Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2026, 14(8), 2901-2916
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Formulas for the Kinetic Energy and Momentum of an Electron in a Hydrogen Atom, and the Relationship between the Two

Koshun Suto

Chudai-Ji Temple, Isesaki, Japan
Email: hunza9980@nifty.com

How to cite this paper: Suto, K. (2026) Formulas for the Kinetic Energy and Momentum of an Electron in a Hydrogen Atom, and the Relationship between the Two. *Journal of Applied Mathematics and Physics*, **14**, 2901-2916.
https://doi.org/10.4236/jamp.2026.148143

Received: July 2, 2026
Accepted: August 10, 2026
Published: August 13, 2026

Abstract

The formula for kinetic energy in classical mechanics holds when the velocity of a body is low. However, the formula no longer holds when the velocity of the body increases, and the theory of relativity can no longer be ignored. In classical mechanics, the mass of a body is constant and does not depend on the velocity of the body. Einstein and Sommerfeld defined the kinetic energy of a body as the difference between the relativistic energy of the body mc^2 and its rest mass energy m_0c^2. However, Einstein's energy-momentum relationship, which holds in an isolated system in free space, is not applicable in a hydrogen atom, where potential energy is present. The author has already derived an energy-momentum relationship for an electron in a hydrogen atom, but this paper derives previously unknown formulas for the kinetic energy and momentum of an electron by taking that previous relationship as a departure point.

Keywords

Einstein's Energy-Momentum Relationship, Energy-Momentum Relationship in a Hydrogen Atom, Relativistic Kinetic Energy, Potential Energy, Phase Velocity, Group Velocity

1. Introduction

According to the special theory of relativity (STR), the following relation holds between the energy and momentum of a body moving in free space [1].

$$\left(mc^2\right)^2 = \left(m_0c^2\right)^2 + c^2p^2. \tag{1}$$

Here, m_0c^2 is the rest mass energy of the body. And mc^2 is the relativistic energy.

In the STR, there is the following relationship between m_0c^2 and mc^2.

$$mc^2 = m_0c^2\left(1-\beta^2\right)^{-1/2}, \quad \beta = v/c. \tag{2}$$

When β is extremely small, Formula (2) can be expanded as a power series in β, as indicated below.

$$m_0c^2\left(1-\beta^2\right)^{-1/2} = m_0c^2\left(1+\frac{1}{2}\beta^2+\frac{3}{8}\beta^4+\cdots\right). \tag{3}$$

It is known that, if a body is moving at low velocity, Formula (3) can be approximated at high precision using the first two terms. Thus,

$$mc^2 \approx m_0c^2+\frac{1}{2}m_0v^2. \tag{4}$$

The second term on the right side of Formula (4) is the kinetic energy in Newtonian mechanics.

The formula for kinetic energy in classical mechanics holds when the velocity of a body is low. However, the formula no longer holds when the velocity of the body increases, and the theory of relativity can no longer be ignored. In classical mechanics, the mass of a body is constant and does not depend on the velocity of the body.

Einstein and Sommerfeld defined the relativistic kinetic energy K_{re} as follows [2].

$$K_{\mathrm{re}} = mc^2 - m_0c^2. \tag{5}$$

The "re" subscript of K_{re} stands for "relativistic."

Now, Formula (1) is rewritten as follows.

$$\left(mc^2\right)^2 = m_0^2c^4+\left(m^2c^4-m_0^2c^4\right) = \left(m_0c^2\right)^2 + c^2p^2. \tag{6}$$

Comparing Formulas (5) and (6), the relativistic momentum p_{re} can be defined as follows.

$$p_{\mathrm{re}}^2 = m^2c^2 - m_0^2c^2. \tag{7}$$

Hence,

$$p_{\mathrm{re}} = \left(m^2-m_0^2\right)^{1/2}c. \tag{8}$$

Also, Formula (7) can be written as follows.

$$\begin{aligned} p_{\mathrm{re}}^2 &= \left(m+m_0\right)\left(mc^2-m_0c^2\right) \\ &= \left(m+m_0\right)K_{\mathrm{re}}. \end{aligned} \tag{9}$$

Hence,

$$K_{\mathrm{re}} = \frac{p_{\mathrm{re}}^2}{m+m_0}. \tag{10}$$

Based on the above discussion, it was found that the relativistic kinetic energy of a body moving in isolated systems in free space can be described with Formulas

(5) and (10).

However, Einstein's relationship is not applicable in a hydrogen atom, where potential energy is present.

2. The Relationship between the Energy and Momentum of an Electron in a Hydrogen Atom

Formula (1), which is called Einstein's energy-momentum relationship, holds when the energy absorbed by a body is all converted to the kinetic energy of that body.

However, an electron in an atom acquires kinetic energy through the emission of photon energy. Therefore, Einstein's relationship (1) cannot be applied to an electron in an atom.

Consider the case where an electron at rest in an isolated system in free space is attracted by the electrostatic attraction of the proton (hydrogen atom nucleus), and forms a hydrogen atom.

The electron at rest has a rest mass energy of $m_e c^2$. When this electron is taken into the region of the hydrogen atom, it acquires an amount of kinetic energy K_{re} equivalent to the emitted photon energy.

