

# A Continuous Dynamical Model of Collaborative Learning

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## Abstract

We present a continuous-time model describing how two interacting individuals acquire, forget, and exchange knowledge during collaborative processes. The formulation extends the classical learning with decaying memory framework by introducing interaction terms that depend on the state difference between the two individuals. In the linear case, we derive the equilibrium and fully characterize stability, showing that interaction reduces state differences without affecting the mean level. A coupling threshold is identified, marking the transition from individual to collective dynamics. To reflect that interaction is effective only within a suitable range, we introduce a nonlinear, state-dependent coupling function. This leads to a convergence zone, where equalization is rapid, and a decoupling regime when differences are large. The model provides a tractable dynamical systems framework with potential applications to learning processes

## Keywords

Dynamical Systems, Coupled Systems, Stability Analysis, Consensus Dynamics, Nonlinear Coupling, Learning Dynamics

## 1. Introduction

Collaborative learning can be viewed as a dynamical process in which knowledge evolves over time under the combined influence of acquisition, decay, and interaction mechanisms. Mathematical modeling of such processes provides a systematic and rigorous framework for understanding how individual cognitive dynamics give rise to collective learning behavior. In the classical learning in the presence of decaying memory [1], this evolution is typically described by linear ordinary differential equations that balance knowledge acquisition and loss, leading to well-

defined equilibrium states and characteristic convergence rates. These models capture essential features of individual learning, including boundedness of knowledge and asymptotic stabilization.

A natural extension of this framework involves multiple interacting learners. In this context, knowledge exchange introduces coupling terms that transform individual trajectories into a collective dynamical system. Collective dynamics refers to the regime in which interaction significantly influences the evolution of both learners, leading to coordinated behavior and reduced knowledge differences. Interaction is commonly modeled as proportional to the knowledge difference between individuals, leading to diffusive-type dynamics that promote synchronization or equalization. Such formulations are closely related to consensus models in multi-person systems [2]. Despite their analytical tractability, linear interaction models rely on a strong simplifying assumption: That collaboration is equally effective regardless of the cognitive difference between learners. However, educational theory, most notably the concept of the Zone of Proximal Development (ZPD) introduced by Vygotsky [3], suggests that the effectiveness of interaction depends critically on the knowledge difference. Interaction is most productive when learners are sufficiently close in knowledge, while it becomes ineffective when the cognitive difference is either too small or excessively large.

Similar ideas arise in bounded confidence models of opinion dynamics, such as those proposed by Deffuant *et al.* [4], and Hegselmann and Krause [5], where interaction occurs only when persons are sufficiently close in state space. However, these models are typically formulated in discrete time and rely primarily on numerical simulations, providing a discrete insight into the underlying dynamical mechanisms.

In contrast, continuous time dynamical systems offer a powerful analytical framework that enables explicit characterization of equilibria, stability, and convergence properties. Within this setting, it becomes possible to identify critical thresholds and structural properties that govern the system's behavior features that are often difficult to extract from purely simulation-based approaches. The critical coupling threshold denotes the value of the coupling parameter above which interaction effects dominate the intrinsic learning dynamics and knowledge equalization becomes significantly faster.

Motivated by these considerations, the present work introduces a nonlinear continuous-time model of collaborative learning that incorporates state-dependent interaction. Specifically, we extend the classical learning-with-decaying memory framework to a system of two interacting learners in which the coupling strength depends explicitly on the cognitive difference between learners. This formulation provides a direct mathematical representation compatible with existing pedagogical principles within a system of differential equations.

The analysis proceeds in two stages. First, we investigate a linear baseline model, deriving closed-form expressions for equilibrium states and providing a complete characterization of stability through eigenvalue analysis. By introducing a mean-

difference transformation, the system is decomposed into two fundamental components: the evolution of average knowledge and the dynamics of knowledge difference. This decomposition reveals a key structural property: collaboration acts as a dissipative mechanism that reduces cognitive differences while leaving the mean knowledge invariant. Building on this result, we identify a critical coupling threshold that separates two qualitatively distinct regimes: one dominated by individual learning dynamics and another governed by collective convergence.

We then extend the model to a nonlinear setting by introducing a state-dependent coupling function, leading to qualitatively richer dynamics. In particular, the nonlinear model gives rise to a convergence zone in which interaction is effective, as well as a decoupling regime in which sufficiently large knowledge difference suppress collaboration. The convergence zone is defined as the set of initial conditions for which the interaction remains sufficiently strong to promote rapid reduction of cognitive differences between learners.

The main contributions of this work can be summarized as follows:

- (i) the formulation of a continuous-time nonlinear model of collaborative learning with state-dependent interaction,
- (ii) the derivation of analytical results for equilibrium, stability, and convergence dynamics,
- (iii) the identification of a critical coupling threshold separating distinct dynamical regimes, and
- (iv) the introduction of a convergence-zone concept linking dynamical systems theory with educational interpretation.

The proposed framework establishes a bridge between dynamical systems theory and educational modeling, providing a tractable and extensible basis for the analysis of collaborative learning processes. Unlike existing models of collaborative learning and consensus dynamics, the present work provides an analytical framework for the linear model, together with a qualitative nonlinear extension based on state-dependent interaction. The framework explicitly characterizes equilibrium structure, global stability, and convergence mechanisms for the linear system, while the nonlinear model is investigated through qualitative dynamical analysis. In particular, the proposed formulation introduces a critical coupling threshold separating individual and collective learning regimes, as well as a nonlinear state-dependent interaction that gives rise to a convergence-zone phenomenon. This approach offers a unified mathematical interpretation of collaborative learning processes, bridging dynamical systems theory with fundamental concepts from educational science.

## Modeling Assumptions

The collaborative learning model is based on the following assumptions:

- 1) The system consists of two interacting learners with knowledge levels  $y_i(t)$ ,  $i = 1, 2$ .
- 2) Both learners share the same total knowledge target  $M$ .

3) Knowledge evolves in continuous time and is governed by a deterministic system of ordinary differential equations (ODEs).

4) Each learner is characterized by a learning rate  $A_i > 0$  and a forgetting rate  $L_i > 0$ ,  $i = 1, 2$ .

