Journal of Applied Mathematics and Physics, 2026, 14(8), 3225-3235
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Viscous Time Theory Frame for Birch-Swinnerton-Dyer Conjecture

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How to cite this paper: Bianchetti, R. and Danesh, P. (2026) Viscous Time Theory Frame for Birch-Swinnerton-Dyer Conjecture. *Journal of Applied Mathematics and Physics*, **14**, 3225-3235.
https://doi.org/10.4236/jamp.2026.148158

Received: July 9, 2026
Accepted: August 24, 2026
Published: August 27, 2026

Abstract

We formulate an arithmetic Viscous Time Theory framework around Selmer complexes of elliptic curves. The construction replaces the Sobolev field space by a finite split model of a Selmer complex, treats local conditions as anchors, and eliminates a finite memory block by a Schur complement. Under the stated splitting and invertibility hypotheses, the active soft sector is isomorphic to the Bloch-Kato Selmer group. Standard determinant-line bookkeeping identifies the finite correction with the p-primary Tate-Shafarevich, Tamagawa, and torsion terms, while normalized Arakelov and nonarchimedean Green energies recover the Néron-Tate height pairing. These results establish an operator framework for organizing the arithmetic ingredients of the Birch-Swinnerton-Dyer formula. They do not prove the Birch-Swinnerton-Dyer conjecture. The rank and leading-term conclusions in Section 6 are obtained only under an explicit Comparison Hypothesis asserting that the active VTT determinant germ agrees with the central germ of the Hasse-Weil L-function with the prescribed normalization. That comparison is not established here and remains the principal open problem.

Keywords

Birch-Swinnerton-Dyer Conjecture, Viscous Time Theory, Elliptic Curves, Selmer Complexes

1. Introduction

The arithmetic of elliptic curves links the finitely generated group $E(\mathbb{Q})$ with the analytic behavior of its Hasse-Weil L-function. Mordell established finite generation [1], Cassels developed the Selmer and Tate-Shafarevich groups [2], Birch and Swinnerton-Dyer formulated the rank and leading-term conjecture [3], and Tate clarified the local arithmetic entering the refined formula [4]. The Gross-Zagier height formula [5], the Bloch-Kato special-value framework [6], Kato's Euler system [7], and Nekovář's Selmer complexes [8] provide the principal arithmetic tools relevant to the present construction.

For the analytic rank, let $E/\mathbb{Q}$ be an elliptic curve with conductor N:

$$r_{\text{an}}(E) = \operatorname{ord}_{s=1} L(E,s), \tag{1}$$

where $L(E,s)$ is the Hasse-Weil L-function. The algebraic rank is the rank of the free part of $E(\mathbb{Q})$. BSD predicts that these two ranks agree and that the first nonzero Taylor coefficient is determined by the period, regulator, Tamagawa factors, torsion, and the Tate-Shafarevich group.

The present paper uses the VTT variational architecture developed in [9]-[11] to organize these arithmetic objects. It constructs a finite split model of the Selmer complex, realizes the active operator as a Schur complement, and identifies its soft sector with Selmer cohomology under explicit hypotheses. It also records how canonical heights and the finite arithmetic factors enter the same determinant-line bookkeeping. The paper does not prove BSD or any new case of BSD. Section 6 isolates an unproven Comparison Hypothesis between the active determinant germ and $L(E,s)$; Theorem 8 states only the formal conditional implication that this hypothesis would yield the BSD rank and leading-term formulas.

2. VTT for Arithmetic

The original VTT construction begins with a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, a Sobolev phase space $H_0^1(\Omega)$, and finitely many anchored measurements. Point evaluation is not continuous on $H_0^1(\Omega)$ in general dimensions, so VTT replaces point anchors with mollified averages [10] [11]. If $a_i \in \Omega$, $s_i \in \mathbb{R}$, and ρ_ε is a standard mollifier, the anchor functional is

$$\ell_{i,\varepsilon}(\Phi) = \int_\Omega \rho_\varepsilon(x-a_i)\Phi(x)\,\mathrm{d}x. \tag{2}$$

This small regularization is mathematically important. It makes each anchor a bounded linear functional on the Sobolev space.