Now, the following relationship holds if the energy of a photon emitted from an electron is taken to be $h\nu$.

$$h\nu = K_{re}. \tag{11}$$

In classical quantum theory, the total mechanical energy of a hydrogen atom is defined as the sum of the potential energy and kinetic energy of the electron. That is,

$$E_n = V(r_n) + K_n, \quad E_n < 0. \tag{12}$$

Also, the potential energy of an electron is given by the following formula.

$$V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \tag{13}$$

According to the Virial theorem, $2K = -V(r)$ in the case of a circular orbit, the energy can be written as follows.

$$E_n = V(r_n) + K_n = \frac{V(r_n)}{2} = -K_n = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n}. \tag{14}$$

Now, if $E_{ph,n}$ is used to represent the photon energy emitted when an electron placed an infinite distance away from the atomic nucleus (proton) of the hydrogen atom is taken into the hydrogen atom, then the following law of energy conservation holds for the electron.

$$V(r_n) + K_n + E_{ph,n} = 0. \tag{15}$$

Here, the "ph" subscript of $E_{ph,n}$ stands for "photon."

Normally, the energy of a photon is written as $h\nu$, but when multiple photons are emitted, $E_{ph,n}$ indicates the total energy of those photons.

Formula (15) shows that the energy source for the kinetic energy acquired by an electron and the photon energy emitted by the electron is the potential energy of the electron.

The relationship between the rest mass energy of the electron $m_{\mathrm{e}}c^2$ and the relativistic energy of the electron $m_n c^2$ is as follows.

$$m_n c^2 = m_{\mathrm{e}}c^2 + V(r_n) + K_{\mathrm{re},n} = m_{\mathrm{e}}c^2 - K_{\mathrm{re},n} = m_{\mathrm{e}}c^2 + E_{\mathrm{re},n}, \quad n = 1, 2, \cdots. \tag{16}$$

Here, n is the principal quantum number.

Also, $m_n c^2$ is the sum of the residual part of the rest mass energy of the electron $\left(m_{\mathrm{e}}c^2 + V(r_n)\right)$ and the relativistic kinetic energy $K_{\mathrm{re},n}$. $E_{\mathrm{re},n}$ are the relativistic energy levels of a hydrogen atom [3].

The relationship between $E_{\mathrm{re},n}$ and other energy is as follows.

$$E_{\mathrm{re},n} = -E_{\mathrm{ph},n} = -K_{\mathrm{re},n}. \tag{17}$$

Incidentally, the author has previously pointed out that the reduction in rest mass energy of an electron corresponds to the potential energy of the electron.

Here, if the reduction in rest mass energy of the electron is represented as $-\Delta m_{\mathrm{e}}c^2$, then the potential energy of the electron can be defined as follows [3]-[5].

$$V(r) = -\Delta m_{\mathrm{e}}c^2. \tag{18}$$

In classical quantum theory, it was promised that the potential energy of an electron placed at the position $r = \infty$ would be zero. It was thought that the energy of an electron in this state would also be zero.

However, that idea is mistaken. There is a problem with classical quantum theory, which describes the energy levels of a hydrogen atom in relative terms, ignoring the STR.

The view of the author is that the potential energy of an electron placed at the position $r = \infty$ will actually be zero. Also, this electron has a rest mass energy of $m_{\mathrm{e}}c^2$.

These energies can be illustrated as follows (**Figure 1**):

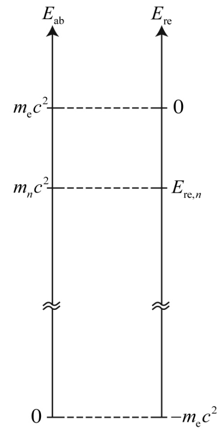

Figure 1. This paper defines $E_{\mathrm{re},n}$ as the relativistic energy levels of the hydrogen atom.

For $E_{\mathrm{re},n}$, the potential energy when the electron is at a position infinitely distant from the atomic nucleus (proton) is set to zero. $E_{\mathrm{ab},n}$, in contrast, takes into account the existence of the rest mass energy of the electron. The "ab" subscript of E_{ab} stands for "absolute."

The relativistic energy of an electron in a hydrogen atom becomes smaller than the rest mass energy. That is,

$$m_n c^2 < m_{\mathrm{e}} c^2. \tag{19}$$

The behavior of an electron inside an atom, where there is potential energy, cannot be described with the relationship of Formula (1).

Now, referring to Formula (5), it is natural to define the relativistic kinetic energy of an electron in a hydrogen atom as follows.

$$K_{\mathrm{re},n} = -E_{\mathrm{re},n} = m_{\mathrm{e}} c^2 - m_n c^2. \tag{20}$$

Next, the relativistic kinetic energy of an electron in a hydrogen atom is defined as follows by referring to Formula (10).

$$K_{\mathrm{re},n} = \frac{p_{\mathrm{re},n}^2}{m_{\mathrm{e}} + m_n}, \quad p_{\mathrm{re},n} = m_n v_n. \tag{21}$$

In this way, two formulas have been obtained for the relativistic kinetic energy of the electron in a hydrogen atom (Formulas (20) and (21)).

The following equation can be derived from Formulas (20) and (21).

$$m_{\mathrm{e}} c^2 - m_n c^2 = \frac{p_{\mathrm{re},n}^2}{m_{\mathrm{e}} + m_n}. \tag{22}$$

Rearranging this, the following relationship can be derived.

$$\left(m_n c^2\right)^2 + c^2 p_{\mathrm{re},n}^2 = \left(m_{\mathrm{e}} c^2\right)^2. \tag{23}$$

Formula (23) is the energy-momentum relationship applicable to the electron in a hydrogen atom.

The author has previously derived Formula (23) using five methods [6]-[11].