5) The interaction is symmetric, *i.e.*, the influence of learner 1 on learner 2 is equal to the influence of learner 2 on learner 1, and is governed by a common coupling coefficient  $k$ .

6) The interaction depends only on the knowledge difference between the learners.

## 2. Mathematical Model of Learning Dynamics

### 2.1. Individual Learning Model with Decaying Memory

The classical learning-with-decaying memory model is described by the ordinary differential equation:

$$\frac{dy}{dt} = A(M - y(t)) - Ly(t), \quad t > 0 \quad (1)$$

with initial condition

$$y(0) = \varepsilon M, \quad 0 \leq \varepsilon \leq 1 \quad (2)$$

where:

- $M$ , denotes the total amount of knowledge to be acquired,
- $y(t)$ , represents the knowledge level at time  $t$ ,
- $\varepsilon$ , is the proportion of initially known material,
- $A > 0$ , is the learning parameter,
- $L > 0$ , the decaying memory rate, reflecting the individual's ability to retain acquired knowledge over time.

The long-term equilibrium of the system is given by

$$y^* = \frac{A}{A+L} M \quad (3)$$

which represents the maximum attainable knowledge level under constant learning and forgetting conditions.

The model expresses a balance between knowledge acquisition and decay. The term  $A(M - y)$  represents the remaining learning part of  $M$ , while the term  $Ly$  captures the loss of acquired knowledge over time. This structure ensures boundedness of solutions and convergence to a finite equilibrium.

### 2.2. Extension to Two Interacting Learners

We extend the model to two interacting persons with knowledge levels  $y_1(t)$  at time  $t$  and  $y_2(t)$ . The initial conditions are given by

$$y_1(0) = \varepsilon_1 M, \quad y_2(0) = \varepsilon_2 M \quad (4)$$

with  $0 \leq \varepsilon_1, \varepsilon_2 \leq 1$ , and

$$0 \leq y_i(t) \leq M, \quad i = 1, 2 \quad (5)$$

The admissible region

$$\Omega = \{(y_1, y_2) : 0 \leq y_1 \leq M, 0 \leq y_2 \leq M\}$$

is positively invariant under the dynamics of system (7), provided that  $A_i > 0$ ,  $L_i > 0$ ,  $k > 0$ , and the initial conditions satisfy  $0 \leq y_i(0) \leq M$ ,  $i = 1, 2$ .

Indeed, on the boundary  $y_i = 0$ , the corresponding derivative satisfies

$$\frac{dy_i}{dt} = A_i M + k y_j \geq 0,$$

where  $j \in \{1, 2\}$  and  $j \neq i$ . Similarly, on the boundary  $y_i = M$ ,

$$\frac{dy_i}{dt} = -L_i M + k(y_j - M) \leq 0,$$

since  $0 \leq y_j \leq M$ .

Therefore, trajectories cannot leave the region  $\Omega$ , and the bounds

$$0 \leq y_i(t) \leq M, \quad i = 1, 2,$$

are preserved for all  $t \geq 0$ .

### 2.3. Modeling Interaction

Interaction between persons is modeled through knowledge exchange proportional to their differences. The coupling terms are defined as

$$k(y_2 - y_1) \text{ for person 1, } k(y_1 - y_2) \text{ for person 2}$$

This interaction tends to reduce knowledge differences between learners, driving the system toward cognitive equalization. It should be emphasized that the interaction term is not intended to represent a literal transfer of knowledge from one learner to another. Rather, it provides an idealized consensus-type mechanism that models the tendency of interaction to reduce cognitive disparities between learners.

In this interpretation, the conservation property refers only to the interaction contribution and should not be interpreted as implying that knowledge is literally transferred or depleted through collaboration. Knowledge acquisition continues to occur through the external learning terms, while the interaction acts as a mathematical equalization mechanism.

The interaction satisfies the conservation property

$$k(y_2 - y_1) + k(y_1 - y_2) = 0 \quad (6)$$

which ensures that knowledge is redistributed within the system without being artificially created or destroyed.

This diffusive coupling structure reflects peer-learning mechanisms and depends only on relative knowledge differences.

### 2.4. The Collaborative Dynamical System

Combining individual learning dynamics with interaction, we obtain the system

$$\begin{cases} \frac{dy_1(t)}{dt} = A_1(M - y_1(t)) - L_1 y_1(t) + k(y_2(t) - y_1(t)) \\ \frac{dy_2(t)}{dt} = A_2(M - y_2(t)) - L_2 y_2(t) + k(y_1(t) - y_2(t)) \end{cases} \quad (7)$$

Equivalently, the system can be written as,

$$\begin{cases} \frac{dy_1(t)}{dt} = A_1 M - (A_1 + L_1 + k)y_1(t) + ky_2(t) \\ \frac{dy_2(t)}{dt} = A_2 M - (A_2 + L_2 + k)y_2(t) + ky_1(t) \end{cases} \quad (8)$$

In matrix form, we write

$$\frac{dY(t)}{dt} = AY(t) + b \quad (9)$$

where

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} -(A_1 + L_1 + k) & k \\ k & -(A_2 + L_2 + k) \end{pmatrix}, \quad b = \begin{pmatrix} A_1 M \\ A_2 M \end{pmatrix} \quad (10)$$

The system (9) is a non-homogeneous linear dynamical system admitting a unique equilibrium. Its structure allows a complete analytical treatment, including explicit computation of equilibrium states and stability properties. In the next section, we exploit this formulation to derive equilibrium solutions, analyze eigenvalues, and characterize convergence dynamics in terms of the model parameters.

### 3. Analysis of the Linear Collaborative Learning Model

We present here a complete analytical and qualitative study of the linear collaborative system (9). In addition to formulated results on equilibrium and stability, particular emphasis is placed on the interpretation of the dynamics and their visualization through phase portraits.

#### 3.1. Equilibrium and Long-Term Behavior

We begin by determining the steady state of the system (9).

**Theorem 3.1 (Existence and uniqueness of equilibrium)** *The system (9) admits a unique equilibrium point  $Y^* = (y_1^*, y_2^*)$  given by*

$$Y^* = -A^{-1}b \quad (11)$$

where  $A$  and  $b$  are defined in (10).

*Proof.* The equilibrium is obtained by setting  $\frac{dY(t)}{dt} = 0$  in the system (9).

