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Common Fixed Point Theorems for Random Multivalued Contractive Mappings with Weak Compatibility

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Abstract

In this paper, we introduce more general random multivalued contractive mappings, and we establish the existence and uniqueness of random common fixed points for these random multivalued contractive mappings with weak compatibility. Our results extend, improve and unify the corresponding literatures and single value results.

Keywords

Random Common Fixed Point, Random Multivalue Mapping, Measurable Mapping

1. Introduction

The application of fixed point theory in different branches of mathematics, statistics, engineering and economics relating to problems associated with approximation theory, theory of differential equations, theory of integral equations etc. has been recognized in the existing literature [1]-[3]. Progress in the study on fixed points of non-expansive mappings, contractive mappings in various spaces like a metric space, a Banach space, a fuzzy metric space, a cone metric space etc. has been saturated at large. After the initial impetus given by the Prague school of Probability in 1950s, considerable attention has been given to the study of random fixed point theorems. This arises because of the significance of fixed point theorems in probabilistic functional analysis and probabilistic models along with several applications. Issues relating to measurability of solutions, probabilistic and statistical aspects of random solutions have arisen due to the introduction of randomness. It is no denying the fact that random fixed point theorems are stochastic generalizations of classical fixed point theorems that have been described as deterministic results.

Špaček [4] and Hanš [5] first proved random fixed point theorems for random contraction mappings on separable complete metric spaces. The article by Fierro *et al.* [6] in 2009 attracted the attention of several mathematicians and led to the development of this theory. Špaček and Hanš's theorems have been extended to multivalued contraction mappings by Itoh [7]. A random version of Schaduer's fixed point theorem on an atomic probability measure space has been provided by Mukherjee [8]. Itoh [7] obtained random fixed point theorems with an application to random differential equations in Banach spaces. Several random fixed point theorems including random analogue of the classical results have been obtained by many authors [9]-[20]. Kumam [19] proved some random coincidence points and random common fixed point theorems for nonlinear multivalued random operators. They also proved the existence of a random coincidence point for a pair of reciprocally continuous and compatible single-valued and multivalued operators. Later, many scholars have obtained a lot of multi-valued and single valued operator of fixed point results [21]-[27].

Motivated and inspired by the above results, our main objective is to prove some random fixed point theorems in a separable complete metric space for a certain class of random multivalued contractive mappings. The results are stochastic generalizations of deterministic fixed point theorems of single value mappings. The result obtained in this paper will also be useful in application to a random nonlinear integral equation.

Remark 1.1 (Motivation for the contraction condition). The rational expressions m_1, m_2, m_3 in (3.2) are introduced to handle situations where the mappings A, B do not necessarily commute with S, T, and where the distances between image sets may vanish. These terms are inspired by the classical rational contraction inequalities of Ćirić and others, and they allow for a unified treatment of both single-valued and multivalued mappings under weak compatibility. The term m_3 captures the product of distances between cross-images, which is essential when the usual triangle inequality fails to provide a contraction. These expressions are not arbitrary; they represent a natural generalization of the standard rational contractive conditions in the random multivalued setting.

In order to make the paper self-contained, we state some important definitions and Lemmas relate to our paper.

2. Preliminaries

Let (Ω, Σ) be a measurable space with Σ being a σ-algebra of subsets of Ω, and let (X, d) be a complete separable metric space. We denote by $CB(X)$ the family of all nonempty closed bounded subset of X, and by H the Hausdorff metric on $CB(X)$ induced by d, that is

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

for $A,B \in CB(X)$, where $d(x,E) = \inf\{d(x,y) \mid y \in E\}$ is the distance from x to $E \subset X$.

Definition 2.1 ([18]). *A measurable operator* $\xi : \Omega \to X$ *is said to be a random fixed point of an operator* $F : \Omega \times X \to CB(X)$ *if* $\xi(\omega) \in F(\omega, \xi(\omega))$, *for all* $\omega \in \Omega$.

Definition 2.2 ([18]). *A measurable operator* $\xi : \Omega \to X$ *is said to be a random coincidence point of* $g : \Omega \times X \to X$ *and* $F : \Omega \times X \to CB(X)$ *if*

$$g(\omega, \xi(\omega)) \in F(\omega, \xi(\omega)), \text{ for all } \omega \in \Omega.$$

Definition 2.3 ([18]). *A measurable operator* $\xi : \Omega \to X$ *is said to be a random common fixed point of* $g : \Omega \times X \to X$ *and* $F : \Omega \times X \to CB(X)$ *if*

$$\xi(\omega) = g(\omega, \xi(\omega)) \in F(\omega, \xi(\omega)), \text{ for all } \omega \in \Omega.$$

Definition 2.4. *The random mapping* $F : \Omega \times X \to CB(X)$ *and* $g : \Omega \times X \to X$ *are said to be weakly random compatible, if for all* $\omega \in \Omega$ *and* $x \in X$, *the condition* $g(\omega, x) \in F(\omega, x)$ *implies*

$$g(\omega, F(\omega, x)) := \{g(\omega, y) : y \in F(\omega, x)\} \subseteq F(\omega, g(\omega, x)).$$

Remark 2.1. The notion of "weakly random compatible" is the random analogue of the deterministic "weakly compatible" condition. In this paper, both terms are used interchangeably in the context of random mappings, and they refer to the same compatibility condition.

Lemma 2.1 ([28]). A mapping $T : \Omega \times X \to CB(X)$ is called a multi-valued random operator if for every $x \in X$, $T(\cdot, x)$ is measurable. A mapping $s : \Omega \to X$ is called a measurable selector of a measurable multifunction $T : \Omega \times X \to CB(X)$ if s is measurable and $s(\omega) \in T(\omega)$, for all $\omega \in \Omega$.

Lemma 2.2 ([28]). Let $S, T : \Omega \to CB(X)$ be measurable mappings, there is a measurable selector $u : \Omega \to X$ of S, then for any measurable real valued function $\alpha : \Omega \to (1, \infty)$, there exists measurable selector $v : \Omega \to X$ of T such that $d(u(\omega), v(\omega)) \le \alpha(\omega) H(S(\omega), T(\omega))$.

Lemma 2.3 ([6]). A multivalued operator $T : \Omega \times X \to CB(X)$ is called (Σ)-measurable, if for any open subset B of X, $T^{-1}(B) = \{\omega \in \Omega : T(\omega) \cap B \ne \varnothing\} \in \Sigma$. The set $F(\omega)$ be the fixed point set of $T(\omega, \cdot)$, if $F(\omega) := \{x \in T(\omega, x)\} \ne \varnothing$. Then T has a random fixed point.

Recall that $T : \Omega \times X \to CB(X)$ is continuous, if for each fixed $\omega \in \Omega$, the operator $T : (\omega, \cdot) \to CB(X)$ is continuous.

Throughout this paper, we assume that $R^+ = [0, +\infty)$, $R = (-\infty, +\infty)$, $N_0 = \{0\} \cup N$, where N denotes the set of all positive integers and we give the following definition.

Definition 2.5. $\Psi = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is semi-continuous upper on $\mathbb{R}^+ \setminus \{0\}$, if and only if $\lim_{n \to \infty} r^n = 0$ and $\psi(t) < t$ for each $t > 0\}$, $\Psi_1 = \{\psi : \mathbb{R}^+ \to \mathbb{R}^+$ does not decrease on $\mathbb{R}^+$, $\psi(t) < t$ and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0\}$.

3. Main Results

Now we show two random common fixed point theorems for random multivalued contractive mappings in metric spaces.

