Journal of Applied Mathematics and Physics, 2026, 14(8), 2917-2966
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Resolving the Einstein, Rosen, Podolsky, Bohm, and Aharonov Paradox with Non-Relativistic Quantum Mechanics

Lincoln Stoller

Independent Researcher, Victoria, Canada
Email: LS@mindstrengthbalance.com

How to cite this paper: Stoller, L. (2026) Resolving the Einstein, Rosen, Podolsky, Bohm, and Aharonov Paradox with Non-Relativistic Quantum Mechanics. *Journal of Applied Mathematics and Physics*, **14**, 2917-2966.
https://doi.org/10.4236/jamp.2026.148144

Received: May 31, 2026
Accepted: August 16, 2026
Published: August 19, 2026

Abstract

The results of the Einstein, Rosen, Podolsky, Bohm, and Aharonov (ERPBA) experiment are shown to be fully predicted using standard quantum mechanics and local quantum variables obeying existing quantum rules. The keys to this result are the disentanglement of the two-particle wavefunction at the point of the decay of the singlet state, a correct mapping of the quantum variables defined in the 2-dimensional Hilbert space of SU(2) spinors onto the classical space of 3-dimensional observables described by vectors in SO(3), and the reminder that Bell's Theorem does not apply to theories involving non-commuting variables. The classic paradox is resolved using a disentangled wavefunction along with a careful mapping of the quantum spinor topology.

Keywords

EPRB Experiment, Spinors, Nonlocality, Local Realism, Decoherence, Quantum Variables, Bell's Theorem

1. Historical Context

Bohm and Aharonov's version [1] of the Einstein, Rosen, Podolsky [2] contention about the completeness of quantum mechanics is referenced as a demonstration of nonlocal effects in quantum mechanics. In conjunction with the theorems of Bell, regarding classical theories, and demonstrations arising from the observations of Greenberger, Horne, and Zeilinger [3], it is often concluded that no local theories exist that can explain this nonlocal, quantum mechanical behavior [4]-[7]. Because this appears to undermine our notions of causality, the ERPBA experiment is referred to as the ERPBA paradox.

The ERPBA paradox begins with the description of two entangled electrons, referred to as electrons A and B, in a singlet state. The electrons have overlapping spatial wavefunctions that are symmetric with respect to the interchange of A and B, and spin wavefunctions must be antisymmetric with respect to the interchange of A and B.

This resonant state is then presumed to decay into a state where the two electrons move in opposite directions while the spin portion of the wavefunction retains the entangled character of the bound state, namely that the wavefunction of the two particles continues to be antisymmetric with respect to their interchange.

The spin part of the two-particle wavefunction, for particles 1 and 2, is represented by $|A, B\rangle \equiv (|A_1B_2\rangle - |A_2B_1\rangle)$. Where A_1, A_2, B_1, and B_2 are either the +1, +2, −1, or −2, the single particle 1 or 2's eigenvalues of $\boldsymbol{S}_z$, the operator that gives the z-component of the spin of the electron with respect to some identically chosen z-axis orientation at both A and B.

The ERPBA paradox then asserts that a measurement of the z-component of the magnetic moment of one of the particles either at position A or B will show the magnetic moment to either be aligned with or against the gradient of an inhomogeneous magnetic field oriented along any locally selected z-axis, while a similar measure of the opposing particle with respect to the same z-axis orientation will record an alignment opposite to the first particle, as required by conservation of zero angular momentum of the two-particle singlet state.

The paradox arises because, in this picture, when the measurement is made at location A on the state $(|A_1B_2\rangle - |A_2B_1\rangle)$, only one of the two equally likely states, that state that's either positively or negatively correlated with the magnetic field, is observed for the particle at A, and the other state disappears.

That is, the wavefunction "collapses" into either the $|A_1B_2\rangle$ or the $|A_2B_1\rangle$ state. At the same time, that part of the wavefunction at B, separated from A by a space-like interval, instantaneously and without any mechanism, collapses to the correctly correlated state. As a result, the particle at A is found to be positively oriented, and the particle at B is negatively correlated with the external magnetic fields, or *vice versa*.

This collapse of the wavefunction from an anti-symmetrized superposition of two 2-particle states to a single 2-particle state violates special relativity because it occurs simultaneously at locations separated by a space-like interval. Some claim this is not a violation since this does not provide a mechanism for the passing of information, but that seems like a specious argument since the information about the state vector has been passed from one location to the other. Additionally, this collapse violates causality because it occurs without any mechanism.

What we will show here is that with a consistent application of the rules of quantum mechanics, there is no need to invoke non-locality in order to reproduce the theoretically required and experimentally observed correlations between the spins of the electrons. Quantum mechanics does not detail the mechanism that couples the electron's spin to the magnetic moments we observe, but whatever the mechanism, it is local.

With the resolution of the correlations between spins as a local effect, and the direction of the spins as local values carried with each electron, the ERPBA effect ceases to be a paradox.

2. Model Structure

We argue that the correct wavefunction describing this situation is a mixture of pure states. However, we point out that there is no way, in theory or practice, to determine how to assign the pure states, so if we want a wavefunction from which we can generate predictions, we must use a mixed-state form. The pure states we use, and the mixed states we construct, are described in Section 4, titled Disentanglement.

We argue that the commonly used entangled state, a superposition also called a Bell state, should be replaced by a mixed state. We consider three of the traditional rationales for using a superposed state and show each to be incomplete and unconvincing. The reasons supporting the use of the entanglement wavefunction are that it:

1) Preserves the irreducible entanglement of the bound state. This is not compelling because we should not expect entanglement to survive the fission of the bound state.

2) Successfully predicts observable statistical outcomes that cannot be predicted by hidden variable theories. This is not compelling because quantum mechanics can be phrased as a hidden variable theory that predicts the observed outcome without entanglement, as demonstrated in section 4.

3) Mathematically ensures the conservation of angular momentum. This is not compelling because the pure state also conserves angular momentum, as shown in section 11.

We consider five arguments in support of using mixed states and show these to be robust. These arguments are:

1) Mixed states are the correct model for a singlet wavefunction decoherence processes.

2) When state symmetry is broken, and states or paths become distinguishable, quantum mechanics forbids us from modeling them as self-interfering, entangled states. With reference to the Young Double-Slit experiment, distinguishable states do not interfere.

3) Expectation values of pure states do conserve angular momentum.

4) The quantum nature of spin is not Einsteinian. Theories that include quantum spin, namely quantum mechanics itself, are not subject to Bell-CHSH inequalities.

5) Consistent inclusion of the influence of quantum spin successfully predicts observed statistical outcomes.

The primary reason for using a superposed state, which is the lack of a local and causal alternative, is obviated by our presentation of a local, causal alternative. This model is consistent and in agreement with the observational tests of the inequalities of Bell, CHSH, and others.

3. State Structure

The pure state of a system is given by a product of eigenstates of the subsystems of which it is composed. A mixed state is not a physical description of a system; it's a description of our knowledge of the system. The inclusion of exclusive alternatives in a mixed state reflects our knowledge of these states' likelihood. There is no collapse of a mixed state, so there is no action-at-a-distance.

An entangled, Bell, or superposed state is a combination of product states that are simultaneously present and which can be made to interfere with each other. If the terms of an entangled state are orthogonal, and if the state is stationary with respect to a particular measurement, then the measurement will not create interference, meaning that all cross terms in the expectation value will be zero. In that case, the entangled state will generate the same expectation values as a mixed state.

Pure states can appear as entangled states when represented in a rotated basis. A pure product state composed of spin-up and spin-down particles 1 and 2 appears entangled when represented in a spin-left and spin-right basis. Consider $\left|+_j\right\rangle$ and $\left|-_j\right\rangle$ as eigenstates of S_z for particles $j = 1$ and 2. In matrix notation $\left|+\right\rangle$ and $\left|-\right\rangle$ are given by the column vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. These can be represented using the $\left|L_j\right\rangle$ and $\left|R_j\right\rangle$ eigenstates of S_x as: $\left|+_j\right\rangle = \left(\left|L_j\right\rangle + \left|R_j\right\rangle\right)\frac{1}{\sqrt{2}}$, and $\left|-_j\right\rangle = \left(\left|L_j\right\rangle - \left|R_j\right\rangle\right)\frac{1}{\sqrt{2}}$. In matrix notation $\left|L\right\rangle$ and $\left|R\right\rangle$ are given by the column vectors $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$.

The pure product state $\left|+_j\right\rangle, \left|-_j\right\rangle$ appears entangled in the $\left|L_j\right\rangle, \left|R_j\right\rangle$ basis as:

$$\begin{aligned}\left|+_1\right\rangle\left|-_2\right\rangle &= \left(\left|L_1\right\rangle + \left|R_1\right\rangle\right)\left(\left|L_2\right\rangle - \left|R_2\right\rangle\right)/2 \\ &= \left(\left|L_1\right\rangle\left|L_2\right\rangle - \left|L_1\right\rangle\left|R_2\right\rangle + \left|R_1\right\rangle\left|L_2\right\rangle - \left|R_1\right\rangle\left|R_2\right\rangle\right)/2\end{aligned} \tag{1}$$

The left-hand side of Equation (1) is a pure state; the right-hand side of Equation (1) is an entangled state. The right-hand side is not a mixed state because all the components apply to the same particles, share the same spatial wavefunctions, and occur at the same time. This is not to imply that all entangled states can be rotated into pure states, but some can.

This is important because we can derive a certain result for a system represented by a pure state, but we cannot for a mixed or an entangled state. It is essential to understand that the state of the system and the state we use to describe the system may have to be different. As De Zela [8] states, "Experiments designed to violate (locality)… have much more to do with entanglement than with local realism."

4. Disentanglement

The ERPBA paradox begins with the initial singlet state that describes a two-electron state before its disintegration. This entangled $|A, B\rangle$ state is required in order to satisfy the requirement that the wavefunction be anti-symmetric under the operation of interchanging indistinguishable fermions. The particles are distinguishable by their position when separate, but they are indistinguishable when their spatial wavefunctions overlap.

The first reason supporting the use of the entangled state, after the decoupling of the bound state, is that the bound state was entangled, and the Schrodinger equation is linear. Presuming that linearity continues to hold through the process of separation implies the entangled contributions survive and the unbound state remains entangled.

This is not compelling because the fission process is not governed by a linear Schrodinger equation. In fact, we have no equation at all to describe the fission process, but as the process can result in the creation or destruction of particles, we can presume it can create or remove contributions from the wavefunction.

The decoupling of a bound state is a non-unitary, open-state process caused by or resulting in a change of state. When the particles separate, their spatial wavefunctions no longer overlap, and they are distinguishable. When occurring spontaneously, this is called "a collapse of the wavefunction." Buks [9] argues this disentanglement is driven dynamically by the addition of a dissipative term in the Schrodinger equation.

Once spatially separated, the wavefunction is no longer required to be in the singlet state, and a disentangled state is allowed. Beyond being allowed, the wavefunction should be disentangled because the entangled state leads to unphysical, action-at-a-distance effects.

The second reason supporting the use of the entangled state after decoupling is that we obtain the correct correlation between the spins at remote locations by our use of the ansatz of action-at-a-distance. In other words, a statistically correct physical prediction is used to justify an absolutely unphysical assumption.

A similar unphysical assumption justified the infamous "swimming test," in which people who sank and drowned were assumed not to be witches because the blessed element water would accept the innocent. On the other hand, water would reject the unholy, causing witches to float like corks. And it is true that some people float while others sink.

If we accept the entangled two-particle system after fission, then the wavefunction is given by a "Bell state" of the form:

$$\left|\Psi_{1,2}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|+_{A,1}\right\rangle\left|-_{B,2}\right\rangle - \left|-_{A,1}\right\rangle\left|+_{B,2}\right\rangle\right) \tag{2}$$

for spin states + and –, positions A and B, and identical particles labeled 1 and 2. The spin component of the wavefunction for the two-particle bound state singlet satisfies the requirement that the state have a net spin of zero. The superposed wavefunction is interpreted as two exclusive but simultaneously present spin states. This is unlike a mixed state, which is a statistical combination of two exclusive states whose combination represents our uncertainty.

Before decay, the full, singlet state is given in terms of position wavefunctions

$\left|\varphi_{A,1}\right\rangle$ and $\left|\varphi_{B,2}\right\rangle$, for particles 1 and 2 centered over their respective 3-dimensional position ***A*** and ***B***, and spin component wavefunctions $\left|+_1\right\rangle$ or $\left|-_1\right\rangle$ and $\left|+_2\right\rangle$ or $\left|-_2\right\rangle$, as:

$$\begin{aligned}\left|\Psi^{\circ}_{1,2}\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|\varphi_{A,1}\right\rangle\left|\varphi_{B,2}\right\rangle+\left|\varphi_{A,2}\right\rangle\left|\varphi_{B,1}\right\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(\left|+_1\right\rangle\left|-_2\right\rangle-\left|-_1\right\rangle\left|+_2\right\rangle\right)\\ &= \frac{1}{2}\left(\left|\varphi_{A,1}\right\rangle\left|\varphi_{B,2}\right\rangle\otimes\left|+_1\right\rangle\left|-_2\right\rangle-\left|\varphi_{A,1}\right\rangle\left|\varphi_{B,2}\right\rangle\otimes\left|-_1\right\rangle\left|+_2\right\rangle\right.\\ &\quad\left.+\left|\varphi_{A,2}\right\rangle\left|\varphi_{B,1}\right\rangle\otimes\left|+_1\right\rangle\left|-_2\right\rangle-\left|\varphi_{A,2}\right\rangle\left|\varphi_{B,1}\right\rangle\otimes\left|-_1\right\rangle\left|+_2\right\rangle\right)\end{aligned}$$

$\left|\Psi^{\circ}_{1,2}\right\rangle$ emerges from the previous expression for $\left|\Psi_{1,2}\right\rangle$ when the symmetrized position components are included.

In the model being proposed here, two of the terms in $\left|\Psi^{\circ}_{1,2}\right\rangle$ vanish during decomposition, leaving us with either of two nonentangled, properly anti-symmetrized wavefunctions $\left|\Psi'_{1,2}\right\rangle$ or $\left|\Psi''_{1,2}\right\rangle$ given by:

$$\left|\Psi'_{1,2}\right\rangle=\frac{1}{\sqrt{2}}\left(\left|\varphi_{A,1}\right\rangle\left|\varphi_{B,2}\right\rangle\left|+_1\right\rangle\left|-_2\right\rangle-\left|\varphi_{A,2}\right\rangle\left|\varphi_{B,1}\right\rangle\left|-_1\right\rangle\left|+_2\right\rangle\right) \tag{3}$$

$$\text{Or}\quad\left|\Psi''_{1,2}\right\rangle=\frac{1}{\sqrt{2}}\left(\left|\varphi_{A,2}\right\rangle\left|\varphi_{B,1}\right\rangle\left|+_1\right\rangle\left|-_2\right\rangle-\left|\varphi_{A,1}\right\rangle\left|\varphi_{B,2}\right\rangle\left|-_1\right\rangle\left|+_2\right\rangle\right)$$

$\left|\Psi'_{1,2}\right\rangle$ represents + spin at position ***A*** and – spin at position ***B***, while $\left|\Psi''_{1,2}\right\rangle$ reverses the locations of the spins. One or the other of these two wavefunctions replace the sum given in Equation (2); only one of these two wavefunctions survives.

The entangled wavefunction of Equation (2) represents a superposition of possibilities that are indistinguishable in practice. Whereas the combination of the two wavefunctions in Equation (3), given as $\left|\Psi'_{1,2}\right\rangle+\left|\Psi''_{1,2}\right\rangle$ are distinguishable in practice, but the presence of one or the other cannot be predicted. They reflect a system in which the particles are in definite spin states, but we don't know the spins of the particles at the respective locations.

Buks [9] shows this fission can be modeled by the addition of a nonlinear term to the Schrodinger equation, active only during the disintegration process. This term explains the process of wavefunction collapse as a local phenomenon, but it does not tell us which outcome prevails. The physical state is given by a pure state wavefunction, but because the result is uncertain, a mixed state represents our best tool for prediction.

It bears repeating that while we believe the localized particle is represented by a pure-state wavefunction, our description of its wavefunction must be given by a mixture of two exclusive, indeterminable possibilities.

Presuming entanglement after separation, which is the assumption that these two states are not exclusive but coexist, compels us to assert that measurement forces an unphysical wavefunction collapse. This involves an acausal, action-at-a-distance effect which lacks any mechanism. To continue to presume entanglement also ignores the previously unrecognized mechanisms provided here, which are physical, causal, and consistent with everything we know.

Whereas quantum mechanics requires the superposition of identical wavefunctions in a given situation, or the superposition of indistinguishable paths, it forbids the superposition of distinguishable components. This holds regardless of what our theory says or what measurements we undertake. We do not have to observe a distinguishable path; it only needs to be distinguishable in concept, whether we have the technology to distinguish it or not.

Our contention is that the actual wavefunction of the separated particles is never given by an entangled state; it is given by a pure product state, which, for predictive purposes, we must describe as a mixed state. This implies that some physical process has distinguished these components, though they appear symmetric to us. This is what locality is telling us. Rather than escaping the issue through the use of action-at-a-distance, it behooves us to uncover the distinguishing process.

There has never been a definitive argument for the entangled, separated state, nor has there been evidence for it. We return to the question of this lack of evidence of entanglement in **Appendix II**.

5. Measurement

When a spin measurement is made at one of the locations, the spin orientations are embodied in the basis in which we represent the entangled wavefunctions. The angular momentum correlations are enforced, zero net spin is maintained, the wavefunction remains anti-symmetric, and the collapse is complete. To use the entangled version of the wavefunction after the separation of the particles represents what Blokhintsev ([10], Section 139) refers to as "an insufficiently sound analysis of some quantum mechanics consequences, appearing paradoxical."

