}_{K_{j+1}}(t_j^+) - \mathcal{H}_{K_j}(t_j^-) = \frac{1}{2}q(t_j)^{\mathrm{T}}(K_{j+1} - K_j)q(t_j). \tag{49}$$

This exact interface identity prevents local wall, filament, or void invariants from being misused as a single global conserved energy.

Table 1. Exact invariants for a frozen conservative cell in the physical-coordinate representation. Each row applies to an invariant spectral subspace of the full K_{phys}; comoving cells must be classified by L_{com} or its propagator.

Localized Sector	Curvature and Invariance Condition	Exact Invariant	Stability Information
Void-Like Hyperbolic Channel	Spectral channel, $\kappa = -\mu^2 < 0$	$I_v = p^2 - \mu^2 q^2 = a_+ a_-$	Stable/unstable phase-space splitting; no boundedness claim
Wall Normal	$Kn = \kappa_n n$, $\kappa_n > 0$	$I_w = p_n^2 + \kappa_n q_n^2$	Bounded normal coordinate
Filament Transverse Plane	$K_\perp = P_\perp K P_\perp > 0$, $[P_\perp, K] = 0$	$I_f = p_\perp^{\mathrm{T}} p_\perp + q_\perp^{\mathrm{T}} K_\perp q_\perp$	Bounded transverse displacement
Positive-Definite Node Cell	$K > 0$	$I_n = p^{\mathrm{T}} p + q^{\mathrm{T}} Kq$	Bounded full configuration deviation
Neutral Channel	Spectral channel, $\kappa = 0$	$p_0 = \mathrm{const}$	Possible linear drift of q_0

3. Exact Dynamics in a Frozen Cell

3.1. Hyperbolic Scalar Mode and Its Modal Exponent

For a scalar principal curvature $\kappa = -\mu^2 < 0$, the frozen physical separation mode satisfies

$$\ddot{q} = \mu^2 q, \quad p = \dot{q}. \tag{50}$$

The Chetaev function $C(q, p) = qp$ has

$$\dot{C} = p^2 + \mu^2 q^2 > 0 \tag{51}$$

in the forward-invariant cone $q > 0$, $p > 0$, establishing instability of the origin [37]. The phase-space modal coordinates

$$a_+ = p + \mu q, \quad a_- = p - \mu q \tag{52}$$

obey

$$\dot{a}_+ = \mu a_+, \quad \dot{a}_- = -\mu a_-. \tag{53}$$

Therefore, $\mu = \sqrt{-\kappa}$ is the exact exponent of the growing phase-space mode a_+, or of data restricted to $p = \mu q$. In addition,

$$I_v = a_+ a_- = p^2 - \mu^2 q^2 \tag{54}$$

is conserved. In the scalar case, its zero level contains the stable and unstable separatrices $p = \mp \mu q$, while nonzero levels are hyperbola.

It is not a universal pointwise upper bound for the coordinate projection $|q|$. For $q(0) = 0$, $p(0) = 1$,

$$q(t) = \frac{\sinh(\mu t)}{\mu}, \quad \frac{\mathrm{d}}{\mathrm{d}t} \log|q(t)| = \mu \coth(\mu t) > \mu. \tag{55}$$

Arbitrary projections must therefore be treated by the propagator of the full state.

In the isotropic physical void approximation $\mathcal{E}_0 \simeq 0$, Equation (24) gives

$$\mu_{\text{phys}} = \left(\frac{\Lambda c^2 - 4\pi G \rho_v}{3} \right)^{1/2}, \quad \Lambda c^2 > 4\pi G \rho_v. \tag{56}$$

This is a physical-separation modal exponent. It includes the background term and is not, by itself, a peculiar growth rate.

3.2. Positive Curvature Gives Confinement, Not a Common Decay Rate

For $\kappa = \omega^2 > 0$,

$$\ddot{q} + \omega^2 q = 0, \quad E = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2, \quad \dot{E} = 0. \tag{57}$$

Hence $|q(t)| \le \sqrt{2E(0)}/\omega$. This is Lyapunov confinement [35] [36] [38]. It is not a monotonic contraction of a fixed Euclidean scaling. For

$$A^2 = q^2 + \tau_*^2 p^2, \tag{58}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t} A^2 = 2qp\left(1 - \tau_*^2 \omega^2\right). \tag{59}$$

Unless $\tau_* \omega = 1$ for that particular cell, the sign can be positive or negative. A scaling matched to one frequency is not generally matched to the next wall or filament cell.

For the unscaled state $(q, p)^{\mathrm{T}}$, the exact scalar cell propagator is

$$U_{\kappa}(\Delta)=\begin{cases}\begin{pmatrix}\cos(\omega\Delta) & \omega^{-1}\sin(\omega\Delta)\\ -\omega\sin(\omega\Delta) & \cos(\omega\Delta)\end{pmatrix}, & \kappa=\omega^2>0,\\ \begin{pmatrix}\cosh(\mu\Delta) & \mu^{-1}\sinh(\mu\Delta)\\ \mu\sinh(\mu\Delta) & \cosh(\mu\Delta)\end{pmatrix}, & \kappa=-\mu^2<0.\end{cases} \tag{60}$$

These matrices retain the phase information that a signed scalar "growth minus stabilization" budget discards.

3.3. Wall, Filament, and Node Cells

For a constant symmetric 3 × 3 Hessian, an orthogonal transformation reduces the frozen physical problem to three scalar blocks of the form (60). Exact localization requires an invariant subspace of the complete Hessian, not merely positive directed curvature. If n is a fixed T-web wall normal, $q_n = n^{\mathrm{T}} q$, $p_n = n^{\mathrm{T}} p$ and $\kappa_n = n^{\mathrm{T}} K n > 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_n^2+\kappa_n q_n^2\right)=-2p_n n^{\mathrm{T}}K\left(I-nn^{\mathrm{T}}\right)q.$$

Hence, the wall-normal energy is exact for all states if and only if $Kn \parallel n$. For a filament transverse projector $P_{\perp}$, the absence of coupling to the axial complement requires

$$P_{\perp}K\left(I-P_{\perp}\right)=0,\ \text{equivalently}\ \left[P_{\perp},K\right]=0.$$

When the classifier tensor and K are constructed with the same smoothing and background convention in Equations (30) and (31), this alignment is automatic because K is an affine combination of T_R and the identity. If morphology and the stability Hessian are reconstructed independently, the commutator must be checked, and the exact invariants must be formulated with the spectral projectors of K. Under this condition, positive wall-normal or filament-transverse curvatures bound the associated configuration coordinates, while the remaining directions may be neutral or hyperbolic. Node intervals are ordinary dynamical intervals and cannot be deleted from an itinerary.

