

Vortical and Self-Similar Flows of 2D Inviscid Rotating Shallow Water System

Pengfei Zhou

School of Mathematics and Statistics, Shandong Normal University, Jinan, China

Email: 2024212039@stu.sdnu.edu.cn

How to cite this paper: Zhou, P.F. (2026) Vortical and Self-Similar Flows of 2D Inviscid Rotating Shallow Water System. *Journal of Applied Mathematics and Physics*, **14**, 2192-2200.
<https://doi.org/10.4236/jamp.2026.146107>

Received: May 23, 2026

Accepted: June 15, 2026

Published: June 18, 2026

Copyright © 2026 by author(s) and Scientific Research Publishing Inc. This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

Rotating shallow water system is an important fluid model that describes the fluid behavior of large scale geophysical fluid motion under the action of Coriolis force. In this paper, we construct a kind of vortical and self-similar solution for 2D inviscid rotating shallow water system and describe its time periodic and blow up phenomena via providing different parameter values.

Keywords

Rotating Shallow Water System, Self-Similar Flows, Time Periodic, Blow up

1. Introduction

The two-dimensional inviscid rotating shallow water system

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} = 0, \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial h}{\partial x} - v = 0, \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial h}{\partial y} + u = 0, \end{cases} \quad (1)$$

can be used to describe the fluid behavior of large scale geophysical fluid motion under the action of Coriolis force [1]-[3]. In the system (1), h denotes the height of the fluid, u and v are the velocity in x and y directions, respectively.

As a typical representative of quasi-linear hyperbolic system, (1) usually formates singularity in finite time for a large class of C^1 initial data, see [4] for example. Some studies have shown that rotation has a stabilizing effect on the lifespan of classical solutions. [5] obtained critical threshold for the initial data of a two dimensional pressureless model with rotating force and proved global existence

for subcritical initial data. The results in [6] indicated that when the pressure is present and dominated by rotating force, the lifespan of classical solutions can be extended. Basing on an important feature of rotating shallow water system—the relative vorticity vanishes all the time if it vanishes initially—[7] provided the global existence and asymptotic behavior of classical solutions for two-dimensional inviscid system with small initial data and zero relative vorticity. When the cylindrical symmetric solution has a form of variable separation, the global existence for the classical solution was considered in [4]. We also refer to [8] [9] for the other results on rotating shallow water equations.

It is interesting to study some special solutions with vorticity. Inspired by the method in [10], we construct a kind of vortical and self-similar flows of two-dimensional rotating shallow water system in this paper. Our main results are:

Theorem 1. For the 2D rotating shallow water system (1), there exist a family of vortical flows solution

$$\begin{cases} h(t, x, y) = \frac{\alpha}{a^2(t)} - \frac{\lambda(x^2 + y^2)}{2a^4(t)}, \\ u(t, x, y) = \frac{a'(t)}{a(t)}x - \left(\frac{\xi}{a^2(t)} - \frac{1}{2}\right)y, \\ v(t, x, y) = \frac{a'(t)}{a(t)}y + \left(\frac{\xi}{a^2(t)} - \frac{1}{2}\right)x. \end{cases} \quad (2)$$

Here $\lambda \leq 0$, $\alpha \geq 0$ and $\xi \in R$ are arbitrary constants, while $a(t)$ satisfies

$$\begin{cases} a''(t) - \frac{\xi^2 + \lambda}{a^3(t)} + \frac{a'(t)}{4} = 0, \\ a(0) = a_0 > 0, \\ a'(0) = a_1, \end{cases} \quad (3)$$

for some constants a_0 and a_1 , and the maximal interval of (2) is determined by the parameter values in (3).

Remark. It is clear that (2) are vortical because

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{2\xi}{a^2(t)} - 1 \neq 0.$$

Theorem 2. For the 2D rotating shallow water system (1), we have:

1) When $\xi^2 + \lambda > 0$, the solution (2) exists globally in time. Moreover,

(i) (2) is time-periodic except for $a_0 = \left[4(\xi^2 + \lambda)\right]^{\frac{1}{4}}$ and $a_1 = 0$;

(ii) if $a_0 = \left[4(\xi^2 + \lambda)\right]^{\frac{1}{4}}$, $a_1 = 0$, (2) is steady.

