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A Structural Compactification Map and a Quotient-Geometric Representation of the Minkowski Null Cone

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Abstract

We formulate a mathematically explicit geometric construction that begins with a normalized cubic domain, passes through a planar disk, a cylindrical immersion and a toroidal closure, and terminates in a representation of a subset of the Minkowski null cone. Two distinct terminal maps are emphasized. The independent parametrization P_J is surjective onto the complete double null cone, whereas the torus-to-cone map Q_J has bounded radial parameter and therefore reaches only a proper subset of the cone. Explicit transition maps are introduced so that the domains and codomains of the successive stages are compatible. The six-petal rosette is retained only as an auxiliary planar symmetry visualization and is not treated as a necessary member of the continuous map chain. We compute the standard induced metrics of the cylindrical and toroidal stages and analyze the non-bijective topology of Q_J through its fibers, singular circles, and quotient factorization. The construction preserves the standard Minkowski metric and is intended as an auxiliary geometric representation rather than a modification of special relativity.

Keywords

Minkowski Null Cone, Geometric Construction, Toroidal Immersion, Induced Metrics, Quotient Topology

1. Introduction

Minkowski spacetime is the affine space $\mathbb{R}^{1,3}$ equipped with the Lorentzian metric

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{1}$$

The null structure associated with this metric is fundamental to special relativity [1] [2]. Standard treatments of Lorentzian and semi-Riemannian geometry provide the mathematical framework for null curves, causal cones, and their differential-geometric properties [3] [4].

The purpose of this article is not to alter Equation (1). Instead, we study an auxiliary sequence of geometric maps beginning with a normalized cube and ending with a non-bijective representation on the standard Minkowski null cone. The intermediate Euclidean geometry is treated with standard surface theory [5], while the global map structure is formulated using the language of smooth maps, quotient spaces, and fibers [6] [7].

The principal clarification developed here is that two terminal constructions play different roles. First, an independent map

$$P_J : [0,\infty) \times S^2 \times \{-1,+1\} \to \mathcal{N}$$

parametrizes the complete double null cone. Second, a map

$$Q_J : \mathbb{T}^2 \to \mathcal{N}$$

constructed from toroidal coordinates has bounded radial image and is therefore not surjective onto the full cone. Its significance lies instead in its non-bijective quotient structure.

Accordingly, the mathematically compatible structural chain studied below is

$$G \xrightarrow{\kappa_J^{(q)}} G_q \xrightarrow{\Pi_J} D^2 \xrightarrow{A_{DC}} S_R \xrightarrow{C_J} C_R \xrightarrow{A_{CT}} \mathbb{T}^2 \xrightarrow{T_J} T_{R_T, r_T} \xrightarrow{Q_J} \mathcal{N}_Q \subsetneq \mathcal{N}. \tag{2}$$

The full cone is recovered separately by P_J.

2. Structural Rank Reduction

Let

$$G = [-1,1]^3 \subset \mathbb{R}^3, \quad \Delta_J = (0,0,0). \tag{3}$$

For $q \in [0,1]$, define

$$\kappa_J^{(q)}(x,y,z) = ((1-q)x, (1-q)y, z). \tag{4}$$

Its Jacobian is

$$D\kappa_J^{(q)} = \begin{pmatrix} 1-q & 0 & 0 \\ 0 & 1-q & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Hence

$$\operatorname{rank} D\kappa_J^{(q)} = \begin{cases} 3, & 0 \le q < 1, \\ 1, & q = 1. \end{cases} \tag{6}$$

Proposition 1. *For every* $q \in [0,1]$, *the image* $G_q = \kappa_J^{(q)}(G)$ *is compact, path connected, and contains* Δ_J. *At* $q = 1$, *the map undergoes a controlled rank reduction from three dimensions to the* z *-axis.*

Proof. The cube G is compact and path connected, and $\kappa_J^{(q)}$ is continuous.

Therefore its image is compact and path connected. Moreover, $\kappa_J^{(q)}(\Delta_J) = \Delta_J$. The rank statement follows directly from the Jacobian.

3. Planar Projection and Auxiliary Rosette Symmetry

Let

$$D^2 = \left\{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \le 1\right\}.$$

Define $\Pi_J : G \to D^2$ by

$$\Pi_J(x,y,z) = \begin{cases} \rho(x,y,z)\dfrac{(x,y)}{\sqrt{x^2+y^2}}, & x^2+y^2 > 0, \\ (0,0), & x = y = 0, \end{cases} \tag{7}$$

where

$$\rho(x,y,z) = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}+|z|+1}. \tag{8}$$

Since $0 \le \rho < 1$, the image lies in D^2. The apparent singularity at $x = y = 0$ is removable.