If we're only concerned with the eigenfunctions of spin, then we can write the wavefunctions more compactly in terms of spin and position by ignoring the spatial components, antisymmetry, and particle identities. We then have:

$$\left|\Psi_{+-}\right\rangle \equiv \left|\Psi'_{1,2}\right\rangle \equiv \left|+_A\right\rangle\left|-_B\right\rangle \tag{4}$$

$$\text{Or} \quad \left|\Psi_{-+}\right\rangle \equiv \left|\Psi''_{1,2}\right\rangle \equiv \left|-_A\right\rangle\left|+_B\right\rangle$$

These disentangled wavefunctions represent anti-symmetrized states that provide a local, relativistic, causal description which fully predicts the observed correlations. We derive these correlations by properly extending the two-dimensional internal properties of spin to three dimensions using a deterministic procedure that is free of additional parameters.

The post-separation state is one of two pure-product states, but there is no way in current theory or practice to know which. We model the post-separation, pre-measurement state with a mixed state wavefunction, knowing that the prediction of the pure state is statistical. There is no wavefunction collapse, and the outcome of the measurement is determined by one of the two pure-state wavefunctions. This is the heart of the present result.

This is an entirely quantum mechanical model. It does not involve classical variables. As such, it avoids the locality prohibitions of Bell, von Neumann, CHSH, Hardy, Kochen-Specker, and related theorems which apply to classical descriptions. We will show that the Bell-CHSH inequality is not a restriction on or statement of locality or nonlocality, but a result that derives from the unnecessarily restrictive local factoring of the correlation function, which is explored in section 9.

6. Detection

Westlund *et al.* [11] make the point that spins whose up and down components are constants of the motion assume a magnetic field only in the z-direction, which is parallel or anti-parallel to this spin component. However, such a magnetic field does not describe a detection machine based on the inhomogeneous magnetic fields of a Stern-Gerlach machine, and therefore, such a machine does not deflect and cannot detect the spins of the particles under consideration.

In order to detect these particles, which is necessary to say anything about their spin orientations, the magnetic field to which they are subjected needs a perpendicular component. When this is added, the z-component is no longer a constant of the motion, and the entangled wavefunction, assuming there is an entangled wavefunction, undergoes decoherence. Potel [12] comes to a similar conclusion, stating "all the asymmetries (of beam deflection in the Stern-Gerlach experiment) are nonvanishing." Stated without the double negative, this says that some of the symmetries of the observed system vanish as a consequence of the Stern-Gerlach measurement.

It should also be noted that the z-component of the measuring device can never be aligned with the z-component of the particle because the z-component of the particle is unknowable before it's been measured. We don't know the particle's z-component until after we've measured it. Even then, any subsequent measurement of the spin at a different angle will create a mixed state.

The situation addressed by Westlund *et al.* pertains to changes in the wavefunction that occur when the particles come into their respective detection devices. It does not address decoherence at the point where the particles separate, so it does not address the question of how the pre-measurement wavefunction carries information.

While we can calculate expectation values for entangled states, as Mochizuki [13] does for this system in his equations #49 through #51, we still agree with Chen [14] that the concept of entanglement itself has no support as a realistic, independent quantum state. In applying these decoherence arguments, we are assuming that the entangled singlet state, which describes the energy levels of bound state particles, no longer describes the free particles to which the ERPBA paradox applies.

After the decoherence of the bound state, the entangled wavefunction of the correlated particles in Equation (2), given by $\left(\left|+_1\right\rangle\left|-_2\right\rangle-\left|-_2\right\rangle\left|+_1\right\rangle\right)$, emerges as either of the pure states $\left|+_A\right\rangle\left|-_B\right\rangle$ or $\left|-_A\right\rangle\left|+_B\right\rangle$, given by Equation (4). That is to say, the "spin up" state is either carried by one or the other of the particles in location A, while the spin-down state is carried by the particle in location B.

It is essential to understand that while the states of these particles, "spin up" and "spin down," must be opposite to satisfy conservation of angular momentum, these directions are not observable. They are "hidden variables" in the observational sense, as any measurement of them along any axis will show that they are aligned parallel and antiparallel with the axis measured. But since these axes are arbitrary, they do not reflect properties of the particles before they are measured.

There is no way in current theory to determine the original basis—that is, the z-orientation before measurement—with respect to which up and down are defined. All we can say is that something like an axis—a direction but not a vector—was established at the time the particles separated, and one of the particles' spins was in this direction, and the other particle's spin was in the opposite direction. However, as this direction is unknowable, we cannot call this direction "an element of reality".

We cannot represent the spin in a manner that satisfies Einstein's criteria for being "elements of physical reality." This is the nature of quantum spinors, as demonstrated by the GBH inequalities [3], and is unrelated to any assumptions, constructions, or models used here.

The only purpose of referring to the existence of each particle's eigenvalue, established with respect to one of the Pauli rotation operators, is to ensure that the two particles are oppositely correlated. This being the case, it is fair to ask whether the notion of there being a preexisting axis has any physical meaning at all.

7. Quantum Spinors

Much is said about the spinor character of electrons, but it is rarely emphasized that these are quantum, not classical spinors. In particular, they are 2-dimensional objects. They can be represented as complex vectors in SU(2), but they cannot be represented as real vectors in SO(3). They have a length, but their direction does not correspond to an observable point on the surface of a sphere in three dimensions.

Despite the lack of simultaneously observable values of spin in directions perpendicular to their one measurable (but not observable without disrupting the system) z-axis, we can still construct an ansatz that involves these non-stationary components for the purpose of including their contributions. That is, we will formulate the quantum spinor as a classical SO(3) vector in order to integrate the contributions of these quantum components and determine the SU(2) spinors' effect on our SO(3)-obeying measuring instrument.

When represented in SO(3), electron spinors have well-defined total and z-component of angular momentum, but they have no well-defined vectorial components in the directions perpendicular to the axis of the basis vectors that define "up" and "down." Just as the electron does not "spin," so too these "up" and "down" Hilbert space basis vectors have no predefined direction in 3-space.

Our ansatz does not equate a quantum spinor with a classical vector. Rather, we are equating the results of a quantum spinor with what results from the integration of a classical vector uniformly distributed over a range of directions. This is a path integral approach in which a singular value of particle spin is just a semi-classical approximation. The full effect of spin is calculated by integrating over all spin values consistent with the particle's positive or negative value of S_z.

The positive and negative basis vectors with respect to some z-axis are often represented as upward and downward pointing cones that have a definite height along their axis of symmetry but no definite azimuthal angle, the angle in the plane perpendicular to the z-axis. That is, they have no preferred direction along the equatorial perimeter of the cones that define them.

These cones are a hybrid concept. Their azimuthal components are not constants of the motion, and their polar directions are unobservable. The cones depict what cannot be seen or measured directly. For all intents and purposes, they are hidden but not classical variables.

Picturing quantum spinors as a cone in 3-space is a throwback to the confusion of whether the wavefunction represents statistically uncertain measurements or whether the quantities that are uncertain fundamentally exist. This confusion has been answered by Greenberger, Horne, Shimony, and Zeilinger [15], who showed that definite values of more than one component of the electron's angular momentum are theoretically inconsistent with the commutator algebra describing these states. From this, we can conclude that only the S_z component of the spinor exists as an independent, measurable quantity in 3-space. At the same time, these perpendicular components exist in the sense that they carry angular momentum and contribute to the effects we measure.

The electron's state can be described by an eigenvector of S^2, S_z, and $\left(S_x + S_y\right)^2$. No other spatial components of the spin "exist" in the sense of being measured simultaneously and without disturbing and entangling the eigenstate. However, the S_x and S_y spatial components contribute to the total spin and, as they couple to the particle's internal magnetic field, they interact with the 3-vector components of external magnetic fields [16].

It is this quality of a quantum spinor, distinct from a classical spinor, that allows it to play the role of a quantum hidden variable and to escape the inequalities proposed by Bell, which prohibit a classical hidden variable from reproducing the quantum result.

8. Bell's Inequality

Bell's inequality asserts that any hidden variable theory "which is emphatically not quantum mechanical," as noted by Clauser [17], must satisfy Bell's Inequality. The argument then goes on to demonstrate that quantum mechanics violates this inequality. Many experiments have subsequently verified that observations violate Bell's inequality [18].

The derivation of Bell's inequality requires equating the expectation value of a sum of observations with the sum of the expectation values performed separately.

Bell's derivation equates the sum of averages with the average of sums, an operation that is fundamental to statistics and which presumes the sums are taken over the same ensemble values. Bell's and related inequalities are not statements about locality; they are statements about expectation values.

In quantum systems, some observed values are incompatible with other observed values, such as observing position and momentum. When the expectation values are obtained by averaging over different states, which is the same as averaging over different ensembles, the sum of the averages over different states is not the average of the sums.

For example, for $x(1)$ and $x(2)$, the position operators of particles 1 and 2, and $p(1)$ and $p(2)$, the momentum operators for the same particles, we can form the correlations

$$x(1,2) = x(1) - x(2), \text{ and } p(1,2) = p(1) - p(2).$$

We can form the expression that gives the sum of these correlations as $x(1,2) + p(1,2)$. We can define the expectation value of this sum of correlations, and assert that it's equal to the sum of the expectation values of the correlations:

$$\langle x(1,2) + p(1,2) \rangle = \langle x(1,2) \rangle + \langle p(1,2) \rangle$$

But this requires the expectation values to be defined with respect to the same sets of states. If they are not so defined, and if we can redefine the position and momentum state correlations differently, then we can make both $x(1,2)$ and $p(1,2)$ as small as we like. Doing so will result in their sum being as small as we like. But the uncertainty relations for position and momentum indicate this is not possible with respect to any common basis set.

In this case, we cannot equate the expectation value of the sum of terms involving incompatible quantum variables with the sum of the expectation values of those terms. This plainly contradicts classical statistics, for which the average of a sum of terms is always equal to the sum of the terms separately averaged.

Bell's theorem is derived by considering correlations between identical, differently oriented detectors measuring correlated spins at different locations. In Bell's case, the correlations are between the detection of a particle of "spin up"—meaning the magnetic moment is found to cause the particle to be deflected upwards—in a measuring apparatus whose "up" direction is an angle a, a', b, or b' with respect to some commonly chosen vertical direction.

In Bell's construction, a value of +1 is recorded when an upward deflection is observed with respect to detection angle "a" from the common vertical. A value of −1 is recorded when a downward deflection is observed with respect to the same angle.

These observations are recorded at either location A or B. This gives us the detection values $A(\boldsymbol{a})$ and $B(\boldsymbol{a})$, both of which only take the values +1 or −1, according to some theory of correlation that is to be determined. Because the detectors always record a particle in either the up or down state regardless of the orientation of the detector's axis relative to the vertical, the detection results are always either +1 or −1.

When these measurements are made with respect to anti-parallel directions, such as the detection orientation $\boldsymbol{a} = -\boldsymbol{b}$, the expectation values are computed with respect to the same eigenstates. In this case, the averages of both measurements are taken with respect to the same ensemble or, equivalently, with respect to the same density matrix.

A correlation function $E(A, \boldsymbol{a}, B, \boldsymbol{b})$—where A and B are locations and $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors—is defined as the product of $A(\boldsymbol{a})B(\boldsymbol{b})$. $E(A, \boldsymbol{a}, B, \boldsymbol{b}) = 1$ when an upward deflection is detected at both A and B with respect to the angles of $\boldsymbol{a}$ and $\boldsymbol{b}$. A correlation value of −1 is recorded when an upward deflection at one location is recorded along with a downward deflection at the other. This correlation can be normalized by summing the results over repeated trials, given by the subscript i, and dividing by the number of trials:

$$\left\langle E\left(A, \boldsymbol{a}, B, \boldsymbol{b}\right)\right\rangle = \frac{1}{N}\sum_{i=1}^{N} E_i\left(A, \boldsymbol{a}, B, \boldsymbol{b}\right)$$

$\left\langle E\left(A, \boldsymbol{a}, B, \boldsymbol{b}\right)\right\rangle$ depends only on the angles of $\boldsymbol{a}$ and $\boldsymbol{b}$, and ranges from +1 to −1. It gives the value zero when there is an equal number of positive and negative correlations.

The correlation between the particles found to be in the "up" state at A—using a detector whose up-axis is set to the angle equal to that of $\boldsymbol{a}$ from the vertical—with the measurement of the correlated particle at B—which is also in the "up" state when measured as a different angle as given by $\boldsymbol{b}$ with respect to the vertical—equals either −1 or +1. A value of −1 means that the magnetic moments of the particles at the two locations are consistently anti-correlated, while a value of +1 means the two particles are consistently correlated, such that their magnetic moments point in the same direction.

9. Classical Hidden Variable Correlations

If the detection values are determined by the direction in which a classical unit vector is pointing—so that a vector with any upward component leads to an "up" event and any downward component leads to a "down" event—then rotating the detector relative to the particle is a simple 3D rotation $R(\theta)$ around some axis.

If the particles are traveling in the x-direction and the yz-plane is perpendicular to this, then a simple polar rotation of angle θ away from the vertical in the yz-plane is represented by a rotation matrix operating on the (x, y, z) components as:

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

When acting on a collection of particles, each with its own orienting unit vector, the effect of $R(\theta)$ can be seen as a rotation of unit vectors on a 2-sphere, an ordinary sphere in three dimensions. Each detection value is then separately shifted, and the shifted value of their products is equal to the product of their values. That is

$$R(\theta)\left[E(a,b)\right]=R(\theta)\left[A(a)B(b)\right]=R(\theta)\left[A(a)\right]R(\theta)\left[B(b)\right]$$

In the construction considered by Bell, the correlations at different detector angles, which are detecting up and down counts from a stream of unrotated particles, are given by the rotation of the detector by various angles in the yz-plane (around the x-axis). If $\boldsymbol{a'}$ is rotated from $\boldsymbol{a}$ by an angle θ and $\boldsymbol{b'}$ is rotated from $\boldsymbol{b}$ by an angle φ, then $A(\boldsymbol{a'}) = R(\theta)[A(\boldsymbol{a})]$ and $B(\boldsymbol{b'}) = R(\varphi)[B(\boldsymbol{b})]$. In particular, $E(\boldsymbol{a'}, \boldsymbol{b'}) = R(\theta$ at $A)A(\boldsymbol{a})R(\varphi$ at $B)B(\boldsymbol{b})$, and $E(\boldsymbol{a'}, \boldsymbol{b'}) = A(\boldsymbol{a'})B(\boldsymbol{b'})$.

In motivating Bell's inequality, A. Aspect [19] then asserts:

$$\begin{aligned} s &\equiv E(\boldsymbol{a},\boldsymbol{b})-E(\boldsymbol{a},\boldsymbol{b}')+E(\boldsymbol{a}',\boldsymbol{b})+E(\boldsymbol{a}',\boldsymbol{b}') \\ &= A(\boldsymbol{a})B(\boldsymbol{b})-A(\boldsymbol{a})B(\boldsymbol{b}')+A(\boldsymbol{a}')B(\boldsymbol{b})+A(\boldsymbol{a}')B(\boldsymbol{b}') \end{aligned} \tag{5}$$

This result requires the correlation of the product of measurements at A and angle θ, and at B and angle φ, as the product of the correlations $A(\boldsymbol{a})$ and $B(\boldsymbol{b})$, allowing us to separate the $A(\boldsymbol{a})$ and $A(\boldsymbol{a'})$ terms. The expression for s can then be rewritten as:

$$s=A(\boldsymbol{a})\left[B(\boldsymbol{b})-B(\boldsymbol{b}')\right]+A(\boldsymbol{a}')\left[B(\boldsymbol{b})+B(\boldsymbol{b}')\right]$$

Since A and B only take the values ±1, either the first or the second term will vanish. This then implies

$$s=\pm 2\,.$$

Notice that we reach this conclusion with the implicit assumption that all of these expectation values are being taken with respect to the same sample space, as this is what allows us to factor s, given in Equation (5), into products and sums of expectation values. This is the assumption of classical statistics.

There is also a local causality assumption that's evident in factoring the correlation function $E(\boldsymbol{a}, \boldsymbol{b})$, the joint expectation value at locations a and b, into a product of measurements made separately at a and b, as given by $A(\boldsymbol{a})$ and $B(\boldsymbol{b})$.

We next see that quantum correlations violate the ±2 bounds on s, because this result was derived under two assumptions; its violation does not imply that quantum mechanics is nonlocal. Quantum mechanics could just as well escape these bounds by violating the classical statistics assumption. As we will show, this is the case: quantum mechanics violates the ±2 result because its statistics differ from the separability assumed in Equation (5), not because it violates local causality. This conclusion is reiterated by Hance and Hossenfelder [20].

Stated more plainly, separating $E(\boldsymbol{a}, \boldsymbol{b})$ into $A(\boldsymbol{a})B(\boldsymbol{b})$ is sufficient to enforce locality, but it is not necessary. Measurements at both A and B continue to share a common notion of the vertical, as embedded in the S_z component of their common internal angular momenta. Because of this, the measures at A are not entirely disconnected from the measurement at B as they both share a common axis. Each measurement is local, and both measurements are correlated.

10. Quantum Correlations

The rotations just considered are rotations of the detectors with respect to the observed states and a common notion of the vertical. Since we are interested in the change only with respect to the relative angles of the detectors, we can derive the general relation for the correlation between two detectors as a function of their relative angles of rotation.

We now present the standard, quantum mechanical calculation for the projection of spin states onto detection states. We'll arrive at the standard result. In doing this, we can assume one detector (which determines the basis of the particles that pass through it) is aligned with the vertical z-axis.

The other detector is rotated by an angle θ in the yz-plane, with the x-axis being the axis along which the particles move. The effect of the rotated detector is to rotate the states to a basis that is an eigenstate of the new, rotated z-axis. The rotation being applied is a rotation around the x-axis by the angle θ.

