Journal of Applied Mathematics and Physics, 2026, 14(9), 3502-3515
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

The Nonlinear Schrödinger Equation Derived from the Higher Order Sawada-Kotera (SK) Equation Using Multiple Scales Method

Murat Koparan

Division of Mathematics Education, Department of Mathematics and Science Education, Faculty of Education, Anadolu University, Yunusemre Campus, Eskişehir, Türkiye
Email: mkoparan@anadolu.edu.tr

How to cite this paper: Koparan, M. (2026) The Nonlinear Schrödinger Equation Derived from the Higher Order Sawada-Kotera (SK) Equation Using Multiple Scales Method. *Journal of Applied Mathematics and Physics*, **14**, 3502-3515.
https://doi.org/10.4236/jamp.2026.149174

Received: June 26, 2026
Accepted: September 13, 2026
Published: September 16, 2026

Abstract

The mathematical models of problems that arise in many branches of science are nonlinear equations of evolution (NLEE). For this reason, nonlinear equations of evolution have served as a language in formulating many engineering and scientific problems. Although the origin of nonlinear evolution equations dates back to ancient times, significant developments have been made in these equations up to the present day. The main reason for this situation is that nonlinear equations of evolution involve the problem of nonlinear wave propagation. Therefore, many different and effective techniques have been developed regarding nonlinear evolution equations and solution methods. Studies conducted in recent years show that evolution equations are becoming increasingly important in applied mathematics. This study is about the multiple scales methods, known as the perturbation method, for nonlinear equations of evolution (NLEE). In this report, the multiple scales method was applied for the analysis of the (1 + 1) dimensional higher order nonlinear Sawada-Kotera (SK) equations, and the nonlinear Schrödinger (NLS) equation was obtained. Also, the approximate solution of the (1 + 1) dimensional higher-order nonlinear Sawada-Kotera (SK) equation is obtained.

Keywords

Perturbation Method, Multiple Scales Method, Sawada-Kotera (SK) Equation, NLS Type Equation

1. Introduction

The studies of science and engineering demand visualization of the mathematical model for real life phenomena and develop a formula. A wide variety of physical, chemical, and biological phenomena are described by nonlinear evolution equations (NLEEs). The study of exact solutions of nonlinear evolution equations (NLEEs) plays a crucial role in the study of nonlinear physical phenomena. Recently, NLEEs have emerged as an important field of study in applied mathematics. Furthermore, nonlinear equations modeling these phenomena in other branches of science have long been a significant area of research interest. An analytical study of these NLEEs is important because the data from their exact solutions facilitates the validation of numerical solvers and supports decision-making analysis of the solutions. This not only helps us better understand the solutions, but also helps us understand the phenomena they describe. [1]-[5]. The studies of engineering and science demand visualization of the mathematical model for real-life phenomena and the development of a formula. Numerous real-world problems are modeled using higher-order differential equations (DEs), either as ordinary or partial differential equations (PDEs). For instance, higher-order differential equations are used to examine the effects of unsafe contact with an infected corpse by the NiV virus and to study the dynamics of an accelerated mass-spring system [6] [7]. They are also employed to investigate chaos, bifurcations, signal processing, heat distribution, secure communications, plasma physics, and the behavior of surface water waves in fluid dynamics, and describe the propagation of electrostatic waves in plasmas [8]-[11].

Known for its distinguished role in the development of nonlinear physics, the Korteweg-de Vries (KdV) equation has been derived in a variety of physical contexts. For example, it can be thought of as modeling the unilateral propagation of long-wavelength gravitational waves of small amplitude in a shallow channel. In any case, the Korteweg-de Vries equation is obtained with a certain degree of approximation and, therefore, cannot be considered to represent physical reality with perfect accuracy. Therefore, an important question arises about what would happen to the solutions of Korteweg-de Vries' equation when perturbations from neglected terms in the derivation come into play. For instance, would the perturbed Korteweg-de Vries equation still have a well-localized single-wave solution? The answer, sure, would depend on the nature and physical origin of the perturbation. Currently, studies are underway on the nonlinear fifth-order KdV-type equations as they can describe real properties in various scientific applications and engineering fields, and have practical and physical significance.

The nonlinear Schrödinger (NLS) equation is an example of a universal nonlinear model because it describes a wide variety of physical systems. As a result, the equation may be used to describe a wide range of nonlinear physical events [12]-[14]. It is renowned that a multiscale analysis of the KdV equation gives rise to the NLS equation for modulated amplitude [15]-[19]. In [15] Zakharov and Kuznetsov showed that multiple scale analysis of the Schrödinger spectral problem yields the Zakharov-Shabat problem for the NLS equation, demonstrating a much deeper relationship between these integrable equations, not only at the equation level but also at the linear spectral problem level. Zakharov and Kuznetsov to reduce the KdV equation into the NLS equation for completeness sake and then apply this to various classes of nonlinear evolution equations. They showed that using this method, integrable systems are reduced to other integrable systems. If the initial system is nonintegrable, the result can be either integrable or nonintegrable. This is our main purpose in the application of this method to integrable systems. Özer and Dağ showed a similar connection between NLS and integrable fifth-order nonlinear evolution equations [20]. Furthermore, similar results were obtained using multiple scale analysis on different equations [21]-[24].