3. Solution of Formula (23)

In the past, Dirac derived the following negative solution from Formula (1).

$$E = \pm mc^2 = \pm m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{24}$$

If the same logic is applied to Formula (23), then the following formula can be derived.

$$E = \pm m_n c^2 = \pm m_{\mathrm{e}} c^2 \left(1 + \frac{v_n^2}{c^2}\right)^{-1/2}. \tag{25}$$

However, Formula (25) does not incorporate the discontinuity peculiar to the micro world.

Therefore, Formula (25) must be rewritten into a relationship where energy is discontinuous.

The author has previously derived the following relationship as a new quantum condition to replace the quantum condition of Bohr [12].

$$\frac{v_n}{c}=\frac{\alpha}{n}. \tag{26}$$

Here, α is the following fine-structure constant.

$$\alpha=\frac{e^2}{4\pi\varepsilon_0\hbar c}. \tag{27}$$

Incidentally, there are positive and negative solutions to Einstein's relationship (1). In the same way, Formula (23) also has the following positive and negative solutions [13] [14].

$$E_{\text{ab},n}^{+}=m_n c^2=m_\text{e}c^2+E_{\text{re},n}=m_\text{e}c^2\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2},\quad n=0,1,2,\cdots. \tag{28}$$

$$E_{\text{ab},n}^{-}=-m_n c^2=-m_\text{e}c^2\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2},\quad n=0,1,2,\cdots. \tag{29}$$

It has already been pointed out that a state with $n = 0$ exists in the energy levels of a hydrogen atom [15] [16].

The relativistic kinetic energy of the electron can be expressed as follows.

$$\begin{aligned}K_{\text{re},n}&=-E_{\text{re},n}=m_\text{e}c^2-m_n c^2\\&=m_\text{e}c^2\left[1-\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}\right].\end{aligned} \tag{30}$$

Next, when the part of Formula (30) in parentheses is expressed as a Taylor expansion,

$$\begin{aligned}E_{\text{re},n}&\approx m_\text{e}c^2\left[\left(1-\frac{\alpha^2}{2n^2}+\frac{3\alpha^4}{8n^4}-\frac{5\alpha^6}{16n^6}\right)-1\right]\\&\approx\frac{\alpha^2 m_\text{e}c^2}{2n^2}.\end{aligned} \tag{31}$$

Incidentally, the nonrelativistic energy levels of a hydrogen atom, derived by Bohr, are given by the following formula.

$$E_{\text{BO},n}=-\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{m_\text{e}e^4}{2\hbar^2}\cdot\frac{1}{n^2},\quad (n=1,2,\cdots). \tag{32}$$

The "BO" subscript of $E_{\text{BO},n}$ stands for "Bohr."

However, when discussing the energy levels of a hydrogen atom, it is best if the rest mass energy of the electron is included in the formula. Thus, Formula (27) is rewritten as follows.

$$\begin{aligned}E_{\text{BO},n}&=-K_{\text{BO},n}=-\frac{m_\text{e}c^2}{2}\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)^2\frac{1}{n^2}\\&=-\frac{\alpha^2 m_\text{e}c^2}{2n^2},\quad (n=1,2,\cdots).\end{aligned} \tag{33}$$

From this, it is evident that Formula (32) is an approximation of Formula (30).

Next, the following table summarizes the energies of a hydrogen atom obtained from Formulas (30) and (32) (Table 1).

Table 1. Comparison of the energies of a hydrogen atom predicted by Bohr's classical quantum theory and this paper.

n	Bohr's Energy Levels, $E_{\mathrm{BO},n}$	This Paper, $E_{\mathrm{re},n}$
0	—	−0.511 MeV
1	−13.6057 eV	−13.6052 eV
2	−3.4014 eV	−3.4014 eV
3	−1.51174 eV	−1.51174 eV

Next, if the electron orbital radii corresponding to the energy levels in Formulas (28) and (29) are taken to be, respectively, $r_{\mathrm{re},n}^{+}$ and $r_{\mathrm{re},n}^{-}$ [17].

$$r_{\mathrm{re},n}^{+}=\frac{r_{\mathrm{e}}}{2}\left[1+\frac{n}{\left(n^{2}+\alpha^{2}\right)^{1/2}-n}\right]. \tag{34}$$

$$r_{\mathrm{re},n}^{-}=\frac{r_{\mathrm{e}}}{2}\left[1-\frac{n}{\left(n^{2}+\alpha^{2}\right)^{1/2}+n}\right]. \tag{35}$$

Here, the subscript "re" is attached to r for consistency with $E_{\mathrm{re},n}$.

Here, r_{e} is the classical electron radius, defined as follows.

$$r_{\mathrm{e}}=\frac{e^{2}}{4\pi\varepsilon_{0}m_{\mathrm{e}}c^{2}}. \tag{36}$$

In Formula (35), the electron approaches toward $r_{\mathrm{e}}/4$ as n increases.

The domain of the ordinary hydrogen atom that we all know starts from $r=r_{\mathrm{e}}/2\ \left(E_{\mathrm{ab}}=0\right)$.

4. Limits on Application of the Formula for Potential Energy of an Electron in a Hydrogen Atom

This section explains why it is necessary to discuss the limits of applying the formula for the potential energy of an electron in a hydrogen atom.

The first reason is that the author showed that the minimum energy level of a hydrogen atom is not the ground state ($n = 1$) predicted by quantum mechanics. Since ultra-low energy levels exist in a hydrogen atom, an explanation must be considered that enables the electron to approach the point $r=r_{\mathrm{e}}/4$ thought to be the proton radius.

