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A Finite Failure Rate

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Abstract

We study a stochastic truncated inverse power-law waiting-time distribution, originally motivated by bounded note durations in musical structure. The model generates waiting times $\tau \in [T_{\min}, T_{\max}]$ from a uniform random variable and produces a normalized inverse power-law probability density with exponent $\mu > 1$. For this bounded renewal process, we derive expressions for the Cumulative Distribution Function, the Probability Density Function, the Survival Probability, and the corresponding Failure Rate (hazard function) in the sense of Cox. The failure rate is finite at $\tau = T_{\min}$ but diverges as $\tau \to T_{\max}$, mathematically enforcing the certainty that an event occurs before the maximal time scale. We also obtain the first moment explicitly, including the $\mu \to 2$ limit via L'Hôpital's rule, and emphasize that all moments are finite due to truncation. These results clarify the structure of bounded heavy-tailed renewal statistics and provide a framework for applications in complex systems where hard lower and upper time-scale constraints are present.

Keywords

Statistical Distribution, Truncations, Inverse Power Law, Stochastic, Waiting Time, Failure Rate, Renewal Events

1. Introduction

Inverse power-law waiting-time distributions commonly appear in complex systems. They characterize phenomena that range from anomalous diffusion and renewal processes to cognitive dynamics and biological rhythms. In many physical and biological systems, the probability that an event occurs after a waiting time, τ, follows a heavy-tailed distribution of the form $\tau^{-\mu}$, with $\mu > 1$.

Pure power-law distributions, however, are idealizations. In real systems, waiting times cannot be arbitrarily small or arbitrarily large. Physical constraints impose a minimum time scale $T_{\min}$ below which events cannot occur and a maximum time scale $T_{\max}$ by which the event must occur. These truncations fundamentally alter the statistical and dynamical properties of the process, rendering all statistical moments finite and introducing a natural termination of survival.

This work is devoted to a state of the art stochastic truncated inverse power-law waiting-time distribution which was originally motivated by musical structure, where notes must exist between minimal and maximal durations to form coherent compositions. Despite this origin, the distribution represents a general renewal process with bounds and an inverse power-law structure. Its mathematical properties extend far beyond musical applications.

In this work, we examine the structure of this truncated distribution. In particular, we derive the Cumulative Distribution Function, the Probability Density Function, the Survival Probability, the Failure Rate, and the first & n^{th} moment. A central result is that the failure rate remains finite at early times but diverges as $\tau \to T_{\max}$, enforcing the certainty of failure before the upper bound. This finite-divergent structure distinguishes the truncated inverse power-law from both exponential (memory-less) and unbounded heavy-tailed processes.

In Section 3, we discuss other inverse power-law distributions as well as a distribution that includes an exponential decay. All of these distributions differ from the Adams-Grigolini distribution in the key sense that the Adams-Grigolini distribution is the only distribution with a hard lower and hard upper bound.

By analyzing this distribution through the lens of renewal theory and failure rates, we show that bounded heavy-tailed processes possess a mathematically compelling and physically interpretable structure.

2. The Adams-Grigolini Distribution

We construct a stochastic truncated inverse power-law waiting-time distribution generated by a uniform stochastic variable [1].

This stochastic truncated inverse power-law waiting-time distribution (Adams-Grigolini Distribution) gives rise to a waiting time, τ (2), that varies between a minimum constant value, $T_{\min}$, and a maximum constant value, $T_{\max}$, due to the variance of a uniform stochastic variable, y (1), while under the influence of power μ.

The associated waiting time, τ (2), was constructed by Paolo Grigolini and David Adams [2] to reflect the idea that musical notes must exist between a minimal and a maximum duration in order to comprise a musical piece. To create a stochastic distribution of notes that the brain will perceive in a musical score, this distribution was introduced.

Although this distribution was introduced for musical purposes, it has vast implications; we shall discuss further in future sections.