Theorem 3.1. Let (X,d) be a complete separable metric space, let (Ω,Σ) be a measurable space, and let $S,T:\Omega\times X\to CB(X)$ and $A,B:\Omega\times X\to X$ be mappings such that

(i) $A(\omega,\cdot),B(\omega,\cdot),T(\omega,\cdot),S(\omega,\cdot)$ are continuous for all $\omega\in\Omega$;

(ii) $A(\cdot,x),B(\cdot,x),T(\cdot,x),S(\cdot,x)$ are measurable for all $x\in X$;

(iii) $S(\Omega\times X)\subseteq B(\Omega\times X)$ and $T(\Omega\times X)\subseteq A(\Omega\times X)$;

(iv) the pairs $\{A,S\}$ and $\{B,T\}$ are weakly random compatible;

(v) one of $A(\Omega\times X),B(\Omega\times X),S(\Omega\times X)$ and $T(\Omega\times X)$ is complete and

$$H\left(S(\omega,x),T(\omega,y)\right)\le\psi\left(\max\left\{m_i(x,y):1\le i\le 4\right\}\right),\ \forall x,y\in X, \tag{1}$$

where $\psi\in\Psi$ and

$$\begin{aligned}
m_1(x,y)&=d\left(B(\omega,y),T(\omega,y)\right)\cdot\frac{1+d\left(A(\omega,x),S(\omega,x)\right)}{1+d\left(A(\omega,x),B(\omega,y)\right)},\\
m_2(x,y)&=d\left(A(\omega,x),S(\omega,x)\right)\cdot\frac{1+d\left(B(\omega,y),T(\omega,y)\right)}{1+d\left(A(\omega,x),B(\omega,y)\right)},\\
m_3(x,y)&=\frac{d\left(B(\omega,y),S(\omega,x)\right)}{1+d\left(A(\omega,x),B(\omega,y)\right)}\cdot d\left(A(\omega,x),T(\omega,y)\right),\\
m_4(x,y)&=\max\Big\{d\left(A(\omega,x),B(\omega,y)\right),\\
&\qquad d\left(A(\omega,x),S(\omega,x)\right),d\left(T(\omega,y),B(\omega,y)\right),\\
&\qquad \frac{1}{2}\left[d\left(S(\omega,x),B(\omega,y)\right)+d\left(A(\omega,x),T(\omega,y)\right)\right]\Big\}.
\end{aligned} \tag{2}$$

Then A,B,S and T have a unique random common fixed point in X.

Proof. Let $\Theta=\{\xi:\Omega\to X\}$ be a family of measurable mappings. Define a function $g:\Omega\times X\to R^+$ as follows:

$$g(\omega,x)=d\left(x,T(\omega,x)\right).$$

Since $x\to T(\omega,x)$ is continuous for all $\omega\in\Omega$, we conclude that $g(\omega,\cdot)$ is continuous for all $\omega\in\Omega$. Also, since $\omega\to T(\omega,x)$ is measurable for all $x\in X$, we conclude that $g(\cdot,x)$ is measurable (see Wagner ([29], page 868, 31, 32)) for all $\omega\in\Omega$. Thus $g(\omega,x)$ is the Caratheodory function. Therefore, if $\xi:\Omega\to X$ is a measurable mapping, then $\omega\to g\left(\omega,\xi(\omega)\right)$ is also measurable (see [29]). Let $\xi_0\in\Theta$ be arbitrary. Then the multifunction $F,G:\Omega\to CB(X)$ defined by $F(\omega)=B\left(\omega,\xi_0(\omega)\right),G(\omega)=A\left(\omega,\xi_0(\omega)\right)$ is measurable [30].

Now we will construct sequences of measurable mappings $\{y_n(\omega)\},\{x_n(\omega)\}\in\Theta$, the sequence

$$\left\{A\left(\omega,x_n(\omega)\right)\right\},\left\{B\left(\omega,x_n(\omega)\right)\right\}\subseteq X,$$
$$\text{and }\left\{T\left(\omega,x_n(\omega)\right)\right\},\left\{S\left(\omega,x_n(\omega)\right)\right\}\subseteq CB(X).$$

By the Kuratowski-Ryll-Nardzewski selector theorem [28], there exists a measurable selector $y_1(\omega)\in S\left(\omega,x_0(\omega)\right)$. Then, using Lemma 2.2 with $\alpha(\omega)\equiv 2$, we obtain a measurable selector $y_2(\omega)\in T(\omega,x_1(\omega))$ such that

$$d\left(y_1(\omega),y_2(\omega)\right)\le 2H\left(S(\omega,x_0(\omega)),T(\omega,x_1(\omega))\right).$$

Since $S(\Omega\times X)\subseteq B(\Omega\times X)$ and $T(\Omega\times X)\subseteq A(\Omega\times X)$, there exist measurable mappings $x_1,x_2\in\Theta$ such that

$$y_1(\omega)=B(\omega,x_1(\omega)),\ y_2(\omega)=A(\omega,x_2(\omega)).$$

Continuing this process inductively, and at each step applying Lemma 2.2 to select the next point with a controlled distance estimate, we construct sequences of measurable mappings $\{y_n(\omega)\},\{x_n(\omega)\}\subset\Theta$ satisfying

$$y_{2n+1}(\omega)=B(\omega,x_{2n+1}(\omega))\in S(\omega,x_{2n}(\omega)), \tag{3}$$

$$y_{2n+2}(\omega)=A(\omega,x_{2n+2}(\omega))\in T(\omega,x_{2n+1}(\omega)),\ n\in\mathbb{N}. \tag{4}$$

The distance $d(y_n,y_{n+1})$ is then controlled by the Hausdorff metric via Lemma 2.2, which justifies the subsequent inequalities involving $H(S,T)$. Put $d_n=d(y_n(\omega),y_{n+1}(\omega))$ for each $n\in\mathbb{N}$.

Firstly, we show that $A(\omega,x),B(\omega,x),S(\omega,x)$ and $T(\omega,x)$ have at most a random common fixed point in X, for all $\omega\in\Omega$, $x\in X$. Suppose that $u(\omega)$ and $v(\omega)$ are two different random common fixed points of $A(\omega,x),B(\omega,x),S(\omega,x)$ and $T(\omega,x)$. It follows from (3.1), (3.2), we have

$$\begin{aligned}
&m_1(u(\omega),v(\omega))\\
&=d\left(B(\omega,v(\omega)),T(\omega,v(\omega))\right)\cdot\frac{1+d\left(A(\omega,u(\omega)),S(\omega,u(\omega))\right)}{1+d\left(A(\omega,u(\omega)),B(\omega,v(\omega))\right)}\\
&=0,\\
&m_2(u(\omega),v(\omega))\\
&=d\left(A(\omega,u(\omega)),S(\omega,u(\omega))\right)\cdot\frac{1+d\left(B(\omega,v(\omega)),T(\omega,v(\omega))\right)}{1+d\left(A(\omega,u(\omega)),B(\omega,v(\omega))\right)}\\
&=0,\\
&m_3(u(\omega),v(\omega))\\
&=\frac{d\left(B(\omega,v(\omega)),S(\omega,u(\omega))\right)d\left(A(\omega,u(\omega)),T(\omega,v(\omega))\right)}{1+d\left(A(\omega,u(\omega)),B(\omega,v(\omega))\right)}\\
&=\frac{d^2(u(\omega),v(\omega))}{1+d(u(\omega),v(\omega))},\\
&m_4(u(\omega),v(\omega))\\
&=\max\Big\{d\left(A(\omega,u(\omega)),B(\omega,v(\omega))\right),\\
&\quad d\left(A(\omega,u(\omega)),S(\omega,u(\omega))\right),d\left(T(\omega,v(\omega)),B(\omega,v(\omega))\right),\\
&\quad \frac{1}{2}\left[d\left(S(\omega,u(\omega)),B(\omega,v(\omega))\right)+d\left(A(\omega,u(\omega)),T(\omega,v(\omega))\right)\right]\Big\}\\
&=d(u(\omega),v(\omega)).
\end{aligned} \tag{5}$$

and

$$
\begin{aligned}
d(u(\omega),v(\omega)) &\le H(S(\omega,u(\omega)),T(\omega,v(\omega))) \\
&\le \psi\left(\max\{m_i(u(\omega),v(\omega)):1\le i\le 4\}\right) \\
&= \psi\left(\max\left\{0,0,\frac{d^2(u(\omega),v(\omega))}{1+d(u(\omega),v(\omega))},d(u(\omega),v(\omega))\right\}\right) \\
&= \psi(d(u(\omega),v(\omega))) < d(u(\omega),v(\omega)),
\end{aligned} \tag{6}
$$

which is a contradiction. Hence $A(\omega,x(\omega)),B(\omega,x(\omega)),S(\omega,x(\omega))$ and $T(\omega,x(\omega))$ have at most a random common fixed point in X.