A basic passive VTT action has the form

$$\mathcal{A}_\varepsilon(\Phi) = \frac{1}{2}\int_\Omega |\nabla\Phi|^2\,\mathrm{d}x + \frac{\theta}{2}\int_\Omega \Phi^2\,\mathrm{d}x + \frac{1}{2}\sum_{i=1}^{M}\alpha_i\left(\ell_{i,\varepsilon}(\Phi)-s_i\right)^2 - \int_\Omega C\Phi\,\mathrm{d}x, \tag{3}$$

where $\theta > 0$, $\alpha_i > 0$, and C is a source density. The first two terms stabilize the field. The anchor terms impose coherent data without leaving the Hilbert-space setting.

The second variation of the passive action is the bilinear form

$$B_{\text{pass}}(\phi,\psi) = \int_\Omega \nabla\phi\cdot\nabla\psi\,\mathrm{d}x + \theta\int_\Omega \phi\psi\,\mathrm{d}x + \sum_{i=1}^{M}\alpha_i\ell_{i,\varepsilon}(\phi)\ell_{i,\varepsilon}(\psi). \tag{4}$$

The last summand is nonnegative. Hence, the passive Hessian is coercive. This is the first structural lesson for BSD: a coercive passive form has no rank-producing kernel.

Theorem 1. The passive VTT Hessian associated with B_{pass} is bounded, symmetric, and coercive on $H_0^1(\Omega)$. In particular, the passive problem has a unique minimizer.

Proof of Theorem 1. Boundedness follows from the continuity of the mollified anchors and the Cauchy-Schwarz inequality. Symmetry is immediate from the definition of B_{pass}. Since $\theta > 0$, the Poincaré inequality gives a constant $c > 0$ such that

$$B_{\text{pass}}(\phi,\phi) \ge c\|\phi\|_{H_0^1(\Omega)}^2. \tag{5}$$

The Lax-Milgram theorem gives the unique critical point, and strict convexity identifies it as the unique minimizer.

The active step introduces memory. We set X be the visible phase space and M a finite-dimensional memory space. By supposing $P: X \to X^\vee$ is the visible Hessian block, $Q: M \to M^\vee$ is positive definite, and $L: X \to M^\vee$ loads visible perturbations into memory. Eliminating the memory variable gives the active operator

$$H_{\text{act}} = P - L^\vee Q^{-1} L. \tag{6}$$

The sign is the point. Memory does not merely add stiffness. After elimination, it subtracts a nonnegative finite-rank correction. This is how an initially stable passive system can develop isolated soft modes. In Figure 1, the passive spectrum remains uniformly positive. The active spectrum contains three small eigenvalues,

$$0.025, 0.061, 0.112,$$

all below the displayed cutoff of 0.15. The next active eigenvalue is approximately 1.53, so the soft sector is spectrally separated. This is the numerical picture behind the arithmetic claim that the active kernel should be identified with a Selmer space, while the passive form should remain stable.

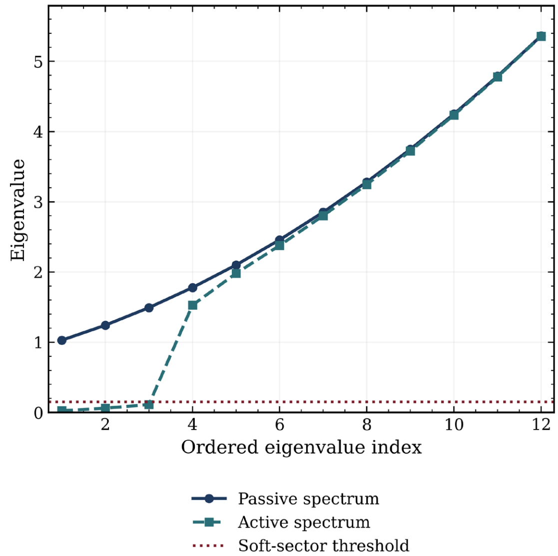

Figure 1. Isolation of the active soft sector.