In this paper, we apply a multiple scale method following Zakharov and Kuznetsov [15] related to the connection of the KdV and NLS equations. This is an important derivation because the KdV flow equations follow from the NLS equation. The strength of this method is that for each degree of coefficients in epsilon, the equations contain no secular terms. Therefore, there is no freedom in choosing the coefficients and the expansion is uniquely determined. The derivation of this hierarchy is not a simple case of algorithms, but basically relies on two facts. First, different time flows must commute; *i.e.*: $u_{t_i t_j} = u_{t_j t_i}$. Second, in each order in epsilon, the coefficient equations contain secular terms. Eliminating the secular terms requires u_{t_i} to have a certain high symmetry (flow) of the hierarchy, and this way all coefficients of expansion are fixed and no arbitrariness is left.

After the introduction, in Section 2, we briefly expressed the fifth-order Korteweg-de Vries Equation (KdV5) flow equations, the seventh-order Korteweg-de Vries Equation (KdV7) flow equations and the multiple scales method, respectively. In Section 3, we applied the method given in Section 2 to the (1 + 1) dimensional fifth-order and seventh-order Sawada-Kotera (SK) equation. The last part consists of the conclusion part.

2. Background Materials

In this section, we present some background material on the best-known fifth-order, seventh-order Sawada-Kotera (SK) equation and the multi-scale method.

2.1. The Fifth Order Korteweg-De Vries Equation (fKdV) Flow Equations

The best-known fifth-order KdV equations look like this

$$u_t = \omega u_{xxxxx} + \alpha u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x \tag{1}$$

where α, β, γ and ω are arbitrary nonzero and real parameters, and $u = u(x,t)$ is a smooth enough function. Since the parameters α, β, γ and ω are arbitrary and take different values, this will greatly change the properties of the fKdV Equation (1). Changing the actual values of the parameters allows you to generate many different variations of the fKdV equation. The fKdV equations, which are widely used in nonlinear optics and quantum physics, are an important mathematical model. Characteristic examples are commonly utilized in many domains, including plasma physics, quantum field theory, solid-state physics, and fluid physics [25] [26].

Some important special cases of the Equation (1) are:

Kaup-Kupershmidt (KK) equation [27]-[31]

$$u_t = u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x, \tag{2}$$

Sawada-Kotera (SK) equation [32] [33]

$$u_t = u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x, \tag{3}$$

Caudrey-Dodd-Gibbon (CDG) equation

$$u_t = u_{xxxxx} + 30u_{xxx} + 30u_x u_{xx} + 180u^2 u_x, \tag{4}$$

Lax equation [25]

$$u_t = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x, \tag{5}$$

Ito equation [34] [35]

$$u_t = u_{xxxxx} + 3u_{xxx} + 6u_x u_{xx} + 2u^2 u_x, \tag{6}$$

As the values of α, β, and γ vary, the characteristics of the Equation (1) change drastically. For example, the KK equation with $\alpha = 10$, $\beta = 25$, and $\gamma = 20$, which is known to be integrable [29], and has bilinear representations [29] [31], but the obvious form of the N-soliton solutions is not known. Another example is the SK equation where $\alpha = \beta = \gamma = 5$, and the Lax equation with $\alpha = 10$, $\beta = 20$, and $\gamma = 30$, are both fully integrable. These two equations have N-soliton solutions and an endless set of conserved densities. One last equation in this class is the Ito equation, with $\alpha = 3$, $\beta = 6$, and $\gamma = 2$, which cannot be fully integrated but has a limited number of special conserved densities [35].

2.2. The Seventh Order Korteweg-De Vries Equation (sKdV) Flow Equations

The best-known fifth-order KdV equations look like this

$$u_t + au^3 u_x + bu_x^3 + cuu_x u_{2x} + du^2 u_{3x} + eu_{2x} u_{3x} + fu_x u_{4x} + guu_{5x} + u_{7x} = 0 \tag{7}$$

where a, b, c, d, e, f and g are are constant parameters also these parameters cannot be zero and $u_{ix} = \left(\partial^i / \partial x^i\right) u$. Pomeau *et al.* [36] have introduced the seventh-order KdV equation for discussing the structural stability of KdV equation under a singular perturbation.

When $a = 252$, $b = 63$, $c = 378$, $d = 126$, $e = 63$, $f = 42$ and $g = 21$, we have:

$$\begin{aligned} &u_t + 252u^3 u_x + 63u_x^3 + 378u_x u_{2x} + 126u^2 u_{3x} \\ &+ 63u_{2x} u_{3x} + 42u_x u_{4x} + 21u_{5x} + u_{7x} = 0 \end{aligned} \tag{8}$$

This equation is the well-known seventh-order Sawada-Kotera equation [37]-[39].

When $a = 140$, $b = 70$, $c = 280$, $d = 70$, $e = 70$, $f = 42$ and $g = 14$, we have:

$$\begin{aligned} & u_t + 140u^3u_x + 70u_x^3 + 280u_xu_{2x} + 70u^2u_{3x} \\ & + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14u_{5x} + u_{7x} = 0 \end{aligned} \tag{9}$$

This equation is the well-known seventh-order: Lax equation [37]-[39].