For visualization only, define the six-petal polar curve

$$r(\theta) = R_0 \left|\cos(3\theta)\right|, \quad 0 \le \theta < 2\pi. \tag{9}$$

Its enclosed area is

$$A_R = \frac{1}{2}\int_0^{2\pi} r^2(\theta)\,\mathrm{d}\theta = \frac{\pi R_0^2}{2}. \tag{10}$$

***Remark* 1.** The rosette is not used as a domain of the cylindrical immersion and is not required for the composition in Equation (2). It records a six-sector planar symmetry only. This separates the decorative symmetry representation from the mathematically necessary transition maps. The auxiliary six-petal symmetry visualization is shown in Figure 1.

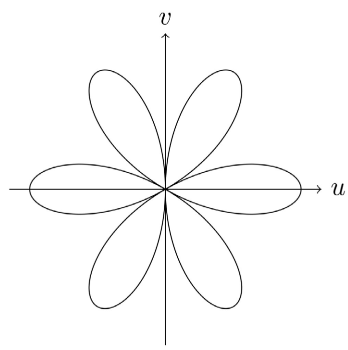

Figure 1. The six-petal rosette is retained as an auxiliary visualization and is not an indispensable transition surface in the structural map chain.

4. Compatible Transition Maps

The original schematic passage from a disk to cylindrical strip coordinates and then to angular torus coordinates requires explicit changes of variables. We now provide them.

Let $D_*^2 = D^2 \setminus \{(0,0)\}$. Write a point of D_*^2 in polar coordinates as

$$(u,v) = (r\cos\alpha, r\sin\alpha),\ 0 < r \le 1,\ 0 \le \alpha < 2\pi.$$

For fixed $R > 0$ and $h > 0$, define the strip

$$S_R = [0, 2\pi R) \times [-h, h].$$

The disk-to-strip transition map is

$$A_{DC} : D_*^2 \to S_R,\quad A_{DC}(r,\alpha) = (R\alpha, h(2r-1)). \tag{11}$$

The angular coordinate is undefined at the disk center, so the center is excluded from this chart. This is a standard coordinate singularity rather than a singularity of the disk itself. A second local chart may be introduced around the center if a full atlas is required [6].

Next define

$$A_{CT} : S_R \to \mathbb{T}^2,\quad A_{CT}(u,v) = \left(\phi = \frac{u}{R} (\operatorname{mod} 2\pi), \theta = \pi\frac{v+h}{h} (\operatorname{mod} 2\pi)\right). \tag{12}$$

Thus the longitudinal strip coordinate becomes the major angular coordinate and the vertical strip coordinate becomes the minor angular coordinate. Endpoint identification in u produces the cylindrical closure, while endpoint identification in v completes the toroidal parameter domain.

***Remark* 2.** Equations (11) and (12) make the structural composition type-compatible. They are chosen transition maps; they are not claimed to be canonical.

5. Cylindrical Immersion and Local Isometry

Define

$$C_J : S_R \to \mathbb{R}^3,\quad C_J(u,v) = \begin{pmatrix} R\cos(u/R) \\ R\sin(u/R) \\ v \end{pmatrix}. \tag{13}$$

The tangent vectors are

$$\partial_u C_J = \begin{pmatrix} -\sin(u/R) \\ \cos(u/R) \\ 0 \end{pmatrix},\quad \partial_v C_J = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore

$$\partial_u C_J \cdot \partial_u C_J = 1,\quad \partial_v C_J \cdot \partial_v C_J = 1,\quad \partial_u C_J \cdot \partial_v C_J = 0.$$

Proposition 2. C_J *is a local isometric immersion of the Euclidean strip*:

$$C_J^*\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) = \mathrm{d}u^2 + \mathrm{d}v^2. \tag{14}$$

This is the familiar developable geometry of a circular cylinder [5]. The statement is local; global closure requires the identification $u \sim u + 2\pi R$.

