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Renormalized Solutions of a Nonlinear Elliptic-Parabolic Equation with L^1 Data and Finite-Element Numerical Simulation

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Abstract

We study the existence and uniqueness of renormalized solutions for a nonlinear degenerate elliptic-parabolic problem of the form

$$\frac{\partial M(u)}{\partial t} - \operatorname{div}\left(\nabla u + K(u)\boldsymbol{e}_z\right) = f \quad \text{in } Q_T = (0,T)\times\Omega,$$

with L^1 data $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$, arising from the Richards model of unsaturated flow in porous media. Because classical weak solutions fail to be unique at this low-regularity level, we work within the framework of renormalized solutions, which select the physically relevant solution through an energy-dissipation condition at infinity. Existence is established via a double approximation scheme $(\psi_{m,n})$ combined with nonlinear semigroup theory, and uniqueness follows from a comparison principle obtained by the doubling-of-variables technique. We further present a conforming finite-element /implicit-Euler/Picard-iteration scheme validated on the Brooks-Corey soil model, for which we prove an $O(h^2+\tau)$ error estimate in $L^2(\Omega)$ and unconditional stability.

Keywords

Renormalized Solutions, Elliptic-Parabolic Equation, Porous Media, Richards Equation, Finite Elements, Picard Iteration, L^1 Data, Semigroup Theory

1. Introduction

The movement of water in variably saturated soils is governed by the Richards equation [1], which, after the Kirchhoff transformation $u = \int_0^h K_r(\theta(s))\mathrm{d}s$, takes the form

$$\frac{\partial M(u)}{\partial t} - \operatorname{div}\left(\nabla u + K(u)\boldsymbol{e}_z\right) = f \text{ in } Q_T = (0,T)\times\Omega, \tag{1}$$

supplemented with homogeneous Dirichlet boundary conditions and initial datum $M(u)(\cdot,0) = v_0$ in $\Omega \subset \mathbb{R}^3$ bounded. Here $M:\mathbb{R}\to\mathbb{R}$ is continuous, non-decreasing with $M(0)=0$, and $K:\mathbb{R}\to\mathbb{R}$ is Lipschitz with $K(0)=0$.

When $f \in L^1(Q_T)$ and $v_0 \in L^1(\Omega)$, classical weak solutions are not unique in general [2]. Three equivalent notions have been introduced to restore uniqueness: SOLA [3], entropy solutions [4], and renormalized solutions [5]. The latter, first used for the Boltzmann equation [5] and later adapted to elliptic-parabolic problems by Ammar-Wittbold [6] and Carrillo-Wittbold [7], provide the most flexible framework for handling degenerate diffusions with rough data.

Contributions of this paper.

- We prove existence (Theorem 3.4) and uniqueness (Theorem 3.5) of a renormalized solution of (1) with L^1 data, via a two-parameter approximation $\psi_{m,n}(r) = \frac{1}{m}r^+ - \frac{1}{n}r^-$ and nonlinear semigroup theory (Section 3).
- We design and analyse a finite-element/implicit-Euler/Picard scheme (Section 4), establish an $O(h^2+\tau)$ error estimate, and validate it on a Brooks-Corey soil model.

Throughout the paper we work under the following standing assumptions:

- $M:\mathbb{R}\to\mathbb{R}$ is continuous, non-decreasing, $M(0)=0$.
- $K:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous, $K(0)=0$.
- $f \in L^1(Q_T)$, $u_0 \in L^1(\Omega)$.

2. Functional Setting and Preliminary Results

We write $Q_T = (0,T)\times\Omega$ and denote by $T_k(s) = \max(-k,\min(s,k))$ the truncation at level $k>0$, by H_0 the Heaviside function, and by sign the sign function. All integrals without explicit domain are over Q_T unless stated otherwise.

2.1. Truncation Operators and Gradient Lemma

Lemma 2.1 (Gradient of truncations, [8]). *Let* u *be measurable on* Ω *with* $T_k(u) \in W_0^{1,p}(\Omega)$ *for every* $k>0$. *Then there exists a unique measurable function* $\boldsymbol{v}:\Omega\to\mathbb{R}^N$ *such that*

$$\nabla T_k(u) = \boldsymbol{v}\mathbf{1}_{\{|u|<k\}} \quad \text{a.e.,} \quad \forall k>0.$$

Moreover, if $u \in W_0^{1,1}(\Omega)$ then $\boldsymbol{v} = \nabla u$.