When $a = 2016$, $b = 630$, $c = 2268$, $d = 504$, $e = 252$, $f = 147$ and $g = 42$, we have:

$$\begin{aligned} & u_t + 2016u^3u_x + 630u_x^3 + 2268u_xu_{2x} + 504u^2u_{3x} \\ & + 252u_{2x}u_{3x} + 147u_xu_{4x} + 42u_{5x} + u_{7x} = 0 \end{aligned} \tag{10}$$

This equation is the well-known seventh-order: Kaup-Kupershmidt equation [37]-[39].

2.3. The Multiple Scales Method

The multiple scales method is a perturbation method. Most nonlinear differential equations may not be solved exactly, but their approximate solutions can be obtained by using numerical methods or perturbations. The latter give asymptotic expansions with respect to a small or a large parameter or coordinate. A straightforward perturbation method, the most naive approach is to simply assume that the solution has the form of a power series in ϵ as:

$$u(t) = \sum_{n=0}^{\infty} \epsilon^n U_n(\tau)$$

Then each component in the power series is found by equating like powers of ϵ [40]. In the multiple scales method first recommended by Zakharov and Kuznetsov [15], Zakharov and Kuznetsov used this method to decrease the KdV equation to the NLS equation and apply it to a class of nonlinear evolution equations. Using this method, they showed that integrable systems can be decreased to integrable systems. If the system we have taken at the beginning is not an integrable system, it has been seen that the reduced system as a result of the application of the method is either integrable or non-integrable. However, if the method is applied to a suitable integrable system, it is seen that the system obtained as a result of the analysis is always an integrable system. This is the master purpose of applying the multi-scale expansion method to integrable systems.

In this section, multiple scales method of nonlinear evolution equations is discussed. By applying the Zakharov and Kuznetsov (1986) technique, the steps of the multi-scale method in obtaining NLS type equations from KdV equations are shown in order.

Let consider the general evolution equation in the following form

$$u_t = K(u, u_x, u_y, \cdots). \tag{11}$$

where $K[u]$ is a function of u and its derivatives with respect to the x-spatial variables. The well known of this type equations is KdV equation.

$L[\partial_x, \partial_y]u$ is the linear part of $K[u]$. So, using $K[u]$ we can reach the dispersion relation for the Equation (11). Substituting the wave solution space

$$u_k = A e^{i(kx + ry - \omega(k,r)t)} \equiv A e^{i\theta}, \tag{12}$$

into the linear part of Equation (11)

$$u_t = L\left[\partial_x, \partial_y\right]u, \tag{13}$$

we get a dispersion relation, which can be solved for ω as $\omega = \omega(k, r)$ for a given k and r. Here θ is the phase, k and r are the wave number and ω is the frequency.

$$\omega(k, r) = iL[ik, ir]. \tag{14}$$

Then, dispersion relation (14) is substituted in Equation (11). We assume the following series expansions for the solution of the Equation (11):

$$u(x, y, t) = \sum_{n=1}^{\infty} \epsilon^n U_n(x, y, t, \xi, \tau),$$

Based on this solution, we also define slow spaces ξ and multiple time variable τ with respect to the scaling parameter $\epsilon > 0$ respectively as follows:

$$\begin{aligned} \xi &= \epsilon\left(x - \frac{\mathrm{d}\omega(k)}{\mathrm{d}k}t\right), \\ \tau &= -\frac{1}{2}\epsilon^2 \frac{\mathrm{d}^2\omega(k)}{\mathrm{d}k^2}t. \end{aligned} \tag{15}$$

A nonlinear equation modulates the amplitude of this plane wave solution in such a way that may consider it depend upon the slow variables. If we choose the slow variables different forms, we can derive higher order NLS equations. The multiple scales analysis starts with the assumption:

$$u(x, t) = U(x, y, t, \xi, \tau), \tag{16}$$

and solution of U is in the form

$$U(x, y, t, \xi, \tau) = \epsilon U_1 + \epsilon^2 U_2 + \epsilon^3 U_3 + \cdots \tag{17}$$

In this case, considering the transformation (16) and solution (17), using (14) and (15), the terms included derivative in Equation (11) are obtained. Substituting these terms with (16) and (17) into the Equation (11), we get a polynomial in ε. We obtain a series of algebraic equations by equalizing each coefficient of this polynomial to zero. Using wave solution space (12) and dispersion relation (14), these equations may be solved by iteration and Reduce. Thus, we can obtain NLS type equations from Equation (11). Furthermore, this approach allows us to obtain approximate solution to KdV-type problems [41].