Thus, the equilibrium satisfies

$$AY^* + b = 0,$$

which gives

$$Y^* = -A^{-1}b,$$

provided that  $A$  is invertible. The determinant of  $A$  is

$$\det(A) = (A_1 + L_1)(A_2 + L_2) + k(A_1 + L_1 + A_2 + L_2) > 0, \quad (12)$$

Since all parameters are positive, it follows that  $\det(A) > 0$ , hence  $A$  is invertible. Therefore, the equilibrium exists and is unique.  $\square$

The equilibrium represents the maximum attainable knowledge levels of the two persons under the combined influence of learning, decaying memory, and interaction.

### 3.2. Stability and Phase Portrait

We examine the stability of the equilibrium point of system (9).

**Theorem 3.2 (Global asymptotic stability)** *The equilibrium point  $Y^* = (y_1^*, y_2^*)$  of system (9) is globally asymptotically stable.*

*Proof.* The eigenvalues  $\lambda_1, \lambda_2$  of matrix  $A$  are

$$\begin{aligned} \lambda_1 &= -\frac{1}{2} \left[ (A_1 + L_1) + (A_2 + L_2) + 2k + \sqrt{d^2 + 4k^2} \right], \\ \lambda_2 &= -\frac{1}{2} \left[ (A_1 + L_1) + (A_2 + L_2) + 2k - \sqrt{d^2 + 4k^2} \right] \end{aligned} \quad (13)$$

where

$$d = (A_1 + L_1) - (A_2 + L_2) \quad (14)$$

Since  $\text{tr}(A) < 0$  and  $\det(A) > 0$ , both eigenvalues are strictly negative.  $\square$

Therefore, the system converges monotonically to equilibrium without oscillations, forming a stable node.

The explicit form of the eigenvalues provides additional insight into the convergence dynamics. In particular, the eigenvalues are always real and negative, implying that the system converges monotonically toward equilibrium without oscillations. Geometrically, this behavior corresponds to a *stable node* in the phase plane. All trajectories are directed toward the equilibrium point and no spiraling occurs. This behavior is illustrated in **Figure 1**, where all trajectories converge toward  $Y^*$  and the vector field points inward.

### 3.3. Convergence Dynamics and Time Scales

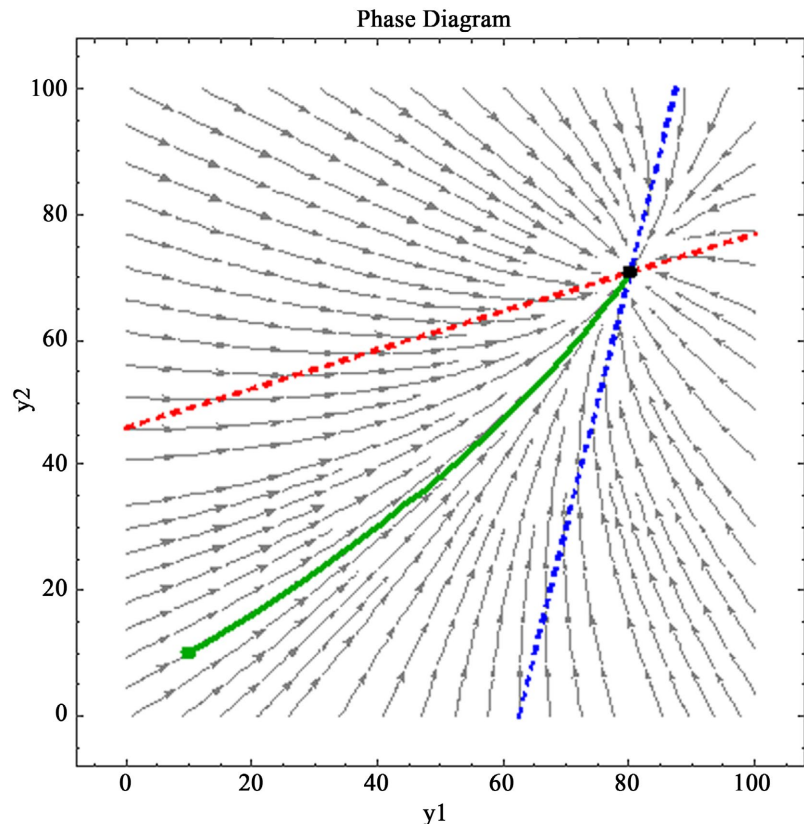
The solution of system (9) can be expressed in terms of the eigenvalues and eigenvectors of matrix  $A$ . In particular, the general solution takes the form

$$Y(t) = Y^* + c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2, \quad (15)$$

where  $\lambda_1, \lambda_2$  are the eigenvalues of matrix  $A$ ,  $\mathbf{v}_1, \mathbf{v}_2$  the corresponding eigenvectors, and  $c_1, c_2$  constants determined by the initial conditions. Since both eigenvalues are real and negative, the system exhibits monotone convergence toward equilibrium without oscillations.

#### Fast and slow dynamics

The solution naturally decomposes into two distinct modes associated with the eigenvalues:



**Figure 1.** Phase portrait of the linear system. Trajectories converge monotonically toward the equilibrium point, forming a stable node.

- A *fast mode*, corresponding to the eigenvalue with larger absolute value,
- A *slow mode*, corresponding to the eigenvalue closest to zero.

The fast mode decays rapidly and governs the initial transient behavior, while the slow mode determines the long-term convergence toward equilibrium.

#### Characteristic time scale

The dominant eigenvalue (*i.e.*, the one closest to zero) determines the characteristic convergence time of the system, defined as

$$\tau = \frac{1}{|\lambda_{\text{slow}}|} \quad (16)$$

A larger absolute value of the dominant eigenvalue corresponds to faster convergence.

#### Geometric interpretation

This behavior admits a clear geometric interpretation in the phase plane. Trajectories initially align with the direction of the fast eigenvector, corresponding to rapid decay of one component. Subsequently, they evolve along the direction of the slow eigenvector, which governs the final approach to equilibrium. This two-stage convergence mechanism is reflected in the trajectories shown in **Figure 1**, which exhibit an initial rapid motion followed by a slower asymptotic approach. This separation of time scales will be further analyzed in Section 3.5, where the



role of the coupling parameter is examined in detail.