2.2. Integration-by-Parts Formula

Lemma 2.2 ([9] [10]). *Let* $M:\mathbb{R}\to\mathbb{R}$ *be continuous and monotone with* $M(0)=0$. *Suppose* $u \in L^2(0,T;H_0^1(\Omega))$ *with* $M(u) \in L^1(Q_T)$,

$\partial_t M(u) \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q_T)$, *and* $u_0 \in L^1(\Omega)$. *Then for every* $h \in C_c^\infty(\mathbb{R})$ *with* $h(0)=0$, $\eta \in H_0^1(\Omega)$, *and* $\varphi \in C_c^\infty(Q_T)$ *with* $\varphi(T)=0$,

$$-\int_0^T \left\langle \frac{\partial M(u)}{\partial t}, h(u-\eta)\varphi \right\rangle \mathrm{d}t = \int_{Q_T} \frac{\partial \varphi}{\partial t} \int_u^{u_0} h(r-\eta)\,\mathrm{d}M(r)\,\mathrm{d}x\mathrm{d}t.$$

2.3. Perturbed Problem and Semigroup Approximation

For $\psi:\mathbb{R}\to\mathbb{R}$ continuous and strictly increasing with $\psi(0)=0$, and $M_k(r) = M(r) + \frac{1}{k}r$ (strictly increasing), define the operator

$$A_{M_k}^{\psi} := \left\{ \left(M_k(u), -\mathrm{div}\, a(u,\nabla u) + \psi(u) \right) \right\}$$

on the domain $\left\{ u \in H_0^1(\Omega) \cap L^\infty(\Omega) : \mathrm{div}\, a(u,\nabla u) \in L^\infty(\Omega) \right\}$, where $a(u,\nabla u) = \nabla u + K(u)\boldsymbol{e}_z$.

Proposition 2.3 ([6] [8]). *Under* (H1)-(H2), $A_{M_k}^{\psi}$ *is* m *-accretive and densely defined in* $L^1(\Omega)$. *Moreover, as* $M_k \to M$ *uniformly on compacts,* $A_M^{\psi} \subset \underline{\lim}_{k\to\infty} A_{M_k}^{\psi}$.

By nonlinear semigroup theory [11], Proposition 2.3 yields a unique mild solution $v_k = M_k(u_k) \in C([0,T];L^1(\Omega))$ of the Cauchy problem $\dot{v}_k + A_{M_k}^{\psi}(v_k) \ni f$, $v_k(0) = v_{0k}$. As $k\to\infty$ and $v_{0k} \to v_0$ in $L^1(\Omega)$, we have $v_k \to v = M(u)$ in $C([0,T];L^1(\Omega))$.

3. Renormalized Solutions: Existence and Uniqueness

3.1. Definition

Definition 3.1 (Renormalized solution). *A measurable function* $u: Q_T \to \mathbb{R}$ *is a renormalized solution of* (1) *if*

(i) $M(u) \in L^1(Q_T)$,

(ii) $T_k(u) \in L^2(0,T;H_0^1(\Omega))$ for every $k>0$,

(iii) for every $h \in C_c^1(\mathbb{R})$ and $\varphi \in C_c^\infty((0,T)\times\Omega)$,

$$\begin{aligned} &\int_{Q_T} \frac{\partial\varphi}{\partial t} \int_u^{u_0} h(r)\,\mathrm{d}M(r)\,\mathrm{d}x\mathrm{d}t + \int_{Q_T} f h(u)\varphi\,\mathrm{d}x\mathrm{d}t \\ &= \int_{Q_T} (\nabla u + K(u)\boldsymbol{e}_z)\cdot\nabla(h(u)\varphi)\,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2}$$

(iv) the following *energy condition at infinity* holds:

$$\int_{Q_T\cap\{k\le|u|\le k+1\}} (\nabla u + K(u)\boldsymbol{e}_z)\cdot\nabla u\,\mathrm{d}x\mathrm{d}t \xrightarrow{k\to+\infty} 0. \tag{3}$$