The one-mode model used in Figure 2 is

$$\lambda_{\mathrm{act}}(\rho)=1.20-\rho^2, \mathcal{R}(\rho)=\frac{1}{\lambda_{\mathrm{act}}(\rho)+0.03}. \tag{7}$$

The soft threshold is $\rho_*=\sqrt{1.20}\approx 1.095$. At $\rho=0$, the response is $1/1.23\approx 0.813$. Near the threshold, the regularized response approaches $1/0.03\approx 33.33$. The active mode softens because of the Schur-complement correction.

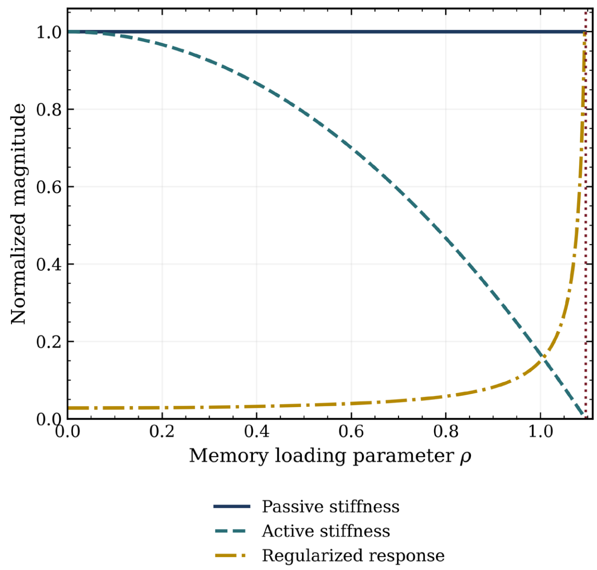

Figure 2. Schur-complement softening of one active mode.

For arithmetic purposes, the VTT dictionary is listed in Table 1. This table fixes the construction used in the rest of the paper.

Table 1. VTT objects and their arithmetic replacements.

VTT Object	Arithmetic Replacement
Sobolev Phase Field	Degree-One Selmer Cochain
Mollified Anchor	Local Bloch-Kato or NÉRon Condition
Passive Hessian	Stable Anchored Arithmetic Form
Memory Variable	Local Obstruction and Dual Selmer Data
Schur Complement	Active Selmer Hessian
Soft Sector	Bloch-Kato Selmer Group
Tensor Response	Height-Pairing Matrix
Finite Memory Determinant	Tate-Shafarevich, Tamagawa, and Torsion Factor
Determinant Germ	Central Expansion of $L(E,s)$ after Comparison

The entries in Table 1 arise from specific constructions. Degree-one Selmer cochains are the visible arithmetic variables because the mapping cone in Equation

(9) records global cochains together with their local-condition defects. The local maps act as anchors because their simultaneous vanishing is precisely the Selmer condition. The soft-sector identification is proved, rather than assumed: after the finite split model is constructed in Section 3, Lemma 2 identifies the Schur-complement kernel with the projected harmonic subspace, and Proposition 3 identifies that subspace with first Selmer cohomology. The finite-memory entry comes from torsion cohomology in the integral Selmer complex and from the local Néron component groups, whose orders supply the Tate-Shafarevich and Tamagawa factors; the Mordell-Weil torsion normalization is explained in Proposition 5. The tensor-response entry is justified by Theorem 6, where the Green-energy pairing is identified with the Néron-Tate height. The last row is deliberately conditional: the determinant germ becomes an L-function germ only under the Comparison Hypothesis of Section 6.