### 3.4. Eigenvectors and Direction of Convergence

The eigenvectors of matrix  $A$  determine the directions along which trajectories approach the equilibrium in the phase plane. In particular, the eigenvectors associated with the eigenvalues  $\lambda_1$  and  $\lambda_2$ , given by (13), define the principal directions of convergence. The corresponding eigenvectors are given by

$$\mathbf{v}_1 = \left( -\frac{d + \sqrt{d^2 + 4k^2}}{2k}, 1 \right), \quad (17)$$

$$\mathbf{v}_2 = \left( -\frac{d - \sqrt{d^2 + 4k^2}}{2k}, 1 \right), \quad (18)$$

where  $d$  is defined in (14). The eigenvector corresponding to the dominant eigenvalue (*i.e.*, the one closest to zero) determines the asymptotic direction of convergence. Denoting by  $m$  the slope of this eigenvector, we obtain

$$m = -\frac{d \pm \sqrt{d^2 + 4k^2}}{2k}. \quad (19)$$

This quantity provides a direct measure of the relative variation of the two knowledge levels during convergence. If  $m > 1$ , then  $y_2$  exhibits a stronger variation than  $y_1$ , indicating that the second person undergoes a larger adjustment. Conversely, if  $m < 1$ , the first person adjusts more significantly. In this way, the interaction induces a stronger response in the dynamically weaker person.

To illustrate this behavior, consider, for example  $m = 3$ . Along this direction, an increase of 10 units in  $y_1$ , corresponds to an increase of approximately 30 units in  $y_2$ , indicating a significantly stronger response of the second person.

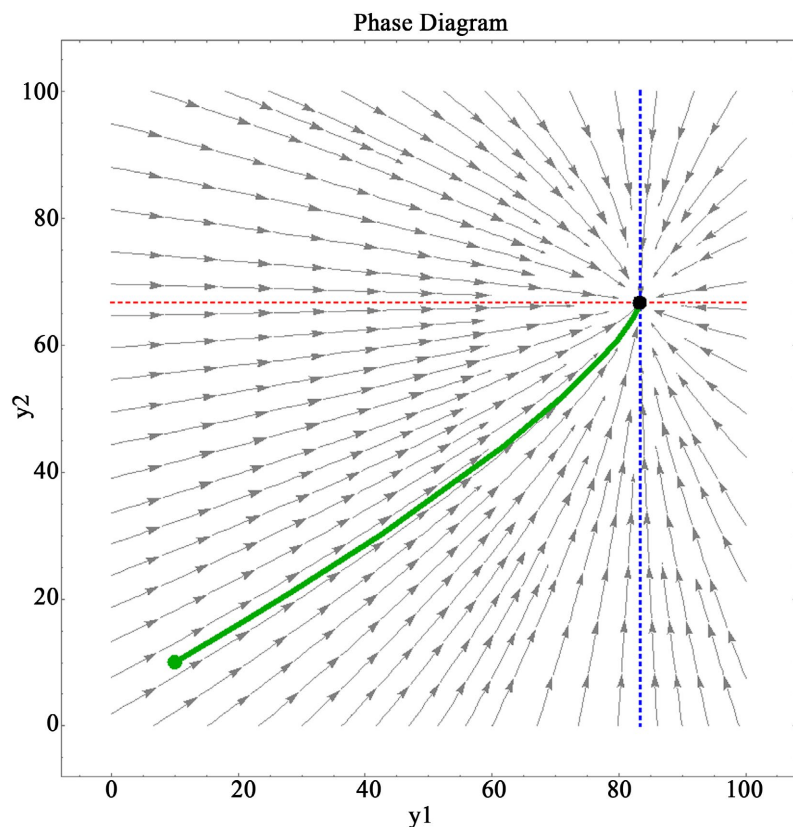
For large values of the coupling parameter  $k$ , the slope satisfies  $m \rightarrow 1$ , implying that the two variables evolve symmetrically. In this regime, the system approaches a state of near-complete equalization of knowledge.

### 3.5. Effect of the Coupling Parameter

The coupling parameter  $k$  plays a central role in shaping the qualitative behavior of the system. When the coupling parameter  $k = 0$ , the system decouples and the two persons evolve independently according to their individual learning decaying memory dynamics. In this case, the eigenvectors coincide with the coordinate axes  $y_1$  and  $y_2$  (**Figure 2**), and their slopes take the limiting values 0 and  $+\infty$ . This indicates that changes in the knowledge of one person have no influence on the other, and each person converges to its individual equilibrium at a rate determined by the parameter  $(A_i + L_i)$ ,  $i = 1, 2$ .

As  $k$  increases, the interaction between persons becomes progressively stronger, and the system transitions from independent evolution to cooperative dynamics. This transition is reflected in the geometry of the phase plane: the eigenvectors rotate away from the coordinate axes, and the trajectories bend toward a common

direction. The slope  $m$  of the dominant eigenvector depends explicitly on both  $k$  and the parameter difference  $d = (A_1 + L_1) - (A_2 + L_2)$ , which represents the difference in the intrinsic dynamical responses of the two persons. When this difference is significant, the slope becomes highly sensitive to variations in  $k$ , indicating that interaction strongly amplifies the adjustment of the dynamically weaker person. As the coupling strength increases ( $k \rightarrow +\infty$ ), the influence of the individual parameters  $A_i$  and  $L_i$  on the system dynamics becomes progressively weaker. In this regime, the coupling parameter  $k$  dominates the evolution, effectively bending the individual learning trajectories and forcing them to converge toward a common direction. As a result, individual learning behavior is transformed into collective stability. Overall, the coupling parameter  $k$  acts as a synchronization mechanism that converts individual evolution into collective dynamics, driving the system toward a common trajectory and reducing knowledge disparities between the persons.



**Figure 2.** Effect of the coupling parameter  $k$  on the phase-plane geometry. As  $k$  increases, trajectories bend toward a common direction, reflecting the transition from independent to cooperative dynamics.