We are looking to calculate the correlation between a state at A aligned with the z-axis and a state at B aligned with the rotated z-axis. This rotation is applied to the 2-dimensional spinor state $|\Psi\rangle$ using the SU(2) rotation operator $R(\theta)$, for details see van der Waerden ([21]: p. 92).

$$R(\theta)=\begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right)\\ i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{pmatrix} \tag{6}$$

We assume the state has been prepared as an equal mixture of spin up and down components, and that each particle recorded at A is in a pure up or down state. We can write it as $|\Psi_{up}\rangle=|1\rangle\equiv\begin{pmatrix}1\\0\end{pmatrix}$, and $|\Psi_{dn}\rangle=|2\rangle\equiv\begin{pmatrix}0\\1\end{pmatrix}$, where $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are the up and down basis vectors in the eigenbasis defined with respect to the z-axis.

Defining the rotated up-state $R(\theta)|\Psi_{up}\rangle$ as $|\Psi'_{up}\rangle$, the probability amplitude for detecting the rotated up-state in the unrotated (z-aligned) up state is:

$$\begin{aligned}\langle\Psi_{up}|R(\theta)|\Psi_{up}\rangle &\equiv \langle\Psi_{up}|\Psi'_{up}\rangle\\ &=(1\quad 0)\begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right)\\ i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\ &=(1\quad 0)\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ i\sin\left(\frac{\theta}{2}\right)\end{pmatrix}\\ &=\cos\left(\frac{\theta}{2}\right)\end{aligned}$$

Squaring the amplitude gives the probability as $\cos^2\left(\frac{\theta}{2}\right)$.

The probability amplitude for detecting the rotated up-state in the unrotated (z-aligned) down state is:

$$\left\langle \Psi_{dn} \middle| \Psi'_{up} \right\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ i\sin\left(\frac{\theta}{2}\right) \end{pmatrix} = i\sin\left(\frac{\theta}{2}\right)$$

The absolute square of this amplitude, $\sin^2\left(\frac{\theta}{2}\right)$, gives the probability of detecting the rotated up-state as a down-state.

The correlation for measuring the same states at both locations A and B is the probability of measuring both rotated states to be in the up-state minus the probability of measuring them in up and down states. The correlation measuring up states at A and B with a relative rotation of θ is defined as $E(up, A, B, \theta)$, where

$$\begin{aligned} E(up, A, B, \theta) &= \frac{\left|\left\langle \Psi_{up} \middle| \Psi'_{up} \right\rangle\right|^2 - \left|\left\langle \Psi_{dn} \middle| \Psi'_{up} \right\rangle\right|^2}{\left|\left\langle \Psi_{up} \middle| \Psi'_{up} \right\rangle\right|^2 + \left|\left\langle \Psi_{dn} \middle| \Psi'_{up} \right\rangle\right|^2} \\ &= \frac{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)} \\ &= \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) \end{aligned}$$

However, we're not focusing on either up or down states in particular, but only symmetric or anti-symmetric combinations. We want to include the correlations before and after the rotation of the down states. That is, we want to add, $\left|\left\langle \Psi_{dn} \middle| R(\theta) \middle| \Psi_{dn} \right\rangle\right|^2 - \left|\left\langle \Psi_{up} \middle| R(\theta) \middle| \Psi_{dn} \right\rangle\right|^2$ which has the same probability as $E(up, A, B, \theta)$. Consequently, we multiply the numerator in the previous result by 2.

We normalize this result by dividing by the sum of the probabilities of all events, up and down, which is

$$\begin{aligned} &\left|\left\langle \Psi_{up} \middle| \Psi'_{dn} \right\rangle\right|^2 + \left|\left\langle \Psi_{dn} \middle| \Psi'_{dn} \right\rangle\right|^2 + \left|\left\langle \Psi_{up} \middle| \Psi'_{up} \right\rangle\right|^2 + \left|\left\langle \Psi_{dn} \middle| \Psi'_{up} \right\rangle\right|^2 \\ &= 2\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right) \\ &= 2 \end{aligned}$$

Adding the separate correlations for up and down gives the combined correlation for symmetric and antisymmetric observed states as,

$$\begin{aligned} E(A,B,\theta) &= E(up,A,B,\theta) + E(dn,A,B,\theta) \\ &= \frac{2\left(\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right)}{2\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right)} \\ &= \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) \\ &= \cos(\theta) \end{aligned} \tag{7}$$

This is not a classical statistical result because we are calculating expectation values for the rotated detectors using rotated basis states. That is, the expectation values at the different locations are computed with respect to sums over different (rotated) inner product states.

Cosine(θ) is the probability that two spin-1/2 particles with the same initial orientations will be measured as having the same orientation with respect to a z-axis that defines similarly directed inhomogeneous magnetic fields, given that one of the particles is subjected to a rotation of θ from this z-axis.

Note that in the EPRBA construction, the two particles begin with oppositely directed spins, so the outcome that results from no rotation ($\theta = 0$) is that cosine θ represents the probability of their measured spins being anti-correlated (or anti-parallel).

This derivation of the cos(θ) form of the correlation between a single pair of spins is made possible by our assumption of the disentanglement of the spin states at the two locations. This result is obtained by an entirely quantum mechanical argument.

This same expression for the correlation is used in the case where the spin states are entangled, but, according to Gu [22], "the result was only obtained by guess, because the concept of a direct product for multi-particle systems is not reasonably defined." Indeed, I have found no derivation of a simple cosine or sine dependence for this correlation probability based on an entangled state.

The simple derivation of this formula for the pure state considered here, and the lack of it for the entangled singlet state that (almost) everyone accepts as describing the post-separated wavefunction (see Bohm [1]), is another reason to doubt the entangled particle wavefunction is the correct wavefunction to represent the decayed singlet state.

In this regard, the many experiments presented as confirmation of nonlocality [23] [24]—because they confirm the cosine correlation—actually do the opposite. They confirm the local causality that follows from the derivation given here. Coming to a similar conclusion, Fritsche and Haugk ([25]: p. 45) state, "it is hard to see how this expectation value (derived from the entangled state wavefunction) can have anything to do with the experiment except that it happens to yield the same $-\cos(\boldsymbol{a}, \boldsymbol{b})$."

Annila and Wilström ([26], Equation 2) show that the same result as given in Equation (7) is obtained classically using the Pearson correlation of an ensemble average using a vector hidden variable, though not for a single measurement. Annila and Wilström are performing the same integrations as done here, but their results are based on the physically impossible and practically unattainable limit of an infinite sum of separate observations.

The integrations done here, in Section 17, pertain to the observation of a single correlated pair. They are integrations over a continuous variable, since the parameter space of the integration is the set of all possible values of the unobservable spin contributions, distributed over the surface of a 2-sphere. The central issue of the nature of quantum correlations, as pertains to local causality and wavefunction collapse, fundamentally involves the physics of single measurements of correlated pairs. These questions cannot be answered by a consideration of ensemble averages.

11. Pure Spin States Conserve Angular Momentum

A final and important note on the topic of decoherence concerns the argument presented against the use of pure states that was given by Bohm and Aharonov [1] in rebuttal to a suggestion by Furry [27] [28]. The gist of their argument is that a pure state wavefunction does not obey conservation of angular momentum. It's a strange argument and, with what we know now, it leads to the wrong conclusion.

They recast the spin version of the EPRBA in terms of photon polarization as they were making the point that photon polarization could be measured while, at that time, spin orientations could not. They considered a pair of photons originally correlated as the result of electron-positron annihilation traveling in opposite directions with opposite polarizations. They express the pure state wavefunctions representing the different polarizations of two photons as,

$$\Psi_1 = C_1^x C_2^y \left|\Psi_0\right\rangle,$$

$$\Psi_2 = C_1^y C_2^x \left|\Psi_0\right\rangle,$$

$$\Psi_3 = C_1^x C_2^x \left|\Psi_0\right\rangle,$$

$$\Psi_4 = C_1^y C_2^y \left|\Psi_0\right\rangle,$$

where $\left|\Psi_0\right\rangle$ is the ground state of the electromagnetic field and C_1^x and C_2^y are creation operators that create a photon polarized in the x and the y directions at locations 1 and 2. They then say:

$$\text{“}\,\Psi_1 = \left(C_1^{x'}\cos\theta + C_1^{y'}\sin\theta\right)\left(-C_2^{x'}\sin\theta + C_2^{y'}\cos\theta\right)\Psi_0$$

$$= -\sin\theta\cos\theta\Psi_{3'} + \sin\theta\cos\theta\Psi_{4'} + \cos^2\theta\Psi_{1'} - \sin^2\theta\Psi_{2'}$$

With similar expressions for Ψ_2, Ψ_3, and Ψ_4. It is clear from the above equation that in a rotated system of axes, the wavefunction no longer represents (as it did in the original system) a state in which the two photons have orthogonal directions of excitation. Rather, we see that it is possible for these directions either to be orthogonal or parallel.”

What they have done is rotate the coordinates in SO(3) and place the Hilbert space creation operators on new axes, but this is not a quantum mechanical rotation. They then use the early idea that expectation values represent the statistical results of observations to conclude that, because we now have combinations of photons with wavefunctions representing parallel polarization, as appear in Ψ_3 and Ψ_4, the resulting wavefunction has nonzero net polarization and does not conserve angular momentum.

Applying the same logic to the spin system, we can replace x and y with + and −, for spins up and down, and their wavefunctions Ψ_1 and Ψ_2 with spin wavefunctions S_1, S_2. Then replace their parallel polarization states x, x and y, y, given by Ψ_3 and Ψ_4, with $S_3 \equiv C_1^+ C_2^R$ and $S_4 \equiv C_1^L C_2^-$, where C_2^R and C_1^L are creation operators for spins rotated 90 degrees to the left and right relative to the up and down orientations. This gives us:

$$S_1 = C_1^+ C_2^- \left| S_0 \right\rangle,$$

$$S_2 = C_1^- C_2^+ \left| S_0 \right\rangle,$$

$$S_3 = C_1^+ C_2^R \left| S_0 \right\rangle,$$

$$S_4 = C_1^L C_2^- \left| S_0 \right\rangle.$$

Replacing coordinates as they have done, our rotated spin wavefunctions, for rotations in the range $-\pi/2 < \theta < \pi/2$, have the form:

$$\begin{aligned} S_1 &= \left(C_1^+ \cos\theta + C_1^L \sin\theta \right)\left(-C_2^- \cos\theta - C_2^R \sin\theta \right) S_0 \\ &= -\cos^2\theta C_1^+ C_2^- + \sin\theta\cos\theta C_1^+ C_2^R + \sin\theta\cos\theta C_1^L C_2^- - \sin^2\theta C_1^L C_2^R. \end{aligned}$$

This demonstrates, according to the mechanism put forward by Bohm and Aharonov, that the rotated wavefunction now contains terms $C_1^+ C_2^R$ and $C_1^L C_2^-$ that have non-zero net angular momentum. Interpreting the appearance of these terms as indicators of a non-zero probability of non-zero angular momentum states, we would conclude that the pure wavefunction cannot represent a zero-angular momentum state.

They go on to equate this failure to maintain orthogonality in photon polarization under SO(3) rotation with what would be a similar failure to maintain opposite spins in the case of an anti-correlated fermion pair. They say,

> "The wavefunction for a pair of photons evidently resembles the wavefunction for the spins of a pair of electrons. In both cases, we form a special linear combination of product wavefunctions, which guarantees that the two particles will be in opposite states, in relation to a group of rotated coordinate frames… Thus, the paradox of ERP can equally well be tested by polarization properties of pairs of photons."

But rotations for photons, like electrons, are governed by SU(2), not SO(3). The rotations presented above are not how you represent a rotated state for SU(2). It is incorrect to apply an SO(3) rotation to SU(2) operators; you apply an SU(2) rotation to the SU(2) states. Neither do you equate the expectation values of an observation with the amplitudes of the coefficients of a component state.

Furthermore, and quite separately, it is not the state vector that needs to obey rotational invariance but the observables derived from it. As we learned from the change in phase of the vector potential around a closed loop without the addition of any force, angular momentum can remain invariant even under conditions when the state vector, which is not directly observable, is not invariant.

This example, known as the Ehrenberg-Siday-Aharonov-Bohm effect [29], demonstrates that it is the observables that must obey the conservation laws, not the terms in the equations for them. To establish the invariance of angular momentum under rotation, we therefore need to establish the rotational invariance of the expectation value of the total angular momentum.

The correct statement of anti-correlated spins is the expectation value of the product of the eigenvalues in a space where their angular momenta are constants of the motion. This is initially given by the expectation value of the product of the spin operators for the two particles, namely

$$\begin{aligned}\langle S_z(1)S_z(2)\rangle &= \langle +_1|\langle -_2|S_z(1)S_z(2)|+_1\rangle|-_2\rangle \\ &= \langle +_1|S_z(1)|+_1\rangle\langle -_2|S_z(2)|-_2\rangle \\ &= \langle +_1|(+1)|+_1\rangle\langle -_2|(-1)|-_2\rangle \\ &= -1\end{aligned}$$

where $S_z(1)$ and $S_z(2)$ are the spin operators for particles 1 and 2, measured in a direction we call "the z-axis," but which we now understand is no particular direction in SO(3) until it has been measured. A rotation of this expectation value is accomplished by applying a SU(2) rotation to the eigenvectors.

Any rotation can be given by a rotation $U(\theta)$ where θ is an angle around some new axis. This new axis can be defined as a combination of rotations around the axes in the original rectilinear coordinate system. That is, $U(\theta)$ is a sequence of some angular rotation around the x, y, and z axes of the SU(2) coordinates.

We want to determine whether or not $\langle S_z(1)S_z(2)\rangle$ is invariant under an arbitrary rotation. We can consider rotations around each axis separately in order to see whether rotations around any axis change this expectation value.

A rotation in SU(2) is given by $U(\boldsymbol{\theta}) = \exp(-i/2(\boldsymbol{\sigma}\cdot\boldsymbol{\theta}))$, where $\boldsymbol{\sigma}$ are the Pauli matrices σ_x, σ_y, σ_z, and $\boldsymbol{\theta}$ are rotations θ_x, θ_y, θ_z around the x, y, and z axes. In units where $\hbar/2 = 1$, where $|+_1\rangle$ and $|-_2\rangle$ are the up and down states at locations 1 and 2, and the new orientation of the z-axis is given by z′, the rotated expectation value is given by:

$$\langle S_{z'}(1)S_{z'}(2)\rangle = \langle +_1|\langle -_2|U^\dagger(\theta_z)S_z(1)S_z(2)U(\theta_z)|+_1\rangle|-_2\rangle$$

A rotation $R(\theta_z)$ around the z-axis is manifestly invariant because S_z commutes with σ_z and all powers of it, so S_z commutes with $U(\sigma_z\theta_z)$. Using the unitarity of the rotation, $U^\dagger(\theta)U(\theta) = 1$, the expectation value remains unchanged.

A SO(3) rotation $R(\theta_x)$ around the x-axis is given by

$U(\theta_x) = \exp\left(-\frac{i}{2}(\sigma_x\cdot\theta_x)\right) = \sigma_x(\cos(\theta_x) + i\sin(\theta_x))$. The rotated expectation value is:

$$\begin{aligned}\left\langle S_{z'}(1)S_{z'}(2)\right\rangle &= \langle +_1|\langle -_2|U^\dagger(\theta_z)S_z(1)S_z(2)U(\theta_z)|+_1\rangle|-_2\rangle \\ &= \langle +_1|\sigma_x^\dagger(\cos\theta_x - i\sin\theta_x)S_z(1)\sigma_x(\cos\theta_x + i\sin\theta_x)|+_1\rangle \\ &\quad \times\langle -_2|\sigma_x(\cos\theta_x - i\sin\theta_x)S_z(2)\sigma_x(\cos\theta_x + i\sin\theta_x)|-_2\rangle \\ &= (\cos\theta_x - i\sin\theta_x)(\cos\theta_x + i\sin\theta_x)\langle -_1|(-1)|-_1\rangle \\ &\quad \times(\cos\theta_x - i\sin\theta_x)(\cos\theta_x + i\sin\theta_x)\langle +_2|(+1)|+_2\rangle \\ &= (-1)(+1)\left(\cos^2\theta_x + \sin^2\theta_x\right)^2\langle -_1|-_1\rangle\langle +_2|+_2\rangle \\ &= -1\end{aligned}$$

Rotation $R(\theta_y)$ around θ_y also leaves the expectation value invariant. Since the expectation value is invariant under rotation around any axis, it remains invariant after any sequence of rotations around any axes, which is to say it remains invariant after any general rotation. We conclude $\left\langle S_z(1)S_z(2)\right\rangle$, defined with respect to the pure states $|+_1\rangle|-_2\rangle$, is invariant under all rotations. We have proved that the total $S_z = 0$ pure spin state expectation value conserves angular momentum.

Consequently, the spins remain anti-correlated, and Bohm and Aharonov's conclusion is incorrect. Their conclusion that the pure state violates conservation of angular momentum is also incorrect.

If we were to apply Gell-Mann's "Totalitarian Principle (that) anything that is not forbidden is compulsory" [30], the pure state not only can be used but must be used!

12. Comparing Classical and Quantum Correlations

In quantum mechanics, a probability amplitude is given by the inner product of an operator evaluated between orthonormal states. The Born Rule then requires the absolute square of this amplitude to find the probability. Consequently, when the probability is classical, a rotation is a linear operation applied only to the factor being rotated. Note that the values a and b, appearing below, now represent coplanar angles, not vectors. θ is the angle between a and b, that is $\theta = a - b$. We can think of the angle b as the angle a rotated by θ, that is $b = a - \theta$. A classical correlation between detectors at A and B with a relative orientation of θ is given by:

$$E_{cl}(A,B,\theta) = A(a)R(\theta)\left[B(a)\right] = A(a)B(b,\theta)$$

While quantum mechanically, we have from Aspect [19],

$$E_{qm}(A,B,a-b) = \cos\left(2(a-b)\right) \neq A(a)B(b) \tag{8}$$

where $A(a)$ and $B(b)$ are any two separate functions of a and b. Which is to say the cosine of the difference of two angles does not equal the product of two functions, each of which depends only on one angle or the other. But, as we will see, this does not mean that there is a nonlocal interaction between the systems at locations A and B.