3. Selmer Complexes as the Arithmetic VTT Phase Space

We fix a prime p

$$T = T_p E, V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{8}$$

Consider S contain p, the infinite place, and every prime of bad reduction. Let $G_{\mathbb{Q},S}$ be the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S. For each $v \in S$, let G_v be the local absolute Galois group.

The Selmer complex is the arithmetic phase space. In Nekovář's language, it is represented by a mapping cone

$$C_E^{\bullet} = \mathrm{Cone}\Big(R\Gamma\big(G_{\mathbb{Q},S}, V\big) \oplus \bigoplus_{v\in S} U_v \to \bigoplus_{v\in S} R\Gamma\big(G_v, V\big)\Big)[-1]. \tag{9}$$

The local complex U_v imposes the chosen local condition. Away from pN, it is unramified. At p, it is the Bloch-Kato finite condition. At bad primes, it records the Néron local condition and component-group correction. At ∞, it records the real compatibility condition.

The first cohomology of $C_E^{\bullet}$ is the Bloch-Kato Selmer group,

$$H^1\big(C_E^{\bullet}\big) = H_f^1\big(\mathbb{Q}, V\big). \tag{10}$$

Under the standard finiteness hypotheses for Selmer complexes, $C_E^{\bullet}$ is perfect over $\mathbb{Q}_p$ [8]. Choose a bounded complex $F_E^{\bullet}$ of finite-dimensional $\mathbb{Q}_p$-vector spaces and a quasi-isomorphism $F_E^{\bullet} \to C_E^{\bullet}$. Write $B_E^i = \operatorname{im} d_E^{i-1}$ and $Z_E^i = \ker d_E^i$. Finite-dimensionality permits complements $\mathcal{H}_E^i$ and A_E^i with $Z_E^i = B_E^i \oplus \mathcal{H}_E^i$ and $F_E^i = Z_E^i \oplus A_E^i$; the restriction $d_E^i : A_E^i \to B_E^{i+1}$ is then an isomorphism. In Equation (11), $\big(d_E^i\big)^*$ denotes the inverse of this restriction on B_E^{i+1} and zero on $\mathcal{H}_E^{i+1} \oplus A_E^{i+1}$. Thus $*$ is the splitting operator determined by these complements, not a canonical p-adic metric adjoint. Any norm on the finite-dimensional terms makes the maps continuous; the VTT term "Hilbert model" refers to this finite normed realization.

$$\Delta_E^1 = d_E^0\big(d_E^0\big)^* + \big(d_E^1\big)^* d_E^1. \tag{11}$$

By construction, Δ_E^1 acts as the identity on $B_E^1 \oplus A_E^1$ and as zero on $\mathcal{H}_E^1$. Consequently, $F_E^1 = B_E^1 \oplus \mathcal{H}_E^1 \oplus A_E^1$ is a degree-one Hodge decomposition, $\ker \Delta_E^1 = \mathcal{H}_E^1$, and the quotient map $Z_E^1 \to Z_E^1 / B_E^1$ identifies $\mathcal{H}_E^1$ with $H^1\left(F_E^\bullet\right) \cong H^1\left(C_E^\bullet\right)$. The representative harmonic subspace depends on the chosen complements, but the resulting cohomology group and its dimension do not.

Lemma 2. Let X and M be finite-dimensional vector spaces over a field K, equipped with fixed nondegenerate pairings, and let L^* denote the corresponding transpose. Set

$$\mathcal{B} = \begin{pmatrix} P & L^* \\ L & Q \end{pmatrix}$$

where P, Q, and L are the indicated block maps and Q is invertible. Then, the projection of $\ker \mathcal{B}$ onto X is an isomorphism onto $\ker\left(P - L^* Q^{-1} L\right)$.