2.1. Stochastic Variable

The stochastic variable, y (1), gives rise to a random number which is equally probable between and including 0 & 1. There are an infinite number of options between 0 and 1 to choose from.

$$y \in [0,1] \tag{1}$$

2.2. Waiting Time

The time it takes for an event to occur, τ (2), is determined by the following function that depends on the stochastic variable, y (1) [2]. This intuitive expression generates values from a minimum limit to a maximum limit with an inverse power-law distribution that corresponds to the value μ.

$$\tau = \left[T_{\max}^{1-\mu} - y\left(T_{\max}^{1-\mu} - T_{\min}^{1-\mu}\right)\right]^{\frac{1}{1-\mu}},\ T_{\min} > 0,\ T_{\max} < \infty,\ \mu > 1 \tag{2}$$

Note that the limits of y give rise to $T_{\min}$ & $T_{\max}$.

$$\begin{cases} y = 0: & \tau = \left[T_{\max}^{1-\mu}\right]^{\frac{1}{1-\mu}} = T_{\max} \\ y = 1: & \tau = \left[T_{\min}^{1-\mu}\right]^{\frac{1}{1-\mu}} = T_{\min} \end{cases}$$

Although the generation of y is uniform [$y \sim \mathcal{U}[0,1]$], the resultant values for τ follow an inverse power-law μ.

2.3. Cumulative Distribution Function (CDF)

The Cumulative Distribution Function (CDF) of the stochastic variable, y (1), gives the probability that an event **has** occurred by time τ (2). Next, we solve for y (1), in τ (2) to find the Cumulative Distribution Function (3). Consequently, we will see that the Cumulative Distribution Function is the complement of the Survival Probability. They both sum up to 1.

$$y = \frac{\tau^{1-\mu} - T_{\max}^{1-\mu}}{T_{\min}^{1-\mu} - T_{\max}^{1-\mu}},\ T_{\min} \le \tau \le T_{\max} \tag{3}$$

2.4. Probability Density Function (PDF)

Looking at (2), we solve for (3) as a function of the variable τ. Then, we use the following relation:

$$f(y)\mathrm{d}y = f(\tau)\mathrm{d}\tau \Rightarrow f(\tau) = f(y)\left|\frac{\mathrm{d}y}{\mathrm{d}\tau}\right| = \left|\frac{\mathrm{d}y}{\mathrm{d}\tau}\right|,\ \text{since } f(y) = 1. \tag{4}$$

Taking the absolute value of $\left|\frac{\mathrm{d}y}{\mathrm{d}\tau}\right|$ yields

$$\left|\frac{\mathrm{d}y}{\mathrm{d}\tau}\right| = \frac{\mu - 1}{T_{\min}^{1-\mu} - T_{\max}^{1-\mu}} \cdot \frac{1}{\tau^{\mu}} = \psi(\tau).$$

This normalized Probability Density Function (which is already normalized in all-space from $T_{\min}$ to $T_{\max}$), $\psi(\tau)$ (5), is represented below.

$$\psi(\tau) = \frac{\mu - 1}{T_{\min}^{1-\mu} - T_{\max}^{1-\mu}} \cdot \frac{1}{\tau^{\mu}},\ T_{\min} \le \tau \le T_{\max} \tag{5}$$

2.5. Survival Probability (SP)

The Survival Probability (SP), $\Psi(\tau)$ (7), gives the probability that an event **has not** occurred by time τ (2).

The survival probability $\Psi(\tau)$ (7) is defined as:

$$\Psi(\tau) = 1 - \int_{T_{\min}}^{\tau} \psi(t)\,\mathrm{d}t,\ T_{\min} \le \tau \le T_{\max}. \tag{6}$$

Substitute the expression for $\psi(t)$, which we get from (5), into (6):

Thus, the survival probability is as follows.

$$\Psi(\tau) = 1 + \frac{\tau^{1-\mu} - T_{\min}^{1-\mu}}{T_{\min}^{1-\mu} - T_{\max}^{1-\mu}} \tag{7}$$

2.6. Failure Rate $g(\tau)$

The Failure Rate $g(\tau)$, introduced by D. R. Cox [3], is defined as the ratio of the negative time derivative of the SP, $-\dot{\Psi}(\tau)$, to the SP, $\Psi(\tau)$, itself. It answers the question: If we have survived up to time τ, how likely are we to fail in the next instant?

$$g(\tau) = -\frac{\mathrm{d}\Psi(\tau)/\mathrm{d}\tau}{\Psi(\tau)} = \frac{\psi(\tau)}{\Psi(\tau)},\ T_{\min} \le \tau \le T_{\max} \tag{8}$$

