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CMB Temperature as Geometric Mean of Maximum and Minimum Unruh Temperature: $T_{cmb} = \sqrt{T_{U,\min} T_{U,\max}}$

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Abstract

We demonstrate that the CMB temperature can be described as the geometric mean of the minimum and maximum Unruh temperatures: $T_{cmb} = \sqrt{T_{U,\min} T_{U,\max}}$. The geometric-mean relation is an algebraic consequence of the limiting scales adopted below; its interpretation as a possible origin of the CMB temperature is a separate physical hypothesis. One possible interpretation of this result is that the CMB temperature corresponds to the geometric mean of the horizon-induced temperature spectrum associated with vacuum excitations in the observable universe. In this framework, the Unruh effect links proper acceleration to effective temperature over a finite range. The lower bound is fixed by the smallest physically relevant acceleration determined by the cosmic horizon, while the upper bound is set by the largest admissible acceleration near the Planck scale. The minimum corresponds to an observer whose Rindler horizon is comparable to the Hubble radius, whereas the maximum corresponds to a horizon of Planckian size. Viewed this way, the observed CMB temperature emerges as the geometric midpoint of these two extreme scales.

Keywords

CMB Temperature, Unruh Temperature, Rindler Horizon, Hawking Temperature, Geometric Mean

1. Introduction

The Hawking [1] temperature and the Unruh [2] temperature arise from the same underlying physics—quantum fields in the presence of horizons—and differ mainly in what creates the horizon (gravity vs. acceleration). The Hawking temperature is given by:

$$T_{Haw} = \frac{\hbar\kappa}{2\pi k_b c} \tag{1}$$

where $\kappa = \frac{GM}{R_s^2} = \frac{c^2}{2R_s}$ is the surface gravity of a Schwarzschild black hole. The Unruh temperature is given by

$$T_U = \frac{\hbar a}{2\pi k_b c} \tag{2}$$

where a is the constant proper acceleration. An observer with constant acceleration a experiences a thermal bath with a temperature equal to the Unruh temperature. When $a = \kappa$, the Unruh temperature is equal to the Hawking temperature. We will argue that the minimum Unruh temperature is related to the following minimum acceleration:

$$a_{\min} = \frac{GM_c}{R_{H_t}^2} \tag{3}$$

where R_{H_t} is the Hubble radius at time t, *i.e.*, $R_{H_t} = ct$ in cosmology and $M_c = \frac{c^3}{2GH_0}$. This is the proper acceleration with a Rindler horizon $R_{H_t} = ct$. Furthermore, we claim that the maximum acceleration is:

$$a_{\max} = \frac{Gm_p}{\left(2l_p\right)^2} \tag{4}$$

where m_p is the Planck [3] [4] mass and l_p is the Planck length. This is the proper acceleration of a Planck-mass particle with a Rindler horizon $2l_p$.

The motivation for these endpoints is that the Hubble radius is the largest characteristic horizon scale in the present cosmological model, whereas $2l_p$ represents the smallest horizon diameter admitted in the Planck-scale construction. These choices should be understood as physically motivated bounds of the model, rather than as uniquely derived consequences of the Unruh effect.

The minimum Unruh temperature is then:

$$T_{U,\min} = \frac{\hbar a_{\min}}{2\pi k_b c} \tag{5}$$

and the maximum Unruh temperature is

$$T_{U,\max} = \frac{\hbar a_{\max}}{2\pi k_b c} \tag{6}$$

Because the Unruh effect is observer-dependent, $T_{U,\min}$ and $T_{U,\max}$ refer to two different idealized uniformly accelerated observers, with proper accelerations $a_{\min}$ and $a_{\max}$, respectively. They are therefore limiting temperatures within the chosen family of accelerated frames, not temperatures measured simultaneously by one observer; an inertial observer does not detect the same Unruh bath.

The CMB temperature is then given by the geometric mean of the maximum and minimum Unruh temperature in the Hubble sphere:

$$T_{cmb} = \sqrt{T_{U,\min} T_{U,\max}} \tag{7}$$

Thus, the central mathematical result is the equivalence produced by the stated definitions. The further claim that the CMB may emerge from horizon-induced vacuum excitations is an interpretation of this equivalence, not an independent derivation of the observed radiation field.