Proof of Lemma 2. A vector (x, m) lies in $\ker \mathcal{B}$ exactly when $Px + L^* m = 0$ and $Lx + Qm = 0$. Since Q is invertible, the second equation gives $m = -Q^{-1} L x$. Substitution into the first gives $\left(P - L^* Q^{-1} L\right) x = 0$. Conversely, every vector in this Schur-complement kernel determines the unique memory coordinate $m = -Q^{-1} L x$. □

Choose a decomposition $F_E^1 = X_E \oplus M_E$ into visible and memory coordinates. When the memory block Q_E of Δ_E^1 is invertible, define the active arithmetic Hessian to be the Schur complement $H_{\mathrm{act},E} = P_E - L_E^* Q_E^{-1} L_E$ on X_E.

Proposition 3. Let $F_E^\bullet$ be the finite split model constructed above. Assume that $F_E^1 = X_E \oplus M_E$ is chosen so that the memory block Q_E of Δ_E^1 is invertible, and define $H_{\mathrm{act},E}$ as its Schur complement. Then, the projection induces an isomorphism $\ker H_{\mathrm{act},E} \cong H_f^1(\mathbb{Q}, V)$.

Proof of Proposition 3. The split construction gives $\ker \Delta_E^1 = \mathcal{H}_E^1$ and $\mathcal{H}_E^1 \cong H^1\left(F_E^\bullet\right) \cong H^1\left(C_E^\bullet\right) = H_f^1(\mathbb{Q}, V)$. In the visible-memory coordinates, Lemma 2 identifies $\ker H_{\mathrm{act},E}$ with the projection of $\ker \Delta_E^1$ onto X_E. Invertibility of Q_E makes the memory coordinate unique, so that projection is an isomorphism. The representative isomorphism depends on the selected splittings, whereas the cohomology space and its dimension are intrinsic. □

The resulting cohomological identification is

$$\mathcal{S}_E^{\mathrm{act}} \simeq H_f^1(\mathbb{Q}, V). \tag{12}$$

Thus, the active soft modes represent Selmer cohomology classes.

4. Mordell-Weil Rank and Tate-Shafarevich Memory

The Tate-Shafarevich group is the global obstruction group

$$\mathrm{Sha}(E/\mathbb{Q}) = \ker\left(H^1(\mathbb{Q}, E) \to \prod_v H^1(\mathbb{Q}_v, E)\right). \tag{13}$$

An element of $\mathrm{Sha}(E/\mathbb{Q})$ is represented by a torsor under E that has points over every completion of $\mathbb{Q}$, but not necessarily over $\mathbb{Q}$ itself. In the VTT language, Sha is hidden arithmetic memory which satisfies every local test, yet it is not a visible rational point.

The Kummer sequence gives the relation between rational points, Selmer classes, and Sha :

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p \to H^1_f(\mathbb{Q}, V) \to V_p \mathrm{Sha}(E/\mathbb{Q}) \to 0. \tag{14}$$

Here, $V_p \mathrm{Sha}(E/\mathbb{Q})$ is the rational p-adic Tate module of the p-primary part of Sha .

Theorem 4. Let $E/\mathbb{Q}$ and p be as in Section 3, and assume the hypotheses of Proposition 3. Then $\dim_{\mathbb{Q}_p} \ker H_{act,E} = \mathrm{rank}\, E(\mathbb{Q}) + \dim_{\mathbb{Q}_p} V_p \mathrm{Sha}(E/\mathbb{Q})$. If $\mathrm{Sha}(E/\mathbb{Q})[p^\infty]$ is finite, the second term vanishes, and the active soft dimension equals the Mordell-Weil rank.