Now, we substitute in $\psi(\tau)$ and $\Psi(\tau)$, which are (5) and (7):

$$g(\tau) = \frac{\dfrac{\mu-1}{T_{\min}^{1-\mu} - T_{\max}^{1-\mu}}\tau^{-\mu}}{1 + \dfrac{\tau^{1-\mu} - T_{\min}^{1-\mu}}{T_{\min}^{1-\mu} - T_{\max}^{1-\mu}}} = \frac{(\mu-1)\tau^{-\mu}}{T_{\min}^{1-\mu} - T_{\max}^{1-\mu} + \tau^{1-\mu} - T_{\min}^{1-\mu}} = \frac{(\mu-1)\tau^{-\mu}}{\tau^{1-\mu} - T_{\max}^{1-\mu}} \tag{9}$$

Thus, the failure rate can be compactly written as

$$g(\tau) = \frac{\mu-1}{\tau^{\mu}\left(\tau^{1-\mu} - T_{\max}^{1-\mu}\right)},\ T_{\min} \le \tau \le T_{\max},\ \mu > 1 \tag{10}$$

It is clear to see that when $\tau = T_{\min}$, we have a constant value for $g(\tau)$. Alternatively, when $\tau = T_{\max}$, $g(\tau)$ goes to infinity. This mathematical structure stresses that an event **must** occur before time, $T_{\max}$.

2.7. Derivation of the First Moment $\langle\tau\rangle$

The first moment of the Adams-Grigolini distribution is given by

$$\langle\tau\rangle = \int_{T_{\min}}^{T_{\max}} \tau\psi(\tau)\,\mathrm{d}\tau, \tag{11}$$

where the PDF, $\psi(\tau)$, is given by (5).

We get:

$$\langle\tau\rangle = \frac{\mu-1}{T_{\min}^{1-\mu} - T_{\max}^{1-\mu}}\int_{T_{\min}}^{T_{\max}} \tau^{1-\mu}\,\mathrm{d}\tau.$$

Evaluating the integral for $\mu \ne 2$:

$$\int_{T_{\min}}^{T_{\max}} \tau^{1-\mu} \mathrm{d}\tau = \frac{T_{\max}^{2-\mu} - T_{\min}^{2-\mu}}{2-\mu}.$$

Thus,

$$\langle \tau \rangle = \frac{\mu-1}{2-\mu} \cdot \frac{T_{\max}^{2-\mu} - T_{\min}^{2-\mu}}{T_{\min}^{1-\mu} - T_{\max}^{1-\mu}}, \quad \mu > 1, \quad \mu \neq 2. \tag{12}$$

When $\mu \to 2$, the expression becomes the indeterminate form 0/0. To resolve this, we apply the L'Hôpital rule by differentiating the numerator and the denominator with respect to μ.

Differentiating the numerator and denominator with respect to μ:

Numerator:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(\mu-1)\left(T_{\max}^{2-\mu} - T_{\min}^{2-\mu}\right)\right] \\ &= \left(T_{\max}^{2-\mu} - T_{\min}^{2-\mu}\right) - (\mu-1)\left[\ln T_{\max} \cdot T_{\max}^{2-\mu} - \ln T_{\min} \cdot T_{\min}^{2-\mu}\right] \end{aligned} \tag{13}$$

Evaluate at $\mu = 2$:

$$T_{\max}^{0} - T_{\min}^{0} = 1 - 1 = 0, \quad \mu - 1 = 1$$

Thus:

$$\text{Numerator} \to -\left(\ln T_{\max} - \ln T_{\min}\right) = \ln\left(\frac{T_{\min}}{T_{\max}}\right) \tag{14}$$

Denominator:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(2-\mu)\left(T_{\min}^{1-\mu} - T_{\max}^{1-\mu}\right)\right] \\ &= -\left(T_{\min}^{1-\mu} - T_{\max}^{1-\mu}\right) + (2-\mu)\left[\ln T_{\min} \cdot T_{\min}^{1-\mu} - \ln T_{\max} \cdot T_{\max}^{1-\mu}\right] \end{aligned} \tag{15}$$