6. Toroidal Closure and Induced Metric

Let $R_T > r_T > 0$. Define the standard torus embedding

$$T_J(\phi,\theta) = \begin{pmatrix} (R_T + r_T\cos\theta)\cos\phi \\ (R_T + r_T\cos\theta)\sin\phi \\ r_T\sin\theta \end{pmatrix}, \quad (\phi,\theta) \in \mathbb{T}^2. \tag{15}$$

Its first fundamental form is

$$\mathrm{d}\ell_T^2 = (R_T + r_T\cos\theta)^2\,\mathrm{d}\phi^2 + r_T^2\mathrm{d}\theta^2, \tag{16}$$

with area element

$$\mathrm{d}A = r_T(R_T + r_T\cos\theta)\,\mathrm{d}\phi\mathrm{d}\theta. \tag{17}$$

Consequently,

$$A_T = 4\pi^2 R_T r_T. \tag{18}$$

No global isometry between the Euclidean plane, cylinder, and torus is asserted. In particular, the cylinder is developable, whereas the torus has nonconstant Gaussian curvature.

7. Surjective Parametrization of the Minkowski Null Cone

The double null cone with vertex $O = (0,\mathbf{0})$ is

$$\mathcal{N} = \left\{ X = (ct,\boldsymbol{x}) \in \mathbb{R}^{1,3} : -(ct)^2 + \|\boldsymbol{x}\|^2 = 0 \right\}. \tag{19}$$

Its causal interpretation is standard in Lorentzian geometry [8] [9]. Let

$$\boldsymbol{n}(\chi,\phi) = \begin{pmatrix} \sin\chi\cos\phi \\ \sin\chi\sin\phi \\ \cos\chi \end{pmatrix}, \quad \|\boldsymbol{n}\| = 1. \tag{20}$$

For $\lambda \ge 0$ and $\sigma \in \{-1,+1\}$ define

$$P_J(\lambda,\chi,\phi,\sigma) = (\sigma\lambda, \lambda\boldsymbol{n}(\chi,\phi)). \tag{21}$$

Theorem 1 (Exact parametrization of the null cone). *The map P_J is surjective onto $\mathcal{N}$ and every point in its image satisfies the Minkowski null condition exactly.*

Proof. Since $\|\boldsymbol{n}\| = 1$,

$$-(\sigma\lambda)^2 + \|\lambda\boldsymbol{n}\|^2 = -\lambda^2 + \lambda^2 = 0.$$

Hence $\operatorname{Im} P_J \subseteq \mathcal{N}$. Conversely, let $(ct,\boldsymbol{x}) \in \mathcal{N}$. At the vertex choose $\lambda = 0$. Otherwise set $\lambda = \|\boldsymbol{x}\| = |ct|$, $\sigma = \operatorname{sgn}(t)$, $\boldsymbol{n} = \boldsymbol{x}/\|\boldsymbol{x}\|$. Spherical angles represent $\boldsymbol{n}$, so every null point is obtained.

For fixed (χ,ϕ,σ) and $\lambda = c|t|$, a generator obeys

$$\mathrm{d}\boldsymbol{x} = \sigma c\boldsymbol{n}\mathrm{d}t, \quad \|\mathrm{d}\boldsymbol{x}\|^2 = c^2\mathrm{d}t^2,$$

and therefore Equation (1) gives $\mathrm{d}s^2 = 0$.

8. Restricted Torus-to-Cone Map

We now distinguish the independent surjective parametrization P_J from the torus-induced map. Fix $\lambda_0 > 0$ and define

$$\lambda(\theta) = \lambda_0 |\sin\theta|, \tag{22}$$

$$\chi(\theta) = \frac{\theta}{2}, \tag{23}$$

$$\sigma(\theta) = \begin{cases} +1, & 0 \le \theta \le \pi, \\ -1, & \pi < \theta < 2\pi. \end{cases} \tag{24}$$

Define

$$Q_J(\phi,\theta) = P_J\left(\lambda(\theta), \chi(\theta), \phi, \sigma(\theta)\right). \tag{25}$$

Proposition 3. Q_J *is not surjective onto the full null cone.*

Proof. For every $(\phi,\theta) \in \mathbb{T}^2$,

$$0 \le \lambda(\theta) \le \lambda_0.$$

Therefore no null point with radial parameter $\lambda > \lambda_0$ belongs to $\operatorname{Im} Q_J$. Moreover, χ and λ are coupled through the single parameter θ, so the accessible angular-radial combinations form only a restricted subset. Thus

$$\mathcal{N}_Q := \operatorname{Im} Q_J \subsetneq \mathcal{N}. \tag{26}$$

This distinction removes any implication that the torus itself parametrizes the complete Minkowski null cone. The full cone is parametrized by P_J; Q_J is a separate, restricted, non-bijective projection.