Proof of Theorem 4. Proposition 3 identifies the active soft sector with $H^1_f(\mathbb{Q}, V)$. Exactness of Equation (14) gives the stated dimension formula. When $\mathrm{Sha}(E/\mathbb{Q})[p^\infty]$ is finite, its rational p-adic Tate module is zero, so only $\mathrm{rank}\, E(\mathbb{Q})$ remains. □

The finite part is the arithmetic memory determinant. On the integral side, the finite memory contribution is recorded by

$$\mathfrak{m}_E = \frac{|\mathrm{Sha}(E/\mathbb{Q})| \prod_\ell c_\ell}{|E(\mathbb{Q})_{\mathrm{tors}}|^2}, \tag{15}$$

whenever $\mathrm{Sha}(E/\mathbb{Q})$ is finite. The integer c_ℓ is the Tamagawa number at ℓ.

Proposition 5. Let p be a prime. Assume that $\mathrm{Sha}(E/\mathbb{Q})[p^\infty]$ is finite and that the integral Selmer complex is equipped with the standard Néron local conditions and the usual BSD determinant-line normalization. Then the p-adic valuation of the finite arithmetic memory factor is given by Equation (16), where $v_p(|\mathrm{Sha}(E/\mathbb{Q})|)$ means the valuation of its p-primary part.

$$\ell_p(\mathcal{M}_{E,p}) = v_p(|\mathrm{Sha}(E/\mathbb{Q})|) + \sum_\ell v_p(c_\ell) - 2v_p(|E(\mathbb{Q})_{\mathrm{tors}}|). \tag{16}$$

Proof of Proposition 5. For every $n \geq 1$, the Kummer sequence gives $0 \to E(\mathbb{Q})/p^n E(\mathbb{Q}) \to \mathrm{Sel}_{p^n}(E/\mathbb{Q}) \to \mathrm{Sha}(E/\mathbb{Q})[p^n] \to 0$ [2]. After stabilization, the finite global obstruction therefore contributes $v_p(|\mathrm{Sha}(E/\mathbb{Q})[p^\infty]|)$. At a bad prime ℓ, the quotient occurring in the Néron local condition is the p-primary part of the component group $\Phi_\ell(\mathbb{F}_\ell)$; its order is the Tamagawa number c_ℓ, so determinant multiplicativity contributes $v_p(c_\ell)$ [8] [12] [13]. Finally, the BSD determinant-line trivialization uses the Mordell-Weil lattice together with its dual. Replacing $E(\mathbb{Q})$ by its free quotient contributes $-v_p(|E(\mathbb{Q})_{\mathrm{tors}}|)$ in each of these two dual factors, hence the term $-2v_p(|E(\mathbb{Q})_{\mathrm{tors}}|)$ [6] [8] [12]. Additivity of lengths in exact sequences and exact triangles yields Equation (16). This bookkeeping identity does not prove the finiteness of Sha . □

5. Height Energy and the Regulator

The refined BSD formula contains the regulator. In VTT, this regulator is the determinant of the soft height-energy pairing.

First, we let $P \in E(\mathbb{Q})$, and let $D_P = (P) - (O)$. At the Archimedean place, we choose the normalized Green potential on $E(\mathbb{C})$. At a nonarchimedean place, use the corresponding admissible Green function on the regular model or Berkovich skeleton. For rational points P, Q, define the local VTT height energy by

$$\mathcal{E}_v(P,Q) = \int_{\mathcal{X}_v} \mathrm{d}g_{P,v}\, \mathrm{d}g_{Q,v} + \iota_v(D_P, D_Q), \tag{17}$$

where $\mathcal{X}_v$ denotes the analytic surface at $v = \infty$ and the reduction graph or Berkovich skeleton at finite v. The term ι_v is the local intersection correction. When divisors meet, the value is defined by the standard limiting procedure used in Arakelov theory.

The global VTT height pairing is the sum over all places,

$$\langle P, Q \rangle_{\mathrm{VTT}} = \sum_v \mathcal{E}_v(P,Q). \tag{18}$$

Theorem 6. Let $E/\mathbb{Q}$ be an elliptic curve. At the archimedean place, use the normalized Arakelov Green function; at every finite place, use the normalized Néron local height represented on a regular model, equivalently by the admissible Green function on the associated Berkovich skeleton after semistable reduction, together with the local intersection correction. For intersecting divisors, use the standard regularization. Under these normalizations, the pairing in Equation (18) is well defined and equals the Néron-Tate height pairing on $E(\mathbb{Q})$.

