Journal of Applied Mathematics and Physics, 2026, 14(8), 3089-3099
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

An Angular Reformulation of the Kronecker Delta through a Jeno-Type Structural Nullity

—Gram-Schmidt Orthogonalization and a Sliding-Ladder Application

Oscar Mauricio Jeno Soto

Department of Natural Sciences (Physics), Liceo Federico Albert Faupp, Chanco, Chile
Email: oscar.jeno@liceofedericoalbert.cl

How to cite this paper: Soto, O.M.J. (2026) An Angular Reformulation of the Kronecker Delta through a Jeno-Type Structural Nullity. *Journal of Applied Mathematics and Physics*, **14**, 3089-3099.
https://doi.org/10.4236/jamp.2026.148151

Received: July 19, 2026
Accepted: August 21, 2026
Published: August 24, 2026

Abstract

A geometric reformulation of the Kronecker delta is proposed for finite-dimensional real inner-product spaces equipped with normalized basis vectors. The construction separates index identity, represented by δ_{ij}, from off-diagonal angular coupling, represented by $(1-\delta_{ij})\cos\theta_{ij}$. The resulting quantity coincides with the Gram matrix for normalized bases and reduces exactly to the classical Kronecker delta for orthonormal bases. The label structural nullity distinguishes an off-diagonal pair whose metric coupling vanishes by orthogonality; it does not introduce a new arithmetic zero. Two applications are developed. First, Gram-Schmidt orthogonalization is interpreted as the systematic elimination of off-diagonal Gram couplings, quantified by the sign-independent functional $\mathcal{N}(B)=\sum_{i<j}(1-\cos^2\theta_{ij})$. Second, a frictionless sliding-ladder problem is used to show how the angular decomposition exposes the geometric source of a kinematic singularity and the physical loss-of-contact condition. For a homogeneous ladder released from an angle θ_0, contact with the wall is lost when $\sin\theta_s=(2/3)\sin\theta_0$; for release from the vertical, $y_s=2L/3$. The framework is presented as an interpretive decomposition of standard linear algebra, not as a replacement for the Kronecker delta, the Gram matrix, or classical mechanics.

Keywords

Kronecker Delta, Gram Matrix, Gram-Schmidt Orthogonalization, Sliding Ladder, Unilateral Contact, Orthogonality, Structural Nullity

1. Introduction and Scope

Throughout this work, V denotes a finite-dimensional real inner-product space.

Unless explicitly stated otherwise, every vector entering the angular construction is normalized. These assumptions are essential. In a complex inner-product space, the inner product is generally complex and cannot be represented solely by a real angle, whereas for non-normalized vectors, the standard norm factors must be retained.

For an orthonormal basis $\{e_i\}_{i=1}^{n}$,

$$\langle e_i, e_j \rangle = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{1}$$

The off-diagonal zero summarizes two logically distinct statements: the indices differ, and the corresponding vectors are orthogonal. The purpose of the present notation is to keep these mechanisms separate.

Although the Kronecker delta and the Gram matrix already provide a complete mathematical description of orthogonality, they encode index identity and metric coupling in a compact form. The present decomposition does not replace either object. Instead, it separates their logical roles, allowing the geometric source of an off-diagonal contribution to remain explicit in calculations where orthogonality is created, tested, or lost.

The Kronecker delta encodes index identity and orthonormality in linear algebra and tensor notation [1] [2]. For a general basis, inner products are collected in the Gram matrix [3] [4]. To demonstrate utility beyond elementary examples, the manuscript develops two applications: a complete Gram-Schmidt calculation in $\mathbb{R}^3$ and a constrained-mechanics analysis of a homogeneous ladder sliding without friction. The latter distinguishes a purely kinematic divergence from the physically admissible motion governed by unilateral contact.

2. Complementary Off-Diagonal Selector

Definition 2.1 (Off-diagonal selector). *Define*

$$\Delta_{ij}^{\mathrm{J}} = 1 - \delta_{ij}. \tag{2}$$

Thus, $\Delta_{ij}^{\mathrm{J}} = 0$ for $i = j$ and $\Delta_{ij}^{\mathrm{J}} = 1$ for $i \neq j$.

The quantity Δ_{ij}^{J} is not an inner product. It only selects an off-diagonal index pair, indicating where an angular coupling may occur.