Evaluate at $\mu = 2$:

$$T_{\min}^{-1} - T_{\max}^{-1} = \frac{1}{T_{\min}} - \frac{1}{T_{\max}}, \quad 2 - \mu = 0$$

Thus:

$$\text{Denominator} \to -\left(\frac{1}{T_{\min}} - \frac{1}{T_{\max}}\right) = \frac{1}{T_{\max}} - \frac{1}{T_{\min}} \tag{16}$$

Thus, applying the L'Hôpital Rule:

$$\lim_{\mu \to 2} \langle \tau \rangle = \frac{\ln\left(\frac{T_{\min}}{T_{\max}}\right)}{\frac{1}{T_{\max}} - \frac{1}{T_{\min}}}$$

$$\lim_{\mu \to 2} \langle \tau \rangle = \frac{T_{\min} T_{\max}}{T_{\max} - T_{\min}} \ln\left(\frac{T_{\max}}{T_{\min}}\right) \tag{17}$$

It is important to note that all moments, $\langle \tau^n \rangle$, for $n \in \mathbb{Z}^+$, are finite.

3. Other Representations

The Adams-Grigolini Distribution is a distribution that truncates the "All-Space" from the entire positive real line to a finite region (segment on the real line). An example of another inverse power-law waiting-time distribution is the Probability Density Function derived from the Pomeau-Manneville map [4]. The non-linear equation of motion of the Pomeau-Manneville map gives rise to the following Waiting-Time and Probability Density Function with inverse power-law, μ [5].

$$\tau = T\left[\frac{1}{y^{1/(\mu-1)}} - 1\right] \tag{18}$$

$$\psi(\tau) = \frac{(\mu-1)T^{\mu-1}}{(T+\tau)^{\mu}} \tag{19}$$

Fractional Calculus also provides an approach to defining an inverse power-law waiting-time distribution through the Mittag-Leffler approach [6] [7].

Mittag-Leffler (which is similar to the Taylor Series of an exponential) is defined as:

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \tag{20}$$

where,

$$z = -\left(\frac{\tau}{T}\right)^{\alpha} \tag{21}$$

and,

$$\alpha = \mu - 1. \tag{22}$$

We first look at the Survival Probability,

$$\Psi(\tau) = E_{\alpha}\left(-(\lambda\tau)^{\alpha}\right) \tag{23}$$

and we find the following formula:

$$\Psi(\tau) = \sum_{n=0}^{\infty} \frac{(-1)^n (\tau/T)^{\alpha n}}{\Gamma(\alpha n + 1)}. \tag{24}$$

Next, we see the relation that the Probability Density Function has with the Survival Probability:

$$\psi(\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau}\Psi(\tau) \tag{25}$$

We evaluate and find:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Psi(\tau) = \sum_{n=1}^{\infty} \frac{(-1)^n}{T} \frac{(\tau/T)^{\alpha n-1}}{\Gamma(\alpha n)}. \tag{26}$$

Now we take the negative to solve for the Probability Density Function; we find:

$$\psi(\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau}\Psi(\tau) = \frac{1}{T}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(\tau/T)^{\alpha n-1}}{\Gamma(\alpha n)} \tag{27}$$

where, $\alpha = \mu - 1$, and μ represents the inverse power-law.

There also exists an example of a Probability Density Function that is not a pure power-law due to an exponential truncation [8] [9]. This clever approach ensures that the Waiting Times may not be ∞. However, this differs from a truncated distribution with a finite upper bound; here, the truncation is smooth. (28) shows the said Probability Density Function.

$$\psi(\tau) = \frac{\tau^{-1-\alpha}\mathrm{e}^{-\rho\tau}}{\rho^{\alpha}\Gamma(-\alpha,\rho\tau_0)},\ \tau \ge \tau_0 > 0,\ 0 < \alpha < 1,\ \rho > 0 \tag{28}$$

As mentioned, (28) is obtained by tempering a pure power-law with an exponential factor,

$$\psi(\tau) \propto \tau^{-1-\alpha}\mathrm{e}^{-\rho\tau}. \tag{29}$$

For short times ($\tau \ll 1/\rho$), the exponential term is approximately unity and the distribution behaves as a power law. For long times ($\tau \gg 1/\rho$), the exponential term dominates and suppresses the tail, effectively truncating the distribution.