Proof of Theorem 6. The stated local Green functions define the normalized Néron local pairings: they are symmetric and bilinear on degree-zero divisors, are compatible with principal divisors, and incorporate the required intersection correction. The product formula removes the remaining additive ambiguity when the local terms are summed. The resulting global pairing is therefore the canonical Néron-Tate height pairing [13] [14]. □

By letting $P_1, \cdots, P_r$ as a basis of $E(\mathbb{Q})/E(\mathbb{Q})_{\mathrm{tors}}$, the VTT regulator is

$$\mathrm{Reg}_{\mathrm{VTT}}(E) = \det\left(\langle P_i, P_j \rangle_{\mathrm{VTT}}\right)_{1 \le i, j \le r}. \tag{19}$$

Corollary 7. The VTT regulator is the classical Néron-Tate regulator.

Proof of Corollary 7. By Theorem 6, every entry of the matrix in (19) is the corresponding Néron-Tate height pairing. Taking the determinant gives the usual regulator.

This completes the height-theoretic part of the framework. The VTT tensor layer has a precise arithmetic meaning: it is the height matrix on the free Mordell-Weil group.

6. Conditional Determinant Comparison and BSD Implication

Let us $\mathcal{S}_E^{\mathrm{act}}$ be the active soft sector and let $r = \dim \mathcal{S}_E^{\mathrm{act}}$. On the complement of the soft sector, the active Hessian is invertible, and the determinant germ at the central point is

$$D_E^{\mathrm{VTT}}(t) = t^r \det{}^{\#}\left(H_{\mathrm{act},E}\right)\left(1+O(t)\right), t \to 0, \tag{20}$$

where $\det^{\#}$ denotes the determinant of the non-soft component.

The determinant germ has order r at $t=0$. Transferring this spectral order to the analytic rank requires an independent comparison with $L(E,s)$; no such comparison follows from Sections 2 - 5.

Comparison Hypothesis. There is a holomorphic function $u_E(s)$, nonzero at $s=1$, such that

$$L(E,s) = u_E(s) D_E^{\mathrm{VTT}}(s-1), \tag{21}$$

and the nonzero leading factor satisfies

$$u_E(1)\det{}^{\#}\left(H_{\mathrm{act},E}\right) = \Omega_E \mathrm{Reg}_{\mathrm{VTT}}(E)\mathfrak{m}_E. \tag{22}$$

The Comparison Hypothesis is not proved in this paper and is not a consequence of the VTT-Selmer construction. It contains the unresolved analytic step. Accordingly, the result below is a conditional compatibility statement only and does not establish BSD for any elliptic curve.

Figure 3 explains the multiplicative leading coefficient on a logarithmic scale. The plotted template uses

$$-0.35+0.62+0.69+1.10-0.88=1.18,$$

so the corresponding illustrative coefficient is $e^{1.18} \approx 3.25$. In a curve-specific computation, those five numbers must be replaced by the actual logarithms of the period, regulator, Tate-Shafarevich order, Tamagawa product, and torsion denominator.