Note that Aspect's factor of two in the cosine, which doubles the difference of the angles, does not appear in other presentations. Its presence in this case is due to Aspect's experimental configuration, which uses a photon analyzer that does not distinguish photons whose phase differs by $\pm\pi$ ([26]: p. 4).

Using this quantum mechanical function for the correlations between parallel and anti-parallel orientations as a function of the angles a, a', b, and b', measured at locations A and B, we have

$$s_{qm} \equiv E_{qm}(a,b) - E_{qm}(a,b') + E_{qm}(a',b) + E_{qm}(a',b') \tag{9}$$

If we choose the angles $a = 0$, $b = \pi/8$, $a' = 2\pi/8$, and $b' = 3\pi/8$, such that $a - b = \pi/8$, $a - b' = -3\pi/8$, $a' - b = -\pi/8$, and $a' - b' = -\pi/8$. Then, using the symmetry of the cosine function,

$$\begin{aligned} s_{qm} &= \cos(\pi/4) - \cos(3\pi/4) + \cos(\pi/4) + \cos(\pi/4) \\ &= 4\left(\frac{1}{\sqrt{2}}\right) \\ &= 2.828 \end{aligned} \tag{10}$$

This value is greater than 2. From this observation, it is asserted that no classical hidden variable model will reproduce the quantum result.

Extending this conclusion to unjustified generality, Bell's result is commonly misunderstood [31]-[34] to apply to quantum hidden variable models despite Clauser's reminder [17] that Bell's result only applies to models composed of classical variables. Overlooking this distinction leads to fundamental ambiguity in what is meant by the terms "locality" and "realism," and the misconception that Bell's result applies to all local variable models.

Note that the classical limit of $s \le 2$, derived in Equation (5) and exceeded in (10), is the result of assuming a specific functional structure for local theories, not the local nature of variables. To understand the results of Bell and related inequalities, it must be understood that it is not the local nature of the variables that underlies this result, but the functional structure assumed by the theory.

13. The Locality Assumption

The underlying difference between classical and quantum variables arises from the fact that observing a quantum system can change the results of subsequent observations. In the case of quantum spin systems, an observation of the angle of spin establishes a new basis for subsequent observations at that location. This creates situations of variable interdependence. In contrast, measuring a classical variable, hidden or otherwise, does not change the variable's value and so cannot create correlations between variables.

For example, if you find a quantum particle in the "up state," where up is defined as vertical, and then—assuming you have not disturbed any other aspects of the system—measure its state horizontally, it will no longer be in the up state when measured again vertically. In contrast, a particle whose spin is determined by a classical variable will remain in its state without regard to measurements made at different orientations.

In Bell's exploration of the difference between classical and quantum systems, he defined a correlation function, $E(A, a, B, b)$, based on the independent measurement of the spin at two locations, A and B, of a correlated pair of spins measured at two different angles, a at location A and b at location B. This correlation function is defined to give the result +1 when the spins at both locations are recorded as being aligned with respect to the respective detectors oriented at angles a or b, and −1 when the spin at one location is found to be aligned with and the other aligned opposite to the orientation of their respective detectors.

Bell then noted that for any function $E(A, a, B, b)$ to be local, it is sufficient that this correlation function can be factored into the product of two space-like separable functions $E(A, a)$ and $E(B, b)$ such that

$$E\left(A,a,B,b\right)=E\left(A,a\right)E\left(B,b\right).$$

This is referred to as "the locality assumption." In contrast, the correlation between spins measured in accordance with quantum mechanics is proportional to the cosine of the difference between the two angles of measurement, a-b, as shown above in Equation (7).

Cosine(a-b) cannot be factored into the product of two functions such that each is a function of only a or only b. We can find no two functions of classical, scalar variables whose product is expressed as the difference between their values. This is emphasized by Hess [35], who explains: "If we have two spaceships with pilots Alice and Bob, respectively, who know absolutely nothing of each other, no correlation of any physical processes in the spaceships can ever be found."

Correlation between separated events is nonlocal by definition, but that does not mean it is acausal. Bell's locality assumption, written only in terms of separate functions of a and b, is both nonlocal and acausal.

We can, however, define a function of two vector variables that depends on the vector inner product—and hence the cosine of the angle between the vectors—but this function is not "separable" according to the locality assumption. Such a function is not local, since a and b are not co-located, but it also says nothing about causality, one way or the other.

A hidden variable formulation replaces $E(A, a)E(B, b)$ with $E(A, a, \lambda)E(B, b, \lambda)$, where λ is an unrecognized "hidden" variable. In this formulation, the correlation function depends only on the angles a and b at the respective locations, but the additional parameter λ correlates the results at the two locations.

Convolution is a third locality formulation that's not considered in the Bell-CHSH formulations. This is the formulation explored here in Section 17. Equation (14) presents the convolution in the form

$$E\left(A,\boldsymbol{a},B,\boldsymbol{b}\right)=\int_{\Lambda}E\left(A,\boldsymbol{a},\boldsymbol{\lambda}\right)E\left(B,\boldsymbol{b},\boldsymbol{\lambda}\right)\mathrm{d}\boldsymbol{\lambda},$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{\lambda}$ are 3-vectors. Here, it is the integrand that plays the role of a variable that's more than hidden; it is undetectable. The convolution formulation combines Bell's locality assumption with a local weighting function whose value is independent of the time or distance separating A and B. This puts a different perspective on the meaning of a hidden variable since the variable $\boldsymbol{\lambda}$ is not subject to classical or relativistic limitations. It is free of any such conditions because it will not appear in any physical result.

Relying exclusively on Bell's original and limited locality assumption, the majority of physicists have concluded that quantum mechanics is not a local theory. This conclusion does not follow logically since Bell's locality assumption, as a functional structure, is not necessary for a theory's local, causal structure. Hess explains this in more detail.

In this section, we demonstrate that a convolution formulation satisfies locality and causality, uses the unobservable value of S_z as the hidden variable, and reproduces the correct quantum correlation. The observed value of S_z then emerges as something like a path-integrated value of unobservable S_z contributions in the same way that the classical path of a particle is the sum of quantum path contributions.

14. Bell Locality

The basis for the Bell-CHSH inequalities is the definition of the quantity s, given above in Equation (10), constructed as the linear combination of correlation functions:

$$s \equiv E(A,a,B,b) - E(A,a,B,b') + E(A,a',B,b) + E(A,a',B,b')$$

When we call $E(A,a,B,b)$ a classical expectation value, we mean that it describes either an exact or an average measured value. If it's an exact value, then it presumes a specific known, repeatable, and identical configuration being measured for every correlation function. If it's an average value, then it presumes measurement over a known, identical configuration sample for every correlation function.

We can define s_{qm} as a quantum operator, as we have in Equation (9); aside from the name "expectation value," the classical and quantum expressions for s will involve different mechanisms. In the quantum case, the expectation value of s_{qm}, written as $\langle s_{qm} \rangle$, is equal to

$$\langle s_{qm} \rangle = \langle E_{qm}(A,a,B,b) - E_{qm}(A,a,B,b') + E_{qm}(A,a',B,b) + E_{qm}(A,a',B,b') \rangle \quad (11)$$

We would like to write this as the sum of the separate expectation values as

$$= \langle E_{qm}(A,a,B,b) \rangle - \langle E_{qm}(A,a,B,b') \rangle + \langle E_{qm}(A,a',B,b) \rangle + \langle E_{qm}(A,a',B,b') \rangle$$

This is statistically valid only when the expectation value of each term is computed with respect to the same set of basis states. After rewriting the expression as a sum of expectation values and applying Bell's locality assumption, the expression $\langle s_{qm} \rangle$ for becomes:

$$\begin{aligned} \langle s_{qm} \rangle = {} & \langle E_{qm}(A,a) \rangle \left[\langle E_{qm}(B,b) \rangle - \langle E_{qm}(B,b') \rangle \right] \\ & + \langle E_{qm}(A,a') \rangle \left[\langle E_{qm}(B,b) \rangle + \langle E_{qm}(B,b') \rangle \right] \end{aligned} \quad (12)$$

If we then tabulate the resultant values of $\langle s_{qm} \rangle$ using the various combinations of the four different terms involving of $\langle E_{qm} \rangle$ each being either ±1, we conclude that of $\langle s_{qm} \rangle$ has a lower bound of −2 and an upper bound of +2.

Since quantum mechanics can exceed this limit, as shown in Equation (10), much of the physics community has concluded that quantum mechanics violates locality. However, the disparity results from the overly restrictive nature of Bell's locality assumption, not from a quantum mechanical violation of locality.

Expressions of the form $\langle E_{qm}(A,a,B,b) \rangle$ return the probability of finding the spin of both particles at A and B in the "up" state when the particle at A is measured in the "up" state with a detector at an angle a, and the particle at B is measured with a detector rotated by an angle b, and the difference of the two angles is ≤π. If we consider each of the expectation values separately, then global rotational invariance allows us to rotate the states used to define each expectation value so that the first angle is zero.

Expanding the exact quantum mechanical expression for $\langle s_{qm} \rangle$, the expectation values separate into products of expectation values at the respective locations, but this separation does not obey Bell's locality assumption, as the second term, seen below, involves the difference of two angles.

Without loss of generality, we set $a = 0$. The operator $R(\theta)$ is inserted to rotate states at B to align with the state at A. This gives:

$$\langle E_{qm}(A,a,B,b) \rangle = \langle +_A | S_z R(0) | +_A \rangle \langle +_B | S_z R(b-a) | +_B \rangle = \langle +_B | S_z R(b-a) | +_B \rangle$$

And

$$\begin{aligned} E_{qm}(A,a',B,b) = E_{qm}(A,0,B,b-a') \\ &= \langle +_A | S_z R(0) | +_A \rangle \langle +_B | S_z R(b-a') | +_B \rangle \\ &= \langle +_B | S_z R(b-a') | +_B \rangle \end{aligned}$$

Where we've used $\langle +_A | S_z R(0) | +_A \rangle = +1$ by the definition of $|+\rangle$. Equation (12) becomes:

$$\begin{aligned} \langle s_{qm} \rangle &= \langle E_{qm}(A,0,B,b) \rangle - \langle E_{qm}(A,0,B,b') \rangle \\ &\quad + \langle E_{qm}(A,a',B,b) \rangle + \langle E_{qm}(A,a',B,b') \rangle \\ &= \langle +_B | S_z R(b) | +_B \rangle - \langle +_B | S_z R(b') | +_B \rangle \\ &\quad + \langle +_B | S_z R(b-a') | +_B \rangle + \langle +_B | S_z R(b'-a') | +_B \rangle \end{aligned}$$

We can drop the B subscript since all results are taken relative to an angle $a = 0$ at A. Applying the matrix form of $R(\theta)$, given in Equation (6), gives $R(\theta)|+\rangle = \cos\left(\frac{\theta}{2}\right)|+\rangle + i\sin\left(\frac{\theta}{2}\right)|-\rangle$, and

$$\begin{aligned} \langle s_{qm} \rangle &= \langle + | S_z \cos\left(\frac{b}{2}\right) | + \rangle + i \langle + | \sin\left(\frac{b}{2}\right) | - \rangle - \langle + | S_z \cos\left(\frac{b'}{2}\right) | + \rangle \\ &\quad + i \langle + | \sin\left(\frac{b'}{2}\right) | - \rangle + \langle + | S_z \cos\left(\frac{b-a'}{2}\right) | + \rangle + i \langle + | \sin\left(\frac{b-a'}{2}\right) | - \rangle \\ &\quad + \langle + | S_z \cos\left(\frac{b'-a'}{2}\right) | + \rangle + i \langle + | \sin\left(\frac{b'-a'}{2}\right) | - \rangle \end{aligned}$$

We can then drop the terms proportional to $\langle + | - \rangle$, which equal 0, to get

$$\begin{aligned}\left\langle s_{qm}\right\rangle &= \langle +|S_z\cos\left(\frac{b}{2}\right)|+\rangle - \langle +|S_z\cos\left(\frac{b'}{2}\right)|+\rangle \\ &\quad + \langle +|S_z\cos\left(\frac{b-a'}{2}\right)|+\rangle + \langle +|S_z\cos\left(\frac{b'-a'}{2}\right)|+\rangle \\ &= \cos\left(\frac{b}{2}\right) - \cos\left(\frac{b'}{2}\right) + \cos\left(\frac{b-a'}{2}\right) + \cos\left(\frac{b'-a'}{2}\right)\end{aligned}$$

These terms do not factor according to Bell's locality assumption and cannot be put into the form of Equation (12). From this, we see that it is the locality assumption that fails in the case of quantum mechanics, not the locality of the theory.

15. The Quantum Hidden Variable

$$\left\langle E_{qm}\left(A,0,B,b\right)\right\rangle = \frac{\left|\left\langle \Psi_{up}\middle|\rho\middle|\Psi'_{up}\right\rangle\right|^2 - \left|\left\langle \Psi_{dn}\middle|\rho\middle|\Psi'_{up}\right\rangle\right|^2}{\left|\left\langle \Psi_{up}\middle|\rho\middle|\Psi'_{up}\right\rangle\right|^2 + \left|\left\langle \Psi_{dn}\middle|\rho\middle|\Psi'_{up}\right\rangle\right|^2}$$

is a correlation function built from projection operators $\rho \equiv \sum_j \left|\psi_j\right\rangle\left\langle\psi_j\right|$, where j runs over a complete set of orthonormal basis vectors ψ_j. In this expression, the $\langle\Psi|$ states are final states, $|\Psi'\rangle$ are initial states, and the density function ρ provides all the paths between them. $E_{qm}(a,b)$ does not describe the mechanics that underlie the correlations, but we show that the mechanics leading to these correlations are entirely local.

We must extend the theory to map the parameters of the 2-dimensional spinor onto 3-dimensional space, which is the space of observations. As this mapping was missing from the quantum mechanics of 1935, we might claim the quantum description of 1935 was incomplete, as suggested by Einstein *et al.* [2].

Adding this mapping completes the quantum description, at least in terms of restoring causality and a quantum form of local realism. This is not Einstein's version of local realism because it does not ascribe independent, observable values to the spin components S_x, S_y, and S_z. To the contrary, it denies an independent reality to any spin component other than what has been previously measured, since all spin observations are destructive.

Guangye Chen [36] has shown that the correct quantum correlation function $E_{qm}(a,b)$ can be recovered using the entangled singlet state wavefunctions—the anti-symmetric spin components, sometimes called the Bell state, which has the form $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow,A\rangle\otimes|\downarrow,B\rangle - |\downarrow,A\rangle\otimes|\uparrow,B\rangle\right]$, in conjunction with a real-valued vector hidden variable. This would seem to contradict Bell's theorem.

Chen's result works because his hidden variable vector does not behave classically. In particular, it is constructed to behave in whatever way is required to obtain the desired result. While this is effectively an ad hoc result, it is sufficient to demonstrate what is needed in order to get the correct result.

We argue here that if we replace the entangled state with a separable, pure state, and replace Chen's classical hidden variable with an embedding of the z-components of the wavefunctions at A and B, then we arrive at the desired result for

$E_{qm}(a,b)$.

All we're adding is an observation that is already implicit in the quantum formulation, namely that the z-components of the wavefunctions are not classical vectors but quantum spinors. As such, we need a prescription for how the values of these 2-dimensional objects map onto our 3-dimensional reality. That is, we need a mapping of SU(2) objects into SO(3) objects.

Accomplishing this leads to three important conclusions:

1) The dynamics of this experiment are entirely local; there are no nonlocal effects.

2) The mapping of 2-dimensional information onto the 3-dimensional space of observation involves a process we're calling spontaneous symmetry creation.

3) 3-dimensional objects, such as we think our world is composed of, are no more real than the 2-dimensional objects of which our world is equally composed. That is, while we can restore EPR's notion of locality, it's not based on their 3-dimensional notion of reality.

16. Using σ_z as a Hidden Variable

The A and B components of our disentangled wavefunction are eigenvectors of σ_z in some basis, with equal and opposite eigenvalues. Some orientation is chosen by the system that defines this basis when the components disentangle, and this becomes the z-axis along which the particles' eigenvalues are fixed. These are not the arbitrary axes we choose for our measurements.

We know the two components carry opposite values of intrinsic angular momentum, but we don't know the orientation in space that defines them. In fact, the direction of the z-axis can never be known, as every experiment that can measure the z-axis finds the wavefunctions to be diagonal with respect to a basis defined along the measured direction. That is the situation as quantum mechanics currently describes it, but future experiments could change this.

We know there is a component of the intrinsic angular momentum, and this has the value of $\pm\hbar/2$ along an unknown direction for both particles. We know the absolute value of the total angular momentum of each particle is $\hbar\sqrt{3/2}$. The other components of the angular momentum of each part of the wavefunction, what we might call the azimuthal components, cannot be measured without disturbing the system. The GHZ result [3] shows that there are no naturally well-defined azimuthal components in any direction.

As long as the remaining components of angular momentum are symmetrically distributed around the z-axis—whatever it means for things that cannot be well defined to be symmetric—they add no time-dependent change to the wavefunction.

Because the particles are always measured to align in opposite directions when exposed to an inhomogeneous magnet field, as the theory currently presumes, we also know the z-axis is not changing direction over time. If it were executing some angular rotation, and if we measured the direction of the orientation of magnetization at different times for each of the two particles, then there would be times when we would find the two particles no longer anti-correlated.

We can picture the intrinsic angular momenta as oppositely pointing vectors, subject to the caveat that only the z-component has a well-defined value. It might be better to picture the spin variables as oppositely oriented hemispheres that have specified z directions and radius. In terms of Euler angles, the azimuthal angle of the "up" vector is uniformly distributed from 0 to 2π, while the polar angle is uniformly distributed between 0 and $\pi/2$. The azimuthal angle of the "down" component is similarly evenly distributed from 0 to 2π, but its polar component ranges between $\pi/2$ and π.