The second reason is that the author pointed out that the reduction of the rest mass energy of the electron corresponds to the potential energy of the electron in a hydrogen atom.

Formula (13), the existing formula for potential energy, can be written as follows.

$$V\left(r_n\right) = -\frac{e^2}{4\pi\varepsilon_0 m_e c^2}\frac{m_e c^2}{r_n} = -m_e c^2 \cdot \frac{r_e}{r_n}. \tag{37}$$

According to the STR, the rest mass energy of the electron is $m_e c^2$. Inside a hydrogen atom, the rest mass energy of the electron is depleted when the electron approaches the atomic nucleus up to the point $r = r_e$. However, the electron acquires a kinetic energy of $m_e c^2/2$ at this time.

In the finished form of quantum mechanics, there is no discussion of the type of energy possessed by the electron. However, in classical quantum theory, the energy was discriminated.

Thus, this paper also discriminates the energy of the electron into the residual part of the rest mass energy and kinetic energy. In this case, $E_{ab,n}^{+}$ in Formula (28) can be written as follows.

$$\begin{aligned} E_{ab,n}^{+} &= m_n c^2 = m_e c^2 + V\left(r_n\right) + K_n \\ &= m_e c^2 + 2E_{re,n} - E_{re,n}. \end{aligned} \tag{38}$$

The sum of the first and second terms on the right side of Formula (38) is the residual part of the rest mass energy of the electron. The third term is the kinetic energy of the electron.

When the types of energy of the electron are taken into account, an electron that has approached the atomic nucleus to the point $r = r_e$ next approaches the point $r = r_e/2$ by reducing the acquired kinetic energy $m_e c^2/2$.

Therefore, according to this paper, the energy of an electron that has approached the atomic nucleus to the point $r = r_e/2$ is as follows.

$$V\left(r_e/2\right) = -m_e c^2, \quad K_{re} = 0, \quad E_{ab} = 0. \tag{39}$$

However, under these conditions, the electron cannot approach closer than this to the atomic nucleus. We must consider how the electron can reach ultra-low energy levels.

Thus, taking a hint from the idea of renormalization theory, the author has previously assumed that the energy of an electron placed at the point $r = r_e/2$ is not actually zero, and that this electron additionally has a photon energy $m_e c^2$ and a negative energy specific to the electron of $-m_e c^2$ [18] [19] (**Figure 2**).

The photon energy of an electron corresponds to the white rectangle in the diagram (A + B). Energy A is an energy we understand well. This paper asserts the existence of the B part. Also, the negative energy specific to the electron $-m_e c^2$ corresponds to the black rectangle. This figure shows that the original photon energy of an electron with rest mass energy $m_e c^2$ is $2m_e c^2$. (However, this figure is just a conceptual illustration. The *r* coordinate on the *x*-axis is not accurate).

In the state $E_{ab} = 0$, the photon energy $m_{e,B} c^2$ and negative energy $-m_e c^2$ cancel each other out, resulting in a state where energy is zero. An electron in the state where $E_{ab} = 0$ still has photon energy, so it can emit another photon and drop to a negative energy level.

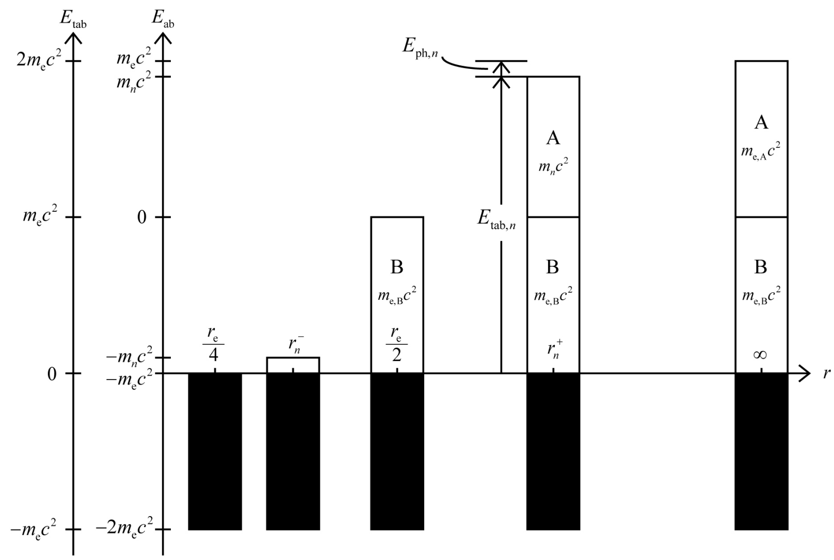

Figure 2. Photon energies of electrons in different states, and negative energy.

The author has previously defined the residual energy $E_{\mathrm{tab},n}$ of an electron that has emitted the photon energy $E_{\mathrm{ph},n}$ as follows [8].

$$E_{\mathrm{tab},n} = \left(m_{\mathrm{e,A}} + m_{\mathrm{e,B}}\right)c^2 - E_{\mathrm{ph},n} = E_{\mathrm{ab},n} + m_{\mathrm{e,B}}c^2 = \left(m_n + m_{\mathrm{e,B}}\right)c^2. \tag{40}$$

The "tab" subscript of this energy indicates the true, absolute photon energy. The descriptor "tab" is applied because absolute energy $E_{\mathrm{ab},n}$ has already been defined.