3. Applications

3.1. The Fifth Order Sawada-Kotera (SK) Equation

Following the method given in Section 2.3 we use a multiple scales method to derive the NLS equations from the fifth-order Sawada-Kotera (SK) Equation (3). To find dispersion relation for (3), we consider the linear part of (3) in the form

$$u_t = u_{xxxxx}, \tag{18}$$

and linear differential Equation (18) satisfies the solution

$$u(x,t)=\mathrm{e}^{i\theta},\ \theta=kx-\omega(k)t. \tag{19}$$

Substituting the solution (19) into the linear differential Equation (18), we get and from this we reach the

$$\omega(k)=-k^5, \tag{20}$$

dispersion relation. Thus the solution of linear differential Equation (18) is as follows:

$$u(x,t)=\mathrm{e}^{i(kx-w(k)t)}, \tag{21}$$

Let the solution of Equation (3) is in the form

$$u(x,t)=U(x,t,\xi,\tau),\ U(x,t,\xi,\tau)=\epsilon U_1+\epsilon^2U_2+\epsilon^3U_3+\cdots \tag{22}$$

ϵ scale parameter and slow variables

$$\xi=\epsilon\left(x-\frac{\mathrm{d}\omega(k)}{\mathrm{d}k}t\right)=\epsilon\left(x-5k^4t\right),\ \tau=-\frac{1}{2}\epsilon^2\frac{\mathrm{d}^2\omega(k)}{\mathrm{d}k^2}t=-10\epsilon^2k^3t. \tag{23}$$

Then we assume the following series expansions for solutions:

$$U(x,y,t,\xi,\tau)=\epsilon U_1+\epsilon^2U_2+\epsilon^3U_3+\cdots. \tag{24}$$

In this case, considering the transformation and solution in (22), using (20) and (23), the terms included derivative in Equation (3) are obtained. Substituting these terms and (22) into the Equation (3), we get a polynomial in ϵ. Equalizing each coefficient of this polynomial to zero, we get a set of algebraic equations. If we let $\varepsilon\to 0$ we find the following:

$$\epsilon: u_{1t}-u_{1xxxxx}=0, \tag{25}$$

$$\epsilon^2: u_{2t}+u_{2xxxxx}-5k^4u_{1\xi}+5u_{1\xi xxxx}+5u_1u_{1xxx}+5u_{1x}u_{1xx}=0, \tag{26}$$

$$\begin{aligned}\epsilon^3: &u_{3t}-u_{3xxxxx}-5k^4u_{2\xi}-10k^3u_{1\tau}+5u_{2\xi xxxx}+10u_{1\xi\xi xxx}\\&+5u_1\left(u_{2xxx}+3u_{1\xi xx}\right)+5u_2u_{1xxx}+5u_{1x}\left(u_{2xx}+2u_{1\xi x}\right)\\&+5u_{1xx}\left(u_{2x}+2u_{1\xi}\right)+u_1^2u_{1x}=0\end{aligned} \tag{27}$$

$$\vdots$$

Then, we can find the solution of (25) as follows

$$u_1(x,t,\xi,\tau)=v_1(\xi,\tau)\mathrm{e}^{i(kx-k^5t)}+c.c. \tag{28}$$

where $c.c.$ is complex conjugate of v_1. Substituting the solution (28) into (26), the solution of (26) is in the form

$$u_2(x,t,\xi,\tau)=v_2(\xi,\tau)\mathrm{e}^{2i(kx-k^5t)}+c.c.+f_0(\xi,\tau), \tag{29}$$

where f_0 is integration constant. Thus we get

$$\begin{aligned}v_2&=\frac{1}{3k^2}v_1^2,\\ v_{-2}&=\frac{1}{3k^2}v_{-1}^2,\end{aligned} \tag{30}$$

where v_{-1} is the complex conjugate of v_1 and v_{-2} is the complex conjugate of v_2. Substituting solutions (28), (29) and (30) into the (27), we find the solution of (27) in the form

$$u_3(x,t,\xi,\tau)=v_3(\xi,\tau)\mathrm{e}^{3i(kx-k^5t)}+c.c.+f_1(\xi,\tau)\mathrm{e}^{2i(kx-k^5t)}+f_2(\xi,\tau)\mathrm{e}^{-2i(kx-k^5t)}, \quad (31)$$

where f_1 is integration constant and v_{-3} is the complex conjugate of v_3. Then we get

$$\begin{aligned}
&v_3=-\frac{10}{123k^4}v_1^3,\\
&v_{-3}=-\frac{10}{123k^4}v_{-1}^3,\\
&f_0=\frac{3}{k^2}v_1v_{-1},\ f_1=\frac{10}{17k^3}iv_{-1}v_{-1\xi},\ f_2=\frac{-10}{17k^3}iv_1v_{1\xi}
\end{aligned} \quad (32)$$

and

$$\begin{aligned}
&iv_{1\tau}=v_{1\xi\xi}-\frac{2}{k^2}v_1v_{-1},\\
&iv_{1\tau}=v_{1\xi\xi}-\frac{2}{k^2}v_1\left|v_1\right|^2.
\end{aligned} \quad (33)$$

Describing as $q=\frac{v_1}{k}$ and $q_{-1}=\frac{v_{-1}}{k}$, from Equation (33) we get the NLS type equations

$$iq_\tau=q_{\xi\xi}-2q\left|q^2\right|. \quad (34)$$

Also, approximate solution of the (1 + 1) fifth-dimensional Sawada-Kotera (SK) Equation (3) is found as

$$\begin{aligned}
u(x,t)=&\epsilon k\left(q(\xi,\tau)\mathrm{e}^{i\theta}+q_{-1}(\xi,\tau)\mathrm{e}^{-i\theta}\right)\\
&+\epsilon^2\left(3q(\xi,\tau)q_{-1}(\xi,\tau)+\frac{1}{3}q^2(\xi,\tau)\mathrm{e}^{2i\theta}+\frac{1}{3}q_{-1}^2(\xi,\tau)\mathrm{e}^{-2i\theta}\right)\\
&+\epsilon^3k\left(\frac{10}{17}iq_{-1}(\xi,\tau)q_{-1}(\xi,\tau)\mathrm{e}^{2i\theta}+\frac{10}{17}iq(\xi,\tau)q(\xi,\tau)\mathrm{e}^{-2i\theta}\right)\\
&-\epsilon^3k\frac{10}{123}\left(q^3(\xi,\tau)\mathrm{e}^{2i\theta}+q_{-1}^3(\xi,\tau)\mathrm{e}^{-2i\theta}\right)
\end{aligned} \quad (35)$$

where q is solution of NLS equation.