Unlike a hard cutoff at a finite $T_{\max}$, this tempering introduces a smooth decay; which ensures that all moments remain finite while retaining fractional characteristics at intermediate times just as the Adams-Grigolini distribution does. The main differences are the renewal properties and the hard-cutoff of the Adams-Grigolini distribution.

4. Is the Adams-Grigolini Distribution Renewal?

A stochastic process is said to be a renewal process if the waiting times between consecutive events are independent and identically distributed random variables [3]. In such a process, after each event occurs, the system "renews" itself and the statistical properties governing the next waiting time are identical to those governing all previous waiting times.

The Adams-Grigolini distribution defines a waiting-time probability density function, $\psi(\tau)$ (5), on the bounded interval $[T_{\min}, T_{\max}]$, generated by an independent realization of a uniform stochastic variable, $y \in [0,1]$ (1), for each event. Because each realization of, y (1), is statistically independent and because the same functional mapping

$$\tau = \left[T_{\max}^{1-\mu} - y\left(T_{\max}^{1-\mu} - T_{\min}^{1-\mu}\right)\right]^{\frac{1}{1-\mu}}$$

is applied identically for each event, the resulting sequence of waiting times

$$\{\tau_1, \tau_2, \tau_3, \cdots\}$$

constitutes a sequence of independent and identically distributed random variables.

Since the Adams-Grigolini hazard is time-dependent, the resulting renewal process exhibits statistical aging and is not Markov in continuous time.

The Adams-Grigolini failure rate

$$g(\tau)=\frac{\psi(\tau)}{\Psi(\tau)}$$

is time-dependent and diverges as $\tau \to T_{max}$. The process therefore retains statistical aging in the sense that the conditional probability of failure depends explicitly on the elapsed time since the last event.

Nevertheless, the absence of inter-event correlations ensures that once an event occurs, the statistical clock resets. Hence, the Adams-Grigolini distribution defines a bounded, heavy-tailed renewal process with finite moments and a divergent terminal hazard.

5. Numerical Simulations

The Adams-Grigolini Waiting-Time, (2), follows an inverse power-law that should be verifiable by a log-log histogram. Figure 1 represents four time series developed using (2) with four different options for the inverse power-law, μ. The number of events chosen were one hundred thousand from $T_{min}=1$ to $T_{max}=10000$.

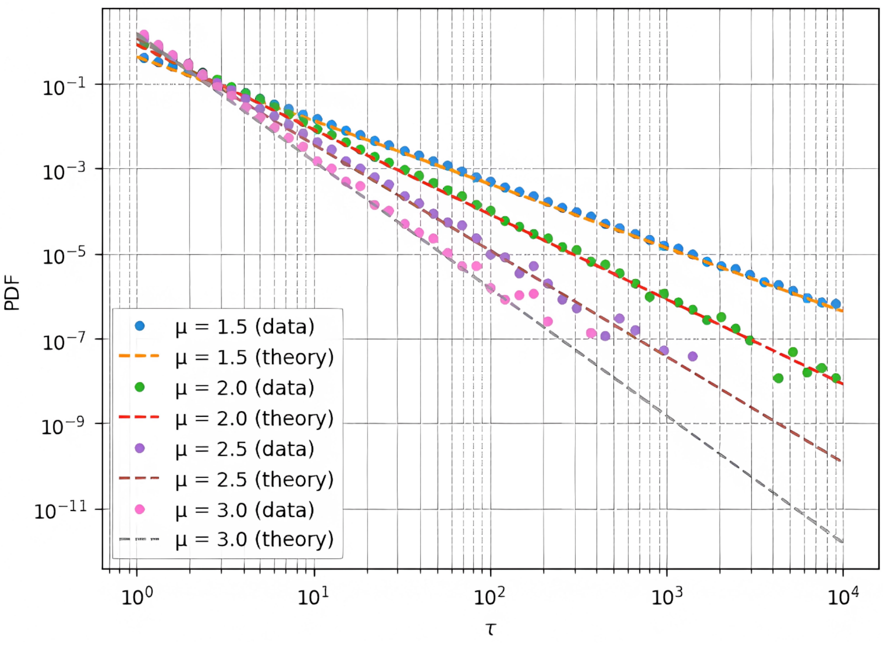

Figure 1. Histogram of Four Adams-Grigolini Time Series with Different Values for μ.