4. Finite-Time Propagators and Morphology-Labeled Reporting

4.1. Maximal Contiguous Report Intervals

All primary gains are computed in the fixed dimensionless state of Equations (18) - (20). The default metric is $W_* = I_6$; the formulas below allow a different fixed $W = W^{\mathrm{T}} > 0$ only to make metric sensitivity explicit:

$$\|z\|_W=\left(z^{\mathrm{T}}Wz\right)^{1/2},\ \|U\|_W=\left\|W^{1/2}UW^{-1/2}\right\|_2. \tag{61}$$

For an interval of duration $\Delta t > 0$, the associated metric Cauchy-Green tensor and the largest finite-time Lyapunov exponent are

$$C_W(U) = W^{-1/2} U^{\mathrm{T}} W U W^{-1/2},$$

$$\sigma_{\max}^{\mathrm{FT}}(U;W) = \frac{1}{2\Delta t} \log \lambda_{\max}\left[C_W(U)\right] = \frac{1}{\Delta t} \log \|U\|_W .$$

This singular-value construction is standard [13]-[15]. The claimed contribution is not a new FTLE or Cauchy-Green measure, but the fixed-metric, maximal-interval, and background-referenced reporting protocol developed below.

Let $m(t)$ be the T-web label after applying the declared threshold and classifier tolerance. A *report interval* $I_k = [s_k, s_{k+1}]$ is a maximal connected interval on which $m(t)$ is constant and resolved. Ambiguous transition intervals are assigned the auxiliary label u and are retained in the propagation. Thus

$$m_k \in \mathcal{M} := \{v, w, f, n, u\}. \tag{62}$$

The numerical integrator may use an arbitrarily fine mesh $s_k = t_{k,0} < \cdots < t_{k,M_k} = s_{k+1}$. Its substep propagators are first composed,

$$U_k = U\left(t_{k,M_k}, t_{k,M_k-1}\right) \cdots U\left(t_{k,1}, t_{k,0}\right),\ G_k^{\mathrm{raw}}(W) = \|U_k\|_W . \tag{63}$$

Only the norm of the composed propagator over the full maximal interval is reported as an interval gain. Integration substeps are not counted as separate morphology contributions.

Proposition 1 (Finite-time report bound). *For either the physical system* (32) *or the comoving system* (33), *provided one representation and one fixed matrix* W *are used throughout,*

$$\frac{\|z(T)\|_W}{\|z(0)\|_W} \le U_{K-1} \cdots U_{0W} \le \prod_{k=0}^{K-1} G_k^{\mathrm{raw}}(W). \tag{64}$$

Proof. State continuity gives $z(T) = U_{K-1} \cdots U_0 z(0)$. The first inequality is the definition of the induced norm; the second is submultiplicativity. □

The proof contains no T-web property and would remain true after arbitrary relabeling. This label-blindness is important. For a fixed classifier and the maximal-interval rule, one may define the reproducible bookkeeping sums

$$\Gamma_\alpha^{\mathrm{lab}}(W) = \sum_{k:m_k=\alpha} \log G_k^{\mathrm{raw}}(W),\ \alpha \in \mathcal{M}, \tag{65}$$

but they are *residence-time-labeled log-bound components*, not invariant or causal effects of the environments. Scientific meaning requires an external demonstration that the conditional gain distributions differ from shuffled-label controls.

4.2. Partition Dependence and the Reporting Convention

For one fixed report interval $I = [s_0, s_1]$, let $U_I = U(s_1, s_0)$. If the same interval is represented by m exact substeps, then

$$U_I = U_m \cdots U_1,\ G_I = \|U_I\|_W,\ B_m = \prod_{\ell=1}^{m} \|U_\ell\|_W,\ G_I \le B_m. \tag{66}$$

The exact composed gain G_I is independent of the bookkeeping partition, apart from numerical integration error. The upper product B_m is not: refinement can increase it systematically. Therefore, B_m is an integration error-control bound only; it is never inserted into Equation (65). A convergence report must show separately that the composed propagator U_I converges as the integration mesh is refined and how the auxiliary product B_m behaves.

4.3. Background-Normalized Peculiar Propagator

Raw comoving propagation includes the common Hubble block and can have an induced norm larger than one even when the peculiar Hessian vanishes. To separate this background exactly, write

$$\begin{aligned} &L_{\text{com}}(t) = L_{\text{bg}}(t) + \Delta L(t), \\ &L_{\text{bg}}(t) = \begin{pmatrix} 0 & \tau_*^{-1} I \\ 0 & -2H(t) I \end{pmatrix}, \quad \Delta L(t) = \begin{pmatrix} 0 & 0 \\ -\tau_* K_{\text{pec}}(t) & 0 \end{pmatrix}. \end{aligned} \tag{67}$$

Let $U_{\text{bg}}(t,t_0)$ and $U(t,t_0)$ be the background and full comoving propagators, respectively, with the same $a(t)$ and $H(t)$. Define the interaction-picture propagator

$$R(t,t_0) = U_{\text{bg}}(t,t_0)^{-1} U(t,t_0). \tag{68}$$

It obeys

$$\dot{R} = B(t) R, \quad B(t) = U_{\text{bg}}(t,t_0)^{-1} \Delta L(t) U_{\text{bg}}(t,t_0), \quad R(t_0,t_0) = I. \tag{69}$$

This is an exact change of variables and retains all noncommutativity between the background and peculiar generators.