5. Formula (23) Derived with the Second Method

Consider the Cartesian coordinate system $\mathrm{O}-xy$ [10]. Letting F and F' be the points $x = \pm f$, an ellipse is drawn taking those 2 points as foci. Let A and A' be the points where the ellipse intersects the *x*-axis, and let B and B' be the points where the ellipse intersects the *y*-axis. Also, let $2a$ be the length of the line segment $\overline{\mathrm{AA}'}$, $2b$ be the length of the line segment $\overline{\mathrm{BB}'}$, and $2f$ be the length of the line segment $\overline{\mathrm{FF}'}$ (**Figure 3**).

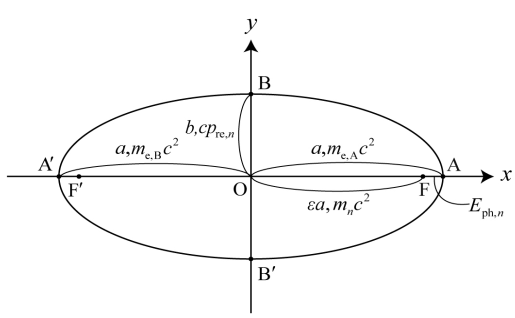

Figure 3. First, the energy $m_{\mathrm{e}}c^2$ is taken to correspond to the line segment $\overline{\mathrm{OA}}$, and then Formula (45) is assumed. Formula (23) can be derived if the Pythagorean theorem is applied to the right triangle OBF.

The eccentricity of the ellipse in this case is defined as follows.

$$\varepsilon = \frac{f}{a}. \tag{41}$$

The eccentricity of the ellipse can also be expressed using the following formula.

$$\varepsilon = \left(1 - \frac{b^2}{a^2}\right)^{1/2}. \tag{42}$$

The following formula can be derived from Formula (42).

$$b = a\left(1 - \varepsilon^2\right)^{1/2}. \tag{43}$$

Here, the line segment $\overline{\mathrm{OA}}$ is placed into correspondence with the energy $m_{\mathrm{e}}c^2$. Let us express this as follows.

$$a = m_{\mathrm{e}}c^2. \tag{44}$$

Also, it is assumed that the shape of an ellipse that can describe the state of an electron is a case satisfying the following conditions.

$$\varepsilon = \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}. \tag{45}$$

Taking Formulas (44) and (45) into account, b can be expressed with the following formula.

$$b = m_{\mathrm{e}}c^2\left(\frac{\alpha^2}{n^2 + \alpha^2}\right)^{1/2}. \tag{46}$$

Also, if Formula (44) and Formula (45) are taken into account, then the f in Formula (41) is as follows.

$$f = a\varepsilon = m_{\mathrm{e}}c^2\left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}. \tag{47}$$

Using Formulas (46) and (47),

$$f^2 + b^2 = \left(\frac{n^2}{n^2 + \alpha^2}\right)\left(m_{\mathrm{e}}c^2\right)^2 + \left(\frac{\alpha^2}{n^2 + \alpha^2}\right)\left(m_{\mathrm{e}}c^2\right)^2 = \left(m_{\mathrm{e}}c^2\right)^2. \tag{48}$$

Here, $\mathrm{BF} = a$ so the following relationship holds.

$$\overline{\mathrm{OA}} = \overline{\mathrm{BF}}. \tag{49}$$

Here, if Formula (28) is also taken into consideration, then Formula (46) can be expressed as follows.

$$b^2 = \left(\frac{\alpha^2}{n^2 + \alpha^2}\right)m_{\mathrm{e}}^2c^2 \cdot c^2 = \left(\frac{\alpha^2}{n^2 + \alpha^2}\right)m_n^2\left(\frac{n^2 + \alpha^2}{n^2}\right)c^2 \cdot c^2. \tag{50}$$

Furthermore, if the relationship in Formula (26) is used for c in Formula (50), the result is as follows.

$$b^2 = \frac{\alpha^2}{n^2} \cdot m_n^2\frac{n^2v_n^2}{\alpha^2}c^2 = c^2\left(m_nv_n\right)^2 = c^2p_n^2. \tag{51}$$

Substituting this result for Formula (51) into Formula (48),

$$f^2+b^2=\left(m_nc^2\right)^2+c^2p_n^2=\left(m_ec^2\right)^2. \tag{52}$$

When m_ec^2 is taken to correspond to the line segment $\overline{\mathrm{OA}}$, and Formula (42) is assumed, then Formula (23) can be derived from the right triangle OBF.

As indicated above, it is evident that Formula (23) can also be derived through a discussion using an ellipse.

However, the lengths of the line segments of the ellipse do not signify the magnitude of the energy and momentum of an electron. The point is simply that Formula (23) can be derived when energy is put into correspondence with the line segment of the ellipse.

In other words, Formula (23) has already been derived with another method, and thus Formula (23) can be derived through a discussion using an ellipse.