### 3.6. Equilibrium and the Cognitive Strength Ratio

In the individual learning model, the equilibrium knowledge level is given by

$$y^* = \frac{A}{A+L} M \quad (20)$$

which represents the maximum level of knowledge that a person can attain under constant learning and decaying memory conditions. In the collaborative model, the equilibrium point ( $Y^*$ ) can be interpreted as a weighted compromise between the corresponding individual equilibria, resulting from the interaction between the two persons through the coupling term. A key role in determining the final knowledge levels is played by the ratio  $\frac{A_i}{L_i}$ ,  $i = 1, 2$ , which can be interpreted as a measure of the cognitive strength of each person. This ratio reflects the ability of a person to retain acquired knowledge relative to the rate of forgetting, and thus determines the level toward which knowledge converges in the long term. In particular, if

$$\frac{A_1}{L_1} > \frac{A_2}{L_2} \quad (21)$$

then person 1 attains a higher equilibrium knowledge level than person 2, even in the presence of strong interaction. This highlights an important structural property of the model: while the coupling parameter ( $k$ ) governs the rate of convergence and promotes the equalization of knowledge levels, it does not fully eliminate the influence of intrinsic individual characteristics. Consequently, although collaboration reduces cognitive differences between persons, the final distribution of knowledge remains strongly determined by the ratio  $\frac{A_i}{L_i}$ ,  $i = 1, 2$ , which encodes the long-term cognitive capacity of each person. Geometrically, this effect can be interpreted as a shift of the equilibrium point in the phase space. Persons with higher values of the ratio  $\frac{A_i}{L_i}$ ,  $i = 1, 2$ , exhibit stronger knowledge retention, leading to higher equilibrium levels. As a result, the equilibrium point is displaced toward the person with greater cognitive strength, as illustrated in **Figure 3**.

### 3.7. Mean-Difference Formulation

To gain deeper insight into the structure of the system, we introduce the transformation

$$S(t) = \frac{y_1(t) + y_2(t)}{2}, \quad D(t) = y_1(t) - y_2(t) \quad (22)$$

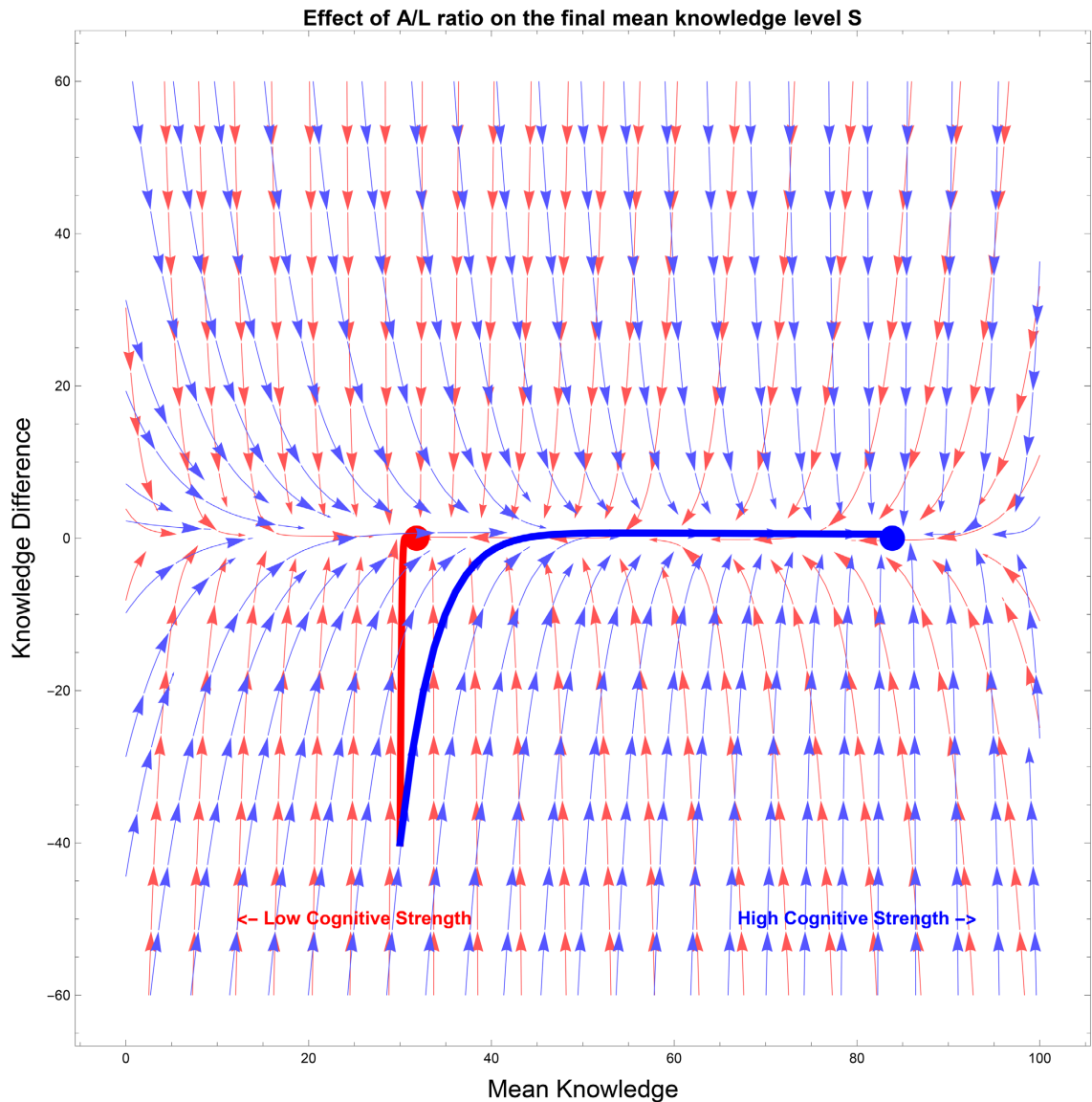
where  $S(t)$  represents the mean knowledge of the system and  $D(t)$  the cognitive difference between the two persons.