Secondly we show that $A(\omega,x(\omega)),B(\omega,x(\omega)),S(\omega,x(\omega))$ and $T(\omega,x(\omega))$ have a random common fixed point $A(\omega,a(\omega))\in X$, if there exist $a(\omega),b(\omega)\in X$ satisfying

$$B(\omega,b(\omega))=A(\omega,b(\omega))\subseteq S(\omega,a(\omega))=T(\omega,a(\omega)). \tag{7}$$

Assume that (3.4) holds for some measurable mappings $a(\omega),b(\omega)\in\Theta$, *i.e.*,

$$B(\omega,b(\omega))=A(\omega,b(\omega))\subseteq S(\omega,a(\omega))=T(\omega,a(\omega)).$$

Put $c(\omega)=A(\omega,a(\omega))\in X$. Note that (iv) implies that

$$A(\omega,c(\omega))=A(\omega,S(\omega,a(\omega)))\subseteq S(\omega,c(\omega))=S(\omega,A(\omega,a(\omega)))$$

and

$$B(\omega,c(\omega))=B(\omega,T(\omega,b(\omega)))\subseteq T(\omega,c(\omega))=T(\omega,B(\omega,b(\omega))).$$

Suppose that $c(\omega)\notin T(\omega,c(\omega))$. In view of (3.1), (3.2), (3.4), (3.5) and $\psi\in\Psi$, we infer that

$$
\begin{aligned}
&m_1(a(\omega),c(\omega)) \\
&= d(B(\omega,c(\omega)),T(\omega,c(\omega)))\cdot\frac{1+d(A(\omega,a(\omega)),S(\omega,a(\omega)))}{1+d(A(\omega,a(\omega)),B(\omega,c(\omega)))} \\
&= 0, \\
&m_2(a(\omega),c(\omega)) \\
&= d(A(\omega,a(\omega)),S(\omega,a(\omega)))\cdot\frac{1+d(B(\omega,c(\omega)),T(\omega,c(\omega)))}{1+d(A(\omega,a(\omega)),B(\omega,c(\omega)))} \\
&= 0, \\
&m_3(a(\omega),c(\omega)) \\
&= \frac{d(B(\omega,c(\omega)),S(\omega,a(\omega)))d(A(\omega,a(\omega)),T(\omega,c(\omega)))}{1+d(A(\omega,a(\omega)),B(\omega,c(\omega)))} \\
&= \frac{d(c(\omega),B(\omega,c(\omega)))d(T(\omega,c(\omega)),c(\omega))}{1+d(c(\omega),B(\omega,c(\omega)))} \\
&= \frac{d^2(c(\omega),T(\omega,c(\omega)))}{1+d(c(\omega),T(\omega,c(\omega)))},
\end{aligned}
$$

$$\begin{aligned}
&m_4\left(a(\omega),c(\omega)\right)\\
&=\max\Big\{d\left(A(\omega,a(\omega)),B(\omega,c(\omega))\right),\\
&\quad d\left(A(\omega,a(\omega)),S(\omega,a(\omega))\right),d\left(T(\omega,c(\omega)),B(\omega,c(\omega))\right),\\
&\quad \frac{1}{2}\left[d\left(S(\omega,a(\omega)),B(\omega,c(\omega))\right)+d\left(A(\omega,a(\omega)),T(\omega,c(\omega))\right)\right]\Big\}\\
&=\max\Big\{d\left(c(\omega),B(\omega,c(\omega))\right),0,0,\\
&\quad \frac{1}{2}\left[d\left(c(\omega),B(\omega,c(\omega))\right)+d\left(T(\omega,c(\omega)),c(\omega)\right)\right]\Big\}\\
&=d\left(c(\omega),T(\omega,c(\omega))\right).
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&d\left(c(\omega),T(\omega,c(\omega))\right)\\
&\le H\left(S(\omega,a(\omega)),T(\omega,c(\omega))\right)\\
&\le \psi\left(\max\left\{m_i\left(a(\omega),c(\omega)\right):1\le i\le 4\right\}\right)\\
&=\psi\left(\max\left\{0,0,\frac{d^2\left(c(\omega),T(\omega,c(\omega))\right)}{1+d\left(c(\omega),T(\omega,c(\omega))\right)},d\left(c(\omega),T(\omega,c(\omega))\right)\right\}\right)\\
&=\psi\left(d\left(c(\omega),T(\omega,c(\omega))\right)\right)\\
&\le d\left(c(\omega),T(\omega,c(\omega))\right).
\end{aligned} \tag{9}$$

which is impossible. Consequently, $c(\omega)=B(\omega,c(\omega))\in T(\omega,c(\omega))$. Similarly we conclude that $c(\omega)=A(\omega,c(\omega))\in S(\omega,c(\omega))$. That is, $c(\omega)$ is a random common fixed point of $A(\omega,c(\omega)),B(\omega,c(\omega)),S(\omega,c(\omega))$ and $T(\omega,c(\omega))$.

Thirdly we show that (3.4) holds for some $a(\omega),b(\omega)\in X$. In order to prove (3.4), we have to consider three possible cases as follows.

Case1. There exists $n\in N$ satisfying $d_{2n_0}=0$. We claim that $d_{2n_0+1}>0$, otherwise $d_{2n_0+1}=0$. Using (3.1)-(3.3) and $\psi\in\Psi$, we deduce that

$$\begin{aligned}
&m_1\left(x_{2n_0}(\omega),x_{2n_0+1}(\omega)\right)\\
&=d\left(B(\omega,x_{2n_0+1}(\omega)),T(\omega,x_{2n_0+1}(\omega))\right)\cdot\frac{1+d\left(A(\omega,x_{2n_0}(\omega)),S(\omega,x_{2n_0}(\omega))\right)}{1+d\left(A(\omega,x_{2n_0}(\omega)),B(\omega,x_{2n_0+1}(\omega))\right)}\\
&=d\left(y_{2n_0+1}(\omega),y_{2n_0+2}(\omega)\right)\cdot\frac{1+d\left(y_{2n_0}(\omega),y_{2n_0+1}(\omega)\right)}{1+d\left(y_{2n_0}(\omega),y_{2n_0+1}(\omega)\right)}\\
&=d_{2n_0+1},\\
&m_2\left(x_{2n_0}(\omega),x_{2n_0+1}(\omega)\right)\\
&=d\left(A(\omega,x_{2n_0}(\omega)),S(\omega,x_{2n_0}(\omega))\right)\cdot\frac{1+d\left(B(\omega,x_{2n_0+1}(\omega)),T(\omega,x_{2n_0+1}(\omega))\right)}{1+d\left(A(\omega,x_{2n_0}(\omega)),B(\omega,x_{2n_0+1}(\omega))\right)}\\
&=d\left(y_{2n_0}(\omega),y_{2n_0+1}(\omega)\right)\cdot\frac{1+d\left(y_{2n_0+1}(\omega),y_{2n_0+2}(\omega)\right)}{1+d\left(y_{2n_0}(\omega),y_{2n_0+1}(\omega)\right)}\\
&=0,
\end{aligned}$$