Remark 3.2. Condition (3) selects the physical solution among (possibly infinitely many) weak solutions: it expresses that no energy is dissipated at infinite amplitude levels. When $\mathrm{supp}\, h \subset [-k,k]$, the term $(\nabla u + K(u)\boldsymbol{e}_z)\cdot\nabla(h(u)\varphi)$ is identified with $(\nabla T_k(u) + K(T_k(u))\boldsymbol{e}_z)\cdot\nabla(h(T_k(u))\varphi)$, which is well-defined by (ii).

3.2. Comparison Lemma

Lemma 3.3 (Monotone comparison). *Let* $u_0, \tilde{u}_0, f, \tilde{f} \in L^\infty(Q_T)$ *and let* $\psi, \tilde{\psi} : \mathbb{R} \to \mathbb{R}$ *be continuous and strictly increasing with* $\psi(0) = \tilde{\psi}(0) = 0$. *Let* $u, \tilde{u}$ *be weak solutions of the perturbed problem* (1) *with perturbations* ψ *and* $\tilde{\psi}$ *respectively. If* $u_0 \le \tilde{u}_0$ *a.e. on* Ω, $f \le \tilde{f}$ *a.e. on* Q_T, *and* $\tilde{\psi}(r) \le \psi(r)$ *for all* $r \in \mathbb{R}$, *then* $u \le \tilde{u}$ *a.e. on* Q_T.

Proof. Taking $\frac{1}{L}\varphi(t)\rho_p(t-s)T_L(u(t,x)-\tilde{u}(s,x))$ as test function and using the identity

$$(\psi(u)-\tilde{\psi}(\tilde{u}))\operatorname{sign}(u-\tilde{u}) \ge (\tilde{\psi}(u)-\tilde{\psi}(\tilde{u}))^+,$$

together with the same doubling argument as in Lemma 2.2 and Proposition 2.3, one obtains $\int_{Q_T}(\tilde{\psi}(u)-\tilde{\psi}(\tilde{u}))^+ \mathrm{d}x\mathrm{d}t \le 0$. Since $\tilde{\psi}$ is strictly increasing, $u \le \tilde{u}$ a.e. ☐

3.3. Existence

Theorem 3.4 (Existence). *Under* (H1)-(H3), *there exists at least one renormalized solution of* (1).

Proof. **Step 1: Approximation.** Set $f_{m,n} = \max(\min(f,m),-n)$, $u_{0,m,n} = \max(\min(u_0,m),-n)$, and $\psi_{m,n}(r) = \frac{1}{m}r^+ - \frac{1}{n}r^-$ for $m,n \in \mathbb{N}^*$. By Proposition 2.3 and nonlinear semigroup theory (applied with $\psi = \psi_{m,n}$, which is strictly increasing), there exists a unique weak solution $u_{m,n}$ of the perturbed problem

$$\frac{\partial M(u)}{\partial t} + \psi_{m,n}(u) - \operatorname{div} a(u,\nabla u) = f_{m,n} \text{ in } Q_T,\ M(u)(\cdot,0) = u_{0,m,n},\ u|_{\partial\Omega} = 0. \quad (4)$$

Step 2: Monotonicity and pointwise limits. By Lemma 3.3,

$$u_{m,n} \uparrow_m u_n \text{ and } u_n \downarrow_n u \text{ a.e. on } Q_T,$$

where u_n and u are measurable and almost-everywhere finite (see [6]).

Step 3: Energy estimate. Choosing test function $T_k(u_{m,n})\varphi_\delta$ in (4), where $\varphi_\delta \in C^1([0,T])$ is a cut-off near T, using Lemma 2.2 and the monotonicity of $\psi_{m,n}$, one obtains

$$C\int_0^T\int_\Omega |\nabla T_k(u_{m,n})|^2 \mathrm{d}x\mathrm{d}t \le k\left(\|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)}\right). \quad (5)$$

By the Poincaré inequality, $(T_k(u_{m,n}))$ is bounded in $L^2(0,T;H_0^1(\Omega))$, so $T_k(u_{m,n}) \rightharpoonup T_k(u)$ weakly in $L^2(0,T;H_0^1(\Omega))$.