2) When $\xi^2 + \lambda \leq 0$, the solution (2) blows up in finite time, *i.e.* there exists a finite time T^* , such that (h, u, v) are all unbounded as $t \rightarrow T^*$.

2. The Construction of Vortical and Self-Similar Flow

At the beginning of this section, we propose a function structure which satisfies

mass conservation Equation (1)₁. The proof can be seen in [11].

Lemma 3. The mass conservation Equation (1)₁ has a classical solution:

$$\begin{cases} h(t, x, y) = \frac{f(s)}{a^2(t)}, \\ u(t, x, y) = \frac{a'(t)}{a(t)}x - \frac{G(t, r)}{r}y, \\ v(t, x, y) = \frac{a'(t)}{a(t)}y + \frac{G(t, r)}{r}x, \end{cases} \quad (4)$$

with $r = \sqrt{x^2 + y^2}$, $s = \frac{r}{a(t)}$, $a(t) \in C^2(R^+)$, $f \in C^1(R^+)$,

$G(t, r) \in C^1(R^+ \times R^+)$ and $f \geq 0$, $a(t) > 0$.

Proof. The function structure

$$\begin{cases} h(t, x, y) = h(t, r), \\ u(t, x, y) = \frac{F(t, r)}{r}x - \frac{G(t, r)}{r}y, \\ v(t, x, y) = \frac{F(t, r)}{r}y + \frac{G(t, r)}{r}x, \end{cases} \quad (5)$$

with arbitrary C^1 functions $F(t, r)$ and $G(t, r)$, can be substituted into the Equation (1)₁ to verify the result:

$$\begin{aligned} & h_t + (hu)_x + (hv)_y \\ &= h_t + h_x \frac{Fx}{r} + F_x \frac{hx}{r} + \frac{hF}{r} + hF_x \left(\frac{1}{r}\right)_x \\ & \quad - h_x \frac{Gy}{r} - G_x \frac{hy}{r} - hG_y \left(\frac{1}{r}\right)_x + h_y \frac{Fy}{r} + F_y \frac{hy}{r} \\ & \quad + \frac{hF}{r} + hF_y \left(\frac{1}{r}\right)_y + h_y \frac{Gx}{r} + G_y \frac{hx}{r} + hG_x \left(\frac{1}{r}\right)_y \\ &= h_t + h_r F \frac{x^2}{r^2} + hF_r \frac{x^2}{r^2} + \frac{hF}{r} - hF \frac{x^2}{r^3} - h_r G \frac{xy}{r^2} \\ & \quad - hG_r \frac{xy}{r^2} + hG \frac{xy}{r^3} + h_r F \frac{y^2}{r^2} + hF_r \frac{y^2}{r^2} + \frac{hF}{r} \\ & \quad - hF \frac{y^2}{r^3} + h_r G \frac{xy}{r^2} + hG_r \frac{xy}{r^2} - hG \frac{xy}{r^3} \\ &= h_t + h_r F + hF_r + \frac{hF}{r}. \end{aligned} \quad (6)$$

Then if the self-similar structure is taken for the height function,

$$h(t, x, y) = h(t, r) = \frac{f(s)}{a(t)^2}, \quad (7)$$

we can see that (6) is zero when $F(t, r) = \frac{a'(t)}{a(t)}r$:

$$\begin{aligned}
 & h_t + (hu)_x + (hv)_y \\
 &= h_t + h_r F + hF_r + \frac{hF}{r} \\
 &= -\frac{a'(t)rf'(s)}{a(t)^4} - \frac{2a'(t)f(s)}{a(t)^3} + \frac{a'(t)rf'(s)}{a(t)^4} \\
 &\quad + \frac{a'(t)f(s)}{a(t)^3} + \frac{a'(t)f(s)}{a(t)^3} \\
 &= 0.
 \end{aligned} \tag{8}$$

The proof is completed. □

Based on the function structure (4), we have

Proof. (Proof of Theorem 1.) Let $G(t, r) = \left(\frac{\xi}{a^2(t)} - \frac{1}{2}\right)r$ and substitute (4)

into (1)₂ to get

$$u_t + uu_x + vu_y - v + h_x = \left(\frac{a''(t)}{a(t)} - \frac{\xi^2}{a^4(t)} + \frac{1}{4} + \frac{2f'(s)}{a^4(t)}\right)x. \tag{9}$$

If we assume $a(t)$ satisfies (3), then the right hand side of (9) turns into $(\lambda + 2f'(s))\frac{x}{a^4(t)}$. Then the momentum is conserved with $f(s) = \alpha - \frac{\lambda}{2}s$.