$$
\begin{aligned}
& m_3\left(x_{2n_0}(\omega), x_{2n_0+1}(\omega)\right) \\
& = \frac{d\left(B\left(\omega, x_{2n_0+1}(\omega)\right), S\left(\omega, x_{2n_0}(\omega)\right)\right)}{1+d\left(A\left(\omega, x_{2n_0}(\omega)\right), B\left(\omega, x_{2n_0+1}(\omega)\right)\right)} \cdot d\left(A\left(\omega, x_{2n_0}(\omega)\right), T\left(\omega, x_{2n_0+1}(\omega)\right)\right) \\
& = \frac{d\left(y_{2n_0+1}(\omega), y_{2n_0+1}(\omega)\right) d\left(y_{2n_0+2}(\omega), y_{2n_0}(\omega)\right)}{1+d\left(y_{2n_0}(\omega), y_{2n_0+1}(\omega)\right)} \\
& = 0, \\
& m_4\left(x_{2n_0}(\omega), x_{2n_0+1}(\omega)\right) \\
& = \max\Big\{ d\left(A\left(\omega, x_{2n_0}(\omega)\right), B\left(\omega, x_{2n_0+1}(\omega)\right)\right), \\
& \quad d\left(A\left(\omega, x_{2n_0}(\omega)\right), S\left(\omega, x_{2n_0}(\omega)\right)\right), d\left(T\left(\omega, x_{2n_0+1}(\omega)\right), B\left(\omega, x_{2n_0+1}(\omega)\right)\right), \\
& \quad \frac{1}{2}\left[d\left(S\left(\omega, x_{2n_0}(\omega)\right), B\left(\omega, x_{2n_0+1}(\omega)\right)\right) + d\left(A\left(\omega, x_{2n_0}(\omega)\right), T\left(\omega, x_{2n_0+1}(\omega)\right)\right)\right]\Big\} \\
& = \max\Big\{ d\left(y_{2n_0}(\omega), y_{2n_0+1}(\omega)\right), d\left(y_{2n_0}(\omega), y_{2n_0+1}(\omega)\right), d\left(y_{2n_0+1}(\omega), y_{2n_0+2}(\omega)\right), \\
& \quad \frac{1}{2}\left[d\left(y_{2n_0+1}(\omega), y_{2n_0+1}(\omega)\right) + d\left(y_{2n_0+2}(\omega), y_{2n_0}(\omega)\right)\right]\Big\} \\
& = \max\left\{d_{2n_0}, d_{2n_0+1}\right\} = d_{2n_0+1}.
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
d_{2n_0+1} &= d\left(y_{2n_0+1}(\omega), y_{2n_0+2}(\omega)\right) \\
&= H\left(S\left(\omega, x_{2n_0}(\omega)\right), T\left(\omega, x_{2n_0+1}(\omega)\right)\right) \\
&\le \psi\left(\max\left\{m_i\left(x_{2n_0}(\omega), x_{2n_0+1}(\omega)\right) : 1 \le i \le 4\right\}\right) \\
&= \psi\left(\max\left\{d_{2n_0+1}, 0, 0, d_{2n_0+1}\right\}\right) < d_{2n_0+1},
\end{aligned} \tag{11}
$$

which is a contradiction. Hence $d_{2n_0+1} = 0$. It follows that

$$y_{2n_0}(\omega) = y_{2n_0+1}(\omega) = A\left(\omega, x_{2n_0}(\omega)\right) \in S\left(\omega, x_{2n_0}(\omega)\right),$$

and

$$y_{2n_0+1}(\omega) = y_{2n_0+2}(\omega) = T\left(\omega, x_{2n_0+1}(\omega)\right) \in B\left(\omega, x_{2n_0+1}(\omega)\right)$$

Put $a(\omega) = x_{2n_0}(\omega)$ and $b(\omega) = x_{2n_0+1}(\omega)$. It is easy to see that (3.4) holds and $y_{2n_0}(\omega)$ is a random common fixed point of $A(\omega, x(\omega))$, $B(\omega, x(\omega))$, $S(\omega, x(\omega))$ and $T(\omega, x(\omega))$.

Case2. There exists $n \in N$ satisfying $d_{2n_0-1} = 0$. As in the proof of Case1, we infer similarly that (3.4) holds for $a(\omega) = x_{2n_0}(\omega)$ and $b(\omega) = x_{2n_0-1}(\omega)$ and $y_{2n_0-1}(\omega)$ is a random common fixed point of $A(\omega, x(\omega))$, $B(\omega, x(\omega))$, $S(\omega, x(\omega))$ and $T(\omega, x(\omega))$.

Case3. $y_n(\omega) \ne y_{n+1}(\omega)$ for all $n \in N$. Now we claim that $d_{2n} \le d_{2n-1}$ for all $n \in N$. Suppose that $d_{2n} \le d_{2n-1}$ for some $n \in N$. By virtue of (3.1), (3.2) and

$\psi \in \Psi$, we arrive at

$$\begin{aligned}
&m_1\left(x_{2n}(\omega), x_{2n-1}(\omega)\right) \\
&= d\left(B\left(\omega, x_{2n-1}(\omega)\right), T\left(\omega, x_{2n-1}(\omega)\right)\right) \cdot \frac{1 + d\left(A\left(\omega, x_{2n}(\omega)\right), S\left(\omega, x_{2n}(\omega)\right)\right)}{1 + d\left(A\left(\omega, x_{2n}(\omega)\right), B\left(\omega, x_{2n-1}(\omega)\right)\right)} \\
&= d\left(y_{2n-1}(\omega), y_{2n}(\omega)\right) \cdot \frac{1 + d\left(y_{2n}(\omega), y_{2n+1}(\omega)\right)}{1 + d\left(y_{2n}(\omega), y_{2n-1}(\omega)\right)} \\
&= d_{2n-1} \frac{1 + d_{2n}}{1 + d_{2n-1}} < d_{2n}, \\
&m_2\left(x_{2n}(\omega), x_{2n-1}(\omega)\right) \\
&= d\left(A\left(\omega, x_{2n}(\omega)\right), S\left(\omega, x_{2n}(\omega)\right)\right) \cdot \frac{1 + d\left(B\left(\omega, x_{2n-1}(\omega)\right), T\left(\omega, x_{2n-1}(\omega)\right)\right)}{1 + d\left(A\left(\omega, x_{2n}(\omega)\right), B\left(\omega, x_{2n-1}(\omega)\right)\right)} \\
&= d\left(y_{2n}(\omega), y_{2n+1}(\omega)\right) \cdot \frac{1 + d\left(y_{2n-1}(\omega), y_{2n}(\omega)\right)}{1 + d\left(y_{2n}(\omega), y_{2n-1}(\omega)\right)} \\
&= d_{2n}, \\
&m_3\left(x_{2n}(\omega), x_{2n-1}(\omega)\right) \\
&= \frac{d\left(B\left(\omega, x_{2n-1}(\omega)\right), S\left(\omega, x_{2n}(\omega)\right)\right)}{1 + d\left(A\left(\omega, x_{2n}(\omega)\right), B\left(\omega, x_{2n-1}(\omega)\right)\right)} \cdot d\left(A\left(\omega, x_{2n}(\omega)\right), T\left(\omega, x_{2n-1}(\omega)\right)\right) \\
&= \frac{d\left(y_{2n+1}(\omega), y_{2n-1}(\omega)\right) d\left(y_{2n}(\omega), y_{2n}(\omega)\right)}{1 + d\left(y_{2n}(\omega), y_{2n-1}(\omega)\right)} \\
&= 0, \\
&m_4\left(x_{2n}(\omega), x_{2n-1}(\omega)\right) \\
&= \max\Big\{d\left(A\left(\omega, x_{2n}(\omega)\right), B\left(\omega, x_{2n-1}(\omega)\right)\right), \\
&\quad d\left(A\left(\omega, x_{2n}(\omega)\right), S\left(\omega, x_{2n}(\omega)\right)\right), d\left(T\left(\omega, x_{2n-1}(\omega)\right), B\left(\omega, x_{2n-1}(\omega)\right)\right), \\
&\quad \frac{1}{2}\left[d\left(S\left(\omega, x_{2n}(\omega)\right), B\left(\omega, x_{2n-1}(\omega)\right)\right) + d\left(A\left(\omega, x_{2n}(\omega)\right), T\left(\omega, x_{2n-1}(\omega)\right)\right)\right]\Big\} \qquad (12) \\
&= \max\Big\{d\left(y_{2n}(\omega), y_{2n-1}(\omega)\right), d\left(y_{2n}(\omega), y_{2n+1}(\omega)\right), d\left(y_{2n-1}(\omega), y_{2n}(\omega)\right), \\
&\quad \frac{1}{2}\left[d\left(y_{2n+1}(\omega), y_{2n-1}(\omega)\right) + d\left(y_{2n}(\omega), y_{2n}(\omega)\right)\right]\Big\} \\
&= \max\left\{d_{2n-1}, d_{2n}\right\} = d_{2n}
\end{aligned}$$