Step 4: Strong convergence of truncations. Taking $h(u_n)(T_k(u_n)-T_k(u))\varphi$ as test function in the equation satisfied by u_n, one shows that $I_1 + I_2 + I_3 \to 0$

(where I_1 involves the time derivative, I_2 the gradient cross-terms, and I_3 the convection), yielding $\|T_k(u_n)\|_{L^2(H_0^1)} \to \|T_k(u)\|_{L^2(H_0^1)}$. Combined with the weak convergence, this gives

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^2\left(0,T;H_0^1(\Omega)\right). \tag{6}$$

Step 5: Passage to the limit. Using the strong convergence (6) and the Lebesgue dominated convergence theorem, one passes to the limit in each term of the equation for u_n as $n \to \infty$ to obtain (2).

Step 6: Energy condition at infinity. Taking $\left(T_{k+1}(u_n) - T_k(u_n)\right)\varphi$ as test function and letting $n \to \infty$, then $k \to \infty$ with $\varphi = 1$, yields (3). ☐

3.4. Uniqueness

Theorem 3.5 (Uniqueness and comparison). *Under* (H1)-(H3), *for* $i = 1,2$ *let* $u_{0i} \in L^1(\Omega)$, $f_i \in L^1(Q_T)$, *and* u_i *be renormalized solutions of* (1) *with data* (u_{0i}, f_i). *Then, for all* $t \in (0,T)$,

$$\int_\Omega \left(M(u_1)(t) - M(u_2)(t)\right)^+ \mathrm{d}x \le \int_\Omega (u_{01} - u_{02})^+ \mathrm{d}x + \int_0^t \int_\Omega (f_1 - f_2)^+ \mathrm{d}x\mathrm{d}t. \tag{7}$$

In particular, $u_1 = u_2$ a.e. when $u_{01} = u_{02}$ and $f_1 = f_2$.

Proof sketch. The proof uses the double-variable technique of [7], exploiting the structural condition

$$\left(a(r,\zeta) - a(s,\eta)\right)\cdot(\zeta - \eta) + C(r,s)\left(1 + |\zeta|^p + |\eta|^p\right)|r - s| \ge \Gamma(r,s)\cdot\zeta + \hat{\Gamma}(r,s)\cdot\eta,$$

which holds for $a(u,\nabla u) = \nabla u + K(u)\mathbf{e}_z$ since K is Lipschitz. A doubling of both the space and time variables, combined with the energy condition (3), yields (7). ☐

4. Numerical Scheme and Analysis

Since the domain is one-dimensional, $\Omega = \left]0,L\right[\subset \mathbb{R}$, the vertical coordinate z of the analytical model coincides with the spatial coordinate x used in the finite-element discretisation. Consequently, the gravitational convection term $K(u)\partial_z$ appearing in the flux $\nabla u + K(u)\mathbf{e}_z$ is written in the finite-element coordinates as

$$K(u)\partial_z \equiv K(u)\frac{\partial}{\partial x},$$

so that the weak contribution of this term against a test function φ_l reads

$$\int_\Omega K(u_h)\frac{\partial \varphi_l}{\partial x}\mathrm{d}x.$$

4.1. Spatial Discretisation by Finite Elements

Let $\Omega = \left]0,L\right[$ with $L = 1\text{ m}$. We construct a mesh $\mathcal{T}_h = \{L_1, \cdots, L_{N+1}\}$ with nodes $0 = x_0 < x_1 < \cdots < x_{N+1} = L$ and $h = \max_i (x_{i+1} - x_i)$. The conforming finite-element space is

$$V_h = \left\{ v_h \in C^0\left(\overline{\Omega}\right) : v_h\big|_{L_i} \in \mathcal{P}_1, v_h(0) = v_h(L) = 0 \right\},$$

with nodal basis $\left\{\varphi_j\right\}_{j=1}^{N}$. The semi-discrete problem reads: find $u_h(t) = \sum_{m=1}^{N} u_m(t)\varphi_m \in V_h$ such that

$$\int_\Omega \frac{\partial M(u_h)}{\partial t}\varphi_l \mathrm{d}x + \int_\Omega \nabla u_h \cdot \nabla\varphi_l \mathrm{d}x + \int_\Omega K(u_h)\partial_z\varphi_l \mathrm{d}x = \int_\Omega f\varphi_l \mathrm{d}x, l = 1,\cdots,N. \quad (8)$$