3.2. The Seventh Order Sawada-Kotera (SK) Equation

Following the method given in Section 2.3 we use a multiple scales method to derive the NLS equations from the seventh-order Sawada-Kotera (SK) Equation (8). To find dispersion relation for (8), we consider the linear part of (8) in the form

$$u_t=u_{xxxxxxx}, \quad (36)$$

and linear differential Equation (18) satisfies the solution

$$u(x,t)=\mathrm{e}^{i\theta},\ \theta=kx-\omega(k)t. \quad (37)$$

Substituting the solution (19) into the linear differential Equation (18), we get and from this we reach the

$$\omega(k)=k^7, \tag{38}$$

dispersion relation. Thus the solution of linear differential Equation (18) is as follows:

$$u(x,t)=\mathrm{e}^{i(kx-w(k)t)}, \tag{39}$$

Let the solution of Equation (2) is in the form

$$u(x,t)=U(x,t,\xi,\tau),\ U(x,t,\xi,\tau)=\epsilon U_1+\epsilon^2U_2+\epsilon^3U_3+\cdots \tag{40}$$

ϵ scale parameter and slow variables

$$\xi=\epsilon\left(x-\frac{\mathrm{d}\omega(k)}{\mathrm{d}k}t\right)=\epsilon\left(x-7k^6t\right),\ \tau=-\frac{1}{2}\epsilon^2\frac{\mathrm{d}^2\omega(k)}{\mathrm{d}k^2}t=-21\epsilon^2k^5t. \tag{41}$$

Then we assume the following series expansions for solutions:

$$U(x,y,t,\xi,\tau)=\epsilon U_1+\epsilon^2U_2+\epsilon^3U_3+\cdots. \tag{42}$$

In this case, considering the transformation and solution in (22), using (20) and (23), the terms included derivative in Equation (8) are obtained. Substituting these terms and (22) into the Equation (8), we get a polynomial in ϵ. Equalizing each coefficient of this polynomial to zero, we get a set of algebraic equations. If we let $\varepsilon\to0$ we find the following:

$$\epsilon: u_{1t}+u_{1xxxxx}=0, \tag{43}$$

$$\begin{aligned}\epsilon^2: &u_{2t}-u_{2xxxxxxx}-7k^6u_{1\xi}-7u_{1\xi xxxxxx}-21u_1u_{1xxxxx}\\&-42u_{1x}u_{1xxxx}-63u_{1xx}u_{1xxx}=0,\end{aligned} \tag{44}$$

$$\begin{aligned}\epsilon^3: &u_{3t}-u_{3xxxxxxx}-7k^6u_{2\xi}-21k^5u_{1\tau}-7u_{2\xi xxxxxx}-21u_{1\xi\xi xxxxx}\\&-21u_1\left(u_{2xxxxx}+5u_{1\xi xxxx}\right)-21u_2u_{1xxxxx}-42u_{1x}\left(u_{2xxxx}+4u_{1\xi xxx}\right)\\&-42u_{1xxxx}\left(u_{2x}+u_{1\xi}\right)-63u_{1xx}\left(u_{2xxx}+3u_{1\xi xx}\right)-63u_{1xxx}\left(u_{2xx}+2u_{1\xi x}\right)\\&-126u_1^2u_{2xxx}-378u_1u_{1x}u_{1xx}-63u_{1x}^3=0\end{aligned} \tag{45}$$

$$\vdots$$

Then, we can find the solution of (43) as follows

$$u_1(x,t,\xi,\tau)=v_1(\xi,\tau)\mathrm{e}^{i\left(kx-k^7t\right)}+c.c. \tag{46}$$

where $c.c.$ is complex conjugate of v_1. Substituting the solution (46) into (44), the solution of (44) is in the form

$$u_2(x,t,\xi,\tau)=v_2(\xi,\tau)\mathrm{e}^{2i\left(kx-k^7t\right)}+c.c.+f_0(\xi,\tau), \tag{47}$$

where f_0 is integration constant. Thus we get

$$\begin{aligned}v_2&=\frac{1}{k^2}v_1^2,\\v_{-2}&=\frac{1}{k^2}v_{-1}^2,\end{aligned} \tag{48}$$

where v_{-1} is the complex conjugate of v_1 and v_{-2} is the complex conjugate of v_2. Substituting solutions (46), (47) and (48) into the (45), we find the solution of (45) in the form