Similarly, for the Equation (1)₃, we have

$$v_t + uv_x + vv_y + u + h_y = (\lambda + 2f'(s))\frac{y}{a^4(t)} = 0 \tag{10}$$

Besides, when $\lambda \leq 0$ and $\alpha \geq 0$, the height function is non-negative for any time $t > 0$. □

3. Time Periodic and Blow up Phenomena

In this section, we discuss the time periodic and blow up phenomena in the vortical solution (2). As the foundation, we give the following Lemma.

Lemma 4. For the IVP

$$\begin{cases} a''(t) - \frac{k}{a^3(t)} + \frac{a(t)}{4} = 0, \\ a(0) = a_0 > 0, \\ a'(0) = a_1, \end{cases} \tag{11}$$

1) When $k > 0$,

(i) (11) possesses a time-periodic solution, except for $a_0 = (4k)^{\frac{1}{4}}$, $a_1 = 0$;

(ii) if $a_0 = (4k)^{\frac{1}{4}}$, $a_1 = 0$, there exists a constant solution $a(t) \equiv a_0$.

2) When $k \leq 0$, there exists a finite time T such that $\lim_{t \rightarrow T^-} a(t) = 0$.

Proof. The proof is inspired by the Lemma 3 and Lemma 4 in [11].

1. For $k > 0$, the classical energy method can be applied to the autonomous ODE (11).

Define the total energy function: $E(t) = \frac{a'^2(t)}{2} + \frac{k}{2a^2(t)} + \frac{a^2(t)}{8}$, where the kinetic energy is $F_{kin} = \frac{a'^2(t)}{2}$, and the potential energy is $F_{pot} = \frac{k}{2a^2(t)} + \frac{a^2(t)}{8}$.

Figure 1 is the graph of potential energy function for a special case $k = 1$.

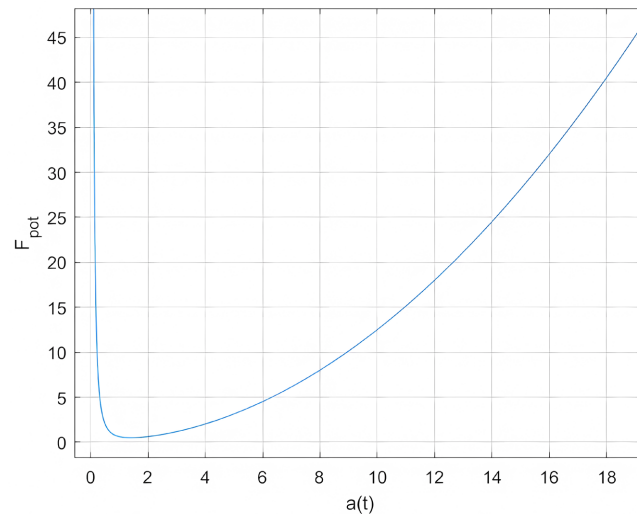


Figure 1. The potential energy function with $k = 1$.

A straight calculation shows that $E(t) = E(0)$ for all $t \geq 0$. Since the potential energy function has only one global minimum at $\bar{a} = (4k)^{\frac{1}{4}}$ and

$\lim_{a(t) \rightarrow 0} F_{pot}(a(t)) = \lim_{a(t) \rightarrow +\infty} F_{pot}(a(t)) = +\infty$, the solution of (11) has a closed trajectory for any $k > 0$, and we can prove the time for traveling the closed orbit is finite using the method in [11]. **Figure 2** shows the phase plane of (11) with some different $k > 0$ and $a_0 = a_1 = 1$.