#### 3.7.1. Dynamical Decomposition

Using the transformation (22), the system can be rewritten in terms of  $S(t)$  and  $D(t)$ . The initial conditions (4) become

$$S(0) = \frac{(\varepsilon_1 + \varepsilon_2)M}{2}, \quad D(0) = (\varepsilon_1 - \varepsilon_2)M \quad (23)$$

where  $D(0)$  quantifies the initial difference in knowledge between the two persons. By adding the two equations of system (9), we obtain



**Figure 3.** Influence of the cognitive strength ratio  $A_i/L_i$  on the equilibrium point. Higher ratios lead to higher equilibrium knowledge levels, shifting the steady state accordingly.

$$\frac{dS(t)}{dt} = \frac{(A_1 + A_2)M - (A_1 + L_1)y_1 - (A_2 + L_2)y_2}{2} \quad (24)$$

Similarly, subtracting the two equations gives

$$\begin{aligned} \frac{dD(t)}{dt} = & (A_1 - A_2)M - (A_1 - A_2 + L_1 - L_2)S(t) \\ & - \frac{A_1 + A_2 + L_1 + L_2 + 4k}{2}D(t) \end{aligned} \quad (25)$$

A key structural property follows directly from Equations (24)-(25): The coupling parameter  $k$  does not appear in the evolution of  $S(t)$ , while it enters explicitly in the decay of  $D(t)$ . This implies that collaboration does not increase the total knowledge of the system, but instead redistributes it by reducing cogni-

tive differences between the learners. It should be emphasized that this statement refers only to the interaction terms. The total knowledge of the system may still increase through the external learning inputs  $A_1 M$  and  $A_2 M$ , which represent exogenous knowledge acquisition. The role of collaboration is therefore redistributive rather than generative.

### 3.7.2. Dissipation of Knowledge Disparity

To obtain an explicit closed-form expression for the difference variable, we consider the symmetric case

$$A_1 = A_2 = A, \quad L_1 = L_2 = L.$$

Under this assumption, Equation (25) reduces to

$$\frac{dD(t)}{dt} = -\left(\frac{A_1 + A_2 + L_1 + L_2}{2} + 2k\right)D(t),$$

Equation (25) admits the explicit solution

$$D(t) = D(0)e^{-\left(\frac{A_1 + A_2 + L_1 + L_2}{2} + 2k\right)t} \quad (26)$$

This expression shows that the knowledge difference decays exponentially over time, with an effective convergence rate given by

$$r(k) = \frac{A_1 + A_2 + L_1 + L_2}{2} + 2k \quad (27)$$

This expression shows that the coupling parameter  $k$  enhances the decay rate linearly, acting as a dissipative mechanism that accelerates the reduction of cognitive differences.

### 3.7.3. Critical Coupling Threshold

The threshold interpretation introduced below is based on the symmetric case considered in Section 3.7.2, namely

$$A_1 = A_2 = A, \quad L_1 = L_2 = L.$$

The expression (27) suggests the definition of a critical coupling value

$$k_c = \frac{A_1 + A_2 + L_1 + L_2}{4} \quad (28)$$

This threshold separates two qualitatively different regimes:

- For  $k \ll k_c$ , convergence is dominated by the intrinsic learning and decaying memory dynamics.
- For  $k \geq k_c$ , the interaction becomes dominant, leading to rapid equalization of knowledge.

Thus,  $k_c$  defines a transition scale between individual and collective learning behavior.

### 3.7.4. Phase-Space Interpretation

It should be emphasized that the complete equalization condition  $D^* = 0$  is obtained only in the symmetric case  $A_1 = A_2$  and  $L_1 = L_2$ . For asymmetric learn-

ers, characterized by different cognitive strengths  $\frac{A_1}{L_1} \neq \frac{A_2}{L_2}$ , the system generally

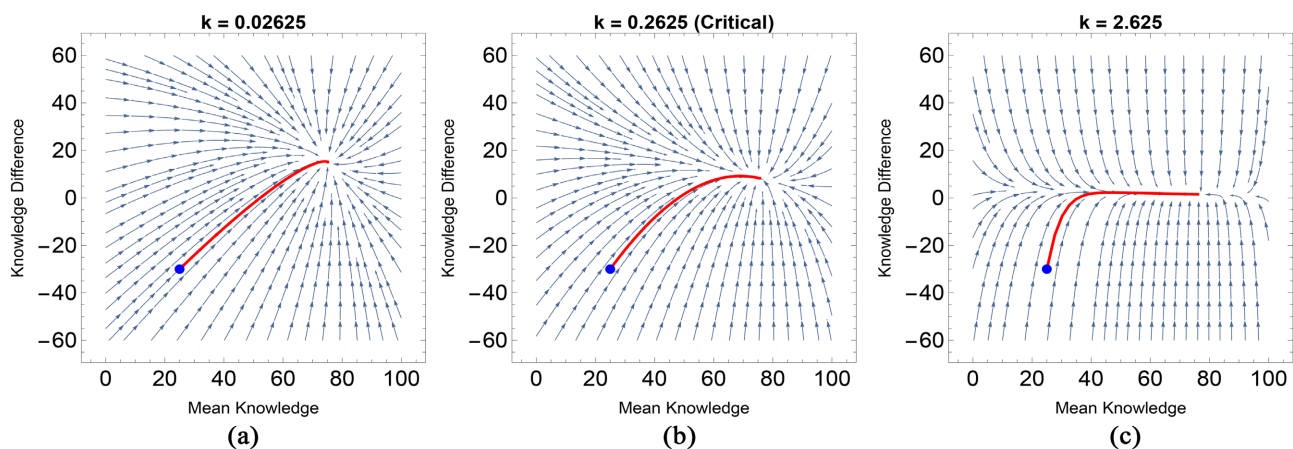
converges to a nonzero equilibrium disparity  $D^* \neq 0$ . In this case, collaboration still reduces cognitive differences and accelerates convergence, but it does not completely eliminate the steady-state disparity induced by intrinsic differences between the learners. The dynamics in the  $(S, D)$  plane are illustrated in **Figure 4**.

- For small  $k$ , trajectories approach the equilibrium gradually, with slow decay of  $D(t)$ .
- Near the critical value  $k_c$ , trajectories exhibit noticeable curvature, reflecting the increasing influence of interaction.
- For large  $k$ , trajectories exhibit rapid reduction of  $D(t)$ , approaching  $D^* = 0$  in the symmetric case and  $D^* \neq 0$  in the asymmetric case.

This representation clearly separates two fundamental processes:

- knowledge accumulation, governed by  $S(t)$ ,
- knowledge equalization, governed by  $D(t)$ .