$$u_3(x,t,\xi,\tau) = v_3(\xi,\tau)\mathrm{e}^{3i(kx-k^7t)} + c.c. + f_1(\xi,\tau)\mathrm{e}^{2i(kx-k^7t)} + f_2(\xi,\tau)\mathrm{e}^{-2i(kx-k^7t)}, \tag{49}$$

where f_1 is integration constant and v_{-3} is the complex conjugate of v_3. Then we get

$$\begin{aligned} &v_3 = \frac{3}{4k^4}v_1^3, \\ &v_{-3} = \frac{3}{k^4}v_{-1}^3, \\ &f_0 = -\frac{3}{k}v_1v_{-1},\ f_1 = -\frac{2}{k^3}iv_{-1}v_{-1\xi},\ f_2 = \frac{2}{k^3}iv_1v_{1\xi} \end{aligned} \tag{50}$$

and

$$\begin{aligned} iv_{1\tau} &= v_{1\xi\xi} - \frac{2}{k^2}v_1v_{-1}, \\ iv_{1\tau} &= v_{1\xi\xi} - \frac{2}{k^2}v_1|v_1|^2. \end{aligned} \tag{51}$$

Describing as $q = \frac{v_1}{k}$ and $q_{-1} = \frac{v_{-1}}{k}$, from Equation (51) we get the NLS type equations

$$iq_\tau = q_{\xi\xi} - 2q\left|q^2\right|. \tag{52}$$

Also, approximate solution of the (1 + 1) seventh-dimensional Sawada-Kotera (SK) Equation (8) is found as

$$\begin{aligned} u(x,t) = {} & \epsilon k\left(q(\xi,\tau)\mathrm{e}^{i\theta} + q_{-1}(\xi,\tau)\mathrm{e}^{-i\theta}\right) \\ & + \epsilon^2\left(-3q(\xi,\tau)q_{-1}(\xi,\tau) + q^2(\xi,\tau)\mathrm{e}^{2i\theta} + q_{-1}^2(\xi,\tau)\mathrm{e}^{-2i\theta}\right) \\ & + \epsilon^3 k\left(-2iq_{-1}(\xi,\tau)q_{-1}(\xi,\tau)\mathrm{e}^{2i\theta} + 2iq(\xi,\tau)q(\xi,\tau)\mathrm{e}^{-2i\theta}\right) \\ & + \epsilon^3 k\frac{3}{4}\left(q^3(\xi,\tau)\mathrm{e}^{2i\theta} + q_{-1}^3(\xi,\tau)\mathrm{e}^{-2i\theta}\right) \end{aligned} \tag{53}$$

where q is solution of NLS equation.

4. Conclusion

The multiple scales method, which is a perturbation method, is used to find approximate solutions of nonlinear evolution equations. Although the origins of nonlinear formation equations date back to ancient times, significant developments have occurred regarding these equations from the past to the present. The main reason for this is that nonlinear formation equations involve the problem of nonlinear wave propagation. This method allows us to find the solution of the given nonlinear equation depending on the solution of the linear part by using multiple scales, which are defined as the slow variables and depend on a parameter, in time and space variables. First, Zakharov and Kuznetsov showed that integrable systems can be reduced to other integrable systems using this method. If the initially taken system is not integrable, it is seen that the reduced system is either integrable or non-integrable as a result of the application of the method. However, if the method is applied to an integrable system, it is seen that the system obtained as a result of the analysis is always integrable. This is the main purpose of applying multiple-scale methods to integrable systems. This study examined how only NLS equation is derived from higher-order nonlinear Sawada-Kotera (SK) equations using the multiple scales method. We hope this conclusion serves as a valuable resource for researchers, scientists, and engineers interested in nonlinear wave theory, inspiring a deeper desire to explore further and reveal the mysteries still held within the intricate language of the NLS equation. At the same time, we think that the obtained results will form the basis of numerical calculations. Also, the method can be applied to many different NLEE equations.

Funding

This research is partly supported by Anadolu University, Scientific Research Project (BAP), Project Number: SBA-2025-2933.

Data Availability

All data generated or analysed during this study are included in this published article.

Conflicts of Interest

The author declares that they have no conflict of interests regarding the publication of this paper.

References

[1] Khusnutdinova, K.R., Stepanyants, Y.A. and Tranter, M.R. (2018) Soliton Solutions to the Fifth-Order Korteweg-de Vries Equation and Their Applications to Surface and Internal Water Waves. *Physics of Fluids*, **30**, Article 022104.

[2] Choudhury, S.R. and Alfonso Rodriguez, R. (2022) Perturbative and Reversible Systems Approaches to New Families of Embedded Solitary Waves of a Generalized Fifth-Order Korteweg-de Vries Equation. *Partial Differential Equations in Applied Mathematics*, **5**, Article 100314. https://doi.org/10.1016/j.padiff.2022.100314

[3] Lee, C.T., Liu, M.L., Lin, J.E. and Lee, C.C. (2019) Some Notes on Numerical Waves of Fifth-Order Korteweg-de Vries Equations. *Physics Essays*, **32**, 127-139. https://doi.org/10.4006/0836-1398-32.1.127