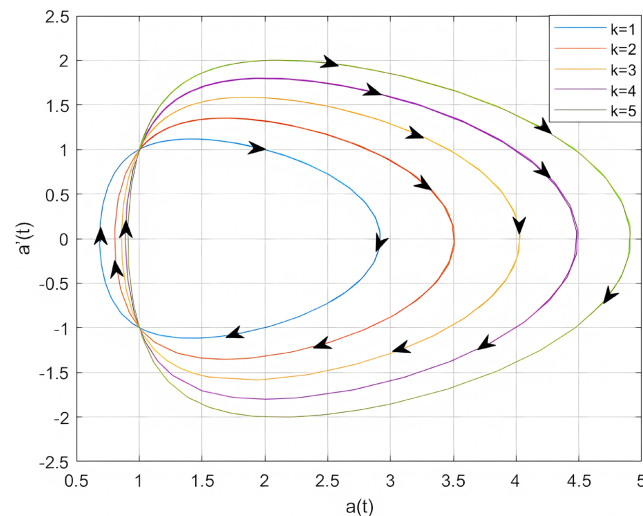


Figure 2. The phase plane for Equation (11) with some $k > 0$ and $a_0 = a_1 = 1$.

And the graph of solution function $a(t)$ is shown in **Figure 3**.

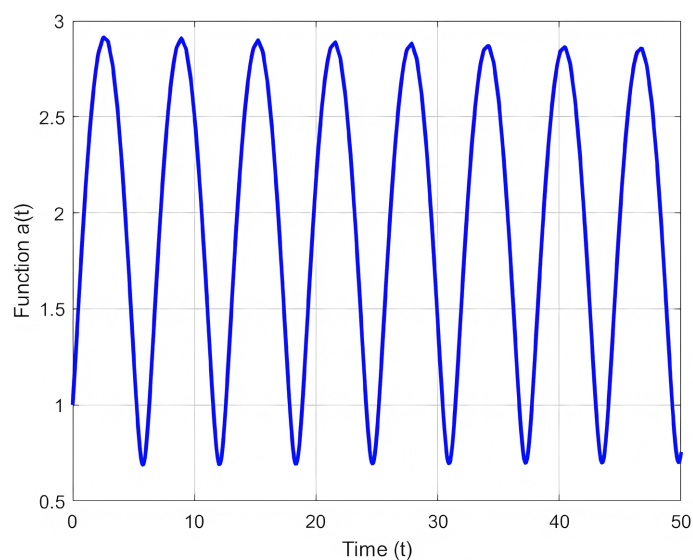


Figure 3. Time-periodic solution $a(t)$ for Equation (11).

Specially, if $a_0 = (4k)^{\frac{1}{4}}$ and $a_1 = 0$, the solution of (11) is steady, as shown in **Figure 4**.

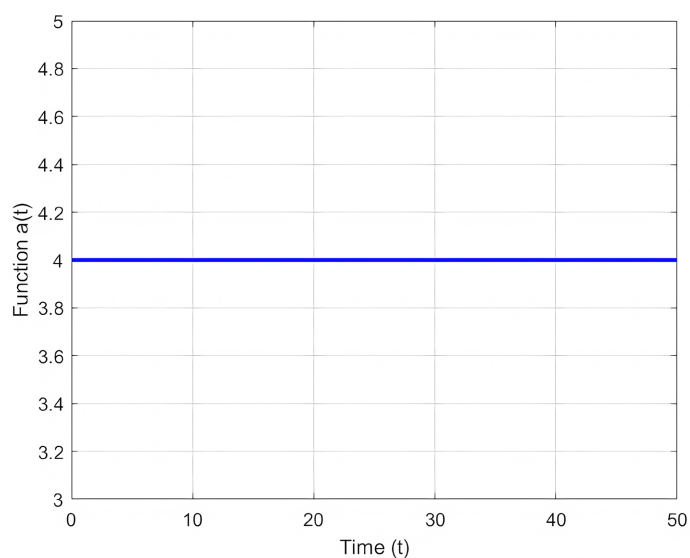


Figure 4. Steady solution $a(t)$ for Equation (11).

2. For $k = 0$, we calculate the exact solution:

$$a(t) = a_0 \cos \frac{t}{2} + 2a_1 \sin \frac{t}{2}. \quad (12)$$

Take $T = 2\left(\pi - \arctan \frac{a_0}{2a_1}\right)$, $T = \pi$, and $T = -2\arctan \frac{a_0}{2a_1}$ for $a_1 > 0$,

$a_1 = 0$ and $a_1 < 0$, respectively. Then we have $\lim_{t \rightarrow T^-} a(t) = 0$ for any $t \in [0, T)$.

