

Theoretical Foundations of Sequence Convergence in Real Analysis

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Abstract

Convergence serves as the cornerstone of Real Analysis, providing the rigorous framework necessary for the formalization of limits, continuity, and calculus. This review paper examines the fundamental properties of real-valued sequences, with particular emphasis on the relationship between boundedness and monotonicity. Key results, including the Monotone Convergence Theorem and the Bolzano-Weierstrass Theorem, are analyzed to demonstrate how sequence behavior reflects the completeness structure of the real number system. In addition to standard definitions and proofs, the paper incorporates recursive and nontrivial examples to illustrate deeper applications of convergence. The discussion further connects classical convergence theory with modern applications in numerical analysis, dynamical systems, chaos theory, and computational mathematics. Through this synthesis, the paper highlights the continuing importance of convergence in both theoretical and applied mathematical research.

Keywords

Real Numbers, Sequences, Convergence, Monotone Convergence Theorem, Bolzano-Weierstrass Theorem

1. Introduction

In mathematics, sequences are ordered lists of numbers that follow a specific rule or pattern. Investigating the behavior of sequences as the index increases indefinitely leads to the concept of convergence, which forms the basis of limits and advanced analytical reasoning. The study of convergence is essential in understanding limits and many advanced topics in analysis [1].

Within the field of Real Analysis, convergence provides a rigorous logical framework necessary for establishing mathematical proofs. Classical theorems concerning limits rely heavily on convergence concepts, making them foundational topics in undergraduate and graduate mathematics curricula [2]. Beyond its traditional role in calculus, convergence has become increasingly important in contemporary mathematical research, where it appears in statistical convergence, computational methods, optimization theory, and functional analysis [3] [4].

This paper is intended as a review and synthesis of foundational convergence theory in Real Analysis. Rather than presenting new theorems, the paper aims to consolidate classical results while demonstrating how these ideas remain relevant in modern mathematical applications. By integrating both elementary and advanced examples, the discussion illustrates how convergence bridges pure and applied mathematics.

This review synthesizes classical and contemporary literature on sequence convergence drawn primarily from standard real analysis texts and recent research articles in analysis and applied mathematics. Sources were selected based on their foundational importance and relevance to modern applications such as numerical analysis and dynamical systems. The paper is organized as follows: Section 2 introduces preliminary concepts, Sections 3-5 discuss convergence, boundedness, and monotonicity, Sections 6 and 7 present major convergence theorems, and Section 8 highlights key applications. This structure emphasizes both theoretical development and practical relevance.

2. Preliminaries

2.1. Real Numbers

The system of real numbers, denoted by $\mathbb{R}$, consists of rational and irrational numbers. One of its most significant properties is completeness, which states that every nonempty subset of $\mathbb{R}$ that is bounded above possesses a least upper bound or supremum [5].

The completeness property distinguishes $\mathbb{R}$ from the rational number $\mathbb{Q}$ and provides the theoretical foundation for convergence theorems. Without completeness, many important analytical results—including the convergence of bounded monotone sequences—would fail.

2.2. Sequences

A sequence is a function from the natural numbers $\mathbb{N}$ to the real numbers $\mathbb{R}$. A sequence is usually written as

$$\{a_n\}_{n=1}^{\infty} \quad (1)$$

where a_n represents the n th term.

Sequences serve as the foundational structures for more advanced analytical objects such as infinite series, function spaces, and iterative numerical algorithms [6]. In many practical applications, sequences arise naturally from repeated ap-

proximation procedures.

Example:

$$a_n = \frac{1}{n} \quad (2)$$

Sequence:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \quad (3)$$

3. Convergence of Sequences

Definition of Convergence

Define A sequence $\{a_n\}$ converges to a real number L if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } |a_n - L| < \epsilon \quad (4)$$

for all $n \geq N$.

This definition, known as the ϵ - N definition of convergence, provides a precise and characterization of the limiting behavior of sequences [7] [8]. It formalizes the intuitive idea that the terms of a convergent sequence become arbitrarily close to a fixed real number.

Example:

Consider the sequence

$$a_n = \frac{1}{n} \quad (5)$$

To show that $a_n \rightarrow 0$, let $\epsilon > 0$. Choose $N > \frac{1}{\epsilon}$. Then for all $n \geq N$,

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon \quad (6)$$

Hence, the sequence converges to 0.

The significance of this definition extends beyond elementary examples. In numerical methods and approximation theory, convergence guarantees that iterative approximations become increasingly accurate representations of the desired quantity.

4. Bounded Sequences

A sequence $\{a_n\}$ is bounded if there exists a real number $M > 0$ such that

$$|a_n| \leq M \quad (7)$$

for all $n \in \mathbb{N}$.

Boundedness is an important property because it prevents divergence to infinity and is often a prerequisite for convergence-related theorems. However, boundedness alone does not guarantee convergence, emphasizing the need for additional structural conditions such as monotonicity [2].