3. Angular Kronecker-Jeno Quantity

Let e_i and e_j be unit vectors, and let $\theta_{ij} \in [0, \pi]$ denote their angle.

Definition 3.1 (Angular Kronecker-Jeno quantity).

$$\delta_{ij}^{\mathrm{KJ}} = \delta_{ij} + \left(1 - \delta_{ij}\right) \cos\theta_{ij}. \tag{3}$$

Equivalently, $\delta_{ii}^{\mathrm{KJ}} = 1$ and $\delta_{ij}^{\mathrm{KJ}} = \cos\theta_{ij}$ for $i \neq j$.

Proposition 3.2 (Recovery of the classical Kronecker delta). *If* $\{e_i\}$ *is orthonormal, then* $\delta_{ij}^{\mathrm{KJ}} = \delta_{ij}$.

Proof. For $i = j$, the definition gives $\delta_{ii}^{\mathrm{KJ}} = 1$. For $i \neq j$, orthogonality gives

$\theta_{ij} = \pi/2$, and therefore $\delta_{ij}^{\mathrm{KJ}} = \cos(\pi/2) = 0$. □

4. Structural Nullity

Theorem 4.1 (Structural nullity). *Let* $i \neq j$ *and let* e_i, e_j *be unit vectors satisfying* $\langle e_i, e_j \rangle = 0$. *Then*

$$\delta_{ij}^{\mathrm{KJ}} = (1 - \delta_{ij}) \cos\theta_{ij} = 1 \cdot 0 = 0_{\mathrm{J}}. \tag{4}$$

Numerically, $0_{\mathrm{J}} = 0$.

Proof. Because $i \neq j$, one has $1 - \delta_{ij} = 1$. Orthogonality implies $\theta_{ij} = \pi/2$, that the angular factor vanishes. The symbol 0_{J} records the mechanism by which the ordinary numerical zero was obtained. □

Remark 4.2. *The distinction is semantic and structural, not arithmetic. Standard algebra continues to use the ordinary real number zero. The label* 0_{J} *carries bookkeeping information only.*

5. Relation to the Gram Matrix and Metric Tensor

For

$$x = \sum_i x^i e_i, \quad y = \sum_j y^j e_j, \tag{5}$$

the inner product is

$$\langle x, y \rangle = \sum_{i,j} x^i y^j g_{ij}, \quad g_{ij} = \langle e_i, e_j \rangle. \tag{6}$$

For normalized vectors, $g_{ij} = \cos\theta_{ij}$ and $g_{ii} = 1$, so

$$\delta_{ij}^{\mathrm{KJ}} = g_{ij} = \delta_{ij} + (1 - \delta_{ij}) g_{ij}. \tag{7}$$

Thus, symmetry, positive semidefiniteness, rank, and basis dependence are inherited directly from the Gram matrix.

Define the sign-independent angular-nullity matrix by

$$N_{ii} = 0, \ N_{ij} = 1 - g_{ij}^2 = \sin^2\theta_{ij}, \ i \neq j. \tag{8}$$

Unlike $1 - \cos\theta_{ij}$, this quantity is invariant under reversal of either vector and reaches its maximum precisely at orthogonality.

6. Restriction to Normalized Real Vectors

For arbitrary nonzero vectors,

$$\langle e_i, e_j \rangle = \|e_i\| \|e_j\| \cos\theta_{ij}. \tag{9}$$

Consequently, the angular definition does not reproduce the full Gram matrix unless $\|e_i\| = 1$ for every i. Restoring the norm factors gives the standard inner product. In particular, for $i \neq j$,

$$\langle e_i, e_j \rangle = \|e_i\| \|e_j\| \delta_{ij}^{\mathrm{KJ}}. \tag{10}$$

The proposal, therefore, does not replace the ordinary inner product. In complex spaces, a real angle alone also loses phase information; a direct extension would require a phase-sensitive construction and lies outside the present scope.

7. Gram-Schmidt Orthogonalization

The Gram-Schmidt process and its numerically stable variants are standard tools in matrix computation and numerical linear algebra [5] [6]. Let $B=(v_1,\cdots,v_n)$ be linearly independent. Gram-Schmidt constructs

$$w_1=v_1, \tag{11}$$

$$w_k=v_k-\sum_{i=1}^{k-1}\operatorname{proj}_{w_i}(v_k), \tag{12}$$

where

$$\operatorname{proj}_{w_i}(v_k)=\frac{\langle v_k,w_i\rangle}{\|w_i\|^2}w_i. \tag{13}$$

If $q_i=w_i/\|w_i\|$, then

$$\operatorname{proj}_{q_i}(v_k)=\|v_k\|\cos\theta_{ik}q_i. \tag{14}$$

Thus, the coefficient removed at each step is exactly the off-diagonal angular coupling with a previously normalized direction.