These hemispheres are an SO(3), vector-based description of the SU(2) spinor that describes each particle. The SU(2) spinors have fixed z-components of $\pm\hbar/2$ along unknown directions, while the SO(3) vectors are constrained to either the upper or lower hemispheres (**Figure 1**).

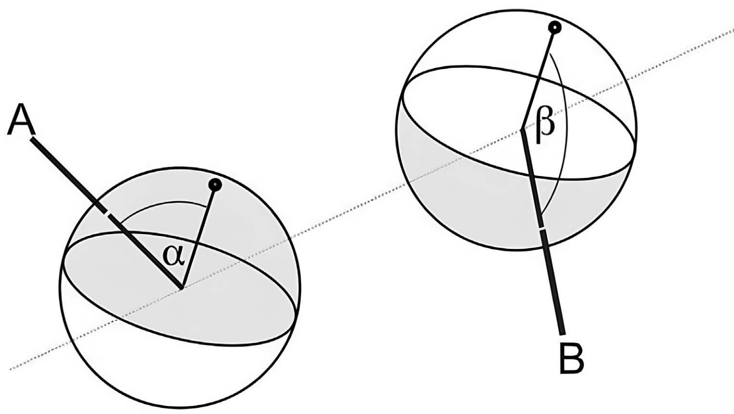

Figure 1. The unobservable directions of the quantum spinors, imagined as the vector axes of two opposing hemispheres, and the unobservable angles α and β that lie between these axes and directions A and B in which the magnetic moments are measured.

A vector direction given to this spinor is just the first of two steps in its use in this calculation. The necessary next step is to integrate over these unobservable vector directions so that we're left with a physical theory that only contains observable values of the square of and the deflection due to the direction of internal angular momentum.

We integrate over these physically nonexistent directions in the same way we integrate over particle positions that can't be observed when we integrate in our path integral in order to obtain an interference pattern. If we attempt to "locate" the particles involved in a path integral, then we destroy the phase relationships that generate the interference. Similarly, here, we cannot observe the spin orientations over which we're integrating without altering the wavefunction that contains the correlation that exists between the particles.

In putting forth this mnemonic, we're advancing the idea that the particle's spinor orientation is the hidden variable that resolves the question of how spin correlation is achieved. We can do this because the eigenvector that corresponds to the stationary component of internal angular momentum is neither observed nor observable. "The particle coordinates remain definitely hidden parameters… this applies to the eigenvalues of Hermitian operators as well" ([25]: p. 45).

This is analogous to the situation we encounter in Young's Double-Slit experiment [37], where we only measure one particle location on the detection screen, but we must include all the possible paths the particle took to get there. Here, we will only measure one direction for the internal spin but we must include contributions from all internal spin directions that satisfy the anti-correlation requirement. A similar sum-over-histories approach has also been developed for calculating the partition function of a quantum spin system [38].

We never know, nor can we ever assign a direction to the internal angular momentum before it has been measured. After it has been measured, it always points with or against the gradient of the inhomogeneous magnetic fields used to measure it.

All we can observe are the directions A and B in which the magnetic moments are measured, and the angles a and b between these vectors and the z-axis in the laboratory frame. The supposition that there exists a well-defined z-component of internal angular momentum before it has been measured is an unobservable construct (**Figure 2**).

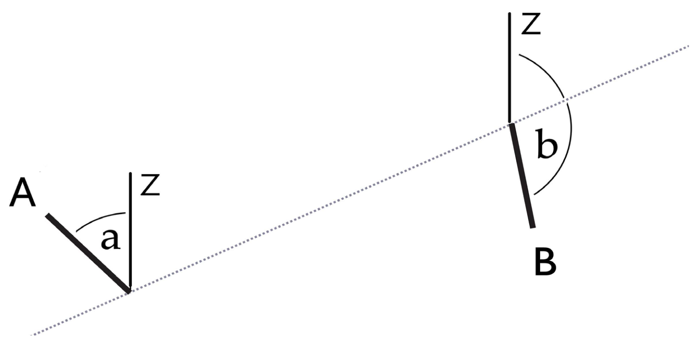

Figure 2. If when it's measured in direction A the first particle registers as parallel to A, then measured in direction B, whose angle is more than π/2 away from A, the second particle registers as parallel to B. We infer that the particle at A was pointing up, while the particle at B was pointing down.

Coddens [16] points out that the lack of a Hilbert space basis that provides eigenvectors for all the rotation generators of SU(2) is a statement about the SU(2) Clifford algebra, not the magnetic dipole. He says, "the spin vector $\boldsymbol{s}$ is not represented by $\boldsymbol{\sigma}$ or $\boldsymbol{\gamma}$ but by $\boldsymbol{s}\cdot\boldsymbol{\sigma}$ or $\boldsymbol{s}\cdot\boldsymbol{\gamma}$, which often remains hidden inside the notation for the spinor ψ." The inner product construction used here is based on a similar assertion that while the Hilbert-space formalism denies simultaneous observation of all spinor components, the components of the magnetic moment vector still interact with external magnetic fields.

17. The Vector Mnemonic

Since we only measure the magnitude and direction of the magnetic moments, and observe them to point in opposite directions when measured according to a common angle, we do not know what directions they point before our measurement. We can picture them more accurately not as vectors but as the opposite sides of two bisected spheres. It is in this sense that we can say the 2-dimensional spinors "spontaneously create" hemispherical symmetry in 3-dimensions.

We cannot measure where on the surface of these hemispheres the angular momentum vector points because there is no such point in 3-dimensions that is a constant of the motion. The z-component is a constant of the motion whose fixed value exists in SU(2) space but has no pre-measurement value in three dimensions.

Directionally defined x and y components cannot exist simultaneously with a z-component in either space. Nevertheless, we can still integrate over different values of these perpendicular (or azimuthal) components. As mentioned, we do this in the same way that we integrate over a particle wavefunction's position variable, even when that particle has no well-defined position.

Let $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ be two unit 3-vectors, that include x and y components, which accompany the z-component of the internal angular momenta. $\boldsymbol{\lambda} = -\boldsymbol{\lambda}'$ because the two particles have net zero angular momentum. $\boldsymbol{\lambda}$ represents both the direction of the z-axis used to define each particle's eigenstate and the perpendicular components that cannot be measured.

Following Gu [39], define $A(\boldsymbol{a}, \boldsymbol{\lambda})$ as the function that tells us whether we measure the particle to have an upward or downward pointing magnetic moment when measured at position A using a machine oriented at an angle given by the unit vector $\boldsymbol{a}$ lying in the yz-plane, perpendicular to motion in the x direction. The angle of the vector $\boldsymbol{a}$ is measured with respect to a common notion of vertical.

Diverging from Gu, $\boldsymbol{\lambda}'$ is assigned to the second particle so that $B(\boldsymbol{b}, \boldsymbol{\lambda}') = -A(\boldsymbol{b}, \boldsymbol{\lambda})$. This indicates the direction the particle's magnetic moment will be measured at B, using a machine oriented at the angle of the unit vector $\boldsymbol{b}$ from the common vertical, as measured in the yz-plane.

The functions A and B only return the values ±1, indicating a momentum in a positive or negative direction with respect to the angles made by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. They return +1 when $\boldsymbol{\lambda}$ projects along the direction of a unit vector in the yz-plane at the angle given by $\boldsymbol{a}$, and −1 when $\boldsymbol{\lambda}$ projects against the direction of this unit vector. As an ansatz, following Bell ([4]: p. 197), we define $A(\boldsymbol{a}, \boldsymbol{\lambda})$ and $B(\boldsymbol{b}, \boldsymbol{\lambda}')$ as having the values

$$\begin{aligned} A(\boldsymbol{a},\boldsymbol{\lambda}) &= \operatorname{sign}(\boldsymbol{a}\cdot\boldsymbol{\lambda}) \\ B(\boldsymbol{a},\boldsymbol{\lambda}') &= \operatorname{sign}(\boldsymbol{b}\cdot\boldsymbol{\lambda}') = -\operatorname{sign}(\boldsymbol{b}\cdot\boldsymbol{\lambda}) \end{aligned} \tag{13}$$

This is substantiated using the following reasoning:

1) The spin contribution must have some representation in SO(3) if its contributions to observables in SO(3) are to be calculated in any form.

2) We know the spin contribution cannot be represented by an observable vector since it has only one stationary component, whereas an observable SO(3) vector requires three.

3) We know there are additional spin contributions besides that given by the

one observable component because $|\boldsymbol{S}|^2 > S_z^2$.

4) We know the two spin contributions are binary, antisymmetric, and exclusive (orthogonal).

5) We know the additional spin contributions must be azimuthally symmetric because there is no observable azimuthal coordinate to break the "virtual" azimuthal symmetry.

We depart from Bell [4] in not equating the polarization vector $\boldsymbol{\lambda}$ with the spin state of a particle because, as mentioned, a spinor cannot be represented by a vector. It is only the integrated value of $\boldsymbol{\lambda}$ over the unit sphere that represents the spin state of the particle, and only in the context of this measurement. Through this process, we obtain the correct quantum mechanical correlation.

Notice that when $\boldsymbol{a} = \boldsymbol{b}$, the recordings at A and B are made along the same axes and $A(a, \boldsymbol{\lambda})$ and $B(b, \boldsymbol{\lambda})$ are of opposite sign, indicating the particles are anti-correlated.

Here, $\boldsymbol{a}\cdot\boldsymbol{\lambda}$ and $\mathbf{b}\cdot\boldsymbol{\lambda}'$ are simple inner products of these pairs of 3-component objects. The function sign($\boldsymbol{x}\cdot\boldsymbol{y}$) returns the value +1 or −1 depending on the sign of the inner product of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Equation (13) is written in terms of classical variables and expresses the quantum result. There is no classical explanation for Equation (13) as it reflects the mapping of the properties of quantum SU(2) spinors onto our classical SO(3) space.

$\boldsymbol{\lambda}$ is the direction in 3-space along which the eigenstate of one of the particles is pointing, and against which the other particle's eigenstate points. $\boldsymbol{\lambda}$ is represented by a vector for computational purposes, but it does not behave like a classical variable. Only its z component, or eigenvector defining component, is well defined. Its behavior derives from the non-commutative properties of the Pauli matrices.

Because we never know or see the values of $\boldsymbol{\lambda}$ or $\boldsymbol{\lambda}'$, they must not appear in any physical result. There is really only one variable here as $\boldsymbol{\lambda}' = -\boldsymbol{\lambda}$ because the two particles are always correlated. As a result, the correlation involves only a single variable $\boldsymbol{\lambda}$. We ensure the disappearance of $\boldsymbol{\lambda}$ from our result by integrating over all values of it.

$$E(A,\boldsymbol{a},B,\boldsymbol{b}) = -\int_\Lambda \rho_A(\boldsymbol{a},\lambda)A(\boldsymbol{a},\lambda)\rho_B(\boldsymbol{b},-\lambda)B(\boldsymbol{b},-\lambda)\mathrm{d}\lambda$$

Since the density functions ρ_A and ρ_B are symmetric in $\boldsymbol{\lambda}$, and the orientation functions $A(\boldsymbol{a}, \boldsymbol{\lambda})$ and $B(\boldsymbol{b}, \boldsymbol{\lambda})$ are anti-symmetric in $\boldsymbol{\lambda}$, we replace $\rho_B(\boldsymbol{b},-\lambda)B(\boldsymbol{b},-\lambda)$ with $-\rho_B(\boldsymbol{b},\lambda)B(\boldsymbol{b},\lambda)$. Chen [36] denotes $E(A, \boldsymbol{a}, B, \boldsymbol{b})$ as $P(a, b)$ in his equation #2. We then have:

$$E(A,\boldsymbol{a},B,\boldsymbol{b}) = \int_\Lambda \rho_A(\boldsymbol{a},\lambda)A(\boldsymbol{a},\lambda)\rho_B(\boldsymbol{b},\lambda)B(\boldsymbol{b},\lambda)\mathrm{d}\lambda \tag{14}$$

Here, $\rho_A(\boldsymbol{a},\lambda)$ and $\rho_B(\boldsymbol{b},\lambda)$ are the densities of $A(\boldsymbol{a}, \boldsymbol{\lambda})$ and $B(\boldsymbol{b}, \boldsymbol{\lambda})$, and Λ is the S2 unit sphere (the unit sphere in three dimensions) over which $\boldsymbol{\lambda}$ is uniformly distributed.

The densities $\rho_A(\boldsymbol{a},\lambda)$ and $\rho_B(\boldsymbol{b},\lambda)$ are equal to the absolute value of the vector inner product of their arguments. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are of unit length. Their arbitrary angles are the measurement angles set by experiment.

$$\rho_A(\boldsymbol{a},\boldsymbol{\lambda}) = |\boldsymbol{a}\cdot\boldsymbol{\lambda}| \text{ and } \rho_A(\boldsymbol{b},\boldsymbol{\lambda}) = |\boldsymbol{b}\cdot\boldsymbol{\lambda}| \tag{15}$$

$\rho_A(\boldsymbol{a},\boldsymbol{\lambda})A(\boldsymbol{a},\boldsymbol{\lambda}) = |\boldsymbol{a}\cdot\boldsymbol{\lambda}|\operatorname{sign}(\boldsymbol{a}\cdot\boldsymbol{\lambda})$, and this equals $\boldsymbol{a}\cdot\boldsymbol{\lambda}$.

$\rho_B(\boldsymbol{b},\boldsymbol{\lambda})B(\boldsymbol{b},\boldsymbol{\lambda}) = |\boldsymbol{b}\cdot\boldsymbol{\lambda}|\operatorname{sign}(\boldsymbol{b}\cdot\boldsymbol{\lambda})$ which equals $\boldsymbol{b}\cdot\boldsymbol{\lambda}$.

These expressions leave unchanged the marginal distributions describing the interaction of a single spin with an external magnetic field. This is because $\boldsymbol{\lambda}$ is symmetrically distributed around the particle's magnetic moment and represents the stochastic value of spin. Seeing spin as contributing a superposition of variable contributions, rather than having a fixed value, is a quantum mechanically more accurate representation of its role as an internal parameter.

Applying Equation (15) to Equation (14) gives:

$$E(A,\boldsymbol{a},B,\boldsymbol{b}) = \int_\Lambda (\boldsymbol{a}\cdot\boldsymbol{\lambda})(\boldsymbol{b}\cdot\boldsymbol{\lambda})\,\mathrm{d}\boldsymbol{\lambda} \tag{16}$$

Rewritten in the spherical coordinates appropriate for integration, $E(A,\boldsymbol{a},B,\boldsymbol{b}) = \int_\Lambda (\boldsymbol{a}\cdot\boldsymbol{\lambda})(\boldsymbol{b}\cdot\boldsymbol{\lambda})\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, where integration over Λ means integration over the azimuthal angle ϕ = from 0 to 2π, and the polar angle θ = from 0 to π.

While $A(\boldsymbol{a}, \boldsymbol{\lambda})$ and $B(\boldsymbol{b}, \boldsymbol{\lambda})$ are observables, $\boldsymbol{\lambda}$ is not. Following Chen, representing $\boldsymbol{\lambda}$ as a random vector of length r uniformly distributed over the surface of a sphere whose radius is r, we write $\boldsymbol{\lambda}$ in rectangular coordinates as

$$\boldsymbol{\lambda} = (r\cos\phi\sin\theta, r\sin\phi\sin\theta, r\cos\theta) \tag{17}$$

We then have

$$\begin{aligned}
&E(A,\boldsymbol{a},B,\boldsymbol{b}) \\
&= r^2\Big(\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(a_1b_1\cos^2\phi\sin^2\theta + a_2b_2\sin^2\phi\sin^2\theta + a_3b_3\cos^2\theta\right)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&\quad + \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(a_1b_2 + a_2b_1)\sin\phi\cos\theta\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&\quad + \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(a_1b_3 + a_3b_1)\cos\phi\sin^2\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&\quad + \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(a_2b_3 + a_3b_2)\sin\phi\sin^2\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\Big)
\end{aligned} \tag{18}$$

where a_i and b_i, i = 1, 2, 3, are components of $\boldsymbol{a}$ and $\boldsymbol{b}$. Equation (18), somewhat amazingly—due to the azimuthal anti-symmetry of the last three integrals, which vanish—reduces to

$$E(A,\boldsymbol{a},B,\boldsymbol{b}) = \frac{4r^2\pi}{3}(a_1b_1 + a_2b_2 + a_3b_3) = \frac{4r^2\pi}{3}(\boldsymbol{a}\cdot\boldsymbol{b}) \tag{19}$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors, this means that $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\omega)$, where ω is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Since $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$ and $\cos(\omega) \le 1$, we can reset r, which is the length of $\boldsymbol{\lambda}$, equal to $(3/4\pi)^{1/2}$ to "normalize" the correlation. The resulting expression for the correlation is,

$$E(A,\boldsymbol{a},B,\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b} = \cos(\omega) \tag{20}$$

Equation (20) is identical to the correlation given in (7), which was derived from quantum expectation values without any reference to hidden variables. Consequently, we see that it is the correlation of the internal angular momentum vector $\boldsymbol{\lambda}$, which is both theoretically and practically unobservable, with the measurement angles given by $\boldsymbol{a}$ and $\boldsymbol{b}$ that explains the cosine relationship we observe.

Note that Equations (15), for the probability densities, are functions of the directions $\boldsymbol{a}$ and $\boldsymbol{b}$ in which measurements are made. Correlation measurements that depend on the orientation of the measurement instruments are considered to open "freedom-of-choice" loopholes [7], [40] in experiments that provide verification of Bell's theorem, so it may appear that these densities exploit this loophole.

However, it is not the inclusion of these orientations in the density formulas that are the loopholes, but the dependence of experimentally observable quantities on these orientations. The density in Equation (15) is not observable, and the result that is observable is given by Equation (16), in which the density is integrated over values of the hidden variable. As Equation (20) shows, the resulting correlation depends only on the relative orientations. Consequently, Equation (20) is orientation-free and does not exploit this loophole.