and

$$\begin{aligned}
d_{2n} &= d\left(y_{2n+1}(\omega), y_{2n}(\omega)\right) \\
&\le H\left(S\left(\omega, x_{2n}(\omega)\right), T\left(\omega, x_{2n-1}(\omega)\right)\right) \\
&\le \psi\left(\max\left\{m_i\left(x_{2n}(\omega), x_{2n-1}(\omega)\right) : 1 \le i \le 4\right\}\right) \qquad (13) \\
&= \psi\left(\max\left\{d_{2n-1} \frac{1 + d_{2n}}{1 + d_{2n-1}}, d_{2n}, 0, d_{2n}\right\}\right) \\
&= \psi\left(d_{2n}\right) < d_{2n},
\end{aligned}$$

which is absurd. Hence $d_{2n} \le d_{2n-1}$ for each $n \in N$. As in the proofs of (3.6) and (3.7), we infer similarly that $d_{2n+1} \le d_{2n}$ for all $n \in N$. Consequently, $\{d_n\}_{n\in N}$ is a nonincreasing positive sequence, which means that there exists a constant $r \ge 0$ with

$$\lim_{n\to\infty} d_n = r \tag{14}$$

Suppose that $r > 0$. Making use of (2.1), (2.2), (2.6), (2.8) and $\psi \in \Psi$, we get that

$$\begin{aligned} d_{2n} &= d\left(y_{2n+1}(\omega), y_{2n}(\omega)\right) \\ &\le H\left(S\left(\omega, x_{2n}(\omega)\right), T\left(\omega, x_{2n-1}(\omega)\right)\right) \\ &\le \psi\left(\max\left\{m_i\left(x_{2n}(\omega), x_{2n-1}(\omega)\right) : 1 \le i \le 4\right\}\right) \\ &= \psi\left(\max\left\{d_{2n-1}\frac{1+d_{2n}}{1+d_{2n-1}}, d_{2n}, 0, d_{2n}\right\}\right), \end{aligned}$$

we take the limit and use the property of ψ ,we get that

$$r = \lim_{n\to\infty} d_{2n} \le \lim_{n\to\infty} \psi\left(d_{2n}\right) \le \psi\left(\limsup_{n\to\infty} d_{2n}\right) = \psi(r) < r,$$

which is a contradiction. Hence $r = 0$. That is,

$$\lim_{n\to\infty} d_n = 0. \tag{15}$$

In order to prove that $\{y_n\}_{n\in N}$ is a Cauchy sequence, by (3.9) we need only to prove that $\{y_{2n}\}_{n\in N}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in N}$ is not a Cauchy sequence. It follows that there exists $\varepsilon Å > 0$ such that for each even integer $2k$ there are even integers $2m_k$, $2n_k$ with $2n_k > 2m_k > 2k$ and

$$d\left(y_{2n_k}(\omega), y_{2m_k}(\omega)\right) \ge \varepsilon. \tag{16}$$

For every even integer $2k$, let $2m_k$ be the least even integer exceeding $2n_k$ satisfying (3.10). It follows that

$$d\left(y_{2n_k}(\omega), y_{2m_k-2}(\omega)\right) < \varepsilon. \tag{17}$$

Note that

$$\begin{aligned} &d\left(y_{2n_k}(\omega), y_{2m_k}(\omega)\right) \le d\left(y_{2n_k}(\omega), y_{2m_k-2}(\omega)\right) + d_{2m_k-2} + d_{2m_k-1}, \\ &\left|d\left(y_{2n_k+1}(\omega), y_{2m_k}(\omega)\right) - d\left(y_{2n_k}(\omega), y_{2m_k}(\omega)\right)\right| \le d_{2n_k}, \\ &\left|d\left(y_{2n_k}(\omega), y_{2m_k-1}(\omega)\right) - d\left(y_{2n_k}(\omega), y_{2m_k}(\omega)\right)\right| \le d_{2m_k-1}, \\ &\left|d\left(y_{2n_k+1}(\omega), y_{2m_k-1}(\omega)\right) - d\left(y_{2n_k+1}(\omega), y_{2m_k}(\omega)\right)\right| \le d_{2m_k-1}. \end{aligned} \tag{18}$$

In terms of (2.9)-(2.12), we know that

$$\begin{aligned} \varepsilon &= \lim_{k\to\infty} d\left(y_{2n_k}(\omega), y_{2m_k}(\omega)\right) = \lim_{k\to\infty} d\left(y_{2n_k+1}(\omega), y_{2m_k}(\omega)\right) \\ &= \lim_{k\to\infty} d\left(y_{2n_k}(\omega), y_{2m_k-1}(\omega)\right) = \lim_{k\to\infty} d\left(y_{2n_k+1}(\omega), y_{2m_k-1}(\omega)\right). \end{aligned} \tag{19}$$