For a normalized ordered family $B=(b_1,\cdots,b_n)$, define

$$\mathcal{N}(B)=\sum_{1\le i<j\le n}\left(1-\cos^2\theta_{ij}\right)=\sum_{1\le i<j\le n}\sin^2\theta_{ij}. \tag{15}$$

Proposition 7.1. *For normalized vectors,*

$$0\le\mathcal{N}(B)\le\binom{n}{2}, \tag{16}$$

and $\mathcal{N}(B)=\binom{n}{2}$ *if and only if the family is pairwise orthogonal.*

Proof. Every term belongs to $[0,1]$. The upper bound is reached exactly when $\cos\theta_{ij}=0$ for every pair $i<j$. □

Remark 7.2 (Interpretation). *Gram-Schmidt may be interpreted as a sequential elimination of the off-diagonal Gram couplings associated with the already constructed orthonormal directions. The final orthonormal family has a Gram matrix* I_n *and maximizes* $\mathcal{N}$. *The functional is not claimed to be monotone for every possible intermediate normalization convention; rather, it provides a global diagnostic of pairwise orthogonality for each normalized family being compared.*

Complete Example in $\mathbb{R}^3$

Take

$$v_1=(1,1,0),\ v_2=(2,0,-1),\ v_3=(1,1,2). \tag{17}$$

Then

$$w_1=(1,1,0), \tag{18}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 = (1, -1, -1), \tag{19}$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2 = \left(-\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right). \tag{20}$$

Direct calculation gives

$$\langle w_1, w_2 \rangle = \langle w_1, w_3 \rangle = \langle w_2, w_3 \rangle = 0. \tag{21}$$

After normalization, the vectors $q_i = w_i / \|w_i\|$ satisfy $G(q_1, q_2, q_3) = I_3$.

The geometric interpretation of one Gram-Schmidt projection step is illustrated in Figure 1.

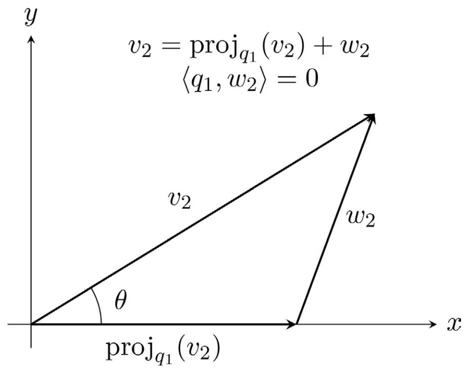

Figure 1. Technical interpretation of one Gram-Schmidt step. The projection carries the angular coupling with the previously normalized direction, while the residual is orthogonal to it.

8. Physical Application: Frictionless Sliding Ladder

The frictionless sliding-ladder problem is a classic example of constrained rigid-body mechanics. The derivation below follows the conventional Newtonian and energy formulations described in standard mechanics texts [7]-[9], and then interprets the relevant geometric factors through the angular decomposition.

Consider a homogeneous rigid ladder of length L and mass m, with its lower endpoint on a frictionless horizontal floor and its upper endpoint on a frictionless vertical wall. Let $\theta(t)$ be the angle measured from the floor. While both contacts are maintained,

$$x = L\cos\theta, \quad y = L\sin\theta, \quad x^2 + y^2 = L^2. \tag{22}$$

8.1. Angular Structure and the Geometric Constraint

Let e_x, e_y be Cartesian unit vectors and let

$$\ell = \cos\theta e_x + \sin\theta e_y \tag{23}$$

be the unit vector along the ladder. Then

$$\delta_{x\ell}^{\mathrm{KJ}} = \cos\theta, \quad \delta_{y\ell}^{\mathrm{KJ}} = \sin\theta. \tag{24}$$

The Gram matrix of $\{e_x, e_y, \ell\}$ is

$$G(\theta)=\begin{pmatrix}1 & 0 & \cos\theta \\ 0 & 1 & \sin\theta \\ \cos\theta & \sin\theta & 1\end{pmatrix}, \ \det G(\theta)=0. \tag{25}$$

The zero determinant records that ℓ is not an independent third direction in the plane. Equivalently, x, y, and θ must not be treated as three independent generalized coordinates.