Overall, the mean-difference formulation reveals that collaborative learning is governed by two distinct mechanisms: the accumulation of knowledge and the dissipation of cognitive differences, with the coupling parameter controlling only the latter.



**Figure 4.** Trajectories in the  $(S, D)$  phase plane for different values of the coupling parameter  $k$ : (a) weak coupling, (b) near-critical coupling, and (c) strong coupling. Increasing  $k$  accelerates the decay of knowledge disparity, leading to stronger synchronization and faster convergence toward equilibrium.

### 3.7.5. Visualization and Trajectory Interpretation

The dynamical behavior of the system is further illustrated in **Figure 4**, where trajectories are represented in the  $(S, D)$  phase plane for different values of the coupling parameter  $k$ . The trajectories originate from the initial condition  $(S(0), D(0))$  and converge toward the equilibrium point.

Three characteristic regimes can be distinguished:

- Weak coupling ( $k \ll k_c$ ): The trajectory approaches equilibrium gradually (**Figure 4(a)**). The cognitive difference  $D$  decays slowly, indicating that the dynamics are dominated by the intrinsic learning and forgetting rates of the

persons. Interaction plays a minor role in this regime.

- Critical coupling ( $k = k_c$ ): The trajectory exhibits noticeable curvature, reflecting the increasing influence of interaction (**Figure 4(b)**). The decay of  $D(t)$  becomes comparable to the evolution of  $S(t)$ , marking the transition between individual and collective dynamics.
- Strong coupling ( $k > k_c$ ): A rapid initial reduction of the cognitive difference  $D(t)$  is observed, corresponding to strong synchronization between the learners. (**Figure 4(c)**). This is followed by a slower evolution along the  $S$  direction toward equilibrium. In this regime, collaboration dominates the dynamics.

These trajectories reveal a two-stage convergence mechanism:

- A fast horizontal (or vertical, depending on representation) motion associated with the decay of cognitive difference,
- Followed by a slower motion toward the final equilibrium governed by the evolution of the mean knowledge.

Thus, the phase-plane representation (**Figure 4**) provides a clear geometric interpretation of the learning process, separating the roles of knowledge accumulation and knowledge equalization.

## 4. Nonlinear Dynamics and the Convergence Zone

In the previous analysis, the coupling parameter  $k$  was assumed to be constant, implying that the interaction strength is proportional to the knowledge difference and independent of its magnitude. While this assumption enables a complete analytical treatment, it represents an idealized approximation of real collaborative learning processes.

Contemporary educational theory most notably the concept of the Zone of Proximal Development (ZPD) introduced by Vygotsky [3] suggests that the effectiveness of interaction depends critically on the cognitive distance between learners. Collaboration is most effective when the knowledge difference lies within an intermediate range, while it becomes ineffective when the difference is either too small or excessively large. Similar mechanisms have been explored in bounded-confidence models of opinion dynamics [4] [5], where interaction occurs only when learners are sufficiently close in state space. However, such models are typically discrete in time and do not provide explicit analytical characterization of convergence behavior.

Motivated by these considerations, we extend the model by introducing a state-dependent coupling function, allowing the interaction strength to depend explicitly on the knowledge difference.

### 4.1. Modeling the Knowledge Difference

We replace the constant coupling parameter  $k$  with a function  $k(D)$ . A natural and analytically convenient choice is the Gaussian form



$$k(D) = k_0 e^{-aD^2} \quad (29)$$

where the coefficient  $k_0 > 0$  represents the maximum interaction strength and the parameter  $a$  controls the sensitivity of the interaction to the knowledge difference. This choice captures the essential feature of the ZPD: interaction is strongest when the persons have comparable knowledge levels and decreases rapidly as the cognitive difference increases.

## 4.2. Emergence of the Convergence Zone

The introduction of state-dependent coupling leads to qualitatively richer dynamics. If the initial cognitive difference  $D(0)$  is sufficiently small, then the coupling function remains close to its maximum value  $k(D) \approx k_0$ , and the system behaves similarly to the strongly coupled linear case. In this regime, the cognitive difference decays rapidly, leading to a fast convergence toward a common knowledge level. This corresponds to an effective collaborative learning process.

In contrast, when the initial cognitive difference is large, the coupling function becomes negligible  $k(D) \approx 0$ , and the system effectively decouples. The persons then evolve almost independently, and the reduction of cognitive difference occurs at a much slower rate. This behavior naturally leads to the notion of a convergence zone, defined as the set of initial conditions for which the interaction remains sufficiently strong to ensure rapid equalization of knowledge.

## 4.3. Critical Disparity and Decoupling Regime

The nonlinear formulation introduces the concept of a critical cognitive difference scale, beyond which interaction becomes ineffective. More specifically:

- For small values of  $|D|$ , the coupling remains strong and the system operates in a collaborative regime.
- For large values of  $|D|$ , the coupling weakens significantly and the system enters a decoupling regime.

Thus, the system exhibits two qualitatively distinct modes of behavior:

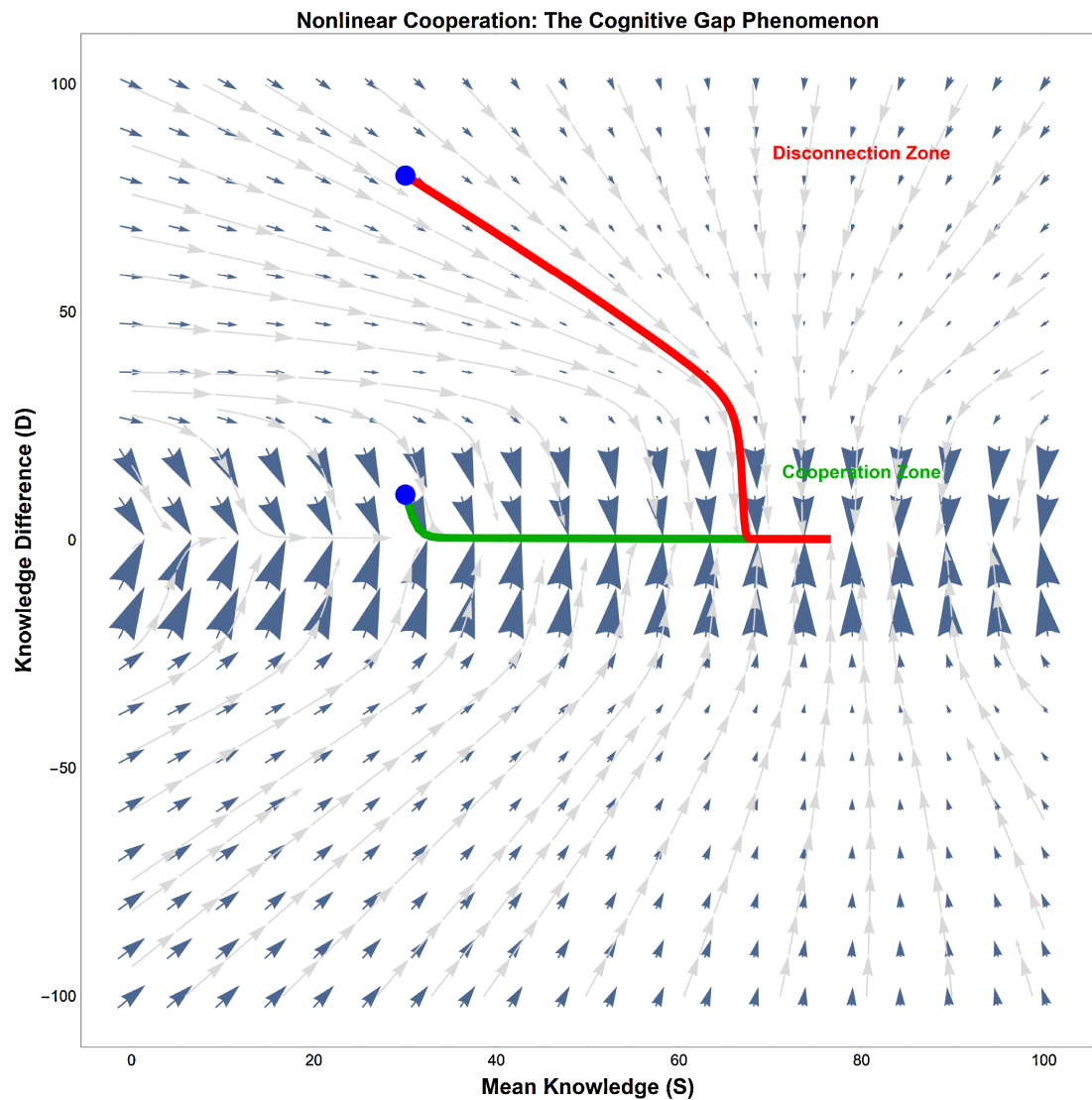
- A collaborative regime, characterized by rapid convergence toward equal knowledge.
- A decoupling regime, in which persons evolve quasi-independently.

Importantly, this transition is not externally imposed, but emerges intrinsically from the state-dependent structure of the interaction.

## 4.4. Phase-Space Interpretation

The qualitative behavior of the nonlinear system is illustrated in **Figure 5**, which depicts representative trajectories in the  $(S, D)$  phase plane. The green trajectory corresponds to an initial condition within the convergence zone. In this case, the coupling remains strong and the trajectory rapidly collapses toward the line  $D = 0$ , indicating efficient knowledge equalization. In contrast, the red trajectory corresponds to a large initial cognitive difference. Here, the coupling function is weak, and the trajectory evolves slowly without significant reduction of disparity,





**Figure 5.** Nonlinear dynamics with state-dependent coupling. The green trajectory corresponds to an initial condition within the convergence zone, leading to rapid equalization of knowledge, while the red trajectory illustrates a decoupling regime where large initial disparity suppresses interaction.

illustrating the decoupling regime.

The phase space can therefore be partitioned into two regions:

- A convergence zone, where interaction is effective and learning is collaborative.
- A decoupling zone, where interaction is suppressed due to excessive disparity.

Overall, the nonlinear extension reveals that collaboration is not uniformly effective, but depends critically on the initial cognitive distance between persons. The model provides a dynamical interpretation of the Zone of Proximal Development, demonstrating that successful interaction requires a sufficient level of cognitive proximity.

## 5. Conclusions and Future Work

In the present work, we developed a continuous-time dynamical framework for

modeling collaborative learning between two interacting persons. Starting from the classical learning with decaying memory model, we introduced a linear coupled system that incorporates knowledge exchange through a diffusive interaction term. The analysis of the linear model provided explicit expressions for the equilibrium state and a complete characterization of stability through eigenvalue analysis. By employing a mean-difference transformation, the system was decomposed into two fundamental components: the evolution of the average knowledge and the dynamics of knowledge disparity. This decomposition revealed a key structural property: Collaboration does not increase the total knowledge of the system, but acts as a dissipative mechanism that reduces disparities between persons.

Furthermore, we identified a critical coupling threshold separating two distinct regimes: one dominated by individual learning dynamics and another governed by collective convergence. This result provides a clear quantitative interpretation of the transition from independent to cooperative learning behavior.

The extension to a nonlinear model with state-dependent coupling led to qualitatively richer dynamics. In particular, the introduction of a Gaussian interaction function allowed us to capture the effect of cognitive distance between learners, leading to the emergence of a convergence zone and a decoupling regime. This formulation provides a dynamical interpretation of the Zone of Proximal Development, highlighting that effective collaboration requires an appropriate level of cognitive proximity.

A natural extension of the present work is the study of collaborative learning in larger groups and networked environments. The two-person framework developed here provides a tractable baseline for analyzing interaction-driven learning dynamics, while future work may consider multiple learners connected through general interaction networks. Such extensions would allow the investigation of collective learning processes in classrooms, educational communities, and online collaborative learning systems.

Overall, the proposed framework establishes a unified and analytically tractable approach to modeling collaborative learning, bridging concepts from dynamical systems theory and educational modeling.

Further analysis within the realm of fractional calculus is currently under investigation. In particular, we investigate the behavior of memory effects through fractional-order dynamics. In real learning processes, knowledge evolution depends not only on the current state but also on past experience. This can be modeled by replacing the classical derivative with a Caputo fractional derivative [6]-[8], allowing the system to capture long-term memory effects. Such an extension is expected to influence convergence rates and transient dynamics, providing a more realistic description of learning behavior and an interesting direction for future research.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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