For each maximal report interval $I_k = [s_k, s_{k+1}]$, two exact background-normalized operators are useful. With one common anchor t_0, define the globally composable factor; with only the interval endpoints, define the local two-point factor:

$$\begin{aligned} &R_k^{\text{ca}} = R(s_{k+1},t_0) R(s_k,t_0)^{-1}, \qquad G_k^{\text{ca}}(W) = \left\| R_k^{\text{ca}} \right\|_W, \\ &\hat{R}_k = U_{\text{bg}}(s_{k+1},s_k)^{-1} U(s_{k+1},s_k), \quad \hat{G}_k(W) = \left\| \hat{R}_k \right\|_W. \end{aligned} \tag{70}$$

Let $B_k = U_{\text{bg}}(s_k,t_0)$. Exact propagator composition gives the similarity identity

$$R_k^{\text{ca}} = B_k^{-1} \hat{R}_k B_k.$$

Thus, R_k^{ca} and $\hat{R}_k$ have the same eigenvalues, but their induced norms in one fixed metric generally differ because B_k is not orthogonal. In particular, $G_k^{\text{ca}}(W)$ depends on the common anchor t_0 (equivalently, on the reported start redshift and preceding background transport) and is not a purely local property of I_k.

The dependence can be written exactly as a transported-metric identity. With

$$W_k = B_k^{-\text{T}} W B_k^{-1},$$

one has

$$G_k^{\mathrm{ca}}(W) = \left\|\hat{R}_k\right\|_{W_k}.$$

The anchor-free quantity $\hat{G}_k(W)$ instead uses the same fixed metric at the two endpoints of the segment. It is suitable for local interval comparisons, but the factors $\hat{R}_k$ do not telescope as a simple product without the omitted background conjugations.

The common-anchor factors do telescope:

$$R(T,t_0) = R_{K-1}^{\mathrm{ca}} \cdots R_0^{\mathrm{ca}},$$
$$\frac{\left\|R(T,t_0)y_0\right\|_W}{\left\|y_0\right\|_W} \le \left\|R(T,t_0)\right\|_W \le \prod_{k=0}^{K-1} G_k^{\mathrm{ca}}(W). \tag{71}$$

When $K_{\mathrm{pec}} = 0$, both R_k^{ca} and $\hat{R}_k$ are the identity, even though $\left\|U_{\mathrm{bg}}\right\|_W$ may exceed one because coordinate and velocity components shear in the chosen full-state norm. Background normalization, therefore, removes the homogeneous dynamics, but it does not make a common-anchor interval norm intrinsic or a morphology label causal. Every report must state t_0; local morphology comparisons should also report $\hat{G}_k(W)$ or, equivalently, the transported metric W_k used to interpret $G_k^{\mathrm{ca}}(W)$.

4.4. Structural Restrictions on Full-State Operator Gains

For the conservative physical system, Equation (45) shows that the canonical propagator is symplectic. Liouville's formula gives $\det U_k = 1$, and the singular values of $W^{1/2}U_kW^{-1/2}$ have product one. Hence

$$G_k^{\mathrm{raw}}(W) \ge 1 \tag{72}$$

for every fixed positive-definite full-state metric. A conservative physical interval cannot be a strict operator contraction in all phase-space directions, although a particular state or coordinate projection can decrease. For noncanonical comoving variables,

$$\det U_{\mathrm{com}}(t_2,t_1) = \exp\left[-6\int_{t_1}^{t_2} H(t)\,\mathrm{d}t\right]. \tag{73}$$

The weighted Wronskian and the canonical system (47) show that this determinant reflects the time-dependent momentum scaling.

4.5. Differential Matrix-Measure Bound

For a continuously varying generator, define the logarithmic norm in the fixed metric W by

$$\mu_W(L) = \frac{1}{2}\lambda_{\max}\left[W^{-1/2}\left(L^{\mathrm{T}}W + WL\right)W^{-1/2}\right]. \tag{74}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\left\|z(t)\right\|_W \le \mu_W\left(L(t)\right),\ \log\frac{\left\|z(T)\right\|_W}{\left\|z(0)\right\|_W} \le \int_0^T \mu_W\left(L(t)\right)\mathrm{d}t. \tag{75}$$

The same formula applies to the interaction generator $B(t)$ for background-normalized evolution. The logarithmic norm is a standard metric-dependent tool for differential error and growth bounds [34] [39] [40]. Its integral is independent of a numerical output partition, although it may be looser than the norm of the exact composed propagator.

4.6. Lower-Rank Observables

Let $P \in \mathbb{R}^{k\times 6}$ be a fixed observation matrix and $A_P(t) = \|Pz(t)\|_2$. If P has a nontrivial kernel, $A_P(T)/A_P(0)$ need not be finite because an initially unobserved component can rotate into the observed subspace. The generally valid estimate is

$$A_P(T) \le \left\|PU(T,0)W^{-1/2}\right\|_2 \left\|z(0)\right\|_W . \tag{76}$$

For a fixed propagator U, a uniform finite ratio relative to $A_P(0)$ requires $\ker P \subseteq \ker(PU)$. Sufficient interval-wise alternatives are invariance of $\ker P$, restriction of initial data to an invariant subspace S on which $P|_S$ is injective, or an explicit observability inequality. The same statements hold with U replaced by the background-normalized propagator R.

4.7. When a Genuine Attenuation Closure Is Added

A reduced non-Hamiltonian closure can produce a negative operator log-gain only after it is specified for the same state and metric. If a common $W > 0$ satisfies

$$L_{\mathrm{red}}^{\mathrm{T}} W + W L_{\mathrm{red}} \le -2\alpha W, \ \alpha > 0, \tag{77}$$

throughout an interval, then

$$\left\|U_{\mathrm{red}}(t_2,t_1)\right\|_W \le \mathrm{e}^{-\alpha(t_2-t_1)}. \tag{78}$$

If different intervals use different Lyapunov matrices, transition factors between the metrics must be included. The principal results of this paper do not assume phenomenological wall or filament attenuation.