The mechanical configuration during the two-contact phase is shown in **Figure 2**.

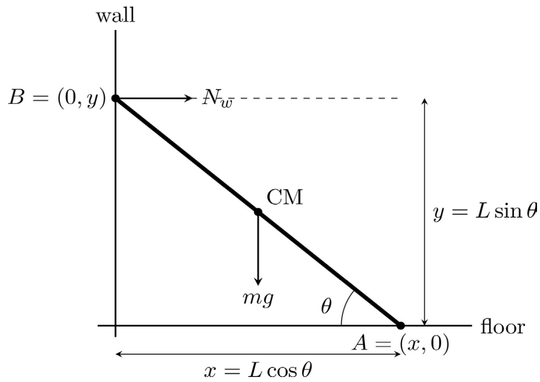

Figure 2. Homogeneous ladder in the two-contact phase. The wall force N_w must remain nonnegative because the wall can push but cannot pull.

8.2. Kinematic Singularity

Differentiating the geometric constraint gives

$$x\dot{x}+y\dot{y}=0, \quad \dot{y}=-\frac{x}{y}\dot{x}=-\frac{\cos\theta}{\sin\theta}\dot{x}. \tag{26}$$

Using the angular quantities,

$$\dot{y}=-\frac{\delta_{x\ell}^{\mathrm{KJ}}}{\delta_{y\ell}^{\mathrm{KJ}}}\dot{x}. \tag{27}$$

If one imposes $\dot{x}=u\neq 0$ all the way to $\theta\to 0$, then $\delta_{y\ell}^{\mathrm{KJ}}=\sin\theta\to 0$ and $|\dot{y}|\to\infty$. This divergence is not a physical prediction. It signals that constant lower-end speed, permanent two-point contact, perfect rigidity, and continuation to $y=0$ are mutually incompatible assumptions.

8.3. Energy and Angular Acceleration

The center of mass is

$$r_C=\left(\frac{L}{2}\cos\theta,\frac{L}{2}\sin\theta\right), \quad v_C^2=\frac{L^2}{4}\dot{\theta}^2. \tag{28}$$

With $I_C=mL^2/12$, the kinetic and potential energies are

$$T=\frac{1}{2}mv_C^2+\frac{1}{2}I_C\dot{\theta}^2=\frac{mL^2}{6}\dot{\theta}^2, \quad V=\frac{mgL}{2}\sin\theta. \tag{29}$$

Released from rest at θ_0, conservation of energy gives

$$\dot{\theta}^2 = \frac{3g}{L}\left(\sin\theta_0 - \sin\theta\right), \tag{30}$$

and differentiation yields

$$\ddot{\theta} = -\frac{3g}{2L}\cos\theta. \tag{31}$$

8.4. Wall Reaction and Loss of Contact

The horizontal coordinate of the center of mass is $x_C = (L/2)\cos\theta$, hence

$$\ddot{x}_C = -\frac{L}{2}\left(\cos\theta\dot{\theta}^2 + \sin\theta\ddot{\theta}\right). \tag{32}$$

Substituting the preceding energy relations gives

$$N_w = m\ddot{x}_C = \frac{3mg}{4}\cos\theta\left(3\sin\theta - 2\sin\theta_0\right). \tag{33}$$

A frictionless wall can exert only a nonnegative normal force. Therefore, contact is lost at the first instant for which $N_w = 0$. Before the horizontal position is reached, $\cos\theta \neq 0$, so

$$\sin\theta_s = \frac{2}{3}\sin\theta_0. \tag{34}$$

Equivalently,

$$\delta_{y\ell}^{\mathrm{KJ}}\left(\theta_s\right) = \frac{2}{3}\delta_{y\ell}^{\mathrm{KJ}}\left(\theta_0\right). \tag{35}$$

For release from the vertical, $\theta_0 = \pi/2$, and therefore

$$\theta_s = \arcsin\left(\frac{2}{3}\right) \approx 41.81^\circ, \quad y_s = L\sin\theta_s = \frac{2L}{3}. \tag{36}$$

For smaller angles, the two-contact formula would predict $N_w < 0$, that the wall would require the ladder to pull. The constrained model must therefore terminate at θ_s and be replaced by the subsequent one-contact dynamics.