Incidentally, the following relationship is known to hold in an ellipse.

$$\overline{\mathrm{A'F}}\cdot\overline{\mathrm{FA}}=\overline{\mathrm{OB}}^2. \tag{53}$$

Here, the line segments $\overline{\mathrm{OA}}$ and $\overline{\mathrm{OA'}}$ are taken to correspond to the energy m_ec^2. Let us express these as follows.

$$\overline{\mathrm{OA}}=m_{\mathrm{e,A}}c^2. \tag{54}$$

$$\overline{\mathrm{OA'}}=m_{\mathrm{e,B}}c^2. \tag{55}$$

If Formula (28) is taken into account, m_nc^2 corresponds to εa. That is,

$$\overline{\mathrm{OF}}=\varepsilon a=m_nc^2. \tag{56}$$

Also, the following energy corresponds to $\overline{\mathrm{FA}}$.

$$\overline{\mathrm{FA}}=a-\varepsilon a=E_{\mathrm{ph},n}=m_{\mathrm{e,A}}c^2-m_nc^2. \tag{57}$$

Taking these points into account, Formula (53) can be written as follows.

$$\left(m_nc^2+m_{\mathrm{e,B}}c^2\right)\left(m_{\mathrm{e,A}}c^2-m_nc^2\right)=c^2p_{\mathrm{re},n}^2. \tag{58}$$

Here, $m_{\mathrm{e,A}}$ is normally expressed by omitting m_{e}.

Rearranging Formula (58), the following relationship can be derived.

$$m_{\mathrm{e,A}}c^2\cdot m_{\mathrm{e,B}}c^2=\left(m_nc^2\right)^2+c^2p_{\mathrm{re},n}^2. \tag{59}$$

Also, Formula (59) can be written as follows because $m_{\mathrm{e,A}}c^2=m_{\mathrm{e,B}}c^2$.

$$\left(m_ec^2\right)^2=\left(m_nc^2\right)^2+c^2p_{\mathrm{re},n}^2. \tag{60}$$

Incidentally, Formula (58) can be written as follows because $E_{\mathrm{ph},n}=K_{\mathrm{re},n}$.

$$\left(m_n+m_{\mathrm{e,B}}\right)c^2\cdot K_{\mathrm{re},n}=c^2p_{\mathrm{re},n}^2. \tag{61}$$

Based on this, the following formula can be derived.

$$K_{\mathrm{re},n}=\frac{p_{\mathrm{re},n}^2}{m_n+m_{\mathrm{e,B}}}. \tag{62}$$

It was thus found that the true nature of m_e in Formula (21) is $m_{\mathrm{e,B}}$.

An ellipse was considered in this section in order to incorporate $m_{\mathrm{e,B}}$ into the formulas for $K_{\mathrm{re},n}$ and $p_{\mathrm{re},n}$.

6. Previously Unknown Formulas for $K_{\mathrm{re},n}$ and $p_{\mathrm{re},n}$

According to Maxwell's electromagnetism, the following relationship holds between the momentum p and energy E of light.

$$E = cp. \tag{63}$$

Also, Einstein asserted, based on consideration of the photoelectric effect, that light has a particle nature, although it had previously been regarded as a wave.

If a photon, as a single particle, is assumed to have a frequency ν, Einstein concluded it has the following energy [18].

$$E = h\nu. \tag{64}$$

Here, h is the Planck constant. Also, Formula (65) can be written as follows using the angular frequency ω.

$$E = \hbar\omega, \quad \hbar = \frac{h}{2\pi}. \tag{65}$$

ω is defined as follows:

$$\omega = 2\pi\nu. \tag{66}$$

The following formula can be derived from Formulas (63) and (64).

$$\lambda = \frac{c}{\nu} = \frac{h}{p}. \tag{67}$$

Also, the wavenumber κ is defined as follows:

$$\kappa = \frac{2\pi}{\lambda}. \tag{68}$$

de Broglie applied Formula (67) to matter. In classical physics, the following relation holds between momentum p and kinetic energy K.

$$K_{\mathrm{cl}} = \frac{1}{2}mv^2 = \frac{p^2}{2m}. \tag{69}$$

Here, if Formulas (65), (67), and (68) are used,

$$\hbar\omega = \frac{p^2}{2m} = \frac{1}{2m}\frac{h^2}{\lambda^2} = \frac{1}{2m}\frac{4\pi^2\hbar^2}{\lambda^2} = \frac{\kappa^2\hbar^2}{2m}. \tag{70}$$

Therefore,

$$\omega = \frac{\hbar k^2}{2m}. \tag{71}$$

The phase velocity v_{phase} and group velocity v_{group} of a material wave are defined as follows (in the following, these may be abbreviated as $v_{\mathrm{p}}, v_{\mathrm{g}}$).

$$v_{\mathrm{phase}} = \frac{\omega}{k}, \quad v_{\mathrm{group}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{72}$$

In light of the above, the phase velocity of the wave is as follows.

$$v_{\mathrm{p}} = \frac{\omega}{k} = \frac{\hbar k}{2m} = \frac{p}{2m} = \frac{v}{2}. \tag{73}$$

Also, the group velocity of the wave is as follows.

$$v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\hbar 2k}{2m} = \frac{p}{m} = v. \tag{74}$$

de Broglie thought that, if light—previously thought to be a wave—has a particle nature, then perhaps the electron—thought to be a particle—has a wave nature. Thus, he applied Formula (67) to matter.

Incidentally, the electron's phase velocity $v_{\mathrm{p},n}$ is given by the following formula.

$$v_{\mathrm{p},n} = \lambda_n \nu_n. \tag{75}$$

Here, $v_{\mathrm{p},n}$ is the phase velocity of the electron wave when the principal quantum number is in the n state. Also, λ_n and ν_n are the wavelength and frequency of the electron wave.