5. Controlled Three-Dimensional Benchmark and Protocol Sensitivities

The purpose of this section is algebraic verification, not cosmological validation. All matrices, time intervals, initial conditions, and evaluation rules are stated explicitly in the manuscript. Unlike a simultaneous-diagonal toy route, the benchmark uses changing eigenframes, nonzero components in all six state directions, a classifier independent of the dynamical Hessian, strong morphology-Hessian misalignment, and a near-neutral spectral pair.

5.1. Definition of the Route

Use dimensionless comoving variables with $\tau_* = 1$, $W = I_6$ and constant

$H = 0.12$. For angles in degrees, define

$$Q(\alpha,\beta,\gamma) = R_z(\gamma) R_y(\beta) R_x(\alpha), \tag{79}$$

where R_x, R_y, R_z are the standard right-handed Cartesian rotations. On report interval I_j, let

$$K_j = Q_j D_j Q_j^{\mathrm{T}},\ L_j = \begin{pmatrix} 0 & I \\ -K_j & -2HI \end{pmatrix},\ U_j = \mathrm{e}^{L_j \Delta t_j}. \tag{80}$$

The dynamical data are listed in Table 2.

Table 2. Dynamical Hessians for the three-dimensional benchmark. The angles specify $Q_j = Q(\alpha_j, \beta_j, \gamma_j)$ in degrees.

Label	Δt_j	diag D_j	$(\alpha_j, \beta_j, \gamma_j)$
Void	0.30	$(-0.80, -0.22, 0.15)$	$(10, -15, 20)$
Wall	0.22	$(0.90, -0.16, -0.04)$	$(35, 10, -25)$
Filament	0.26	$(0.55, 0.012, -0.008)$	$(-20, 40, 15)$
Node	0.20	$(0.65, 0.25, 0.06)$	$(50, -30, 5)$

The classifier tensors are constructed independently:

$$T_j = S_j C_j S_j^{\mathrm{T}},\ S_j = Q(\hat{\alpha}_j, \hat{\beta}_j, \hat{\gamma}_j). \tag{81}$$

Their ordered eigenvalues, rotations, and zero-threshold labels are given in Table 3. The last column is the confidence margin from Equation (29).

Table 3. Independent classifier tensors. The environment follows only from the number N_+ of eigenvalues above the zero threshold.

Label	(τ_1, τ_2, τ_3)	$(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$	N_+	d_{class}
Void	$(-0.08, -0.25, -0.70)$	$(-5, 20, 12)$	0	0.080
Wall	$(0.50, -0.05, -0.20)$	$(20, -25, 30)$	1	0.050
Filament	$(0.45, 0.06, -0.03)$	$(45, 5, -20)$	2	0.030
Node	$(0.40, 0.18, 0.025)$	$(-30, 15, 35)$	3	0.025

The initial state has no zero component:

$$z_0 = \frac{1}{\sqrt{2.50}} (1, -0.6, 0.4, 0.3, -0.5, 0.8)^{\mathrm{T}}. \tag{82}$$

For the interaction-picture calculation, U_{bg} is generated by Equation (67); the common-anchor interval propagators R_j^{ca} are defined by Equation (70) with $t_0 = 0$. Let $y_j = U_{\text{bg}}(s_j, 0)^{-1} z(s_j)$.

5.2. Raw, Common-Anchor and Local Background-Normalized Results

Table 4 reports the raw comoving gains and the background-normalized common-anchor gains. The selected-state columns are distinct from operator norms. For wall and filament intervals, the final column gives

$$c_j = \frac{\left\| \left[P_{m,j}, K_j \right] \right\|_2}{\left\| K_j \right\|_2}, \tag{83}$$

where $P_{m,j}$ is the classifier wall-normal or filament-transverse projector. The large values show that the corresponding morphology-local quadratic energies are not exact in this route; the full propagator must be used.

Table 4. Three-dimensional benchmark results. “Raw state” is $\left\| z_{j+1} \right\|_2 / \left\| z_j \right\|_2$; “CA state” is the common-anchor interaction-state ratio $\left\| y_{j+1} \right\|_2 / \left\| y_j \right\|_2$.

Label	G_j^{raw}	G_j^{ca}	Raw State	CA State	c_j
Void	1.26662	1.13601	1.13667	1.02751	–
Wall	1.10929	1.13252	1.04835	1.00787	0.46419
Filament	1.10864	1.11942	1.07769	1.00698	0.48873
Node	1.07555	1.13870	1.02781	1.04062	0

For the complete route,

$$\prod_j G_j^{\text{raw}} = 1.67537,\ \left\| U_3 U_2 U_1 U_0 \right\|_2 = 1.55892,\ \frac{\left\| z_4 \right\|_2}{\left\| z_0 \right\|_2} = 1.31990, \tag{84}$$

$$\prod_j G_j^{\text{ca}} = 1.63995,\ \left\| R_3^{\text{ca}} R_2^{\text{ca}} R_1^{\text{ca}} R_0^{\text{ca}} \right\|_2 = 1.26201,\ \frac{\left\| y_4 \right\|_2}{\left\| y_0 \right\|_2} = 1.08519. \tag{85}$$

Hence, both verified chains have the required ordering:

$$1.31990 \le 1.55892 \le 1.67537,\quad 1.08519 \le 1.26201 \le 1.63995. \tag{86}$$

Table 5 quantifies the distinction between the common-anchor gain in the fixed metric and the local two-point gain for the same segment. The difference grows along the route because the common-anchor factor is conjugated by a longer background prefix.

Table 5. Common-anchor versus local two-point background-normalized gains for the same benchmark segments. The last column is $100\left(G_j^{\text{ca}} / \hat{G}_j - 1 \right)$.