The normalized wall reaction for release from the vertical is displayed in **Figure 3**.

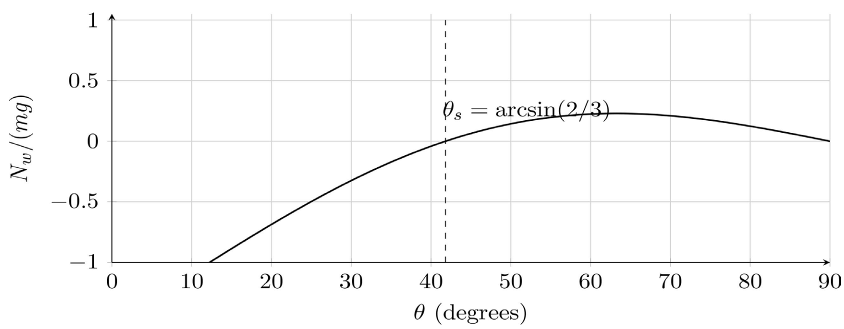

Figure 3. Normalized wall reaction for release from the vertical. The two-contact solution is physically admissible only while $N_w \geq 0$. The zero at $\theta_s \approx 41.81^\circ$ marks loss of wall contact.

Figure 4 shows the evolution of the horizontal and vertical angular couplings together with the loss-of-contact threshold.

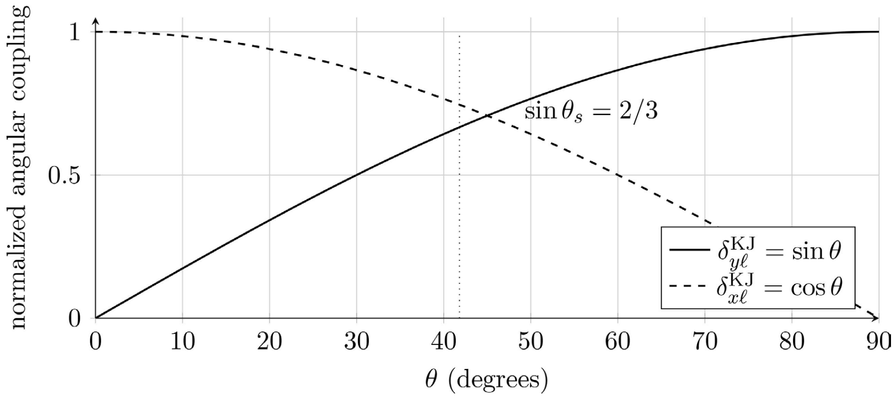

Figure 4. Evolution of the vertical and horizontal angular couplings. The marked value is the loss-of-contact threshold for release from the vertical.

9. Comparison with the Standard Formulation

Table 1 summarizes the interpretive role of the proposed decomposition. The right-hand column does not add new dynamics or algebra; it makes explicit which geometric factor is active in each standard calculation.

Table 1. Classical objects and their angular interpretation.

Standard Formulation	Angular Kronecker-Jeno Interpretation
Kronecker delta δ_{ij}	Index identity is separated from possible off-diagonal angular coupling.
Gram coefficient g_{ij}	For normalized real vectors, $g_{ij} = \delta_{ij}^{\mathrm{KJ}}$.
Orthogonality $g_{ij} = 0$	The off-diagonal selector is active, while the angular coupling vanishes: 0_{J}.
Gram-Schmidt projection coefficient	The removed coefficient is proportional to $\cos\theta_{ik}$.
Ladder kinematic singularity	The divergence arises from division by the vanishing vertical coupling $\sin\theta$.
Loss of wall contact	The physical boundary is imposed separately by the unilateral condition $N_w \geq 0$.

10. Discussion

The Gram-Schmidt application shows that the notation can track which off-diagonal coupling is removed at each projection step. This is mathematically equivalent to standard orthogonalization, but it preserves the angular meaning of the projection coefficient. The functional $\mathcal{N}(B)$ provides a sign-independent scalar diagnostic of pairwise orthogonality for normalized families.