3. For $k < 0$, we consider the problem in two cases:

(i) $a_1 \leq 0$. If the claim is not true, then we have $a(t) > 0$ for all $t \geq 0$, and

$$a''(t) = \frac{k}{a^3(t)} - \frac{a(t)}{4} < 0, \tag{13}$$

which shows that $a'(t)$ is a decreasing function, and $a'(t) \leq a'(t_1) < 0$ for all $t \geq t_1 > 0$. Thus, the solution is bounded by

$$a(t) = \int_{t_1}^t a'(s) ds + a(t_1) \leq a'(t_1)(t - t_1) + a_0. \tag{14}$$

It means that $a(t) < 0$ for all $t > t_1 - \frac{a_0}{a'(t_1)}$. A contradiction is met.

Figure 5 shows the phase plane of (11) with $a_0 > 0$, $a_1 \leq 0$ and different $k < 0$ as example of this case.

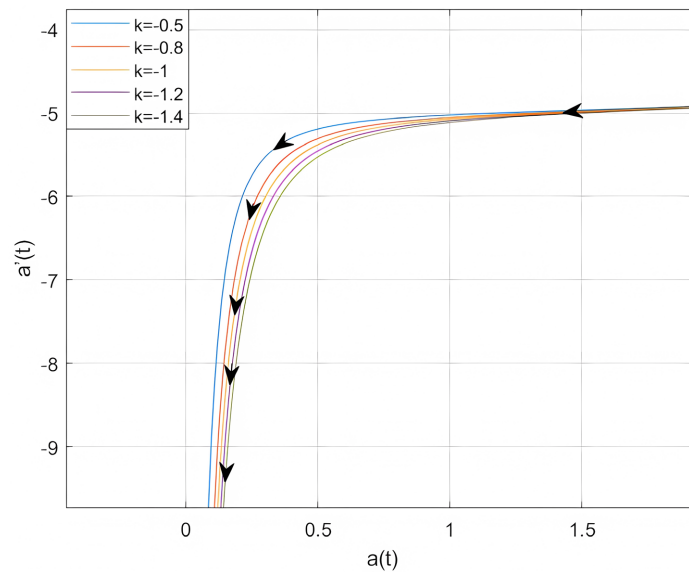


Figure 5. The phase plane for Equation (11) with $a_0 > 0$, $a_1 \leq 0$ and some $k < 0$.

(ii) $a_1 > 0$. If there exists a finite time T^* such that $a'(T^*) = a_1^* \leq 0$, the proof is the same as the situation of $a_1 \leq 0$ with the group property of autonomous systems. Therefore, we only need to prove the existence of T^* .

If we assume $a'(t) > 0$ for any $t \geq 0$, integrating the equation in (11) to obtain:

$$a'(t) = \int_0^t \left[\frac{k}{a^3(t)} - \frac{a(t)}{4} \right] dt + a_1. \tag{15}$$

Since the function

$$H(a(t)) = \frac{k}{a^3(t)} - \frac{a(t)}{4}, \tag{16}$$

have a negative global maximum $-M$ at $a(t) = (-12k)^{\frac{1}{4}}$, we get

$$a'(t) \leq -Mt + a_1, \tag{17}$$

which means there exists a finite time t such that $a'(t) < 0$ after a sufficient large time. A contradiction is met.

Figure 6 shows the phase plane of (11) with $a_0 > 0$, $a_1 > 0$ and different $k < 0$ as example of this case.

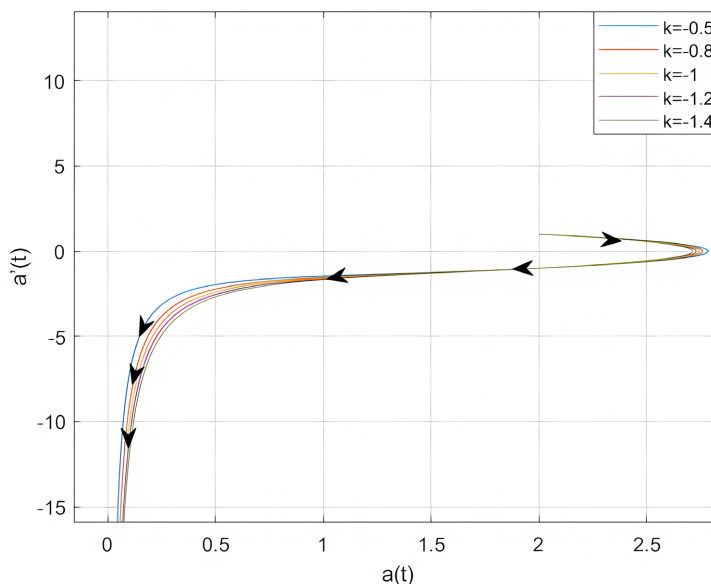


Figure 6. The phase plane for Equation (11) with $a_0 > 0$, $a_1 > 0$ and some $k < 0$.