9. Fibers, Quotient Structure, and Singular Sets

The topologically significant feature of Q_J is not an equality of torus and cone topologies but the information loss generated by its fibers. Define the two toroidal circles

$$F_0 = S^1 \times \{0\}, \quad F_\pi = S^1 \times \{\pi\}. \tag{27}$$

Because $\lambda(0) = \lambda(\pi) = 0$,

$$Q_J(F_0) = Q_J(F_\pi) = \{O\}. \tag{28}$$

Thus entire one-dimensional subsets of $\mathbb{T}^2$ are collapsed to the cone vertex.

Definition 1. Define an equivalence relation $\sim_Q$ on $\mathbb{T}^2$ by

$$p \sim_Q p' \Leftrightarrow Q_J(p) = Q_J(p'). \tag{29}$$

Let

$$X_Q = \mathbb{T}^2 / \sim_Q \tag{30}$$

be the associated quotient space, with quotient projection $\pi_Q : \mathbb{T}^2 \to X_Q$.

Proposition 4 (Quotient factorization). *There exists a unique continuous map*

$$\tilde{Q}_J : X_Q \to \mathcal{N}_Q \tag{31}$$

such that

$$Q_J = \tilde{Q}_J \circ \pi_Q. \tag{32}$$

Moreover, $\tilde{Q}_J$ *is bijective by construction.*

Proof. Q_J is constant on each equivalence class of $\sim_Q$. The universal property of the quotient therefore yields a unique continuous map $\tilde{Q}_J$ satisfying Equation (32). Since the equivalence classes are precisely the fibers of Q_J, two quotient classes have the same image only if they coincide. Surjectivity onto $\mathcal{N}_Q$ follows from the definition $\mathcal{N}_Q = \operatorname{Im} Q_J$. □

***Remark* 3.** The proposition establishes a quotient representation, not a diffeomorphism between the torus and the null cone. In particular, the vertex has a non-discrete preimage containing $F_0 \cup F_\pi$. The quotient therefore records an explicit topological loss of information. The singular behavior here is map-induced: it should not be confused with a physical spacetime singularity.

Away from $F_0 \cup F_\pi$, the radial parameter is positive. The two open annuli

$$U_+ = S^1 \times (0, \pi), \quad U_- = S^1 \times (\pi, 2\pi)$$

map respectively to the future and past sheets selected by σ. The collapsed circles are boundary fibers between these sectors. This gives a simple stratified description: regular two-dimensional parameter regions are attached through fibers that collapse to the zero-dimensional cone vertex.

The causal and global properties of null boundaries are normally analyzed in a much richer Lorentzian setting [10]. The present construction makes no claim to reproduce that theory; the quotient analysis concerns only the topology of the auxiliary map Q_J. The resulting type-compatible structural chain and the distinct roles of P_J and Q_J are summarized in Figure 2.

$$\begin{tikzcd}
G \arrow[r, "\kappa_J^{(q)}"] & G_q \arrow[r, "\Pi_J"] & D^2 \arrow[r, "A_{DC}"] & S_R \arrow[r, "C_J"] & C_R \arrow[d, "A_{CT}"] \\
 & & \mathcal{N}_Q \arrow[d, dash, "\subsetneq"] & T_{R_T, r_T} \arrow[l, "Q_J"'] & \mathbb{T}^2 \arrow[l, "T_J"'] \\
 & & \mathcal{N} & &
\end{tikzcd}$$

Figure 2. Type-compatible structural chain. The independent map P_J parametrizes the full cone $\mathcal{N}$, whereas the torus-induced map Q_J reaches only $\mathcal{N}_Q \subsetneq \mathcal{N}$.

10. Dimensionless Null Residual

For a trajectory with spatial velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$, define

$$\beta = \frac{\|\boldsymbol{v}\|}{c}, \quad \varepsilon_J = 1 - \beta^2. \tag{33}$$

For $\mathrm{d}t \neq 0$,

$$\frac{\mathrm{d}s^2}{c^2 \mathrm{d}t^2} = -1 + \beta^2 = -\varepsilon_J. \tag{34}$$

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Hence

$$\varepsilon_J \begin{cases} >0, & \|\boldsymbol{v}\| < c, \\ =0, & \|\boldsymbol{v}\| = c, \\ <0, & \|\boldsymbol{v}\| > c. \end{cases} \tag{35}$$

This is exactly the standard dimensionless special-relativistic factor $1 - v^2/c^2$, used here only as a diagnostic label. It introduces no new dynamics.