Label	$G_j^{\text{ca}}(W)$	$\hat{G}_j(W)$	Difference (%)
Void	1.13600598	1.13600598	0.00
Wall	1.13251885	1.10690477	2.31
Filament	1.11942273	1.07690204	3.95
Node	1.13870442	1.06896638	6.52

The raw and common-anchor results differ substantially because the former includes the Hubble background. For a pure-background interval of length 0.30

with $K_{\text{pec}} = 0$, the raw operator gain in this metric is 1.12489, whereas both background-normalized propagators are exactly the identity and have a gain of one.

The benchmark is intentionally not aligned: the classifier wall normal and the positive spectral line of K_1 have a principal angle of about 64.0˚; the maximum principal angle between the classifier filament plane and the positive spectral plane of K_2 is about 85.3˚. This checks that the method does not infer exact local energies from the morphology label when the required commutator condition fails.

5.3. Partition Study

The partition issue can be isolated with the frozen physical wall segment

$$K_{\text{p}} = \operatorname{diag}\left(4, -0.15, -0.05\right),\ \Delta t = 0.20,\ W = I_6. \tag{87}$$

Let $U\left(\Delta t\right) = \mathrm{e}^{L\left(K_{\text{p}}\right)\Delta t}$ and split the same interval into m equal exact substeps. Table 6 distinguishes the norm of the composed propagator from the product of substep norms.

Table 6. Partition study for the same wall interval. The exact composed gain is unchanged; only the auxiliary product bound depends on substepping.

Substeps m	$\left\|U\left(\Delta t/m\right)\right\|_2$	$\left\|U\left(\Delta t/m\right)^m\right\|_2$	$\left\|U\left(\Delta t/m\right)\right\|_2^m$
1	1.33384168	1.33384168	1.33384168
2	1.16004186	1.33384168	1.34569711
4	1.07767428	1.33384168	1.34880782
8	1.03818667	1.33384168	1.34959539
16	1.01892378	1.33384168	1.34979291

The product bound approaches $\exp\left[\mu_2\left(L\right)\Delta t\right] = \exp\left(0.30\right) = 1.34985881$, whereas the exact interval gain remains 1.33384168. This is why only the composed maximal-interval gain is used in Equation (65).

5.4. Metric, Threshold, and Inertia-Tolerance Sensitivity

The same wall propagator has different Euclidean gains after changing only the configuration-velocity scale:

$$G_{\text{p}}\left(\tau_* = 0.5\right) = 1.22905,\ G_{\text{p}}\left(1\right) = 1.33384,\ G_{\text{p}}\left(2\right) = 1.96836. \tag{88}$$

The physical trajectory is unchanged; only the reported full-state metric has changed. This numerical spread motivates the fixed scale Equations (18) - (20) and the rule that alternative τ_* values are reported only as sensitivity checks.

The classifier and inertia tolerances are equally consequential near zero. For the tensors in Table 3, thresholds $\tau_{\text{th}} = -0.02$ and 0 give the sequence void-wall-filament-node, whereas $\tau_{\text{th}} = 0.03$ changes the last label to filament because its smallest classifier eigenvalue is 0.025. For the near-neutral dynamical spectrum $\left(0.55, 0.012, -0.008\right)$, the inertia $\left(n_-, n_0, n_+\right)$ is

$$
\begin{aligned}
(n_-, n_0, n_+) &= (1,0,2) \quad \text{for } \kappa_{\mathrm{tol}} = 0.005, \\
&= (0,1,2) \quad \text{for } \kappa_{\mathrm{tol}} = 0.010, \\
&= (0,2,1) \quad \text{for } \kappa_{\mathrm{tol}} = 0.015.
\end{aligned} \tag{89}
$$

A real application must therefore report threshold and tolerance margins and must not attach a sharp physical interpretation to unresolved intervals.

5.5. Rotating-Cell Error Check

To retain a nonzero test of the frozen-cell estimate, consider

$$K_w(t) = R_{12}(0.2t)\operatorname{diag}(4, -0.15, -0.05) R_{12}(0.2t)^{\mathrm{T}}, \quad 0 \le t \le 0.2, \tag{90}$$

where R_{12} rotates the first two coordinates. Direct integration of the laboratory-frame 6×6 propagator gives

$$\left\| U_w - \mathrm{e}^{L_w(0)0.2} \right\|_2 = 0.01658. \tag{91}$$

For the same interval, $\varepsilon = 0.16596$ and $M = \left\| L_w(0) \right\|_2 = 4$, so Equation (35) gives

$$0.01658 \le 0.07387. \tag{92}$$

The test verifies the stated error estimate; it does not substitute for finite-difference validation on cosmological trajectories.

6. Interpretation of the Cosmological Constant

At a fixed matter Hessian in physical coordinates,

$$K_{\mathrm{phys}} = D_x^2 \Phi_0 - \frac{\Lambda c^2}{3} I, \quad \kappa_i^{\mathrm{phys}} = \kappa_i^{(\Phi)} - \frac{\Lambda c^2}{3}. \tag{93}$$

A positive Λ, therefore, moves every physical-separation curvature downward. For a frozen negative physical curvature, the modal exponent is $\mu_{\mathrm{phys}} = \sqrt{-\kappa^{\mathrm{phys}}}$ and

$$\left. \frac{\partial \mu_{\mathrm{phys}}}{\partial \Lambda} \right|_{D^2 \Phi_0} = \frac{c^2}{6 \mu_{\mathrm{phys}}} > 0. \tag{94}$$

Positive wall-normal or filament-transverse physical curvatures are reduced by the same shift. These statements are local parametric sensitivities at a fixed matter Hessian. They do not compare fully evolved cosmologies at fixed comoving initial data.

Peculiar deformation is governed by Equation (11). For a frozen scalar, peculiar curvature κ_{pec} and constant H, the characteristic exponents are

$$r_{\pm} = -H \pm \sqrt{H^2 - \kappa_{\mathrm{pec}}}. \tag{95}$$

If $\kappa_{\mathrm{pec}} < 0$, the growing peculiar exponent is $-H + \sqrt{H^2 + |\kappa_{\mathrm{pec}}|}$. If $\kappa_{\mathrm{pec}} = 0$, the largest exponent is zero, reproducing the constant comoving mode of pure de Sitter space. In this representation, Λ acts through the background functions $a(t)$ and $H(t)$ and through the evolved peculiar Hessian. Consequently, a claim about internal cosmic-web deformation should be based on the comoving propagator, while Equation (93) describes finite-time physical separation at fixed matter Hessian.