Formula (76) can be written as follows using the relationship of Formulas (64) and (67).

$$v_{\mathrm{p},n} = \lambda_n \nu_n = \frac{h}{p_{\mathrm{re},n}} \frac{K_{\mathrm{re},n}}{h} = \frac{K_{\mathrm{re},n}}{p_{\mathrm{re},n}}. \tag{76}$$

Due to the above, the formula for the relativistic kinetic energy of the electron corresponding to Formula (63) is as follows [19].

$$K_{\mathrm{re},n} = v_{\mathrm{p},n} p_{\mathrm{re},n}. \tag{77}$$

$$K_{\mathrm{re},n} = m_n v_{\mathrm{g},n} v_{\mathrm{p},n}. \tag{78}$$

The energy of a photon is found as the product of the photon's momentum and the speed of light. The kinetic energy of an electron, in contrast, is determined by the product of the electron's momentum and its phase velocity.

Next, the following relationship can be derived from Formula (76) by using Formula (62).

$$v_{\mathrm{p},n} = \frac{K_{\mathrm{re},n}}{p_{\mathrm{re},n}} = \frac{p_{\mathrm{re},n}^2}{m_n + m_{\mathrm{e,B}}} \frac{1}{p_{\mathrm{re},n}} = \frac{p_{\mathrm{re},n}}{m_n + m_{\mathrm{e,B}}} = \frac{m_n}{m_n + m_{\mathrm{e,B}}} v_{\mathrm{g},n}. \tag{79}$$

Hence,

$$m_n v_{\mathrm{g},n} = \left(m_n + m_{\mathrm{e,B}}\right) v_{\mathrm{p},n}. \tag{80}$$

The following new formula for momentum can be derived from Formula (80).

$$p_{\mathrm{re},n} = \left(m_n + m_{\mathrm{e,B}}\right) v_{\mathrm{p},n}. \tag{81}$$

The following formula for kinetic energy can be derived from Formulas (77) and (81).

$$K_{\mathrm{re},n} = \left(m_n + m_{\mathrm{e,B}}\right) v_{\mathrm{p},n}^2. \tag{82}$$

7. Conclusions

A. According to Maxwell's electromagnetism, the following relationship holds between the momentum p and energy E of light.

$$E = cp. \tag{83}$$

It was possible to derive the following formula as a formula similar to Formula (83).

$$K_{\mathrm{re},n} = v_{\mathrm{p},n} p_{\mathrm{re},n}. \tag{84}$$

Comparing Formulas (83) and (84), it is evident that the phase velocity of an electron wave $v_{p,n}$ is an intrinsic physical quantity just like the speed of light.

Also, the momentum of an electron can be described by the following formula.

$$p_{\mathrm{re},n} = \left(m_n + m_{\mathrm{e,B}}\right) v_{\mathrm{p},n}. \tag{85}$$

Formula (84) can be written as follows.

$$K_{\mathrm{re},n} = \left(m_n + m_{\mathrm{e,B}}\right) v_{\mathrm{p},n}^2. \tag{86}$$

In addition, the energy of all photons possessed by an electron with kinetic energy $K_{\mathrm{re},n}$ is as follows.

$$E_{\mathrm{tab},n} = \left(m_n + m_{\mathrm{e,B}}\right) c^2. \tag{87}$$

From Formulas (86) and (87), there was found to be a close relationship between the energy of all photons possessed by an electron and the kinetic energy of the electron.

B. It was found that the following relationship holds between $v_{\mathrm{p},n}$ and $v_{\mathrm{g},n}$.

$$v_{\mathrm{p},n} = \frac{m_n}{m_n + m_{\mathrm{e}}} v_{\mathrm{g},n}. \tag{88}$$

Table 2. The formulas for the kinetic energy of an electron in a hydrogen atom derived by classical mechanics, the STR, and this paper.

	Classical Mechanics	STR	This Paper
Kinetic Energy	$K = \frac{1}{2} m_0 v^2$	$K_{\mathrm{re}} = mc^2 - m_0 c^2$	$K_{\mathrm{re},n} = m_{\mathrm{e}} c^2 - m_n c^2$ $K_{\mathrm{re},n} = m_{\mathrm{e}} c^2 \left[1 - \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}\right]$ $K_{\mathrm{re},n} = m_n v_{\mathrm{g},n} v_{\mathrm{p},n}$ $K_{\mathrm{re},n} = \left(m_n + m_{\mathrm{e,B}}\right) v_{\mathrm{p},n}^2$
Momentum	$p = m_0 v$	$p_{\mathrm{re}} = mv$ $p_{\mathrm{re}} = \left(m^2 - m_0^2\right)^{1/2} c$ $p_{\mathrm{re}} = \frac{m_0 v}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}$	$p_{\mathrm{re},n} = m_n v_{\mathrm{g},n}$ $p_{\mathrm{re},n} = \left(m_{\mathrm{e}}^2 - m_n^2\right)^{1/2} c$ $p_{\mathrm{re},n} = m_{\mathrm{e}} c \left(\frac{\alpha^2}{n^2 + \alpha^2}\right)^{1/2}$ $p_{\mathrm{re},n} = \left(m_n + m_{\mathrm{e,B}}\right) v_{\mathrm{p},n}$
Relationship between K and p	$K = \frac{p^2}{2m_0}$	$K_{\mathrm{re}} = \frac{p_{\mathrm{re}}^2}{m + m_0}$	$K_{\mathrm{re},n} = \frac{p_{\mathrm{re},n}^2}{m_n + m_{\mathrm{e,B}}}$ $K_{\mathrm{re},n} = v_{\mathrm{p},n} p_{\mathrm{re},n}$