Example:

Consider the sequence

$$a_n = (-1)^n \quad (8)$$

This sequence is bounded because its values remain between -1 and 1 .

Nevertheless, the sequence does not converge because it oscillates indefinitely between two distinct values.

This example demonstrates that boundedness controls the magnitude of sequence terms but does not necessarily regulate their long-term behavior.

5. Monotone Sequences

A sequence $\{a_n\}$ is monotone increasing if

$$a_n \leq a_{n+1} \quad (9)$$

for all $n \in \mathbb{N}$.

Similarly, a sequence $\{a_n\}$ is monotone decreasing if

$$a_n \geq a_{n+1} \quad (10)$$

for all $n \in \mathbb{N}$.

Monotonicity describes the directional behavior of sequences and plays a critical role in establishing convergence when combined with boundedness [6]. Monotone sequences frequently arise in recursive definitions and iterative approximation methods used in computational mathematics.

Example (Newton's Method)

Consider the recursively defined sequence

$$a_1 = 1, \quad a_{n+1} = \frac{1}{2} \left(a_n + \frac{2}{a_n} \right). \quad (11)$$

This sequence is generated by Newton's Method for approximating $\sqrt{2}$. Computing initial terms:

$$a_2 = 1.5, \quad a_3 \approx 1.4167, \quad a_4 \approx 1.4142. \quad (12)$$

It can be shown that for $n \geq 2$, the sequence is monotone decreasing and bounded below by $\sqrt{2}$. Hence, it converges.

Taking the limits, let $L = \lim a_n$. Then,

$$L = \frac{1}{2} \left(L + \frac{2}{L} \right), \quad (13)$$

which implies $L^2 = 2$, and therefore $L = \sqrt{2}$.

This example illustrates how monotone bounded sequences naturally arise in numerical methods and converge to meaningful limits.

6. Monotone Convergence Theorem

The Monotone Convergence Theorem states:

Every bounded monotone sequence of real numbers converges.

This theorem is a directly from the completeness property of $\mathbb{R}$ [5]. If a sequence is increasing and bounded above, then the set of its terms possesses a su-

premium, and the sequence converges to that supremum. Similarly, a decreasing sequence bounded below converges to its infimum.

The theorem is particularly important in analysis because it guarantees convergence without requiring explicit computation of the limit. In calculus and numerical analysis, iterative procedures often produce monotone bounded sequences whose convergence is ensured by this theorem [1] [4].

Example 1

Consider the sequence

$$a_n = 1 - \frac{1}{n} \quad (14)$$

The sequence is increasing because

$$a_{n+1} - a_n = \left(1 - \frac{1}{n+1}\right) - \left(1 - \frac{1}{n}\right) = \frac{1}{n(n+1)} > 0 \quad (15)$$

and it is bounded above by 1. Therefore, by the Monotone Convergence Theorem, the sequence converges to 1.

Example 2 (Valid Recursive Example)

Consider the recursively defined sequence

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}(a_n + 3). \quad (16)$$

Step 1: Boundedness

We claim that $a_n < 3$ for all n .

Assume $a_n < 3$, then

$$a_{n+1} = \frac{1}{2}(a_n + 3) < \frac{1}{2}(3 + 3) = 3, \quad (17)$$

Thus, the sequence is bounded above by 3.

Step 2: Monotonicity

Observe that

$$a_{n+1} - a_n = \frac{1}{2}(a_n + 3) - a_n = \frac{3 - a_n}{2} > 0, \quad (18)$$

so the sequence is increasing.

Step 3: Convergence

Since the sequence is increasing and bounded above, it converges by the Monotone Convergence Theorem.

Let $L = \lim a_n$. Then

$$L = \frac{1}{2}(L + 3) \Rightarrow L = 3. \quad (19)$$

This example correctly illustrates the theorem using a bounded monotone sequence.

Proof Sketch:

If a sequence is monotone increasing and bounded above, then the set of its terms has a supremum L by completeness of $\mathbb{R}$. It can be shown that the se-

quence approaches L , and hence converges. A complete proof can be found in standard texts such as Bartke & Sherbert (2022) or Rudin (1976) [2] [5].

7. Bolzano-Weierstrass Theorem

The Bolzano-Weierstrass Theorem states:

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

This theorem is fundamental in Real Analysis because it guarantees the existence of convergence even when the original sequence itself fails to converge. It plays a central role in compactness arguments, functional analysis, and optimization theory [3] [8].

Example:

Consider the sequence

$$a_n = (-1)^n. \quad (20)$$

This sequence does not converge because it alternates indefinitely between -1 and 1 . However, it contains the following convergent subsequences:

$$a_{2n} = 1 \rightarrow 1 \quad (21)$$

and

$$a_{2n-1} = -1 \rightarrow -1 \quad (22)$$

The theorem therefore guarantees that bounded oscillatory sequence still contain convergent structure at the subsequence level.