7. Validation Requirements, Uncertainty, and Limitations

The completed result is a formal tangent-flow construction with controlled synthetic checks. It is not an empirical demonstration that T-web morphology predicts finite-time amplification. A cosmological validation must keep integration accuracy, reporting conventions, and scientific inference separate.

A minimum validation workflow is:

(V1) Fix the analysis redshift, R_s, grid spacing, force softening, boundary convention, Poisson solver, Hessian estimator, interpolation rule, τ_{th}, δ_{class}, κ_{tol} and frozen-cell tolerance before examining the gains.

(V2) Use the scaling convention Equations (18) - (20). The common length factor cancels from induced gains; R_s is reported because it changes the smoothed Hessian and labels, while τ_* fixes the relative position-velocity weighting. The primary report uses $W = I_6$ in the scaled state. Alternative τ_* or nonscalar W values belong to a declared sensitivity analysis.

(V3) Construct maximal contiguous morphology intervals. Integrate on as fine a mesh as needed, but compose all substeps before evaluating an interval gain. Report convergence of the composed propagator and, separately, the looser substep product bound.

(V4) Reconstruct the complete Hessian along each trajectory. Record classifier margins, Hessian inertia with tolerance, spectral gaps, the frozen-cell error estimate, and morphology-Hessian commutators. Exact wall-normal or filament-transverse energies are used only when the relevant projector is invariant.

(V5) Integrate the full tangent propagator and the background-normalized interaction propagator. Report the common anchor t_0 (or start redshift), the globally composable gains G_k^{ca}, and the local two-point gains $\hat{G}_k$ or their transported metrics W_k. Compare the tangent prediction with finite differences between independently evolved nearby trajectories over a range of initial separations that demonstrates the linear regime.

(V6) On a calibration sample, estimate conditional distributions such as $p\left(\log \hat{G}_k \mid m_k = \alpha\right)$ and their dependence on residence time, redshift, and local Hessian invariants. Use the local two-point gain for segment comparisons, and use G_k^{ca} only when the globally composable chain is the stated target. Do not fit and test a non-Hamiltonian closure on the same trajectories.

(V7) On a disjoint validation sample, compare morphology-conditioned statistics with a shuffled-label baseline. One possible effect statistic is

$$\Delta_\alpha = \mathbb{E}\left[\log \hat{G}_k \mid m_k = \alpha\right] - \mathbb{E}_{\mathrm{shuffle}}\left[\log \hat{G}_k \mid m_k = \alpha\right]. \tag{96}$$

Confidence intervals must include trajectory-to-trajectory scatter and correlations between adjacent intervals.

(V8) Repeat the analysis across a sensitivity grid in

$$\theta = \left(R_s, \tau_{\text{th}}, \Delta x, \epsilon_{\text{soft}}, \text{Hessian estimator}, \text{boundary rule}, \kappa_{\text{tol}}, \epsilon_{\text{freeze}}, \tau_*, t_0\right). \quad (97)$$

and separate numerical uncertainty from physical scatter. Label stability, projector stability, propagator convergence, and the final conditional statistics must all be reported.

The synthetic benchmark addresses only a subset of these requirements: three-dimensional mode mixing, morphology-Hessian misalignment, a near-zero spectral pair, background normalization, partition convergence, metric sensitivity, and frozen-cell error. It contains no N-body or Vlasov field, no force-resolution study, and no shuffled-label inference. Consequently, the paper does not claim that the T-web labels add predictive power beyond the complete Hessian or that the resulting labeled sums are causal properties of voids, walls, filaments, or nodes.

Other limitations remain explicit. The model treats prescribed-field characteristics, not the self-consistent system (8) and (9). The local inertia of K_{phys} is a frozen physical-coordinate signature, not a finite-time stability theorem for a varying cell. Comoving behaviour is controlled by the full generator or propagator. Lower-rank observables require a full-state normalization or an observability condition. Finally, the local physical Λ shift and the background-normalized peculiar evolution answer different questions and must not be mixed.

Within these restrictions, the construction can be applied to tangent propagators already produced by geodesic-deviation, phase-space-distortion, N-body, or Vlasov calculations. Its output is a reproducible formal report, provided the metric, segmentation, common anchor, local relative convention, background normalization, and tolerances are fixed in advance.

8. Conclusions

The phrase *reduced kinetic stability* is used in the finite-time characteristic-flow sense defined in Section 2. "Kinetic" refers to the phase-space transport that carries the Vlasov distribution, and "stability" covers local stable, unstable, and neutral sectors, invariant boundedness, and finite-time tangent amplification. The term does not claim spectral or nonlinear stability of a self-consistent Vlasov-Poisson solution.

The conclusions supported by the analysis are:

1) Six-dimensional geodesic-deviation and phase-space-distortion integration for dark matter already exists. The contribution here is not a new tangent equation or a new submultiplicativity theorem, but a constrained morphology-labeled reporting protocol for an available propagator.

2) Physical and comoving separations are related by $\xi = a\eta$. Physical growth includes the background scale factor. Peculiar deformation is described by the comoving generator and can be isolated from the homogeneous background through the interaction propagator $R = U_{\text{bg}}^{-1}U$.

3) Density fixes only the Hessian trace. The T-web label and the inertia of the complete effective Hessian are different objects. For a frozen conservative physi-

cal-coordinate cell, the inertia of K_{phys} gives a basis-independent count of hyperbolic, neutral and elliptic configuration directions. Comoving classification requires L_{com} or its propagator.

4) Exact frozen-cell quadratic invariants live on spectral subspaces of the complete Hessian. A morphology-defined wall normal or filament transverse plane inherits such an invariant only when its projector commutes with the full Hessian. Time-dependent cells are controlled by the full propagator, spectral-gap diagnostics and the explicit frozen-cell error estimate.