The ladder application goes further by separating two issues that are often conflated. First, the kinematic relation $\dot{y} = -(\cos\theta/\sin\theta)\,\dot{x}$ contains a denominator that vanishes when the ladder approaches the horizontal configuration. Second, the freely released ladder does not remain in two-point contact until that configuration. The unilateral contact condition forces the two-contact model to terminate earlier, at $\sin\theta_s = (2/3)\sin\theta_0$. The angular decomposition identifies the geometric factor that vanishes, whereas the contact inequality determines the physically admissible interval.

The condition $N_w \geq 0$ is not encoded by $\delta_{ij}^{\mathrm{KJ}}$ itself. It remains a mechanical contact condition. Accordingly, the usefulness of the angular decomposition is diagnostic and interpretive rather than dynamical. The present results do not establish a new tensor, scalar product, or force law. They show that a standard Gram entry can be decomposed into index selection and angular coupling, and that this bookkeeping can clarify both orthogonalization and constrained mechanics.

The framework remains restricted to normalized real vectors unless the usual norm factors are restored. In complex inner-product spaces, phase information cannot be represented by a single real angle. These limitations are not technical afterthoughts; they define the domain in which the proposed notation is mathematically equivalent to familiar objects.

11. Conclusions

The angular quantity

$$\delta_{ij}^{\mathrm{KJ}} = \delta_{ij} + \left(1 - \delta_{ij}\right)\cos\theta_{ij} \tag{37}$$

recovers the classical Kronecker delta in an orthonormal basis and coincides with the Gram matrix in a normalized real basis. Its role is interpretive: it separates index identity from the angular mechanism responsible for an off-diagonal metric contribution.

Two concrete uses were demonstrated. In Gram-Schmidt orthogonalization, off-diagonal Gram couplings are sequentially removed until the Gram matrix becomes the identity, while $\mathcal{N}(B)$ reaches its maximum value. In the sliding-ladder problem, the apparent infinite endpoint speed follows from division by the vanishing vertical coupling $\sin\theta$ under an incompatible constant-speed constraint. In the physically released system, the ladder loses wall contact earlier, at $\sin\theta_s = (2/3)\sin\theta_0$; for release from the vertical, $y_s = 2L/3$.

Thus, the decomposition does not replace classical linear algebra or mechanics. It makes explicit which angular factor vanishes and where a constrained model ceases to be physically admissible. Future work may examine whether the same bookkeeping is useful in constrained dynamics, numerical orthogonalization, finite-element formulations, and tensor calculations in which orthogonality evolves during computation.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Strang, G. (2016) Introduction to Linear Algebra. 5th Edition, Wellesley-Cambridge Press.

[2] Halmos, P.R. (1958) Finite-Dimensional Vector Spaces. 2nd Edition, Van Nostrand.

[3] Horn, R.A. and Johnson, C.R. (2013) Matrix Analysis. 2nd Editon, Cambridge University Press.

[4] Lee, J.M. (2018) Introduction to Riemannian Manifolds. 2nd Edition, Springer. https://doi.org/10.1007/978-3-319-91755-9

[5] Golub, G.H. and Van Loan, C.F. (2013) Matrix Computations. 4th Edition, Johns Hopkins University Press.

[6] Trefethen, L.N. and Bau III, D. (1997) Numerical Linear Algebra. SIAM. https://doi.org/10.1137/1.9780898719574

[7] Taylor, J.R. (2005) Classical Mechanics. University Science Books.

[8] Goldstein, H., Poole, C. and Safko, J. (2002) Classical Mechanics. 3rd Edition, Addison-Wesley.

[9] Kleppner, D. and Kolenkow, R. (2014) An Introduction to Mechanics. 2nd Edition, Cambridge University Press. https://doi.org/10.1017/CBO9781139013963

Appendix: Equivalence with the Normalized Gram Matrix

Let $B=(e_1,\cdots,e_n)$ be a normalized family in a finite-dimensional real inner-product space. Its Gram matrix is $G=(g_{ij})$, where

$$g_{ij}=\langle e_i,e_j\rangle. \tag{38}$$

For $i=j$, normalization gives $g_{ii}=1$. For $i\neq j$, the real inner-product identity gives $g_{ij}=\cos\theta_{ij}$. Hence

$$g_{ij}=\delta_{ij}+(1-\delta_{ij})\cos\theta_{ij}=\delta_{ij}^{\mathrm{KJ}}. \tag{39}$$

Therefore, δ^{KJ} and G are identical matrices under the stated assumptions. Every algebraic property used in this manuscript is consequently inherited from the Gram matrix rather than postulated independently.