In modern analysis, the Bolzano-Weierstrass Theorem is closely related to compactness. For example, in optimization problems, bounded iterative methods often rely on convergent subsequences to establish the existence of optimal solutions.

Proof Sketch:

The proof relies on repeatedly subdividing an interval containing the bounded sequence to extract nested intervals that contain infinitely many terms. This construction yields a convergent subsequence. A detailed proof appears in Tao (2023) and Rudin (1976) [5] [8].

While boundedness alone ensures that a sequence does not diverge to infinity, it does not guarantee convergence, as demonstrated by oscillatory examples such as $a_n = (-1)^n$. In contrast, boundedness combined with monotonicity ensures convergence via the Monotone Convergence Theorem. The Bolzano-Weierstrass Theorem provides a complementary result: even if a bounded sequence is not monotone, it still contains a convergent subsequence. Thus, monotonicity guarantees convergence of the entire sequence, while boundedness alone guarantees only partial convergence behavior.

8. Application of Convergence

Convergence of sequences has wide-ranging applications across both pure and applied mathematics. Beyond serving as a theoretical concept, convergence pro-

vides the analytical foundation for approximation methods, computational algorithms, and mathematical modeling.

8.1. Calculus and Infinite Series

In calculus, convergence is essential for defining limits, derivatives, and integrals. Infinite series such as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad (23)$$

depend on sequence convergence because the sequence of partial sums determines whether the series converges.

Power series representations of functions also rely on convergence intervals. For instance, the exponential function can be expressed as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (24)$$

when convergence guarantees accurate approximation of transcendental functions.

8.2. Numerical Analysis

Many numerical algorithms generate sequences that approximate solutions to equations. Newton's Method, for example, constructs a sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (25)$$

That converges to a root of $f(x) = 0$ under appropriate conditions.

The Monotone Convergence Theorem is frequently applied to verify that iterative approximations remain bounded and converge toward stable solutions. In computational mathematics, convergence analysis determines whether algorithms are reliable and efficient.

This demonstrates how convergence of sequences ensures that iterative algorithms produce increasingly accurate approximations, a key requirement for numerical stability and reliability in computational methods [4].

8.3. Differential Equations

Many Sequence arise naturally in solving differential equations through approximation methods such as Euler's Method and Picard iteration. In Picard iteration, successive approximations form a sequence of functions converging to the solution of a differential equation.

The convergence of these approximations ensures that numerical simulations accurately represent physical systems such as fluid flow, population dynamics, and heat transfer.

The convergence of the sequence of approximations guarantees that the limiting function satisfies the original differential equation, providing both existence and constructive solutions in applied contexts [1].

8.4. Dynamical Systems and Chaos Theory

Recent studies have connected convergence theory with dynamical systems and chaos theory [3]. In these fields, recursively generated sequences describe long-term system behavior.

For example, consider the logistic map:

$$x_{n+1} = rx_n(1 - x_n). \quad (26)$$

When certain parameter values are chosen, the generated sequence converges to stable equilibrium points. For other parameter values, the sequence oscillates chaotically and fails to converge. The study of convergence and divergence in such systems helps researchers distinguish stable dynamical behavior from chaotic behavior.

This demonstrates that convergence theory is not limited to abstract analysis but also provides insight into real-world phenomena such as climate systems, biological populations, and economic models.

In this context, convergence analysis helps determine whether a system evolves toward equilibrium, periodic cycles, or chaotic behavior, making it central to stability analysis in dynamical systems [3].

8.5. Educational and Computational Applications

Modern educational research emphasizes the importance of symbolic, graphical, and verbal representations in helping students understand convergence concepts [9]. Visualization software and computational tools now allow students to explore convergence dynamically through graphs and simulations.

Furthermore, machine learning and optimization algorithms frequently depend on convergent iterative procedures. Gradient descent methods, for example, generate sequences of approximations intended to converge toward optimal parameter values.

These developments illustrate how convergence remains central to both mathematical theory and modern computational practice.

In optimization and machine learning, the convergence of iterative algorithms (such as gradient descent) ensures that solutions approach optimal values, which is crucial for algorithmic effectiveness [10].

9. Conclusions

The study of sequences and their convergence forms a cornerstone of Real Analysis. Through concepts such as boundedness, monotonicity, and subsequences, mathematicians can rigorously analyze the behavior of infinite processes. The Monotone Convergence Theorem and Bolzano-Weierstrass Theorem demonstrate how the completeness of the real numbers system guarantees the existence of limits under appropriate conditions.

This review paper has shown that convergence theory extend far beyond elementary textbook examples. Recursive sequence, iterative numerical methods,

and dynamical systems all rely heavily on convergence principles. By connecting classical theorems with modern applications, the discussion emphasizes the enduring relevance of convergence in contemporary mathematics.

A deep understanding of convergence equips students and researchers with the analytical tools necessary for advanced studies in mathematics, and scientific modeling. As modern research continues to evolve, convergence remains a unifying concept that remains a unifying concept linking pure theory with practical application.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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