5) The smoothing length R_s fixes the field reconstruction and therefore the Hessian and morphology, but its common scalar length factor cancels from the induced norm. The relative position-velocity weighting is fixed by the reference time scale τ_* and any nonscalar metric W. Gains obtained with different τ_* or W are different reported quantities are not compared without an explicit sensitivity analysis.

6) Numerical substeps and morphology report intervals are distinct. A reported interval is a maximal connected region of one resolved label, and its gain is computed only after composing every integration substep. Products of substep norms are auxiliary upper bounds and generally depend on the partition.

7) The finite-time propagator inequality is mathematically valid but label-blind. The quantities $\Gamma_\alpha^{\text{lab}}$ are therefore residence-time-labeled bookkeeping components, not invariant or causal environmental contributions. Morphological predictive content requires conditional statistics, shuffled-label controls, and independent validation data.

8) Raw comoving gains include Hubble evolution. The interaction picture removes the $K_{\text{pec}} = 0$ background exactly while retaining noncommutativity. Common-anchor factors telescope globally, but their induced norms depend on the declared t_0; local two-point factors are anchor-free and are related to the common-anchor factors by a transported metric. Both the anchor and the chosen interval convention must be reported.

9) The three-dimensional benchmark verifies raw and common-anchor product bounds with changing eigenframes, full six-component initial data, classifier-Hessian misalignment, and a near-neutral spectral pair. The partition, metric, threshold, and inertia-tolerance studies demonstrate why the reporting conventions must be fixed.

10) For a conservative physical symplectic propagator, a negative full-state induced operator log-gain in a fixed positive-definite metric is excluded, although a particular state or lower-rank observable may decrease through phase focusing. Any genuine operator contraction requires the actual comoving propagator or an explicitly specified non-Hamiltonian closure.

11) At fixed matter Hessian, positive Λ shifts the local physical curvature spectrum by $-\Lambda c^2 I/3$. This is a local parametric sensitivity, not a stand-alone prediction for the evolution or stability of the cosmic web in different cosmologies.

No N-body or Vlasov validation is presented. Accordingly, the result should be read as a formal reduced kinetic stability construction and a reproducible validation protocol for morphology-labeled characteristic tangent flow, not as an empirically established cosmic-web diagnostic. This restricted meaning is the one intended by the unchanged title.

Statement

The author used AI-assisted language-editing tools during preparation of the manuscript. The scientific content, mathematical formulation, interpretation, and final responsibility for the text remain with the author.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Binney, J. and Tremaine, S. (2008) Galactic Dynamics. 2nd Edition, Princeton University Press.

[2] Mo, H.J., van den Bosch, F.C. and White, S.D.M. (2010) Galaxy Formation and Evolution. Cambridge University Press. https://doi.org/10.1017/cbo9780511807244

[3] Bernardeau, F., Colombi, S., Gaztañaga, E. and Scoccimarro, R. (2002) Large-Scale Structure of the Universe and Cosmological Perturbation Theory. *Physics Reports*, **367**, 1-248. https://doi.org/10.1016/s0370-1573(02)00135-7

[4] Campa, A., Dauxois, T., Fanelli, D. and Ruffo, S. (2014) Physics of Long-Range Interacting Systems. Oxford University Press.

[5] Shandarin, S., Habib, S.F. and Heitmann, K. (2012) Cosmic Web, Multistream Flows, and Tessellations. *Physical Review D*, **85**, Article 083005. https://doi.org/10.1103/physrevd.85.083005

[6] Colombi, S. (2015) Vlasov-Poisson in 1D for Initially Cold Systems: Post-Collapse Lagrangian Perturbation Theory. *Monthly Notices of the Royal Astronomical Society*, **446**, 2902-2920. https://doi.org/10.1093/mnras/stu2308

[7] Rampf, C. (2021) Cosmological Vlasov-Poisson Equations for Dark Matter: Recent Developments and Connections to Selected Plasma Problems. *Reviews of Modern Plasma Physics*, **5**, Article No. 10. https://doi.org/10.1007/s41614-021-00055-z

[8] Vogelsberger, M., White, S.D.M., Helmi, A. and Springel, V. (2008) The Fine-Grained Phase-Space Structure of Cold Dark Matter Haloes. *Monthly Notices of the Royal Astronomical Society*, **385**, 236-254. https://doi.org/10.1111/j.1365-2966.2007.12746.x

[9] White, S.D.M. and Vogelsberger, M. (2009) Dark Matter Caustics. *Monthly Notices of the Royal Astronomical Society*, **392**, 281-286. https://doi.org/10.1111/j.1365-2966.2008.14038.x

[10] Vogelsberger, M. and White, S.D.M. (2011) Streams and Caustics: The Fine-Grained Structure of Λ Cold Dark Matter Haloes. *Monthly Notices of the Royal Astronomical Society*, **413**, 1419-1438. https://doi.org/10.1111/j.1365-2966.2011.18224.x

[11] Abel, T., Hahn, O. and Kaehler, R. (2012) Tracing the Dark Matter Sheet in Phase Space. *Monthly Notices of the Royal Astronomical Society*, **427**, 61-76. https://doi.org/10.1111/j.1365-2966.2012.21754.x

[12] Hahn, O., Abel, T. and Kaehler, R. (2013) A New Approach to Simulating Collision-

less Dark Matter Fluids. *Monthly Notices of the Royal Astronomical Society*, **434**, 1171-1191. https://doi.org/10.1093/mnras/stt1061

[13] Haller, G. (2001) Distinguished Material Surfaces and Coherent Structures in Three-Dimensional Fluid Flows. *Physica D: Nonlinear Phenomena*, **149**, 248-277. https://doi.org/10.1016/s0167-2789(00)00199-8