In light of (3.1), (3.2), (3.9), (3.13), $\psi \in \Psi$, we deduce that

$$
\begin{aligned}
& m_1\left(x_{2n_k}(\omega), x_{2m_k-1}(\omega)\right) \\
& = d\left(B\left(\omega, x_{2m_k-1}(\omega)\right), T\left(\omega, x_{2m_k-1}(\omega)\right)\right) \cdot \frac{1+d\left(A\left(\omega, x_{2n_k}(\omega)\right), S\left(\omega, x_{2n_k}(\omega)\right)\right)}{1+d\left(A\left(\omega, x_{2n_k}(\omega)\right), B\left(\omega, x_{2m_k-1}(\omega)\right)\right)} \\
& = d\left(y_{2m_k-1}(\omega), y_{2m_k}(\omega)\right) \cdot \frac{1+d\left(y_{2n_k}(\omega), y_{2n_k+1}(\omega)\right)}{1+d\left(y_{2n_k}(\omega), y_{2m_k-1}(\omega)\right)} \to 0, \text{ as } k \to \infty, \\
& m_2\left(x_{2n_k}(\omega), x_{2m_k-1}(\omega)\right) \\
& = d\left(A\left(\omega, x_{2n_k}(\omega)\right), S\left(\omega, x_{2n_k}(\omega)\right)\right) \cdot \frac{1+d\left(B\left(\omega, x_{2m_k-1}(\omega)\right), T\left(\omega, x_{2m_k-1}(\omega)\right)\right)}{1+d\left(A\left(\omega, x_{2n_k}(\omega)\right), B\left(\omega, x_{2m_k-1}(\omega)\right)\right)} \\
& = d\left(y_{2n_k}(\omega), y_{2n_k+1}(\omega)\right) \cdot \frac{1+d\left(y_{2m_k-1}(\omega), y_{2m_k}(\omega)\right)}{1+d\left(y_{2n_k}(\omega), y_{2m_k-1}(\omega)\right)} \to 0, \text{ as } k \to \infty, \\
& m_3\left(x_{2n_k}(\omega), x_{2m_k-1}(\omega)\right) \\
& = \frac{d\left(B\left(\omega, x_{2m_k-1}(\omega)\right), S\left(\omega, x_{2n_k}(\omega)\right)\right)}{1+d\left(A\left(\omega, x_{2n_k}(\omega)\right), B\left(\omega, x_{2m_k-1}(\omega)\right)\right)} \cdot d\left(A\left(\omega, x_{2n_k}(\omega)\right), T\left(\omega, x_{2m_k-1}(\omega)\right)\right) \\
& = \frac{d\left(y_{2n_k+1}(\omega), y_{2m_k-1}(\omega)\right) d\left(y_{2m_k}(\omega), y_{2n_k}(\omega)\right)}{1+d\left(y_{2n_k}(\omega), y_{2m_k-1}(\omega)\right)} \to \frac{\varepsilon^2}{1+\varepsilon}, \text{ as } k \to \infty, \\
& m_4\left(x_{2n_k}(\omega), x_{2m_k-1}(\omega)\right) \\
& = \max\left\{d\left(A\left(\omega, x_{2n_k}(\omega)\right), B\left(\omega, x_{2m_k-1}(\omega)\right)\right),\right. \\
& \quad d\left(A\left(\omega, x_{2n_k}(\omega)\right), S\left(\omega, x_{2n_k}(\omega)\right)\right), d\left(T\left(\omega, x_{2m_k-1}(\omega)\right), B\left(\omega, x_{2m_k-1}(\omega)\right)\right), \\
& \quad \left.\frac{1}{2}\left[d\left(S\left(\omega, x_{2n_k}(\omega)\right), B\left(\omega, x_{2m_k-1}(\omega)\right)\right) + d\left(A\left(\omega, x_{2n_k}(\omega)\right), T\left(\omega, x_{2m_k-1}(\omega)\right)\right)\right]\right\} \\
& = \max\left\{d\left(y_{2n_k}(\omega), y_{2m_k-1}(\omega)\right), d\left(y_{2n_k}(\omega), y_{2n_k+1}(\omega)\right), d\left(y_{2m_k-1}(\omega), y_{2n_k}(\omega)\right),\right. \\
& \quad \left.\frac{1}{2}\left[d\left(y_{2n_k+1}(\omega), y_{2m_k-1}(\omega)\right) + d\left(y_{2n_k}(\omega), y_{2n_k}(\omega)\right)\right]\right\} \\
& = \max\left\{d_{2n_k-1}, d_{2n_k}\right\} \to \varepsilon, \text{ as } k \to \infty.
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon & = \lim_{k\to\infty} d\left(y_{2n_k+1}(\omega), y_{2m_k}(\omega)\right) = \lim_{k\to\infty} d\left(S\left(\omega, x_{2n_k}(\omega)\right), T\left(\omega, x_{2m_k-1}(\omega)\right)\right) \\
& \le \lim_{k\to\infty} \psi\left(\max\left\{m_i\left(x_{2n_k}(\omega), x_{2m_k-1}(\omega)\right) : 1 \le i \le 4\right\}\right) \\
& \le \psi\left(\limsup_{n\to\infty} \max\left\{m_i\left(x_{2n_k}(\omega), x_{2m_k-1}(\omega)\right) : 1 \le i \le 4\right\}\right) \\
& = \psi\left(\max\left\{0, 0, \frac{\varepsilon^2}{1+\varepsilon}, \varepsilon\right\}\right) \\
& = \psi(\varepsilon) < \varepsilon,
\end{aligned}
$$

which is a contradiction. Therefore $\{y_n(\omega)\}_{n\in N}$ is a Cauchy sequence, Assume that $A(\Omega\times X)$ is complete. Notice that $\{y_n(\omega)\}_{n\in N} \subseteq A(\Omega\times X)$, which implies that $\{y_n(\omega)\}_{n\in N}$ converges to a point $c(\omega)=A(\Omega\times X)$. Obviously $\lim_{n\to\infty} y_n(\omega)=c(\omega)$. Put $a(\omega)\in A^{-1}(\omega,c(\omega))$. It follows that $c(\omega)\in A(\omega,a(\omega))$. Suppose that $c(\omega)\notin S(\omega,a(\omega))$. In view of (3.1)-(3.3), $\psi\in\Psi$, and $\lim_{n\to\infty} y_n(\omega)=c(\omega)$, we infer that

$$\begin{aligned}
&m_1(a(\omega),x_{2n-1}(\omega))\\
&=d\big(B(\omega,x_{2n-1}(\omega)),T(\omega,x_{2n-1}(\omega))\big)\cdot\frac{1+d\big(A(\omega,a(\omega)),S(\omega,a(\omega))\big)}{1+d\big(A(\omega,a(\omega)),B(\omega,x_{2n-1}(\omega))\big)}\\
&=d\big(y_{2n-1}(\omega),y_{2n}(\omega)\big)\cdot\frac{1+d\big(c(\omega),S(\omega,a(\omega))\big)}{1+d\big(c(\omega),y_{2n-1}(\omega)\big)}\to 0,\ \text{ as } n\to\infty,\\
&m_2(a(\omega),x_{2n-1}(\omega))\\
&=d\big(A(\omega,a(\omega)),S(\omega,a(\omega))\big)\cdot\frac{1+d\big(B(\omega,x_{2n-1}(\omega)),T(\omega,x_{2n-1}(\omega))\big)}{1+d\big(A(\omega,a(\omega)),B(\omega,x_{2n-1}(\omega))\big)}\\
&=d\big(c(\omega),S(\omega,a(\omega))\big)\cdot\frac{1+d\big(y_{2n-1}(\omega),y_{2n}(\omega)\big)}{1+d\big(c(\omega),y_{2n-1}(\omega)\big)}\\
&\to d\big(c(\omega),S(\omega,a(\omega))\big),\ \text{ as } n\to\infty,\\
&m_3(a(\omega),x_{2n-1}(\omega))\\
&=\frac{d\big(B(\omega,x_{2n-1}(\omega)),S(\omega,a(\omega))\big)}{1+d\big(A(\omega,a(\omega)),B(\omega,x_{2n-1}(\omega))\big)}\cdot d\big(A(\omega,x_{2n}(\omega)),T(\omega,x_{2n-1}(\omega))\big)\\
&=\frac{d\big(S(\omega,a(\omega)),y_{2n-1}(\omega)\big)d\big(y_{2n}(\omega),c(\omega)\big)}{1+d\big(c(\omega),y_{2n-1}(\omega)\big)}\to 0,\ \text{ as } n\to\infty,\\
&m_4(a(\omega),x_{2n-1}(\omega))\\
&=\max\Big\{d\big(A(\omega,a(\omega)),B(\omega,x_{2n-1}(\omega))\big),\\
&\quad d\big(A(\omega,a(\omega)),S(\omega,a(\omega))\big),d\big(T(\omega,x_{2n-1}(\omega)),B(\omega,x_{2n-1}(\omega))\big),\\
&\quad \frac{1}{2}\Big[d\big(S(\omega,a(\omega)),B(\omega,x_{2n-1}(\omega))\big)+d\big(A(\omega,a(\omega)),T(\omega,x_{2n-1}(\omega))\big)\Big]\Big\}\\
&=\max\Big\{d\big(c(\omega),y_{2n-1}(\omega)\big),d\big(c(\omega),S(\omega,a(\omega))\big),d\big(y_{2n-1}(\omega),y_{2n}(\omega)\big),\\
&\quad \frac{1}{2}\Big[d\big(S(\omega,a(\omega)),y_{2n-1}(\omega)\big)+d\big(y_{2n}(\omega),c(\omega)\big)\Big]\Big\}\\
&\to d\big(c(\omega),S(\omega,a(\omega))\big),\ \text{ as } n\to\infty,
\end{aligned}$$