After obtaining the Lemma 4, we can prove Theorem 2.

4. Conclusions and Discussion

In this paper, we have constructed a family of explicit vortical and self-similar solutions (2) for the two-dimensional inviscid rotating shallow water system (1). These solutions capture the coupled effects of rotation, nonlinear advection, and height variation, and they are characterized by three parameters: $\lambda \leq 0$, $\alpha \geq 0$ and $\xi \in R$. The function $a(t)$ governing the self-similar scaling obeys a second-order autonomous ODE (3), whose behavior is fully classified by the sign of $\xi^2 + \lambda$.

Several directions merit further investigation. Firstly, the stability of these vortical solutions under perturbations—both within the rotating shallow water system and in more realistic models. Secondly, the present solutions reflect the physical phenomenon at the same latitude on the planet; it would be interesting to extend the results containing Coriolis parameter f , which is normalized to unity in this paper. Finally, the construction relies on a specific separable form (4); one may ask whether more general vortical solutions exist, for instance with non-radial dependence or with a different form of $G(t, r)$.

In summary, the vortical and self-similar solutions presented here provide an explicit example where rotation both regularizes and destabilizes the flow depending on parameter regimes, enriching the theoretical concepts of shallow water dynamics with rotation.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- [1] Gill, A.E. (1982) Atmosphere-Ocean Dynamics: Mathematics in Science and Engineering. Academic Press.
- [2] Majda, A. (2003) Introduction to PDEs and Waves for the Atmosphere and Ocean. American Mathematical Society. <https://doi.org/10.1090/cln/009>
- [3] Pedlosky, J. (1992) Geophysical Fluid Dynamics. Springer.
- [4] Huang, Y. and Xie, C. (2023) Formation of Singularity for the Classical Solutions of the Rotating Shallow Water System. *Journal of Differential Equations*, **348**, 45-65. <https://doi.org/10.1016/j.jde.2022.11.048>
- [5] Liu, H. and Tadmor, E. (2004) Rotation Prevents Finite-Time Breakdown. *Physica D: Nonlinear Phenomena*, **188**, 262-276. <https://doi.org/10.1016/j.physd.2003.07.006>
- [6] Cheng, B. and Tadmor, E. (2008) Long-time Existence of Smooth Solutions for the Rapidly Rotating Shallow-Water and Euler Equations. *SIAM Journal on Mathematical Analysis*, **39**, 1668-1685. <https://doi.org/10.1137/070693643>
- [7] Cheng, B. and Xie, C. (2011) On the Classical Solutions of Two Dimensional Inviscid Rotating Shallow Water System. *Journal of Differential Equations*, **250**, 690-709. <https://doi.org/10.1016/j.jde.2010.09.017>
- [8] Babin, A., Mahalov, A. and Nicolaenko, B. (1997) Global Splitting and Regularity of Rotating Shallow-Water Equations. *European Journal of Mechanics B-Fluids*, **16**, 725-754.
- [9] Chesnokov, A.A. (2009) Symmetries and Exact Solutions of the Rotating Shallow-Water Equations. *European Journal of Applied Mathematics*, **20**, 461-477. <https://doi.org/10.1017/s0956792509990064>
- [10] Yuen, M. (2014) Vortical and Self-Similar Flows of 2D Compressible Euler Equations. *Communications in Nonlinear Science and Numerical Simulation*, **19**, 2172-2180. <https://doi.org/10.1016/j.cnsns.2013.11.008>
- [11] Kwong, M.K. and Yuen, M. (2014) Periodic Solutions of 2D Isothermal Euler-Poisson Equations with Possible Applications to Spiral and Disk-Like Galaxies. *Journal of Mathematical Analysis and Applications*, **420**, 1854-1863. <https://doi.org/10.1016/j.jmaa.2014.06.033>