[14] Shadden, S.C., Lekien, F. and Marsden, J.E. (2005) Definition and Properties of Lagrangian Coherent Structures from Finite-Time Lyapunov Exponents in Two-Dimensional Aperiodic Flows. *Physica D: Nonlinear Phenomena*, **212**, 271-304. https://doi.org/10.1016/j.physd.2005.10.007

[15] Haller, G. (2015) Lagrangian Coherent Structures. *Annual Review of Fluid Mechanics*, **47**, 137-162. https://doi.org/10.1146/annurev-fluid-010313-141322

[16] Libeskind, N.I., van de Weygaert, R., Cautun, M., Falck, B., Tempel, E., Abel, T., *et al.* (2017) Tracing the Cosmic Web. *Monthly Notices of the Royal Astronomical Society*, **473**, 1195-1217. https://doi.org/10.1093/mnras/stx1976

[17] Hahn, O., Porciani, C., Carollo, C.M. and Dekel, A. (2007) Properties of Dark Matter Haloes in Clusters, Filaments, Sheets and Voids. *Monthly Notices of the Royal Astronomical Society*, **375**, 489-499. https://doi.org/10.1111/j.1365-2966.2006.11318.x

[18] Forero-Romero, J.E., Hoffman, Y., Gottlöber, S., Klypin, A. and Yepes, G. (2009) A Dynamical Classification of the Cosmic Web. *Monthly Notices of the Royal Astronomical Society*, **396**, 1815-1824. https://doi.org/10.1111/j.1365-2966.2009.14885.x

[19] Feldbrugge, J., de Weygaert, R.v., Hidding, J. and Feldbrugge, J. (2018) Caustic Skeleton & Cosmic Web. *Journal of Cosmology and Astroparticle Physics*, **2018**, Article 027. https://doi.org/10.1088/1475-7516/2018/05/027

[20] Cautun, M., van de Weygaert, R., Jones, B.J.T. and Frenk, C.S. (2014) Evolution of the Cosmic Web. *Monthly Notices of the Royal Astronomical Society*, **441**, 2923-2973. https://doi.org/10.1093/mnras/stu768

[21] Mao, Q., Berlind, A.A., Scherrer, R.J., Neyrinck, M.C., Scoccimarro, R., Tinker, J.L., *et al.* (2017) A Cosmic Void Catalog of SDSS DR12 BOSS Galaxies. *The Astrophysical Journal*, **835**, Article 161. https://doi.org/10.3847/1538-4357/835/2/161

[22] Malavasi, N., Aghanim, N., Douspis, M., Tanimura, H. and Bonjean, V. (2020) Characterising Filaments in the SDSS Volume from the Galaxy Distribution. *Astronomy & Astrophysics*, **642**, A19. https://doi.org/10.1051/0004-6361/202037647

[23] DESI Collaboration (2026) Data Release 1 of the Dark Energy Spectroscopic Instrument. *The Astronomical Journal*, **171**, Article 285.

[24] Aussel, H., Tereno, I., Schirmer, M. and Alguero, G. (2026) Euclid Quick Data Release (Q1): I. Data Release Overview. *Astronomy & Astrophysics*, **711**, A1.

[25] Springel, V. (2005) The Cosmological Simulation Code Gadget-2. *Monthly Notices of the Royal Astronomical Society*, **364**, 1105-1134. https://doi.org/10.1111/j.1365-2966.2005.09655.x

[26] Vogelsberger, M., Marinacci, F., Torrey, P. and Puchwein, E. (2020) Cosmological Simulations of Galaxy Formation. *Nature Reviews Physics*, **2**, 42-66. https://doi.org/10.1038/s42254-019-0127-2

[27] Nelson, D., Springel, V., Pillepich, A., Rodriguez-Gomez, V., Torrey, P., Genel, S., *et al.* (2019) The Illustristng Simulations: Public Data Release. *Computational Astrophysics and Cosmology*, **6**, Article No. 2. https://doi.org/10.1186/s40668-019-0028-x

[28] Peebles, P.J.E. (1980) The Large-Scale Structure of the Universe. Princeton University Press.

[29] Lahav, O., Lilje, P.B., Primack, J.R. and Rees, M.J. (1991) Dynamical Effects of the

Cosmological Constant. *Monthly Notices of the Royal Astronomical Society*, **251**, 128-136. https://doi.org/10.1093/mnras/251.1.128

[30] Planck Collaboration (2020) Planck 2018 Results. VI. Cosmological Parameters. *Astronomy & Astrophysics*, **641**, A6. https://doi.org/10.1051/0004-6361/202039265

[31] Planck Collaboration (2021) Planck 2018 Results. VI. Cosmological Parameters (Corrigendum). *Astronomy & Astrophysics*, **652**, C4.

[32] Coppel, W.A. (1978) Dichotomies in Stability Theory. Lecture Notes in Mathematics. Vol. 629, Springer.

[33] Davis, C. and Kahan, W.M. (1970) The Rotation of Eigenvectors by a Perturbation. Iii. *SIAM Journal on Numerical Analysis*, **7**, 1-46. https://doi.org/10.1137/0707001

[34] Horn, R.A. and Johnson, C.R. (2013) Matrix Analysis. 2nd Edition, Cambridge University Press.

[35] Arnold, V.I. (1989) Mathematical Methods of Classical Mechanics. 2nd Edition, Springer.

[36] Meyer, K.R., Hall, G.R. and Offin, D. (2009) Introduction to Hamiltonian Dynamical Systems and the N-Body Problem. 2nd Edition, Springer.

[37] Chetaev, N.G. (1961) The Stability of Motion. Pergamon Press.

[38] Lyapunov, A.M. (1992) The General Problem of the Stability of Motion. *International Journal of Control*, **55**, 531-534. https://doi.org/10.1080/00207179208934253

[39] Söderlind, G. (2006) The Logarithmic Norm. History and Modern Theory. *BIT Numerical Mathematics*, **46**, 631-652. https://doi.org/10.1007/s10543-006-0069-9

[40] Higham, N.J. (2008) Functions of Matrices. Society for Industrial and Applied Mathematics. https://doi.org/10.1137/1.9780898717778