It is clear to see that the actual times produced by the stochastic variable, y (1), follow the inverse power-law, μ.

Alternatively, we can generate new time series with the same values for μ, the same number of events, and the same values for T_{min} and T_{max}; subsequently, we can evaluate the Survival Probability. The slope of the Survival Probability goes to $-(\mu-1)$. Figure 2 shows the Survival Probability for those same values of μ.

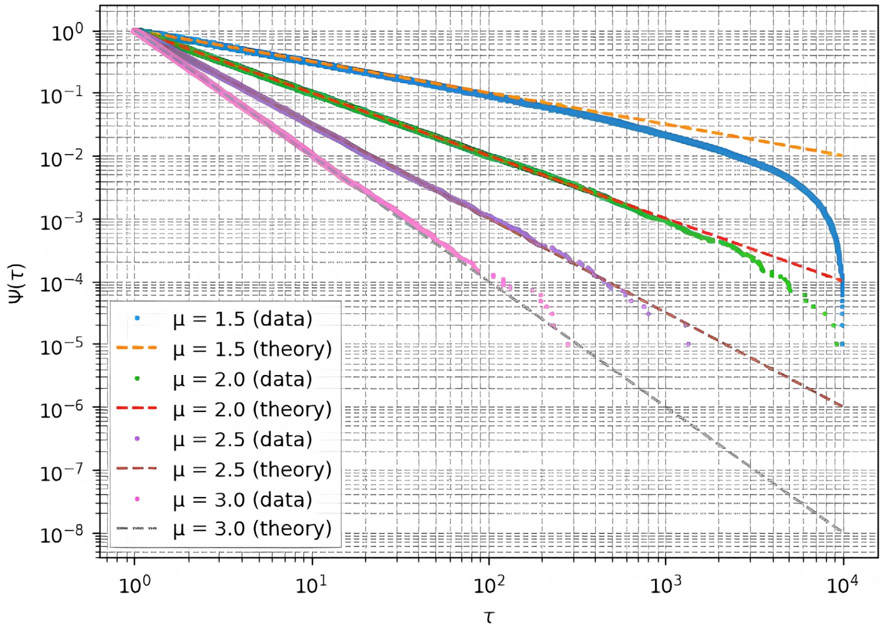

Figure 2. Survival Probability of four Adams-Grigolini time series with different values for μ.

We also show the Failure Rate of the Adams-Grigolini distribution in Figure 3. We have chosen $T_{\min}=1$, and $T_{\max}=100$.

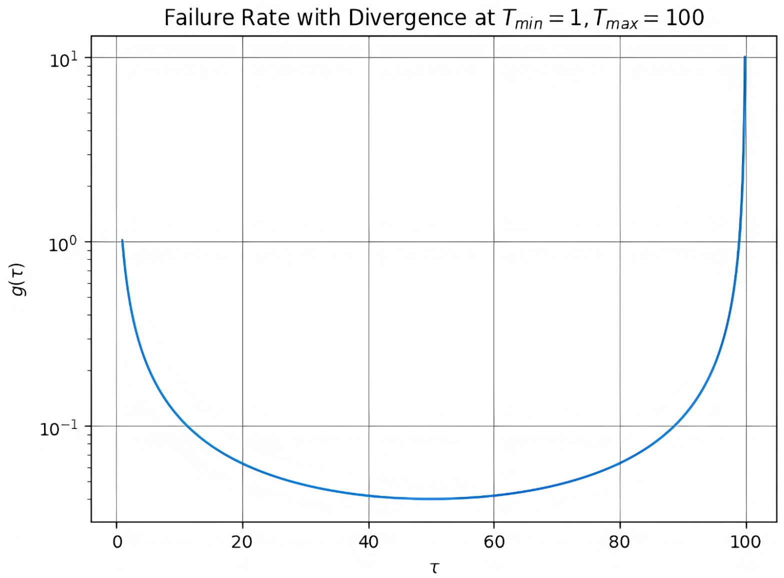

Figure 3. Adams-Grigolini failure rate with $T_{\min}=1$, and $T_{\max}=100$.