[4] Zhao, X., Tian, B., Li, H. and Guo, Y. (2017) Solitons, Periodic Waves, Breathers and Integrability for a Nonisospectral and Variable-Coefficient Fifth-Order Korteweg-de Vries Equation in Fluids. *Applied Mathematics Letters*, **65**, 48-55. https://doi.org/10.1016/j.aml.2016.10.003

[5] Ahmad, H., Khan, T.A. and Yao, S. (2020) An Efficient Approach for the Numerical Solution of Fifth-Order KdV Equations. *Open Mathematics*, **18**, 738-748. https://doi.org/10.1515/math-2020-0036

[6] Baleanu, D., Shekari, P., Torkzadeh, L., Ranjbar, H., Jajarmi, A. and Nouri, K. (2023) Stability Analysis and System Properties of Nipah Virus Transmission: A Fractional

Calculus Case Study. *Chaos*, *Solitons & Fractals*, **166**, Article 112990. https://doi.org/10.1016/j.chaos.2022.112990

[7] Defterli, O., Baleanu, D., Jajarmi, A., Sajjadi, S.S., Alshaikh, N. and Asad, J.H. (2022) Fractional Treatment: An Accelerated Mass-Spring System. *Romanian Reports in Physics*, **74**, Article 122.

[8] Luo, R., Rafiullah, Emadifar, H. and ur Rahman, M. (2023) Bifurcations, Chaotic Dynamics, Sensitivity Analysis and Some Novel Optical Solitons of the Perturbed Non-Linear Schrödinger Equation with Kerr Law Non-Linearity. *Results in Physics*, **54**, Article 107133. https://doi.org/10.1016/j.rinp.2023.107133

[9] Eskandari, Z., Avazzadeh, Z., Khoshsiar Ghaziani, R. and Li, B. (2022) Dynamics and Bifurcations of a Discrete-Time Lotka-Volterra Model Using Nonstandard Finite Difference Discretization Method. *Mathematical Methods in the Applied Sciences*, **48**, 7197-7212. https://doi.org/10.1002/mma.8859

[10] Li, B., Liang, H. and He, Q. (2021) Multiple and Generic Bifurcation Analysis of a Discrete Hindmarsh-Rose Model. *Chaos*, *Solitons & Fractals*, **146**, Article 110856. https://doi.org/10.1016/j.chaos.2021.110856

[11] El-Shorbagy, M.A., Akram, S. and ur Rahman, M. (2024) Propagation of Solitary Wave Solutions to (4+1)-Dimensional Davey-Stewartson-Kadomtsev-Petviashvili Equation Arise in Mathematical Physics and Stability Analysis. *Partial Differential Equations in Applied Mathematics*, **10**, Article 100669. https://doi.org/10.1016/j.padiff.2024.100669

[12] Seadawy, A.R. (2012) Exact Solutions of a Two-Dimensional Nonlinear Schrödinger Equation. *Applied Mathematics Letters*, **25**, 687-691. https://doi.org/10.1016/j.aml.2011.09.030

[13] Liu, W.M. and Kengne, E. (2019) Schrödinger Equations in Nonlinear Systems. Springer.

[14] Al Khawaja, U. and Al Sakkaf, L. (2019) Handbook of Exact Solutions to the Nonlinear Schrödinger Equations. IOP Publishing. https://doi.org/10.1088/978-0-7503-2428-1ch7

[15] Zakharov, V.E. and Kuznetsov, E.A. (1986) Multi-Scale Expansions in the Theory of Systems Integrable by the Inverse Scattering Transform. *Physica D*: *Nonlinear Phenomena*, **18**, 455-463. https://doi.org/10.1016/0167-2789(86)90214-9

[16] Calogero, F., Degasperis, A. and Xiaoda, J. (2000) Nonlinear Schrödinger-Type Equations from Multiscale Reduction of PDEs. I. Systematic Derivation. *Journal of Mathematical Physics*, **41**, 6399-6443. https://doi.org/10.1063/1.1287644

[17] Degasperis, A., Manakov, S.V. and Santini, P.M. (1997) Multiple-Scale Perturbation Beyond the Nonlinear Schroedinger Equation. I. *Physica D*: *Nonlinear Phenomena*, **100**, 187-211. https://doi.org/10.1016/s0167-2789(96)00179-0

[18] Osborne, A.R. and Boffetta, G. (1989) The Shallow-Water Nonlinear Schrödinger Equation in Lagrangian Coordinates. *Physics of Fluids A*: *Fluid Dynamics*, **1**, 1200-1210. https://doi.org/10.1063/1.857343

[19] Osborne, A.R. and Boffetta, G.A. (1991) Summable Multiscale Expansion for the KdV Equation. In: Degasperis, A., Fordy, A.P. and Lakshmanan, M., Eds., *Nonlinear Evolution Equations*: *Integrability and Spectral Methods*, Manchester University Press, 559-571.

[20] Özer, M.N. and Dağ, I. (2001) The Famous NLS Equation from Fifth-Order Integrable Nonlinear Evolution Equations. *Hadronic Journal*, **24**, 195-206.