and

$$\begin{aligned}
&d\big(S(\omega,a(\omega)),c(\omega)\big)\\
&=\lim_{n\to\infty} d\big(S(\omega,a(\omega)),y_{2n}(\omega)\big)\\
&=\lim_{n\to\infty} H\big(S(\omega,a(\omega)),T(\omega,x_{2n}(\omega))\big)\\
&\le\limsup_{n\to\infty}\psi\Big(\max\big\{m_i\big(a(\omega),x_{2n-1}(\omega)\big):1\le i\le 4\big\}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \psi\left(\limsup_{n\to\infty} \max\left\{m_i\left(a(\omega), x_{2n-1}(\omega)\right) : 1 \le i \le 4\right\}\right) \\
&= \psi\left(\max\left\{0, d\left(c(\omega), S\left(\omega, a(\omega)\right)\right), 0, d\left(c(\omega), S\left(\omega, a(\omega)\right)\right)\right\}\right) \\
&= \psi\left(d\left(S\left(\omega, a(\omega)\right), c(\omega)\right)\right) < d\left(S\left(\omega, a(\omega)\right), c(\omega)\right),
\end{aligned}
$$

which is a contradiction. Therefore, $c(\omega) \in S\left(\omega, a(\omega)\right)$, which together with (iii) means that $c(\omega) \in B(\Omega \times X)$. Put $b(\omega) \in B^{-1}\left(\omega, c(\omega)\right)$, that is, $c(\omega) = B\left(\omega, b(\omega)\right)$. Suppose that $c(\omega) \notin T\left(\omega, b(\omega)\right)$. By means of (3.1), (3.2) and $\psi \in \Psi$, we get that

$$
\begin{aligned}
& m_1\left(a(\omega), b(\omega)\right) \\
&= d\left(B\left(\omega, b(\omega)\right), T\left(\omega, b(\omega)\right)\right) \frac{1 + d\left(A\left(\omega, a(\omega)\right), S\left(\omega, a(\omega)\right)\right)}{1 + d\left(A\left(\omega, a(\omega)\right), B\left(\omega, b(\omega)\right)\right)} \\
&= d\left(c(\omega), T\left(\omega, b(\omega)\right)\right), \\
& m_2\left(a(\omega), b(\omega)\right) \\
&= d\left(A\left(\omega, a(\omega)\right), S\left(\omega, a(\omega)\right)\right) \frac{1 + d\left(B\left(\omega, b(\omega)\right), T\left(\omega, b(\omega)\right)\right)}{1 + d\left(A\left(\omega, a(\omega)\right), B\left(\omega, b(\omega)\right)\right)} \\
&= 0, \\
& m_3\left(a(\omega), b(\omega)\right) \\
&= \frac{d\left(B\left(\omega, b(\omega)\right), S\left(\omega, a(\omega)\right)\right)}{1 + d\left(A\left(\omega, a(\omega)\right), B\left(\omega, b(\omega)\right)\right)} \cdot d\left(A\left(\omega, a(\omega)\right), T\left(\omega, b(\omega)\right)\right) \\
&= 0, \\
& m_4\left(a(\omega), b(\omega)\right) \\
&= \max\left\{ d\left(A\left(\omega, a(\omega)\right), B\left(\omega, b(\omega)\right)\right), \right. \\
&\quad d\left(A\left(\omega, a(\omega)\right), S\left(\omega, a(\omega)\right)\right), d\left(T\left(\omega, b(\omega)\right), B\left(\omega, b(\omega)\right)\right), \\
&\quad \left. \frac{1}{2}\left[d\left(S\left(\omega, a(\omega)\right), B\left(\omega, b(\omega)\right)\right) + d\left(A\left(\omega, a(\omega)\right), T\left(\omega, b(\omega)\right)\right)\right]\right\} \\
&= \max\left\{0, 0, d\left(c(\omega), T\left(\omega, b(\omega)\right)\right), \frac{1}{2}\left[0 + d\left(c(\omega), T\left(\omega, b(\omega)\right)\right)\right]\right\} \\
&= d\left(c(\omega), T\left(\omega, b(\omega)\right)\right),
\end{aligned} \tag{20}
$$

and

$$
\begin{aligned}
& d\left(c(\omega), T\left(\omega, b(\omega)\right)\right) \\
&\le H\left(S\left(\omega, a(\omega)\right), T\left(\omega, b(\omega)\right)\right) \\
&\le \psi\left(\max\left\{m_i\left(a(\omega), b(\omega)\right) : 1 \le i \le 4\right\}\right) \\
&= \psi\left(\max\left\{d\left(c(\omega), T\left(\omega, b(\omega)\right)\right), 0, 0, d\left(c(\omega), T\left(\omega, b(\omega)\right)\right)\right\}\right) \\
&= \psi\left(d\left(c(\omega), T\left(\omega, b(\omega)\right)\right)\right) \\
&< d\left(c(\omega), T\left(\omega, b(\omega)\right)\right).
\end{aligned} \tag{21}
$$

which is impossible. That is, $c(\omega)\in T(\omega,b(\omega))$. Hence (3.4) holds.

Assume that $T(\Omega\times X)$ is complete. Notice that $\{y_{2n}(\omega)\}_{n\in N}\subseteq T(\Omega\times X)$, which implies that $\{y_{2n}(\omega)\}_{n\in N}$ converges to a point $c(\omega)\in T(\Omega\times X)$. Obviously $\lim_{n\to\infty} y_n(\omega)=c(\omega)$. Put $b_1(\omega)\in T^{-1}(\omega,c(\omega))$. It follows that $c(\omega)\in T(\omega,b_1(\omega))$. Observe that $T(\Omega\times X)\subseteq A(\Omega\times X)$, which implies that there exists $a(\omega)\in X$ with $c(\omega)=A(\omega,a(\omega))\in T(\omega,b(\omega))$. As in the proof of completeness of $A(\Omega\times X)$, we infer that (3.4) holds. Similarly we conclude that (3.4) holds if one of $B(\Omega\times X)$ and $S(\Omega\times X)$ is complete. This completes the proof.

**Example 3.1 (Concrete application of Theorem 3.1). Let $X=[0,1]$ with the usual metric, $\Omega=[0,1]$ with the Borel σ-algebra, and define

$$S(\omega,x)=\{0\},\ T(\omega,x)=\left\{\frac{x}{2}\right\},\ A(\omega,x)=x,\ B(\omega,x)=\frac{x}{2}.$$

Then $S(\Omega\times X)=\{0\}\subseteq B(\Omega\times X)=[0,1/2]$, and $T(\Omega\times X)=[0,1/2]\subseteq A(\Omega\times X)=[0,1]$. Taking $\psi(t)=t/2$, one can verify that condition (3.1)-(3.2) holds. The unique random common fixed point is $\xi(\omega)=0$ for all $\omega\in\Omega$. This example demonstrates that the rational contraction condition is not vacuous and can be satisfied by simple, non-trivial operators.** As in the proof of Theorem 3.1, we have the following result and omit its proof.