We see an initial value that depends on the value of $T_{\min}$. We also see a divergence at $T_{\max}$, which guarantees that the Failure Rate stays within the upper bound of $T_{\max}$.

6. Application

The main application of the Adams-Grigolini Distribution is to model turbulent systems that follow an inverse power-law of waiting times between turbulent events. These events may not be longer than the observable time in the model/experiment/environment by default, because $T_{\max}$ dictates the hard upper bound of Waiting Times and $T_{\max}$ should not be larger than the entire observable time. These times may also not be "zero", because $T_{\min}$ must be larger than 0 and provides a hard lower bound. Muir *et al.* [10] used the same Probability Density Function, $\psi(\tau)$ (5), to model the distribution of the probabilities of times between earthquakes in a system of several-faults. That work was the first published instance of the Probability Density Function, $\psi(\tau)$ (5). However, the Adams-Grigolini Distribution can also contribute to modeling the waiting times between any renewal events. This includes turbulent time distances between renewal events in truncated systems such as heartbeats, earthquakes, car accidents, musical notes, and even time distances between insurance claims.

The equation representing the distribution of times, τ (2), must have three parameters filled in before choosing a value for the stochastic variable, y (1). $T_{\min}$, $T_{\max}$, & μ all need to be represented by a constant value. As stressed in (2), μ must be greater than 1. $T_{\min}$ represents the concept of the minimal time distance allowed between renewal events. Some systems such as heartbeats may not beat faster than the time it takes for the heart to make a full beat. This conceptual limit allows for a conceptual constant such as $T_{\min}$. $T_{\max}$ has similar conceptual implications. Our sun is a main sequence star with a lifespan of around 10 billion years. It does not make sense for a $T_{\max}$ to exceed times that match the lifespan of a planet that revolves around that star. Systems such as heartbeats also have a conceptual $T_{\max}$ due to the fact that an excessive amount of time between heartbeats renders the system "dead", and these times should not be included.

Real experiments, especially numerical ones due to computational limitations, exist in a finite time range. If one is to represent the Probability Density Function of turbulent events or any other statistical property of their experiment that involves hard cutoffs, one may use the Adams-Grigolini distribution.

7. Conclusions

We have analyzed a stochastic truncated inverse power-law waiting-time distribution generated by a uniform random variable and bounded by hard limits $T_{\min}$ and $T_{\max}$. The resulting Adams-Grigolini Distribution yields a normalized Probability Density Function $\psi(\tau)$ on $[T_{\min}, T_{\max}]$, together with closed-form expressions for the Cumulative Distribution Function, Survival Probability, and the Cox failure rate (hazard function). The important consequence of truncation is that the distribution is *finite in every physically meaningful sense*: the waiting time is guaranteed to lie between two fixed time scales, and therefore all moments $\langle \tau^n \rangle$ exist and are finite.

These boundaries produce a distinctive hazard profile. Unlike exponential waiting times, whose constant hazard reflects memorylessness, the Adams-Grigolini hazard $g(\tau) = \psi(\tau)/\Psi(\tau)$ is time-dependent and displays a finite value at the earliest admissible time $\tau = T_{\min}$ while diverging as $\tau \to T_{\max}$. The divergence is not a pathology; it is the precise mathematical encoding of a hard upper constraint: conditioned on survival up to a late age, failure becomes certain before the maximal time scale. In this way the model reconciles heavy-tailed behavior over intermediate scales with the requirement that events cannot take place too soon or too late.

The Adams-Grigolini distribution is most effective when hard cutoffs exist in contrast to soft cutoffs.

Finally, because successive waiting times are generated independently and identically by repeated draws of the underlying uniform variable, the model defines a renewal process by construction. The finite nature of the Adams-Grigolini Distribution ensures that the renewal statistics remain well-behaved and interpretable across applications. Consequently, the Adams-Grigolini distribution provides a physically motivated framework for bounded renewal events in complex systems, including biological rhythms, seismicity, cognition, and other processes where both a minimum resolvable time scale and a maximum admissible waiting time are necessary.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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