Equation (20) refutes the common but mistaken assumption that the Bell and related theorems "prove" a hidden variable theory cannot generate the correlations that prevail in this quantum system. Such was never the case, but it became widely believed due to a general lack of understanding of the difference between quantum and classical hidden variable theories.

What we have shown is that quantum mechanics is a hidden variable theory of a nonclassical type in which spin plays the role of a hidden variable that is local in this case, and that no additional terms are needed to explain the correlation of originally entangled and subsequently separated states. However, this does not address the process of decoherence by which the entangled singlet state evolves into two pure states, which we assumed in deriving this result. Something in the dissociation process breaks the symmetry between the $|\Psi_{+-}\rangle$ and $|\Psi_{-+}\rangle$ states given in Equation (4).

It is interesting to point out that while we think of the z-component of the internal angular momentum as an observable quantity, it does not appear here, nor in any equation for any observable in any form. Saldanha [41] argues we cannot ascribe an objective reality to any quantum properties even after measurements have been performed.

The angle θ is a mnemonic that defines the direction of $\boldsymbol{\lambda}$. It is defined with respect to an unobservable, internal z-axis. This means the z-component, while a constant of the motion, is as unobservable as the x and y components. Or, to put it another way, the x and y components are as observable as the z component. We already knew this since we can equally well define the internal spin using a basis in which any one of the three components—x, y, or z—is an eigenvalue.

It is a mistake to think of the x, y, and z components of the internal spin as directional components. They correspond to the first, second, and third generators of rotations in the SU(2) group, but they are not defined with respect to any direction in SO(3) until we make a measurement. This is why we take the orienting vector $\boldsymbol{\lambda}$ to be uniformly distributed over the unit sphere in Equation (16). This is also why we refer to the process of measurement as an act of spontaneous symmetry creation.

There is no nonlocal effect between A and B. There is no collapse of an entangled wavefunction. There is a disentanglement of the wavefunction at the point where the particles separate, and there is another change in the wavefunction of each particle when we perform a measurement that forces the particle to align with or against the gradient of the applied magnetic field.

At the end of each particle's transit through a Stern-Gerlach measuring device, after it has interacted with the magnetic field, there is a change in the particle's wavefunction. The Stern-Gerlach device has been analyzed classically [42], semi-classically [43], and quantum mechanically [12] [44] [45]. The general conclusion is that dynamical forces lead to a continuous shift of the orientation of the particles' magnetic moments, and that this change is not a discontinuous collapse as is often supposed.

These "collapses" of the wavefunction are not flaws of quantum mechanics; they are a reflection of our lack of knowledge of the nonunitary processes that represent these changes. We simply don't have the right Hamiltonian to describe what we experimentally induce.

We can safely assume these changes to the wavefunctions do occur. We refer to them as "collapses" because we have no dynamical description of them, but it is likely that these events are described by a local dynamical process, as Buks [9] suggests. Whatever dynamics describe the process we're referring to as a collapse, we expect it will be locally causal.

18. Comparison with Bell

What we've done can be compared with Bell's original formulation [4]. Our equations (13) for $A(\boldsymbol{a}, \boldsymbol{\lambda})$ and $B(\boldsymbol{b}, \boldsymbol{\lambda})$ are identical to his equation #9. Our Equation (14) for $E(A, \boldsymbol{a}, B, \boldsymbol{b})$ as an integral over the hidden variable is almost identical to his equation #2, except that where he has $\rho(\boldsymbol{\lambda})$ as a uniform probability distribution over the unit sphere, we have the product $\rho_A(\boldsymbol{a},\boldsymbol{\lambda})\rho_B(\boldsymbol{b},\boldsymbol{\lambda})$.

Because his distribution $\rho(\boldsymbol{\lambda})$ is uniform, his result has no functional dependence on $\boldsymbol{\lambda}$. As a result, when he integrates the azimuthal angle from 0 to 2π and the polar angle 0 to θ, where θ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ giving the orientation of the two detectors, he's left with a correlation function linear in θ rather than one proportional to cosine θ. This does not form a basis for his eponymous inequality. It is just an example of a classical hidden variable that does not give the observed result. Bell goes on to apply further arguments to conclude that no local theory can reproduce the correlation of Equation (20).

Using the $\boldsymbol{\lambda}$-dependent densities $\rho(\boldsymbol{a}, \boldsymbol{\lambda})$ and $\rho(\boldsymbol{b}, \boldsymbol{\lambda})$ given in Equation (15) as $|\boldsymbol{a}\cdot\boldsymbol{\lambda}|$ and $|\boldsymbol{b}\cdot\boldsymbol{\lambda}|$, we obtain the correlation function of Equation (19), proportional to $\cos(\theta)$, which is consistent with observed results. It may be worth noting that this cosine relation is also the classical result when calculated as an ensemble average of a vector hidden variable model ([26], Equation 2).

Bell says that, "there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorentz invariant." The results shown here indicate that this conclusion is incorrect.

19. Dissociation and Setting Independence

Gu [22] considers the situation described here as one of three probability distributions resulting from factoring the probability of measurements at A and B into the product of two terms. These models are referred to as "setting-independent probabilities" because each depends only on values local to the observations at A or B.

In all of these cases, Gu is working from the entangled singlet wavefunction and suggests the paradox emerges from the distribution function that describes the measurement process. I infer this from his **Figure 3**, which shows $\boldsymbol{\lambda}$, the 3-vector hidden variable, embedded in the entangled wavefunction before the particles have separated.

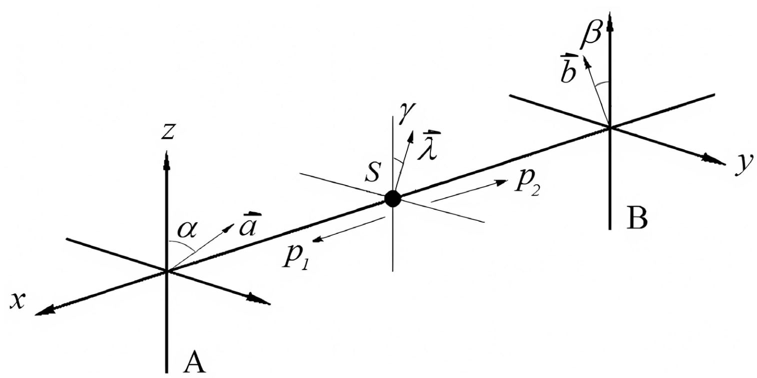

Figure 3. Coordinates of the ERPBA setup shown in **Figure 1** of Gu [22]. Here, the x, y, z coordinates have been cyclically permuted from his original diagram, and angles measured from the z-axis.

Only one of these distributions exactly matches observations carried out and advertised as verifying Bell's theorem and said to confirm nonlocality. Gu says this, "suggests a possibility that some probability might lead to a correlation closer to experimental outcome (sic) than quantum mechanics' prediction."

Figure 4 is a diagram of the situation considered here. The change appears incidental, and it leads to the same algebraic results, but the different meanings of these two figures are significant. **Figure 3** indicates a real vector variable $\boldsymbol{\lambda}$ that exists and is carried in the singlet state wavefunction when the total internal angular momentum is zero and before the particles have separated. **Figure 4** indicates that the variable $\boldsymbol{\lambda}$ exists after the particles have separated, and we emphasize that this variable is only a mnemonic for the spinor states of the correlated particles.

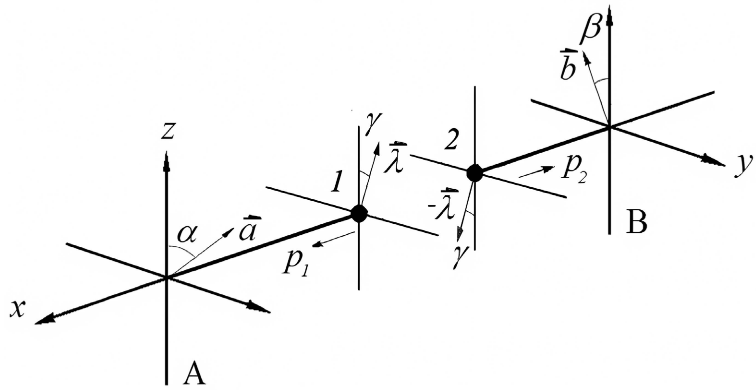

Figure 4. The correlated vector $\boldsymbol{\lambda}$ is an SO(3) representation of the SU(2) orientations of the pure state wavefunction that describes particles disentangled from the singlet state.

The observation made here is that what occurs in fact is not a choice of a probability distribution that needs further explication, but the direct and fully understood result of dissociation and concomitant decoherence of the 2-particle wavefunction. At the point the particles disassociate, their spatial wavefunctions cease to overlap, the symmetry of the combined particle wavefunction is broken, and the separate, correlated spinor wavefunctions diverge in space. At that point, the two-particle wavefunction is described by two correlated, pure, single-particle states.

In short, there is no entanglement in this situation. The "hidden variable" is the result of projecting the 2-dimensional spinor eigenfunction on the 3-dimensional experimental result. This projection, which is achieved by Equations (15) and (16), "completes" the quantum description for this experiment.

20. Conclusions

One might also say there is an incompleteness in the quantum mechanical treatment of this experiment due to our lack of a dynamical description of the decoherence process. Quantum mechanics does not tell us how wavefunctions evolve when their time evolution is not unitary. This is not a flaw in quantum mechanics; it reflects our inability to describe the process.

The linear Schrödinger equation does not define quantum mechanics; it only specifies the time evolution of a system governed by a linear Hamiltonian. Linear Hamiltonians do not describe the nonlinear process of the decay of a composite particle system. This is becoming more obvious as we gain a greater understanding of decoherence [46]-[48].

What is shown here is that while we do not know the dynamics by which the singlet state evolves into a disentangled wavefunction, the kinematics by which the SU(2) parameterization of spin maps on the SO(3) space of observation leads to a function that gives the physical observables presented in Equation (20). This is "an inevitable consequence of the geometry of configuration space" ([49]: p. 312).

We claim that there exists some means by which the pure-state wavefunction develops from the disintegration of the two-state system. The implication is that the two ways that disintegration can occur are distinguishable, and when distinguishability is possible—regardless of whether we can predict it, know it, or observe it—then decoherence always results.

We are not denying superposition, but merely repeating what we already know, which is that superposition requires indistinguishability. Decoherence occurs when there exists a mechanism to distinguish which state has developed. In this case, information contained within the interaction of the spins and the detectors local to them tells us the outcome dynamics. This exemplifies what Greenberger *et al.* ([50]: p. 26) emphasize when they state, "it is potential information, not actual information, that destroys coherence."

We join Mochizuki ([12]: p. 717) in disagreeing with David Bohm's statement ([51]: p. 622) that "quantum theory is inconsistent with the assumption of hidden causal variables" (See Aharonov and Gruss [52] for a different argument for locality based on decoherence). But we agree with Bohm's comment that "there are actually no precisely defined 'elements of reality' belonging to the electron" ([51]: p. 620).

In modern parlance, we say that while quantum mechanics is counterfactually indefinite (see Vaidman [53] for a definition), we can still integrate over counterfactually definite values—such as the nonexistent SO(3) components of spin—that do not survive in the final expressions for observables. We do this in the same way that the path integral includes counterfactually indefinite paths.

The entangled singlet state, which is used to represent the two-particle system before decoupling, does not develop as an entangled state after decoupling. It should instead be represented as a mixed state. Mixed states provide statistical statements that reflect our lack of knowledge.

In this case, the wavefunction is epistemic in that it reflects both the system and our lack of knowledge of it. The shortcoming of the entangled-state model is that it requires a nonlocal, acausal "collapse" of the state in order to extract the prediction of correlated spins. The mixed state model suffers no such shortcoming.

The mixed state, or nonentangled pure state pair, presents the wavefunction as one of two unknowable but "real" models of the particles. "Real" in the ontological sense that this wavefunction fully represents physical reality at some locally defined moments.

To this, we must "manually" add our ignorance regarding which of the two possibilities the bound state decomposes into. When this ignorance is resolved through observation, we learn which of the two possible forms describes the state after decomposition. In this case, what collapses is our ignorance, not the wavefunction.

Our conclusion is that there is no action-at-a-distance effect, no classical hidden variable theory, and no scientifically plausible nonlocal theory to be tested.

The presentation given here, built on several people's work, appears to resolve the ERPBA paradox. It apparently rescues us from nonlocality, frees us from the implications of acausality associated with this experiment, and lets Einstein, Rosen, Podolsky, and Bohm rest in peace. Yakir Aharonov, on the other hand, is still with us, and we hope he finds these results interesting.

Acknowledgements

I thank Herbert Bernstein of Hampshire College for bringing this problem to his attention in 1973. I posthumously thank Eugene P. Wigner for his continued encouragement. I greatly appreciate the 2025 comments from Eyal Buks of the Department of Electrical Engineering, Technion, Israel, and those of the anonymous reviewer, which improved this paper.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Bohm, D. and Aharonov, Y. (1957) Discussion of Experimental Proof for the Paradox of Einstein, Rosen, and Podolsky. *Physical Review*, **108**, 1070-1076. https://doi.org/10.1103/physrev.108.1070

[2] Einstein, A., Podolsky, B. and Rosen, N. (1935) Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? *Physical Review*, **47**, 777-780. https://doi.org/10.1103/physrev.47.777

[3] Greenberger, D.M., Horne, M.A. and Zeilinger, A. (1989) Going beyond Bell's Theorem. In: *Bell's Theorem*, *Quantum Theory and Conceptions of the Universe*, Springer Netherlands, 69-72. https://doi.org/10.1007/978-94-017-0849-4_10

[4] Bell, J.S. (1964) On the Einstein Podolsky Rosen Paradox. *Physics Physique Fizika*, **1**, 195-200. https://doi.org/10.1103/physicsphysiquefizika.1.195

[5] Piceno Martínez, A.E., Benítez Rodríguez, E., Mendoza Fierro, J.A., Méndez Otero, M.M. and Arévalo Aguilar, L.M. (2018) Quantum Nonlocality and Quantum Correlations in the Stern-Gerlach Experiment. *Entropy*, **20**, Article 299. https://doi.org/10.3390/e20040299

[6] Vaidman, L. (2019) Quantum Nonlocality. *Entropy*, **21**, Article 447. https://doi.org/10.3390/e21050447

[7] Shalm, L.K., Meyer-Scott, E., Christensen, B.G., Bierhorst, P., Wayne, M.A., Stevens, M.J., *et al.* (2015) Strong Loophole-Free Test of Local Realism. *Physical Review Letters*, **115**, Article ID: 250402. https://doi.org/10.1103/physrevlett.115.250402

[8] De Zela, F. (2024) Loophole-Free Bell Inequality Violations Cannot Disprove Local Realism. arXiv: 2403.03780.

[9] Buks, E. (2023) Spontaneous Collapse by Entanglement Suppression. *Advanced Quantum Technologies*, **6**, Article ID: 2300103.

[10] Blokhintsev, D.I. and Syrkin, M.I. (1976) Quantum Mechanics (Chapter 25). Reidel Publishing. https://vixra.org/pdf/2207.0018v1.pdf

[11] Westlund, P. and Wennerstrôm, H. (2019) Disentanglement of a Singlet Spin State in a Coincidence Stern-Gerlach Device. *Journal of Modern Physics*, **10**, 1247-1254. https://doi.org/10.4236/jmp.2019.1010083

[12] Potel, G., Barranco, F., Cruz-Barrios, S. and Gómez-Camacho, J. (2005) Quantum Mechanical Description of Stern-Gerlach Experiments. *Physical Review A*, **71**, Article ID: 052106. https://doi.org/10.1103/physreva.71.052106

[13] Mochizuki, R. (2021) Decoherence as an Inherent Characteristic of Quantum Mechanics. *Journal of Modern Physics*, **12**, 700-729. https://doi.org/10.4236/jmp.2021.125045

[14] Chen, G. (2020) Two False States: The Entangled State and the Qubit. https://www.researchgate.net/publication/347916163_Two_false_quantum_states_the_entangled_state_and_the_qubit

[15] Greenberger, D.M., Horne, M.A., Shimony, A. and Zeilinger, A. (1990) Bell's Theorem without Inequalities. *American Journal of Physics*, **58**, 1131-1143. https://doi.org/10.1119/1.16243

[16] Coddens, G. (2021) The Exact Theory of the Stern-Gerlach Experiment and Why It Does Not Imply That a Fermion Can Only Have Its Spin up or down. *Symmetry*, **13**, Article 134. https://doi.org/10.3390/sym13010134

[17] Clauser, J.F., Horne, M.A., Shimony, A. and Holt, R.A. (1969) Proposed Experiment to Test Local Hidden-Variable Theories. *Physical Review Letters*, **23**, 880-884. https://doi.org/10.1103/physrevlett.23.880

[18] Srivastava, A.K., Müller-Rigat, G., Lewenstein, M. and Rajchel-Mieldzioć, G. (2024) Introduction to Quantum Entanglement in Many-Body Systems. In: Aguado, R., Citro, R., Lewenstein, M. and Stern, M., Eds., *New Trends and Platforms for Quantum Technologies*, Springer, 225-285. https://doi.org/10.1007/978-3-031-55657-9_4

[19] Aspect, A. (1983) Experimental Tests of Bell's Inequalities in Atomic Physics. In: Lindgren, I., Rosén, A. and Svanberg, S., Eds., *Atomic Physics* 8, Springer, 103-128. https://doi.org/10.1007/978-1-4684-4550-3_8

[20] Hance, J.R. and Hossenfelder, S. (2022) Bell's Theorem Allows Local Theories of Quantum Mechanics. *Nature Physics*, **18**, 1382-1382. https://doi.org/10.1038/s41567-022-01831-5

[21] Van der Waerden, B.L. (1974) Group Theory and Quantum Mechanics. Springer. https://archive.org/details/grouptheoryquant0000waer/page/n5/mode/2up

[22] Gu, Y. (2020) A Counterexample to Bell's Theorem. https://www.researchgate.net/publication/345006335_A_counterexample_to_Bell's_Theorem

[23] Wiseman, H.M. (2006) From Einstein's Theorem to Bell's Theorem: A History of Quantum Non-Locality. *Contemporary Physics*, **47**, 79-88. https://doi.org/10.1080/00107510600581011

[24] Griffiths, R.B. (2020) Nonlocality Claims Are Inconsistent with Hilbert-Space Quantum Mechanics. *Physical Review A*, **101**, Article ID: 022117. https://doi.org/10.1103/physreva.101.022117

[25] Fritsche, L. and Haugk, M. (2009) Stochastic Foundation of Quantum Mechanics and the Origin of Particle Spin. arXiv: 0912.3442.