[21] Ayhan, B., Özer, M.N. and Bekir, A. (2016) Method of Multiple Scales and Travelling

Wave Solutions for (2+1)-Dimensional KdV Type Nonlinear Evolution Equations. *Zeitschrift für Naturforschung A*, **71**, 703-713. https://doi.org/10.1515/zna-2016-0123

[22] Wazwaz, A. and Kaur, L. (2019) Optical Solitons for Nonlinear Schrödinger (NLS) Equation in Normal Dispersive Regimes. *Optik*, **184**, 428-435. https://doi.org/10.1016/j.ijleo.2019.04.118

[23] Dörfler, W., Lechleiter, A., Plum, M., Schneider, G. and Wieners, C. (2011) The Role of the Nonlinear Schrödinger Equation in Nonlinear Optics. In: Dörfler, W., Lechleiter, A., Plum, M., Schneider, G. and Wieners, C., Eds., *Photonic Crystals*: *Mathematical Analysis and Numerical Approximation*, Springer, 127-162. https://doi.org/10.1007/978-3-0348-0113-3_5

[24] Arnaudon, A. (2016) On a Deformation of the Nonlinear Schrödinger Equation. *Journal of Physics A*: *Mathematical and Theoretical*, **49**, Article 125202. https://doi.org/10.1088/1751-8113/49/12/125202

[25] Lax, P.D. (1968) Integrals of Nonlinear Equations of Evolution and Solitary Waves. *Communications on Pure and Applied Mathematics*, **21**, 467-490. https://doi.org/10.1002/cpa.3160210503

[26] Wazwaz, A. (2007) The Extended Tanh Method for New Solitons Solutions for Many Forms of the Fifth-Order KdV Equations. *Applied Mathematics and Computation*, **184**, 1002-1014. https://doi.org/10.1016/j.amc.2006.07.002

[27] Fordy, A.P. and Gibbons, J. (1980) Some Remarkable Nonlinear Transformations. *Physics Letters A*, **75**, 325-334. https://doi.org/10.1016/0375-9601(80)90829-4

[28] Satsuma, J. and J. Kaup, D. (1977) A Bäcklund Transformation for a Higher Order Korteweg-de Vries Equation. *Journal of the Physical Society of Japan*, **43**, 692-697. https://doi.org/10.1143/jpsj.43.692

[29] Kaup, D.J. (1980) On the Inverse Scattering Problem for Cubic Eigenvalue Problems of the Class $u_{xxx} + 6qu_x + 6ru = ?u$. *Studies in Applied Mathematics*, **62**, 189-216. https://doi.org/10.1002/sapm1980623189

[30] Kupershmidt, B.A. (1984) A Super Korteweg-de Vries Equation: An Integrable System. *Physics Letters A*, **102**, 213-215. https://doi.org/10.1016/0375-9601(84)90693-5

[31] Jimbo, M. and Miwa, T. (1983) Solitons and Infinite Dimensional Lie Algebras. *Publications of the Research Institute for Mathematical Sciences*, **19**, 943-1001. https://doi.org/10.2977/prims/1195182017

[32] Sawada, K. and Kotera, T. (1974) A Method for Finding N-Soliton Solutions of the KdV Equation and KdV-Like Equation. *Progress of Theoretical Physics*, **51**, 1355-1367. https://doi.org/10.1143/ptp.51.1355

[33] Salas, A. (2008) Some Solutions for a Type of Generalized Sawada-Kotera Equation. *Applied Mathematics and Computation*, **196**, 812-817. https://doi.org/10.1016/j.amc.2007.07.013

[34] Ito, M. (1980) An Extension of Nonlinear Evolution Equations of the K-dV (mK-dV) Type to Higher Orders. *Journal of the Physical Society of Japan*, **49**, 771-778. https://doi.org/10.1143/jpsj.49.771

[35] Ito, M. (1986) A Reduce Program for Finding Symmetries of Nonlinear Evolution Equations with Uniform Rank. *Computer Physics Communications*, **42**, 351-357. https://doi.org/10.1016/0010-4655(86)90005-6

[36] Pomeau, Y., Ramani, A. and Grammaticos, B. (1988) Structural Stability of the Korteweg-de Vries Solitons under a Singular Perturbation. *Physica D*: *Nonlinear Phenomena*, **31**, 127-134. https://doi.org/10.1016/0167-2789(88)90018-8

[37] Wazwaz, A. (2008) The Hirota's Direct Method and the Tanh-Coth Method for Multiple-Soliton Solutions of the Sawada-Kotera-Ito Seventh-Order Equation. *Applied Mathematics and Computation*, **199**, 133-138. https://doi.org/10.1016/j.amc.2007.09.034

[38] Jafari, H., Yazdani, A., Vahidi, J. and Ganji, D.D. (2008) Application of He's Variational Iteration Method for Solving Seventh order Sawada-Kotera Equations. *Applied Mathematical Sciences*, **2**, 471-477.

[39] Gotkas, U. and Hereman, W. (1997) Symbolic Computation of Conserved Densities for Systems of Nonlinear Evolution Equations. *Journal of Symbolic Computation*, **24**, 591-622.

[40] Natfeh, A.H. (1980) Introduction to Perturbation Techniques. John Wiley & Sons.

[41] Koparan, M. (2024) The Nonlinear Schrödinger Equation Derived from the Third Order Korteweg-de Vries Equation Using Multiple Scales Method. *Contemporary Mathematics*, **4**, 5087-5097. https://doi.org/10.37256/cm.5420244647