If the velocity of the electron is low,

$$v_{\mathrm{p},n} \approx \frac{1}{2} v_{\mathrm{g},n}. \tag{89}$$

Since the group velocity $v_{\mathrm{g},n}$ of the electron as a wave corresponds to the velocity v_n of the electron as a particle,

$$\frac{1}{2} m_{\mathrm{e}} v_n^2 = \frac{1}{2} m_{\mathrm{e}} v_{\mathrm{g},n}^2. \tag{90}$$

It is evident from Formulas (89) and (90) that the classical formula for kinetic energy of an electron is an approximation of Formula (78). That is,

$$\frac{1}{2} m_{\mathrm{e}} v_{\mathrm{g},n}^2 \approx m_{\mathrm{e}} v_{\mathrm{g},n} v_{\mathrm{p},n} \approx m_n v_{\mathrm{g},n} v_{\mathrm{p},n}. \tag{91}$$

The formulas for the kinetic energy and momentum of an electron in a hydrogen atom, derived in this paper, are shown in the following **Table 2**.

Acknowledgements

I would like to express my thanks to the staff at ACN Translation Services for their translation assistance. Also, I wish to express my gratitude to Mr. H. Shimada for drawing figures.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Einstein, A. (1961) Relativity. Crown, 43.

[2] Sommerfeld, A. (1923) Atomic Structure and Spectral Lines. Methuen & Co. Ltd., 528.

[3] Suto, K. (2018) Potential Energy of the Electron in a Hydrogen Atom and a Model of a Virtual Particle Pair Constituting the Vacuum. *Applied Physics Research*, **10**, 93-101. https://doi.org/10.5539/apr.v10n4p93

[4] Suto, K. (2009) True Nature of Potential Energy of a Hydrogen Atom. *Physics Essays*, **22**, 135-139. https://doi.org/10.4006/1.3092779

[5] Suto, K. (2025) Limits on Application of the Formula for Potential Energy of a Hydrogen Atom and a Previously Unknown Formula. *Journal of High Energy Physics, Gravitation and Cosmology*, **11**, 1039-1051. https://doi.org/10.4236/jhepgc.2025.113067

[6] Suto, K. (2011) An Energy-Momentum Relationship for a Bound Electron Inside a Hydrogen Atom. *Physics Essays*, **24**, 301-307. https://doi.org/10.4006/1.3583810

[7] Suto, K. (2018) Derivation of a Relativistic Wave Equation More Profound than Dirac's Relativistic Wave Equation. *Applied Physics Research*, **10**, 102-108. https://doi.org/10.5539/apr.v10n6p102

[8] Suto, K. (2020) Theoretical Prediction of Negative Energy Specific to the Electron. *Journal of Modern Physics*, **11**, 712-724. https://doi.org/10.4236/jmp.2020.115046

[9] Suto, K. (2023) Previously Unknown Formulas for the Relativistic Kinetic Energy of an Electron in a Hydrogen Atom. *Journal of Applied Mathematics and Physics*, **11**,

972-987. https://doi.org/10.4236/jamp.2023.114065

[10] Suto, K. (2024) The Strange Relationship between the Momentum of a Photon Emitted from an Electron and the Momentum Acquired by the Electron. *Journal of Applied Mathematics and Physics*, **12**, 2652-2664. https://doi.org/10.4236/jamp.2024.127157

[11] Suto, K. (2025) The Photon Energy of an Electron with Rest Mass Energy of $m_e c^2$ Is $2m_e c^2$. *Applied Physics Research*, **17**, 44-56. https://doi.org/10.5539/apr.v17n1p44

[12] Suto, K. (2021) The Quantum Condition That Should Have Been Assumed by Bohr When Deriving the Energy Levels of a Hydrogen Atom. *Journal of Applied Mathematics and Physics*, **9**, 1230-1244. https://doi.org/10.4236/jamp.2021.96084

[13] Suto, K. (2014) Previously Unknown Ultra-Low Energy Level of the Hydrogen Atom Whose Existence Can Be Predicted. *Applied Physics Research*, **6**, 64-73. https://doi.org/10.5539/apr.v6n6p64

[14] Suto, K. (2022) A Compelling Formula Indicating the Existence of Ultra-Low Energy Levels in the Hydrogen Atom. *Global Journal of Science Frontier Research*, **22**, 7-15. https://doi.org/10.34257/gjsfravol22is5pg7

[15] Suto, K. (2014) N = 0 Energy Level Present in the Hydrogen Atom. *Applied Physics Research*, **6**, 109-115. https://doi.org/10.5539/apr.v6n5p109

[16] Suto, K. (2023) An Energy Level with Principal Quantum Number N = 0 Exists in a Hydrogen Atom. *Global Journal of Science Frontier Research*, **23**, 65-79. https://doi.org/10.34257/ljrsvol23is2pg65

[17] Suto, K. (2017) Region of Dark Matter Present in the Hydrogen Atom. *Journal of Physical Mathematics*, **8**, 1-6.

[18] Einstein, A. (1905) On a Heuristic Point of View about the Creation and Conversion of Light. *Annalen der Physik*, **322**, 132-148. (In German) https://doi.org/10.1002/andp.19053220607

[19] Suto, K. (2022) A Surprising Physical Quantity Involved in the Phase Velocity and Energy Levels of the Electron in a Hydrogen Atom. *Applied Physics Research*, **14**, 1-17. https://doi.org/10.5539/apr.v14n2p1