This is mathematically identical to the relation given by Haug [5]

$$T_{cmb} = \sqrt{T_{Haw,\min} T_{Haw,\max}} \tag{8}$$

See also Haug and Tatum [6], who were the first to demonstrate that the CMB temperature is linked to the geometric mean of the shortest and longest possible wavelengths inside the Hubble sphere. The same CMB temperature has also been derived independently from the Stefan-Boltzmann law [7] [8]. This is also consistent with a CMB formula heuristically suggested by Tatum [9] *et al*. However, this is the first time we link the CMB temperature to the geometric mean Unruh temperature. See also the paper by Bhatt and Becker [10] that also works along the lines to somehow connect the Rindler horizon to the CMB temperature.

A possible physical interpretation of this result is that the CMB temperature reflects an equilibrium-like scale emerging from the full range of horizon-induced vacuum excitations accessible in the observable universe. In this framework, the Unruh effect provides a spectrum of effective temperatures associated with different proper accelerations, bounded from below by the smallest meaningful acceleration set by the cosmic horizon and from above by the largest possible acceleration associated with Planck-scale physics. The minimum acceleration corresponds to an observer whose Rindler horizon coincides with the Hubble radius, while the maximum acceleration corresponds to a Planck-scale horizon.

If quantum vacuum fluctuations are fundamentally linked to horizons and their associated accelerations, then the observable vacuum state may effectively encode contributions across this entire range. The geometric mean then arises as a natural scale that is invariant under reciprocal transformations between these extremes, suggesting that the CMB temperature could represent a universal "midpoint" of horizon-induced thermal effects. In this sense, the CMB temperature is not tied to a single horizon, but instead emerges from the interplay between the largest and smallest physically meaningful horizons in the universe.

2. Carnot Optimization and the CMB as the Equilibrium Temperature of the Hubble Sphere

The geometric-mean relation may be further motivated through classical thermodynamics. Consider the Hubble sphere as a finite thermodynamic domain bounded by two limiting temperature scales: a maximum temperature $T_{\max}$, associated with the largest physically meaningful ultraviolet scale, and a minimum temperature $T_{\min}$, associated with the infrared horizon scale. If these two sectors are coupled through an ideal reversible Carnot [11] process, then the resulting equilibrium temperature is naturally the geometric mean of the two bounds.

Assume the hot and cold sectors possess equal effective heat capacities C, and that a reversible engine extracts the maximum possible work while the hot sector cools from $T_{\max}$ to a common final temperature T_*, and the cold sector warms from $T_{\min}$ to the same temperature T_*. Equal effective heat capacities are assumed to impose a symmetric treatment of the ultraviolet and infrared sectors, while reversibility selects the ideal zero-entropy-production limit.

For a reversible process, the total entropy change must vanish,

$$\mathrm{d}S_{\mathrm{tot}} = \mathrm{d}S_h + \mathrm{d}S_c = 0. \tag{9}$$

The entropy decrease of the hot reservoir is

$$\Delta S_h = \int_{T_{\max}}^{T_*} \frac{C\mathrm{d}T}{T} = C\ln\left(\frac{T_*}{T_{\max}}\right), \tag{10}$$

while the entropy increase of the cold reservoir is

$$\Delta S_c = \int_{T_{\min}}^{T_*} \frac{C\mathrm{d}T}{T} = C\ln\left(\frac{T_*}{T_{\min}}\right). \tag{11}$$

Imposing reversibility, $\Delta S_h + \Delta S_c = 0$, gives

$$C\ln\left(\frac{T_*}{T_{\max}}\right) + C\ln\left(\frac{T_*}{T_{\min}}\right) = 0. \tag{12}$$

Combining logarithms yields

$$\ln\left(\frac{T_*^2}{T_{\max}T_{\min}}\right) = 0, \tag{13}$$

hence

$$T_* = \sqrt{T_{\max}T_{\min}}. \tag{14}$$

See also [12]-[14] for a description of the geometric mean temperature in Carnot engines.