Theorem 3.2 Let (X,d) be a complete separable metric space, let (Ω,Σ) be a measurable space, and let $S,T:\Omega\times X\to CB(X)$ and $A,B:\Omega\times X\to X$ be mappings such that

(i) $A(\omega,\cdot),B(\omega,\cdot),T(\omega,\cdot),S(\omega,\cdot)$ are continuous for all $\omega\in\Omega$;

(ii) $A(\cdot,x),B(\cdot,x),T(\cdot,x),S(\cdot,x)$ are measurable for all $x\in X$;

(iii) $S(\Omega\times X)\subseteq B(\Omega\times X)$ and $T(\Omega\times X)\subseteq A(\Omega\times X)$;

(iv) the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible;

(v) one of $A(\Omega\times X),B(\Omega\times X),S(\Omega\times X)$ and $T(\Omega\times X)$ is a complete and

$$H(S(\omega,x),T(\omega,y))\le\psi(m_4(x,y)),\forall x,y\in X. \tag{22}$$

Then A,B,S and T have a unique random common fixed point in X.

Remark 3.1. If in Theorem 3.1, $A(\omega,x)=B(\omega,x)=x$ for all $(\omega,x)\in\Omega\times X$, then we get the following random fixed point theorem.

Theorem 3.3. Let (X,d) be a complete separable metric space, let (Ω,Σ) be a measurable space, and let $S,T:\Omega\times CB(X)\to X$ be mappings such that

(i) $T(\omega,\cdot),S(\omega,\cdot)$ are continuous for all $\omega\in\Omega$;

(ii) $T(\cdot,x),S(\cdot,x)$ are measurable for all $x\in X$;

(iii) one of $S(\Omega\times X)$ and $T(\Omega\times X)$ is a complete and

$$\begin{aligned}&H(S(\omega,x),T(\omega,y))\\&\le\psi\Bigg(\max\Big\{d(x,y),d(S(\omega,x),x),d(T(\omega,y),y),\\&\quad\frac{1}{2}\big[d(S(\omega,x),y)+d(T(\omega,y),x)\big]\Big\}\Bigg),\forall x,y\in X.\end{aligned}\tag{23}$$

Then S and T have a unique random common fixed point in X.

For $S(\omega,X)=T(\omega,X), A(\omega,X)=B(\omega,X)$ we have the following results as a special case of the above theorem.

Corollary 3.1. Let (X,d) be a complete separable metric space, let (Ω,Σ) be a measurable space, and let $T:\Omega\times X\to CB(X)$, $A:\Omega\times X\to X$, be mappings such that

(i) $T(\omega,\cdot), A(\omega,\cdot)$ are continuous for all $\omega\in\Omega$;

(ii) $T(\cdot,x), A(\cdot,x)$ are measurable for all $x\in X$;

(iii) $A(\Omega\times X)$ is a complete and

$$\begin{aligned} &H(T(\omega,x),T(\omega,y)) \\ &\le\psi\Big(\max\Big\{d(A(\omega,x),A(\omega,y)),d(A(\omega,x),T(\omega,x)),d(A(\omega,y),T(\omega,y)), \\ &\quad \frac{1}{2}\big[d(A(\omega,x),T(\omega,y))+d(A(\omega,y),T(\omega,x))\big]\Big\}\Big),\forall x,y\in X, \end{aligned} \tag{24}$$

then T and A have a unique random common fixed point in X.

Remark 3.2. For $\psi(t)=kt$, then we obtain the corresponding theorem of [18].

Now we deduce the results for single-valued self-mappings from Theorem 3.1, 3.2.

Theorem 3.4. Let (X,d) be a complete separable metric space, let (Ω,Σ) be a measurable space, and let $S,T:\Omega\times X\to X$ and $A,B:\Omega\times X\to X$ be mappings such that

(i) $A(\omega,\cdot),B(\omega,\cdot),T(\omega,\cdot),S(\omega,\cdot)$ are continuous for all $\omega\in\Omega$;

(ii) $A(\cdot,x),B(\cdot,x),T(\cdot,x),S(\cdot,x)$ are measurable for all $x\in X$;

(iii) $S(\Omega\times X)\subseteq B(\Omega\times X)$ and $T(\Omega\times X)\subseteq A(\Omega\times X)$;

(iv) the pairs $\{A,S\}$ and $\{B,T\}$ are weakly random compatible;

(v) one of $A(\Omega\times X),B(\Omega\times X),S(\Omega\times X)$ and $T(\Omega\times X)$ is a complete and

$$d(S(\omega,x),T(\omega,y))\le\psi(\max\{m_i(x,y),1\le i\le 4\}),\forall x,y\in X, \tag{25}$$

where $\psi\in\Psi$ and

$$\begin{aligned} m_1(x,y)&=d(B(\omega,y),T(\omega,y))\frac{1+d(A(\omega,x),S(\omega,x))}{1+d(A(\omega,x),B(\omega,y))}, \\ m_2(x,y)&=d(A(\omega,x),S(\omega,x))\frac{1+d(B(\omega,y),T(\omega,y))}{1+d(A(\omega,x),B(\omega,y))}, \\ m_3(x,y)&=\frac{d(B(\omega,y),S(\omega,x))}{1+d(A(\omega,x),B(\omega,y))}\cdot d(A(\omega,x),T(\omega,y)), \\ m_4(x,y)&=\max\Big\{d(A(\omega,x),B(\omega,y)), \\ &\qquad d(A(\omega,x),S(\omega,x)),d(T(\omega,y),B(\omega,y)), \\ &\qquad \frac{1}{2}\big[d(S(\omega,x),B(\omega,y))+d(A(\omega,x),T(\omega,y))\big]\Big\}. \end{aligned} \tag{26}$$

Then A,B,S and T have a unique random common fixed point in X.

Remark 3.3. For $\psi(t)=t$, then we obtain the following Corollary.

Corollary 3.2. Let (X,d) be a complete separable metric space, let (Ω,Σ) be a measurable space, and let $S,T:\Omega\times X\to X$ and $A,B:\Omega\times X\to X$ be mappings such that

(i) $A(\omega,\cdot),B(\omega,\cdot),T(\omega,\cdot),S(\omega,\cdot)$ are continuous for all $\omega\in\Omega$;

(ii) $A(\cdot,x),B(\cdot,x),T(\cdot,x),S(\cdot,x)$ are measurable for all $x\in X$;

(iii) $S(\Omega\times X)\subseteq B(\Omega\times X)$ and $T(\Omega\times X)\subseteq A(\Omega\times X)$;

(iv) the pairs $\{A,S\}$ and $\{B,T\}$ are weakly random compatible;

(v) one of $A(\Omega\times X),B(\Omega\times X),S(\Omega\times X)$ and $T(\Omega\times X)$ is a complete and

$$d\left(S(\omega,x),T(\omega,y)\right)\le \max\left\{m_i(x,y),1\le i\le 4\right\},\forall x,y\in X, \tag{27}$$

where $\psi\in\Psi$ and

$$\begin{aligned}
m_1(x,y)&=d\left(B(\omega,y),T(\omega,y)\right)\frac{1+d\left(A(\omega,x),S(\omega,x)\right)}{1+d\left(A(\omega,x),B(\omega,y)\right)},\\
m_2(x,y)&=d\left(A(\omega,x),S(\omega,x)\right)\frac{1+d\left(B(\omega,y),T(\omega,y)\right)}{1+d\left(A(\omega,x),B(\omega,y)\right)},\\
m_3(x,y)&=\frac{d\left(B(\omega,y),S(\omega,x)\right)}{1+d\left(A(\omega,x),B(\omega,y)\right)}\cdot d\left(A(\omega,x),T(\omega,y)\right),\\
m_4(x,y)&=\max\Big\{d\left(A(\omega,x),B(\omega,y)\right),\\
&\qquad d\left(A(\omega,x),S(\omega,x)\right),d\left(T(\omega,y),B(\omega,y)\right),\\
&\qquad \frac{1}{2}\left[d\left(S(\omega,x),B(\omega,y)\right)+d\left(A(\omega,x),T(\omega,y)\right)\right]\Big\}.
\end{aligned}\tag{28}$$

Then A,B,S and T have a unique random common fixed point in X.

Conflicts of Interest

The author declares that they have no competing interests.

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