11. Relation to Standard Lorentzian Geometry

Table 1 separates standard geometric facts from definitions specific to the structural construction. The causal geometry of the terminal cone remains entirely standard. No gravitational field, curvature of spacetime, or alteration of Lorentz symmetry is inferred from the Euclidean intermediate surfaces.

Table 1. Standard and construction-specific elements.

Element	Standard geometry	Structural construction
Minkowski metric	$\eta = \operatorname{diag}(-1,1,1,1)$	Retained unchanged
Null cone	$\eta(X,X) = 0$	Complete image of P_J
Structural center	No additional object required	Chosen point Δ_J
Cylinder	Developable Euclidean surface	Intermediate immersion
Torus	Standard embedded surface	Auxiliary closure
Q_J	Not required by relativity	Restricted non-bijective map
Quotient	Standard topological construction	Encodes fibers of Q_J
Residual	$1 - v^2/c^2$	Denoted ε_J
Dynamics	Supplied by a physical theory	Not specified by the map chain

12. Discussion

The revised formulation resolves three logically distinct issues.

First, the structural rank reduction in Equation (4) is a property of a Euclidean linear map. At $q = 1$ two directions collapse, but this does not imply a spacetime singularity.

Second, the intermediate maps are now domain-compatible. The explicit transitions A_{DC} and A_{CT} remove the need to interpret the composition only schematically. The disk center remains a coordinate-chart issue because polar angle is undefined there; it can be covered by an additional local chart if a global smooth atlas is desired.

Third, the two terminal maps must not be conflated. The surjectivity theorem applies to P_J, whose independent variables $(\lambda, \chi, \phi, \sigma)$ span the full double null cone. By contrast, Q_J constrains λ and χ through the same toroidal coordinate θ. Its bounded radial image and collapsed fibers make it valuable as a quotient-geometric construction, not as a global parametrization of $\mathcal{N}$.

The most mathematically distinctive feature is therefore the fiber structure of Q_J. The circles F_0 and F_π collapse to the cone vertex, while the complementary annuli map to nonzero future and past null sectors. A deeper treatment could classify all fibers, study the local topology of X_Q, analyze whether alternative choices of $\lambda(\theta)$ and $\chi(\theta)$ change the quotient type, and formulate analogous maps between more general compact parameter manifolds and Lorentzian null hypersurfaces.

No phenomenological Lagrangian is introduced in the present revision. A physical dynamics would require independently motivated measurable degrees of freedom, characteristic scales, and falsifiable observables. Keeping the geometric map separate from such a dynamics prevents an auxiliary representation from being mistaken for a new physical law.

13. Conclusions

We have reformulated the structural compactification construction as a mathematically type-compatible sequence of maps and separated the complete null-cone parametrization from the restricted torus-to-cone projection. The principal conclusions are:

1) $\kappa_J^{(q)}$ has a computable Jacobian and undergoes a controlled rank reduction at $q=1$.

2) The planar rosette is an auxiliary symmetry visualization rather than a necessary step in the continuous map chain.

3) Explicit transition maps connect disk coordinates, strip coordinates, and toroidal angles.

4) The cylindrical stage is locally isometric to a Euclidean strip, while the toroidal stage carries the standard induced torus metric.

5) P_J is surjective onto the complete double Minkowski null cone and satisfies $\mathrm{d}s^2=0$ exactly.

6) Q_J is not surjective onto the complete cone; its image is a proper subset $\mathcal{N}_Q \subsetneq \mathcal{N}$.

7) The circles F_0 and F_π are singular fibers of Q_J in the sense that each is collapsed to the cone vertex.

8) The map Q_J factors naturally through the quotient $\mathbb{T}^2/\sim_Q$, providing a precise mathematical description of the information lost under the non-bijective projection.

The resulting construction is best interpreted as an auxiliary geometric and quotient-topological representation compatible with standard special relativity. It does not modify the Minkowski metric and does not, by itself, define a physical dynamics.

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projection, coordinate compatibility, and quotient topology.

Author Contributions

Oscar Mauricio Jeno Soto: Conceptualization, formal analysis, methodology, visualization, and writing (original draft and revision).

Data Availability

Data sharing is not applicable to this article because no new data were created or analyzed.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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