4.2. Temporal Discretisation and Picard Linearisation

With time step $\tau = T/N_t$ and $u_h^k \approx u_h(\cdot, t_k)$, the implicit Euler step gives:

$$\int_\Omega \frac{M\left(u^k\right) - M\left(u^{k-1}\right)}{\tau}\varphi_l \mathrm{d}x + \int_\Omega \nabla u^k \cdot \nabla\varphi_l \mathrm{d}x + \int_\Omega K\left(u^k\right)\partial_z\varphi_l \mathrm{d}x = \int_\Omega f^k\varphi_l \mathrm{d}x. \quad (9)$$

Since (9) is nonlinear in u^k, we linearise via a first-order Taylor expansion (Picard method):

$$M\left(u^{k,i}\right) \approx M\left(u^{k,i-1}\right) + C_1^{k,i-1}\left(u^{k,i} - u^{k,i-1}\right), K\left(u^{k,i}\right) \approx K\left(u^{k,i-1}\right) + C_2^{k,i-1}\left(u^{k,i} - u^{k,i-1}\right),$$

where $C_j^{k,i-1} = \partial_u\{M, K\}\big|_{u^{k,i-1}}$. This yields, at each Picard step i, the linear system $A^{k,i-1}u^{k,i} = B^{k,i-1}$ with stiffness matrix

$$A_{lm}^{k,i-1} = \int_\Omega C_1^{k,i-1}\varphi_l\varphi_m \mathrm{d}x + \tau\int_\Omega \nabla\varphi_l \cdot \nabla\varphi_m \mathrm{d}x + \tau\int_\Omega C_2^{k,i-1}\varphi_l\partial_z\varphi_m \mathrm{d}x.$$

Algorithm 1 Picard iteration at time step k

```
Input: $u^{k-1}$; set $u^{k,0} \leftarrow u^{k-1}$, tolerance $\varepsilon_{\text{alg}}$
$i \leftarrow 1$
repeat
    Assemble $A^{k,i-1}$ and $B^{k,i-1}$
    Solve $A^{k,i-1}u^{k,i} = B^{k,i-1}$
    $E \leftarrow \|u^{k,i} - u^{k,i-1}\| / \|u^{k,i}\|$
    $i \leftarrow i + 1$
until $E \leq \varepsilon_{\text{alg}}$
Output: $u^k \leftarrow u^{k,i}$
```

4.3. Error Analysis

We work under the additional regularity hypotheses:

- M is Lipschitz with $M' \geq m_0 > 0$ on every compact (non-degenerate regime of Brooks-Corey [12]).
- $u \in L^\infty\left(0,T;H^2(\Omega)\right) \cap H^1\left(0,T;H_0^1(\Omega)\right) \cap H^2\left(0,T;L^2(\Omega)\right)$.
- The Picard iteration converges in a uniformly bounded number N_{Picard} of steps (guaranteed when τ is small relative to the Lipschitz constant of M').

Theorem 4.1 (Convergence order). *Under* (H1)-(H6), *there exists* $C > 0$ *independent of* h *and* τ *such that*

$$\max_{0 \leq k \leq N_t}\left\|u(\cdot,t_k) - u_h^k\right\|_{L^2(\Omega)} \leq C\left(h^2 + \tau\right), \quad (10)$$

$$\left(\tau\sum_{k=1}^{N_t}\left\|u(\cdot,t_k) - u_h^k\right\|_{H_0^1(\Omega)}^2\right)^{1/2} \leq C(h + \tau). \quad (11)$$