[26] Annila, A. and Wikström, M. (2024) Quantum Entanglement and Classical Correlation Have the Same Form. *The European Physical Journal Plus*, **139**, Article No. 560. https://doi.org/10.1140/epjp/s13360-024-05377-8

[27] Furry, W.H. (1936) Note on the Quantum-Mechanical Theory of Measurement. *Physical Review*, **49**, 393-399. https://doi.org/10.1103/physrev.49.393

[28] Furry, W.H. (1936) Remarks on Measurements in Quantum Theory. *Physical Review*, **49**, 476-476. https://doi.org/10.1103/physrev.49.476

[29] Matteucci, G. (2018) The Aharonov-Bohm Effect, Controversial Features of a Long-Standing Debate. *Revista de la Facultad de Ciencias*, **7**, 9-22. https://doi.org/10.15446/rev.fac.cienc.v7n1.68727

[30] Siegfried, T. (2019) Murray Gell-Mann's 'Totalitarian Principle' Is the Modern Version of Plato's Plenitude. Science News. https://www.sciencenews.org/blog/context/murray-gell-mann-totalitarian-principle-modern-plato-plenitude

[31] Khrennikov, A. (2009) Nonlocality as Well as Rejection of Realism Are Only Sufficient (but Non-Necessary!) Conditions for Violation of Bell's Inequality. *Information Sciences*, **179**, 492-504. https://doi.org/10.1016/j.ins.2008.08.021

[32] Thayer, D.R. and Jafari, F. (2015) Understanding the Spin Correlation of Singlet State Pair Particles. *International Journal of Advanced Research in Physical Science*, **2**, 18-26. https://www.arcjournals.org/pdfs/ijarps/v2-i2/2.pdf

[33] Boughn, S. (2017) Making Sense of Bell's Theorem and Quantum Nonlocality. *Foundations of Physics*, **47**, 640-657. https://doi.org/10.1007/s10701-017-0083-6

[34] Laudisa, F. (2018) Stop Making Sense of Bell's Theorem and Nonlocality? *European Journal for Philosophy of Science*, **8**, 293-306. https://doi.org/10.1007/s13194-017-0187-z

[35] Hess, K. (2021) What Do Bell-Tests Prove? A Detailed Critique of Clauser-Horne-Shimony-Holt Including Counterexamples. *Journal of Modern Physics*, **12**, 1219-1236. https://doi.org/10.4236/jmp.2021.129075

[36] Chen, G. (2019) Collapse of Bell's Theorem. *Journal of Modern Physics*, **10**, 1157-1165. https://doi.org/10.4236/jmp.2019.1010076

[37] Murray, A. (2020) Double Slits with Single Atoms. *Physics World*, **33**, 31-34. https://doi.org/10.1088/2058-7058/33/2/34

[38] Karchev, N. (2012) Path Integral Representation for Spin Systems. arXiv: 1211.4509. https://arxiv.org/pdf/1211.4509

[39] Gu, Y. (2022) Conceptual Problems in Bell's Inequality and Quantum Entanglement. *Journal of Applied Mathematics and Physics*, **10**, 2216-2231. https://doi.org/10.4236/jamp.2022.107152

[40] Scheidl, T., Ursin, R., Kofler, J., Ramelow, S., Ma, X., Herbst, T., *et al.* (2010) Violation of Local Realism with Freedom of Choice. *Proceedings of the National Academy of Sciences of the United States of America*, **107**, 19708-19713. https://doi.org/10.1073/pnas.1002780107

[41] Saldanha, P.L. (2020) Inconsistency of a Realistic Interpretation of Quantum Measurements: A Simple Example. *Brazilian Journal of Physics*, **50**, 438-441. https://doi.org/10.1007/s13538-020-00757-8

[42] BO, Z. (2023) A New Interpretation of the Stern-Gerlach Experiment by Classical Physics. *International Journal of Physics*, **11**, 236-241. https://doi.org/10.12691/ijp-11-5-2

[43] Wennerström, H. and Westlund, P. (2012) The Stern-Gerlach Experiment and the Effects of Spin Relaxation. *Physical Chemistry Chemical Physics*, **14**, 1677-1684. https://doi.org/10.1039/c2cp22173j

[44] Wennerström, H. and Westlund, P. (2017) A Quantum Description of the Stern-Gerlach Experiment. *Entropy*, **19**, Article 186. https://doi.org/10.3390/e19050186

[45] Fonseca-Romero, K.M. (2024) A Quantum Treatment of the Stern-Gerlach Experiment. arXiv: 2402.14930. https://arxiv.org/abs/2402.14930

[46] Zeh, H.D. (1996) What Is Achieved by Decoherence? arXiv: quant-ph/9610014.

https://arxiv.org/pdf/quant-ph/9610014

[47] Schlosshauer, M. (2019) Quantum Decoherence. *Physics Reports*, **831**, 1-57. https://doi.org/10.1016/j.physrep.2019.10.001

[48] Schlosshauer, M. and Fine, A. (2007) Decoherence and the Foundations of Quantum Mechanics. In: Schlosshauer, M. and Fine, A., Eds., *Quantum Mechanics at the Crossroads*, Springer, 125-148. https://doi.org/10.1007/978-3-540-32665-6_7

[49] Varadarajan, V.S. (2007) Geometry of Quantum Theory. 2nd Edition, Springer.

[50] Greenberger, D.M., Horne, M.A. and Zeilinger, A. (1993) Multiparticle Interferometry and the Superposition Principle. *Physics Today*, **46**, 22-29. https://doi.org/10.1063/1.881360

[51] Bohm, D. (1989) Quantum Theory. Dover.

[52] Aharonov, A. and Gruss, E. (2005) Two-Time Interpretation of Quantum Mechanics. arXiv: quant-ph/0507269. https://api.semanticscholar.org/CorpusID:117580755

[53] Vaidman, L. (2009) Counterfactuals in Quantum Mechanics. In: Greenberger, D., Hentschel, K. and Weinert, F., Eds., *Compendium of Quantum Physics*, Springer, 132-136. https://doi.org/10.1007/978-3-540-70626-7_40

[54] Gomis, P. and Pérez, A. (2016) Decoherence Effects in the Stern-Gerlach Experiment Using Matrix Wigner Functions. *Physical Review A*, **94**, Article ID: 012103. https://doi.org/10.1103/physreva.94.012103

[55] Larsson, J. (2014) Loopholes in Bell Inequality Tests of Local Realism. *Journal of Physics A*: *Mathematical and Theoretical*, **47**, Article ID: 424003. https://doi.org/10.1088/1751-8113/47/42/424003

[56] Fraser, P.T. and Sanders, B.C. (2018) Loophole-Free Bell Tests and the Falsification of Local Realism. *Journal of Student Science and Technology*, **10**, 23-31. https://doi.org/10.13034/jsst.v10i1.164

[57] Bell, J.S. and Aspect, A. (1977) Free Variables and Local Causality. In: Bell, J.S., Ed., *Speakable and Unspeakable in Quantum Mechanics*: *Collected Papers on Quantum Philosophy*, Cambridge University Press, 100-104.

[58] Wagenaar, W.A. (1972) Generation of Random Sequences by Human Subjects: A Critical Survey of Literature. *Psychological Bulletin*, **77**, 65-72. http://wexler.free.fr/library/files/wagenaar%20%281972%29%20generation%20of%20random%20sequences%20by%20human%20subjects.%20a%20critical%20survey%20of%20literature.pdf

[59] Weihs, G., Jennewein, T., Simon, C., Weinfurter, H. and Zeilinger, A. (1998) Violation of Bell's Inequality under Strict Einstein Locality Conditions. *Physical Review Letters*, **81**, 5039-5043. https://doi.org/10.1103/physrevlett.81.5039

[60] Kwiat, P.G., Mattle, K., Weinfurter, H., Zeilinger, A., Sergienko, A.V. and Shih, Y. (1995) New High-Intensity Source of Polarization-Entangled Photon Pairs. *Physical Review Letters*, **75**, 4337-4341. https://doi.org/10.1103/physrevlett.75.4337

[61] Ou, Z.Y. (2007) Multi-Photon Quantum Interference. Springer.

[62] Hensen, B., Bernien, H., Dréau, A.E., Reiserer, A., Kalb, N., Blok, M.S., *et al.* (2015) Loophole-Free Bell Inequality Violation Using Electron Spins Separated by 1.3 Kilometres. *Nature*, **526**, 682-686. https://doi.org/10.1038/nature15759

[63] Bernien, H., Hensen, B., Pfaff, W., Koolstra, G., Blok, M.S., Robledo, L., *et al.* (2013) Heralded Entanglement between Solid-State Qubits Separated by Three Metres. *Nature*, **497**, 86-90. https://doi.org/10.1038/nature12016

[64] Simon, C. and Irvine, W.T.M. (2003) Robust Long-Distance Entanglement and a

Loophole-Free Bell Test with Ions and Photons. *Physical Review Letters*, **91**, Article ID: 110405. https://doi.org/10.1103/physrevlett.91.110405

[65] Barrett, S.D. and Kok, P. (2005) Efficient High-Fidelity Quantum Computation Using Matter Qubits and Linear Optics. *Physical Review A*, **71**, Article ID: 060310. https://doi.org/10.1103/physreva.71.060310

[66] Ou, Z.Y., Hong, C.K. and Mandel, L. (1987) Relation between Input and Output States for a Beam Splitter. *Optics Communications*, **63**, 118-122. https://doi.org/10.1016/0030-4018(87)90271-9

[67] Barnett, S.M. (2018) Schrödinger Picture Analysis of the Beam Splitter: An Application of the Janszky Representation. *Journal of Russian Laser Research*, **39**, 340-348. https://doi.org/10.1007/s10946-018-9727-z

[68] Braunstein, S.L. and Mann, A. (1995) Measurement of the Bell Operator and Quantum Teleportation. *Physical Review A*, **51**, R1727-R1730. https://doi.org/10.1103/physreva.51.r1727

[69] Wang, K., Hou, Z., Qian, K., Chen, L., Krenn, M., Aspelmeyer, M., *et al*. (2025) Violation of Bell Inequality with Unentangled Photons. *Science Advances*, **11**, eadr1794. https://doi.org/10.1126/sciadv.adr1794

Appendix I. Disentanglement after Separation

The original Stern-Gerlach experiment was performed in a vacuum of around 10^{-5} mm Hg. Using the Ideal Gas Law lets us approximate the number of atoms per cubic millimeter as 3×10^8 atoms/mm^3. We expect 78% of these to be atoms of nitrogen, as is typical of the atmosphere. The collision cross section for nitrogen-to-nitrogen collisions is approximately 1.1×10^{-13} mm^2. We expect an electron-to-nitrogen collision cross section to be half that.

The original Stern-Gerlach device had a gap of 2.5 mm, a width of 1.4 mm, and a length of 350 mm. This corresponds to a volume of about 1.25 cm^3. We can model this as a stack of 350 slabs of atmosphere—each one millimeter thick with a surface area of 2.5 mm × 1.4 mm = 3.5 mm^2—through which the particle must pass while being influenced by the inhomogeneous magnetic field, and before it impacts on the observation screen.

We assume that none of these nitrogen atoms is hiding behind each other, which is reasonable at low densities, so that their collision profiles tile the surface of each of the 350 slabs of evacuated atmosphere (our final answer is independent of their number and width).

Using this non-overlapping approximation, the atoms obscure (3×10^8 obstructions/mm^2)*(1.1×10^{-13} mm^2/obstruction) = 3.3×10^{-5}, which is the ratio of obstructed to unobstructed area in each slab. In this simple approximation, a particle collides with an atom along its path whenever it passes through an obstructed area. Consequently, the ratio 3.3×10^{-5} of obstructed to unobstructed area is also the probability that the particle will encounter an obstruction within each slab. Multiplying this by the 350 slabs gives a final probability of 1.2×10^{-2} of encountering an obstruction in the complete course of transiting the Stern-Gerlach machine.

Stated another way, there is a 1% chance that one or the other of our two correlated particles will encounter an obstacle that may cause the correlation of its orientation, trajectory, and wavefunction to be disturbed.

The vacuum in a Modern Stern-Gerlach machine provides a pressure 1/10th of that of the original experiment. This would lower the number of obstructions/mm^3 to 3×10^7, which lowers each transiting particle's probability of collision to 0.1% per observation.

From this, we can further conclude that between 99% and 99.9% of the particles measured have not had their wavefunctions disturbed in transit. This very approximate conclusion coincides with a more detailed analysis given by Gomis and Pérez, which comes to the same conclusion, namely "that collisions with the air molecules are rare events" ([54]: p. 9).

From the low probability of decoherence in transit, we deduce that the entanglement of the singlet state was lost when the particles separated. Then, the correct wavefunction to use in describing the separated particles is either a pure-state state or a mixed state, depending on whether you want to represent the system itself or the best probabilistic description we have for it.

Appendix II. Loophole-Free Tests of Bell's Inequality

We feel that the Bell-CHSH inequalities have no bearing on the purely quantum theory presented here, but we understand that many readers may think that they do. In this section, we'll analyze a general feature of the loophole-free tests of the Bell-CHSH inequalities, and conclude that they support the pure state model we have proposed.

In order to have confidence in the conclusions drawn from the Bell-CHSH inequalities, violations of which are predicted by quantum mechanics and observed by experiment, it requires that there exist no additional experimental influences to prejudice or control the results. To exclude such sources, various sorts of experimental prejudice are categorized according to their type. These types are referred to as the locality, detection, and freedom of choice loopholes. For additional explanation, see Larsson [55] and Fraser and Sanders [56].

The locality loophole refers to an experimental condition where correlation between measuring devices is effected through some mechanical means external to the detectors and the presumably local systems being measured. Closing the locality loophole means eliminating any alternative communication that could influence the correlation of the devices. "Alternative" means some information channel connected to but not considered part of the systems being measured.

The detection loophole refers to experiments that are biased because they miss or preferentially select certain observations. Issues of correct sample identification, accuracy, filtering, interference, and timing are crucial in closing detection loopholes.

The freedom of choice loophole refers to the experimenters' freedom to independently choose the measurements being taken at the two locations. If the experimenters' choices are correlated through a common human attitude or situation, then the resulting correlation observed between the systems will no longer exclusively represent the systems or the Bell-CHSH inequalities.

We'd like to mention a larger aspect of the freedom-of-choice bias discussed by Scheidl *et al.* [40], and which Bell [57] alluded to when he said, "Here I would entertain the hypothesis that the experimenters have free will." Based on the evidence of 50 years of research in this topic—or actually any topic in which physicists have accepted the assertions of researchers who preceded them—experimenters do not have free will in the statistical sense.

There is a general belief in whatever is generally accepted. This prejudices current and future research projects and the conclusions drawn from them. Human research projects, like human subjects, are not independent: subjective chance is not equal to statistical chance [58].

Subjective prejudice is built into all community-based research, is amplified by peer review, and its effects are uncompensated or are overlooked. This is typical of all community-based belief systems and can be referred to as "the culture effect." One could close this loophole by removing or insisting on the detailed analysis of every assumption, but this is never possible in any absolute sense. The acceptance of assumptions being questioned in this case continues to have uncontrolled and misleading effects on the beliefs physicists have regarding entanglement, non-locality, and acausality.

All of these loopholes refer to the experiment, the experimental setup, or the experimenter. None of these loopholes refers to the quantum mechanical statement of the problem. While there are differences that depend on the mechanics, dynamics, and ingredients of each experiment, all efforts to close the loopholes accept the same presumptions of spinor behavior, quantum entanglement, statistics, and the Hilbert space representations used in quantum mechanics. These experiments do not dispute that the Bell-CHSH inequalities derive from the Bell-CHSH assumptions, nor do we.

The locality loophole is considered to have been closed by Weihs *et al.* [59], who examined pairs of polarized photons. The source of their polarization entangled photon pairs is degenerate type-II parametric down-conversion as described by Kwiat *et al.* [60] and by Ou ([61]: p. 34).

In their work, Kwiat *et al.* report, "Nearly all previous experiments employing photons from parametric down conversion have actually produced product states." In contrast, using their method of type-II phase matching, "down-converted photons are emitted into two cones, one ordinary polarized, the other extraordinary polarized," and "if the angle (at which they are emitted) is increased, the two cones tilt toward the pump, causing... the cones (to) overlap, (so that) the light can be essentially described by an entangled state: $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|H_1, V_2\rangle + e^{i\alpha}|V_1, H_2\rangle\right)$." From this, we see that what they're calling entanglement is created by the direct coincident placement of streams of photons or of a single photon.

It is unclear whether the state produced in this manner is actually entangled or mixed. Entanglement refers to the superposition of properties of a single particle, but photons are not localized particles. In addition, according to quantum field theory, there is a complementarity relation between phase and particle number, so that the more certain we are of the wavefunction's phase, the less certain we are of the particle number. How, then, do we know if physically overlapping beams of photons create entanglements or mixtures?

The distinguishing property of entanglement is self-interference, which means you have the ability to separate entangled components and cause them to interfere with each other. This is what we do in the Young double-slit experiment, where we split the wavefunction into two channels, which correspond to the two slits, and observe interference when they recombine.