Therefore, the equilibrium temperature of an ideal reversible engine operating between two limiting reservoirs is the geometric mean of the two temperatures. Carnot engines in astrophysics are not new; see, for example, [15]-[21]. What is new here is that we demonstrate this leads to a geometric mean temperature.

This result has a natural cosmological interpretation. If the observable universe is bounded by a largest thermal scale and a lowest horizon-defined scale, then the equilibrium background temperature inside the Hubble sphere is expected to take the same form,

$$T_{\mathrm{CMB}} = \sqrt{T_{\max}T_{\min}}. \tag{15}$$

The Cosmic Microwave Background may therefore be interpreted as the stationary thermodynamic temperature of the Hubble sphere itself—the reversible midpoint between ultraviolet and infrared limits.

An equivalent way to express this result is through entropy-balance symmetry,

$$\int_{T_*}^{T_{\max}} \frac{\mathrm{d}T}{T} = \int_{T_{\min}}^{T_*} \frac{\mathrm{d}T}{T}, \tag{16}$$

showing that T_* lies exactly halfway between $T_{\min}$ and $T_{\max}$ in logarithmic temperature space. Thus, the geometric mean is not arbitrary, but the unique scale-invariant equilibrium temperature separating the two extremes.

If $T_{\max}$ and $T_{\min}$ are further identified with the maximum and minimum Unruh temperatures available within the Hubble sphere, then the observed CMB temperature emerges as the natural Carnot equilibrium temperature of the observable universe.

3. Hubble Constant from Planck Scale Unruh Temperature and CMB Temperature

Since we have $T_{U,\min} = \frac{\hbar a_{\min}}{2\pi k_b c} = \frac{\hbar H_t}{k_b 4\pi}$, we can solve the equation below with respect to H_0 and this gives:

$$T_{cmb} = \sqrt{T_{U,\min} T_{U,\max}}$$

$$H_0 = \frac{4\pi k_b}{\hbar} \frac{T_{cmb,0}^2}{T_{U,\max}} = 66.8712 \pm 0.0019\ \mathrm{km/s/Mpc} \tag{17}$$

As the CMB temperature [22] has been measured much more precisely than H_0, this leads to a dramatically increased precision in H_0 predictions and other cosmological parameters such as the radius of the universe and the Hubble time. The Hubble parameter estimated above is based on the CMB temperature given by Dahl *et al.* [23], $T_{cmb,0} = 2.725007 \pm 0.000024\ \mathrm{K}$. In addition, we have taken into account the standard uncertainty in the Planck length, $l_p = (1.616255 \pm 0.000018) \times 10^{-35}\ \mathrm{m}$ (NIST CODATA 2019), which is embedded in:

$$T_{U,\max} = \frac{\hbar a_{\max}}{2\pi k_b c} = \frac{\hbar \frac{c^2}{4 l_p}}{2\pi k_b c} = \frac{\hbar c}{8\pi k_b l_p} \tag{18}$$

The other constants, c, $\hbar$, and k_b, are all defined as exact according to NIST CODATA standards. The high precession in H_0 is fully in line with the geometric mean method CMB of Haug and Tatum, but it has here a different explanation of origin as we link it to the Unruh effect rather than the Hawking temperature, see also [24]-[26]. Further studies should discuss in more detail which of the two possible explanations makes more sense, or whether they are two sides of the same coin.

4. Limitations and Relation to the Standard CMB Account

In the standard cosmological interpretation, the CMB is relic radiation that last scattered around recombination and subsequently cooled through cosmic expansion. The present proposal does not replace that successful account or derive the observed blackbody spectrum, anisotropies, or polarization. Its more limited claim is that the observed temperature scale can be re-expressed, and possibly reinterpreted, as the geometric mean of the two limiting Unruh temperatures defined above. Establishing that horizon-induced vacuum excitations are the actual origin of the CMB radiation would require additional microscopic and observational arguments.

5. Conclusion

Haug and Tatum have discussed how the CMB temperature is linked to the geometric mean of a maximum and minimum Hawking temperature. In this paper, we point out that the CMB temperature can also be understood as the geometric mean of the minimum and maximum possible Unruh temperatures within the Hubble sphere: $T_{cmb} = \sqrt{T_{U,\min} T_{U,\max}}$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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