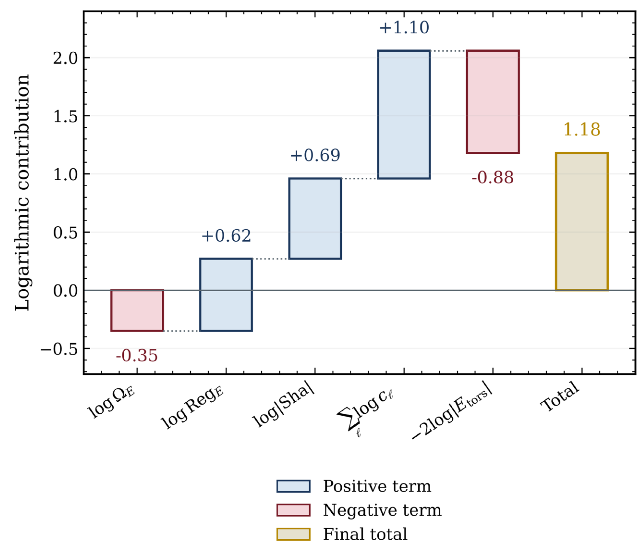

Figure 3. Logarithmic balance of the BSD leading coefficient.

Theorem 8 (conditional implication). Let $E/\mathbb{Q}$ be an elliptic curve. Assume the hypotheses of Proposition 3, the finiteness of $\mathrm{Sha}(E/\mathbb{Q})$, the local-height normalizations of Theorem 6, and the Comparison Hypothesis in Equations (21)-

(22). Then the BSD rank equality and the refined leading-coefficient formula follow for E.

Proof of Theorem 8. Under the Comparison Hypothesis, Equation (20) shows that $D_E^{\mathrm{VTT}}(t)$ has a zero of order r at $t=0$, and Equation (21) transfers that order to $L(E,s)$ at $s=1$. Proposition 3 and Theorem 4 identify r with $\operatorname{rank} E(\mathbb{Q})$ under the stated finiteness assumption. Expanding Equation (21) at $s=1$ and using Equation (22) identifies the first nonzero coefficient with the period, VTT regulator, and finite memory factor.

Corollary 7 replaces the VTT regulator with the Néron-Tate regulator, while Proposition 5 expands the memory term into the Tate-Shafarevich, Tamagawa, and torsion factors. Hence,

$$\frac{L^{(r)}(E,1)}{r!} = \Omega_E \mathrm{Reg}_E \frac{\left|\mathrm{Sha}(E/\mathbb{Q})\right| \prod_\ell c_\ell}{\left|E(\mathbb{Q})_{\mathrm{tors}}\right|^2}. \tag{23}$$

Under the stated hypotheses, this is the refined BSD formula in the normalization fixed in this paper. Without the Comparison Hypothesis, neither the analytic-rank conclusion nor the leading-coefficient identity follows from the preceding sections. The unresolved step is therefore the following open problem.

Open comparison problem. Construct and prove the determinant comparison in Equations (21) and (22) for elliptic curves $E/\mathbb{Q}$, with the prescribed period, regulator, Tate-Shafarevich, Tamagawa, and torsion normalization. Until this problem is solved, the present construction remains a framework with a conditional implication, not a proof of BSD.

7. Conclusion

This paper develops a VTT-inspired operator framework for organizing Selmer cohomology, canonical heights, and the finite arithmetic terms in the BSD leading coefficient. Its unconditional content is the finite split-model construction, the Schur-complement kernel identification under an invertible memory block, the standard local-height realization of the regulator, and the determinant-line bookkeeping of the finite factors under the stated finiteness and normalization hypotheses. The paper does not prove the Birch-Swinnerton-Dyer conjecture or any new case of it. Theorem 8 is a formal conditional implication: BSD follows only after the Comparison Hypothesis in Equations (21) and (22) has been established independently. That comparison remains open and contains the central analytic difficulty. Future work must therefore construct the determinant germ from arithmetic data and prove its equality with the central Hasse-Weil L-function germ before the framework can yield an unconditional BSD result.

Author Contributions

Raoul Bianchetti contributed to the mathematical development, verification of the arithmetic framework, and refinement of the manuscript. Payam Danesh developed the main VTT-BSD concept and overall theoretical structure.

Acknowledgements

We are grateful to the Viscous Time Theory (VTT) team for their valuable feedback on this version. Their comments helped improve the clarity, structure, and presentation of this work.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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