You cannot create self-interference with a mixed state if the components are not coincident, though you can wait for them to arrive together or force them into coincidence. This is an important caveat we discuss below. The answer to the question of whether or not a wavefunction is entangled depends on whether self-interference is observed. However, once it's observed, it's destroyed.

This question is rarely addressed in either the theoretical or experimental papers that discuss the Bell-CHSH theorems, which simply assert entanglement either by assumption or construction. As Kwiat *et al.* note, if "the terms in $|\Psi\rangle$ become, in principle, distinguishable by the order in which the detectors would fire, no interference will be observable." But the question of whether or not the terms are distinguishable, why this might occur, and what it would imply are left unresolved or referred to the results of other research.

The work presented here deals with the states of massive, spin-1/2 particles, which raises the important question of how entangled states of such particles are created, and how this is verified under the conditions of loophole-free tests. This leads us to turn to Hensen *et al.* [62], who are credited with a loophole-free test using electron spins.

Hensen *et al.* employ the spin states associated with single Nitrogen vacancy defect centers in a pair of widely separated diamond chips. Their method is described in Bernien *et al.* [63]. The generation of entanglement between the two distant spins is performed according to the entanglement-swapping method introduced by Simon and Irvine [64] and Barrett and Kok [65].

We consult Simon and Irvine to address the question of whether entanglement-swapping of the photons ensures entangled state creation of the electrons and excludes the possibility of electron mixed states, as we are asserting is actually the case.

According to Simon and Irvine, the entanglement-swapping procedure begins with two ions in triplet or singlet states that are each excited and emit a photon in an entangled state that reflects their triplet or singlet structures, namely the maximally entangled Bell-state two-photon wavefunctions of the form:

$$\frac{1}{\sqrt{2}}\left[\left|s_{1A}\right\rangle a_{1A}^{\dagger}\otimes\left|s_{1B}\right\rangle a_{1B}^{\dagger}+\left|s_{2A}\right\rangle a_{2A}^{\dagger}\otimes\left|s_{2B}\right\rangle a_{2B}^{\dagger}\right]\left|0\right\rangle \quad \text{abbreviated } HH+VV \tag{AII.1}$$

$$\frac{1}{\sqrt{2}}\left[\left|s_{1A}\right\rangle a_{1A}^{\dagger}\otimes\left|s_{1B}\right\rangle a_{1B}^{\dagger}-\left|s_{2A}\right\rangle a_{2A}^{\dagger}\otimes\left|s_{2B}\right\rangle a_{2B}^{\dagger}\right]\left|0\right\rangle \quad HH-VV \tag{AII.2}$$

$$\frac{1}{\sqrt{2}}\left[\left|s_{1A}\right\rangle a_{1A}^{\dagger}\otimes\left|s_{2B}\right\rangle a_{2B}^{\dagger}+\left|s_{2A}\right\rangle b_{2A}^{\dagger}\otimes\left|s_{1B}\right\rangle b_{1B}^{\dagger}\right]\left|0\right\rangle \quad HV+VH \tag{AII.3}$$

$$\frac{1}{\sqrt{2}}\left[\left|s_{1A}\right\rangle a_{1A}^{\dagger}\otimes\left|s_{2B}\right\rangle a_{2B}^{\dagger}-\left|s_{2A}\right\rangle b_{2A}^{\dagger}\otimes\left|s_{1B}\right\rangle b_{1B}^{\dagger}\right]\left|0\right\rangle \quad HV-VH \tag{AII.4}$$

where $\left|s_{1A}\right\rangle$ is the up or "1" spin state (or if "2" then the down spin state) of the particle at site A (or at B), and $a_{1A}^{\dagger}$ and $a_{2B}^{\dagger}$ are creation operators that act on the vacuum state to create a photon polarized in state 1 or 2 emanating from site A or B. In the abbreviated notation, the letters H and V represent any two orthogonal photon polarization states, also given as 1 and 2, not being limited to H for horizontal and V for vertical.

The creation of the entangled state for the massive particles is described [64] as starting with the "emission of a photon by each ion (which) leads to the state (given by Equations (AII.1) through (AII.4)) so that each ion is maximally entangled with the photon it has emitted. The photons from A and B propagate to some intermediate location where a partial Bell-state analysis is performed."

The Bell-state analysis referred to is similar to the experimental procedure that records the Hong-Ou-Mandel photon interference effect [66] in which two photons simultaneously impinge on either side of a 50:50 beam splitter. Each photon is split into a reflected part that undergoes a phase shift and a transmitted part that does not.

If the state of a photon having some polarization and entering the α input channel (or the β channel) is represented by a 2-vector $|\phi\rangle_{\alpha(\text{or }\beta)}$, with components ϕ_1 and ϕ_2, then the transmitted and reflected components can be calculated using a unitary 2 × 2 transmission matrix T. See Barnett [67] for a review. T operates on a state entering one channel to produce a state that exits through two channels:

$$\begin{aligned} T|\phi\rangle_\alpha &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} \phi_{1\alpha} \\ \phi_{2\alpha} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_{1\alpha} + i\phi_{2\alpha} \\ i\phi_{1\alpha} + \phi_{2\alpha} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_c \\ \psi_d \end{pmatrix} = |\psi_\alpha\rangle_c + |\psi_\alpha\rangle_d \\ T|\phi\rangle_\beta &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} \phi_{1\beta} \\ \phi_{2\beta} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_{1\beta} + i\phi_{2\beta} \\ i\phi_{1\beta} + \phi_{2\beta} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_c \\ \psi_d \end{pmatrix} = |\psi_\beta\rangle_c + |\psi_\beta\rangle \end{aligned} \tag{AII.5}$$

The states of the two photons are taken from Equations (AII.1) to (AII.4), where $\frac{1}{\sqrt{2}}\left(a_{1A}^\dagger\right)|0\rangle$ has been replaced with $\phi_{1\alpha}$ and $\frac{1}{\sqrt{2}}\left(a_{1B}^\dagger\right)|0\rangle$ with $\phi_{1\beta}$ to indicate a photon in polarization state 1 (or 2) that originated from vacancy A entering beam-splitter channel α, or from B entering beam-splitter channel β. $|\psi_\alpha\rangle_c$ and $|\psi_\alpha\rangle_d$ are the wavefunctions exiting through the beam-splitter's output channels c and d that result from the wavefunction that was input through channel α, and similarly for channel β.

When two photons simultaneously enter each of the beam-splitter's two input channels, then the wavefunction in each of the output channels is the sum of the contributions from each input channel. That is:

$$T|\phi\rangle_\alpha + T|\phi\rangle_\beta = \left(|\psi_\alpha\rangle_c + |\psi_\alpha\rangle_d\right) + \left(|\psi_\beta\rangle_c + |\psi_\beta\rangle_d\right) = |\psi\rangle_c + |\psi\rangle_d$$

As a result, each of the two output channels carries a superposition of photons from the two input channels, with the polarizations of the reflected photons shifted. In Simon and Irvine's construction, each of the two output channels then impinges on a separate polarizing beam splitter. Each polarizing beam splitter then creates its own two secondary output channels. Each of these four tertiary output channels is then directed to one of four separate photodetectors, as is shown in **Figure A1**.

They then say that **Figure A1**,

> "shows the well-known method for detecting the two states $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(a_1^\dagger b_2^\dagger \pm a_2^\dagger b_1^\dagger\right)|0\rangle$ using only linear optical elements. Detection of the two photons in the state $|\psi_\pm\rangle$ will project the two distant ions into the corresponding $|\psi_\pm\rangle$ state, $\frac{1}{\sqrt{2}}\left(|s_1\rangle_A|s_2\rangle_B \pm |s_2\rangle_A|s_1\rangle_B\right)$."

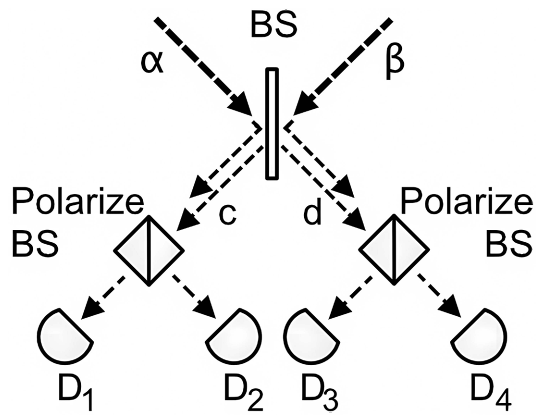

Figure A1. Detectors are used to determine the entangled components of the two incoming particles.

That is, a measurement that confirms one of these entangled photon states "collapses," and in that manner selects a corresponding element of the superposed state at the remote Nitrogen vacancy sites. The problem, of course, is that we cannot measure the state of the electrons without destroying their state, so we have to measure the states of the photons with which they're correlated.

Simon and Irvine comment [64],

> "Only if the two photons are in the antisymmetric Bell state $|\psi_-\rangle$, (given by HV-VH of Equation (AII.4)), there will be one photon in each output mode of the first beam splitter (BS), c and d. Therefore, a coincidence detection between D1 and D3 or D2 and D4 identifies this state. This was first pointed out by Braunstein and Mann [68]. If the photons are in the state $|\psi_+\rangle$, they both go into c or both into d, and are then split by the subsequent polarizing beam splitters (PBS), because they have orthogonal polarizations. Therefore, coincidences between D1 and D2 or between D3 and D4 signify a state $|\psi_+\rangle$. For the two other Bell states $|\varphi_+\rangle$ and $|\varphi_-\rangle$ both photons go to the same detector."

Expressing the entangled antisymmetric state $|\psi_-\rangle$ in terms of the two photons which separately enter each of the input channels α or β, and then applying the beam-splitter transformation given in Equation (AII.5), we get the output state $|\Psi_{\text{output}}\rangle$:

$$\begin{aligned}|\Psi_{\text{entangled output}}\rangle &= T|\Psi_{\text{entangled input}}\rangle = T(HV - VH)\\ &= T(H)_\alpha \otimes T(V)_\beta - T(V)_\alpha \otimes T(H)_\beta\\ &= \frac{1}{2}\left[(H_d + iH_c)(V_c + iV_d) - (V_d + iV_c)(H_c + iH_d)\right]\end{aligned}$$

Recalling that H and V represent photon creation operators $a^\dagger_{(1,2)}$ in states 1 or 2, and applying $\left[a_i^\dagger, a_j^\dagger\right] = 0$ for all i and j, allows us to reorder the H and V terms. The signs of the terms and the factors of i represent phase changes, and equal terms of opposite sign cancel as they represent the combination of photons of opposite phase. The expression for the output wavefunction reduces to:

$$\left|\Psi_{\text{entangled output}}\right\rangle = H_d V_c - H_c V_d$$

This represents the superposition state in which there is one photon of each polarization in each of the two output channels c and d, confirming what Simon and Irvine assert.

Now consider the mixed state that we claim represents the disintegrated EPRBA state. If we assume the same physics applies to the Nitrogen vacancy construction, then the photons emitted from the locations A and B are not in an entangled but in a pure state of HV or VH. These states have a different normalization from the entangled states in Equations (AII.3) and (AII.4):

$$HV = \left|s_{1A}\right\rangle a_{1A}^{\dagger} \otimes \left|s_{2B}\right\rangle a_{2B}^{\dagger}\left|0\right\rangle$$

$$VH = \left|s_{2A}\right\rangle b_{2A}^{\dagger} \otimes \left|s_{1B}\right\rangle b_{1B}^{\dagger}\left|0\right\rangle$$

Because we don't know which of these two states has prevailed, we create a mixed state that combines the two. However, this is not a superposed state as it's not the amplitudes we're combining but the probabilities. That is, there can be no interference between the two components because they are "ontologically" exclusive.

We can write the mixed state in the form "$HV + VH$" only with the caveat that this is not an entangled state and must be treated accordingly. When this state impinges on a beam splitter, we must treat it as if the two states, HV and VH, are not superposed but independent. We are then combining the probabilities of two separate and distinguishable inputs. The effect of the beam-splitter on the HV state can be written as:

$$\begin{aligned}\left|VH_{\text{mixed output}}\right\rangle &= T\left|VH_{\text{mixed input}}\right\rangle \\ &= T(H)_{\alpha} T(V)_{\beta} \\ &= \frac{1}{2}\left(H_d + iH_c\right)\left(V_c + iV_d\right) \\ &= \frac{1}{2}\left[H_d V_c - H_c V_d + i\left(H_d V_d + iH_c V_c\right)\right]\end{aligned}$$

Each of these terms represents a separate, orthogonal, and distinguishable outcome, the probability of which is the absolute square of the amplitude of the corresponding terms. This leads to a 25% probability for each of the incoming photon combinations exiting each of the indicated output channels.

But the HV state only has a 50% probability of occurring. To this we add, not superimpose but include as an exclusive alternative, the mixed state contribution given by $\left|VH\right\rangle$, namely, photons of opposite phase entering into each of the channels of the beam splitter.

$$\begin{aligned}\left|VH_{\text{mixed output}}\right\rangle &= T\left|VH_{\text{mixed input}}\right\rangle \\ &= \frac{1}{2}\left[V_d H_c - V_c H_d + i\left(V_d H_d + iV_c H_c\right)\right]\end{aligned}$$

This alternative results in different polarizations in the respective output channels. Since there is a 50% chance of realizing either the HV or VH states we're left with probabilities of 1/8 for each of the polarization pairs in each of the two channels. The probabilities combine to indicate a 50% chance of photons appearing in the same output channel, both in channel c or channel d, and a 50% chance they'll appear in different output channels. Incidentally, this is what is predicted by classical wave theory.

This method of distinguishing a superposed state from a mixed state presents the following problems:

1) The assumption that a collapse of the photon state generates a collapse of the electron state is the assumption we are trying to test. This is known as the "begging the question" fallacy that occurs when your premise presupposes the truth of your conclusion.

2) The method is not designed to distinguish superpositions from mixtures.

3) The result does not distinguish the Bell-CHSH entangled states from mixed states since both can result in photons exiting the beam-splitter through separate channels. If you experimentally select those cases where photons exit through separate channels, as this method suggests, then this method alone does not distinguish an entangled state from a pure state mixture.

According to the above process involving pairs of photons produced by parametric down-conversion, used by many Bell loophole-excluding experiments, the measurement of the correlated photons does not distinguish interfering entangled states from non-interfering pure or mixed states. That is, correlated photons that exit separate channels of the beam-splitter are predicted 50% of the time for pure states, and 25% of the time from anti-symmetric entangled states.

In fact, it is only the observation of bunching (both photons exiting randomly from one or the other of the two channels) or anti-bunching (photons only exiting the two separate channels but never the same channel) that distinguishes entanglement from the classical prediction. In selecting only those experimental runs in which both photons exit separate channels, we're not even distinguishing quantum from classical mechanics! Of course, this is not a test to distinguish quantum from classical mechanics. It should only be noted that these tests only allow us to draw limited conclusions.

Selecting only states that correlate simultaneous input photons that exit from different output channels provides no distinction between pure or entangled states. We claim that none of these photon observations confirm the nonlocal and acausal collapse postulate put forward to explain Bohm's version of the EPR paradox. Such nonphysical explanations can always be put forward, of course, but they are of a religious rather than scientific nature (**Figure A2**).

Barrett and Kok [65] confirm this is the only method they are using to confirm the creation of massive, entangled particle states. They say,

> "If zero or two photo-detection events are observed on either round of the procedure, the scheme failed, and the qubits must be newly prepared before

re-attempting the entangling procedure. On the other hand, if one (and only one) photo-detection event is observed on each round of the protocol, the scheme has succeeded."

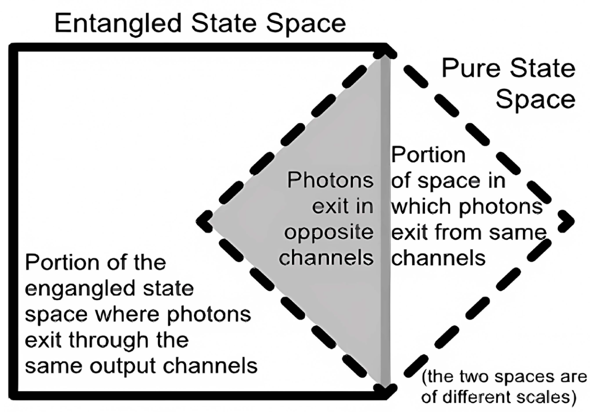

Figure A2. A Venn diagram showing 25% of Bell-entangled states and 50% of our similar pure states have photons exiting the beam-splitter through opposite channels. Loophole-free tests of the Bell-CHSH theorems examine only those states in the shaded intersection representing states that exit the beam-splitter through the two, distinguishable output channels.

Their reference to "one and only one" photon in one output arm of the beam splitter refers to the simultaneous detection of photons in detectors D_1 and D_2, or D_3 and D_4. They say, but they do not mean, that only one photon exists in the beam-splitter. Two photons will always exit when two are entering, and two must be entering if there is to be a measure of correlation between two systems.

Where Barrett and Kok claim the proper observation of these output states confirms entanglement, we say it is just as likely that they confirm a pure state. Wang *et al.* [69] also dispute that these violations of the Bell-CHSH limits are due to entanglement and, instead, suggest they are due to indistinguishability of photon pure states. These arguments show that the above state selection process, typical of methods for preparing entangled states, produces disputed and perhaps misleading results.

At this point, the reader may believe we are pointing out a selection loophole in the tests of the Bell-CHSH theorems, but we are not. The Bell-CHSH theorems do not apply to the present theory that asserts quantum locality because the present theory is not "Einstein real." As such, it is not subject to the assumptions implicit in these theorems.

In short, once you accept that the nature of quantum spin violates Einstein reality—which is proven by the GHZ results—Bell-CHSH theorems no longer have anything relevant to say about theories based on quantum spin, and ours is such a theory. From this, we can conclude that the myriad efforts at closing Bell-CHSH loopholes have no bearing on the correct, quantum, local, causal, and Einsteinian unreal theory proposed here.