Proof sketch. Decompose $u(\cdot,t_k)-u_h^k=\eta^k+\theta^k$ where $\eta^k=u(\cdot,t_k)-\Pi_h u(\cdot,t_k)$ is the elliptic projection error and $\theta^k=\Pi_h u(\cdot,t_k)-u_h^k$. By standard interpolation theory and H^2-regularity of $\Omega=]0,L[$,

$$\|\eta^k\|_{L^2}\le Ch^2\|u(\cdot,t_k)\|_{H^2},\ \|\eta^k\|_{H^1}\le Ch\|u(\cdot,t_k)\|_{H^2}.$$

Testing the error equation for θ^k with θ^k and using the monotonicity of M (hypothesis (H4)) gives

$$\|\theta^k\|_{L^2}^2\le \mathrm{e}^{Ct_k}\|\theta^0\|_{L^2}^2+C\sum_{j=1}^{k}\tau\left(\tau^2+h^4\right).$$

Since $\theta^0=0$, we obtain (10). The estimate (11) follows analogously without the Aubin-Nitsche duality argument. ☐

Proposition 4.2 (Unconditional stability). *Under* (H1)-(H2), *for every* $\tau>0$ *and* $h>0$,

$$\max_{0\le k\le N_t}\|u_h^k\|_{L^2(\Omega)}^2+\tau\sum_{k=1}^{N_t}\|\nabla u_h^k\|_{L^2(\Omega)}^2\le C\left(\|u_0\|_{L^2(\Omega)}^2+\|f\|_{L^2(Q_T)}^2\right).$$

No CFL condition is required.

5. Numerical Experiment: Brooks-Corey Soil Model

5.1. Physical Parameters

We validate the scheme on a one-dimensional infiltration problem with Brooks-Corey constitutive relations [12] (see Table 1 for the physical parameters and numerical settings):

$$M(u)=\theta(h)=\begin{cases}\theta_r+(\theta_s-\theta_r)\left(\dfrac{h}{h_c}\right)^{-\beta} & \text{if } h<h_c\\ \theta_s & \text{if } h\ge h_c\end{cases},\quad K_r(\theta)=\left(\frac{\theta-\theta_r}{\theta_s-\theta_r}\right)^{3+2/\beta}.$$

Table 1. Brooks-Corey parameters and numerical settings.

Parameter	Symbol	Value	Unit
Residual water content	θ_r	0.064	–
Saturated water content	θ_s	0.126	–
Air-entry potential	h_c	−0.078	m
Pore-size index	β	24.774	–
Saturated conductivity	K_{sat}	0.0095	m/s
Time step	τ	0.01	s
Mesh size	h	0.01	m

5.2. Initial and Boundary Conditions

Reduction to homogeneous Dirichlet data. The analytical model (1) is set with homogeneous Dirichlet boundary conditions $u|_{\partial\Omega}=0$ and the finite-element space V_h enforces $v_h(0) = v_h(L) = 0$. In the numerical experiment the physical boundary data are $u(0,t) = 0$ and $u(1,t) = h_c$. To reconcile these with the analytical framework, we introduce the affine lifting

$$\tilde{u}(x,t) = u(x,t) - h_c x, x \in [0,1],$$

which satisfies $\tilde{u}(0,t) = 0$ and $\tilde{u}(1,t) = 0$, so that $\tilde{u}$ belongs to the homogeneous space V_h for all t. The equation for $\tilde{u}$ retains the same form as (1) with a modified right-hand side $\tilde{f} = f + \Delta(h_c x) = f$ (since the lifting is affine), thereby preserving the structure of the analytical model. In the remainder of this section, u denotes the lifted variable $\tilde{u}$ and all results of Sections 3-4 apply without modification.

Initial condition. The initial condition is chosen as

$$u_0(x) = h_c\left(1 - \cos^2\left(\frac{\pi x}{2}\right)\right), \quad x \in [0,1],$$

which satisfies $u_0(0) = 0$ (surface at saturation) and $u_0(1) = h_c$ (unsaturated base), consistently with the boundary data above. Its derivative

$u_0'(x) = \frac{\pi}{2} h_c \sin(\pi x) \le 0$ reflects a physically realistic suction profile increasing with depth.

5.3. Results and Discussion

Figure 1 shows the evolution of the matric potential $u(x,t)$ over ten time steps.

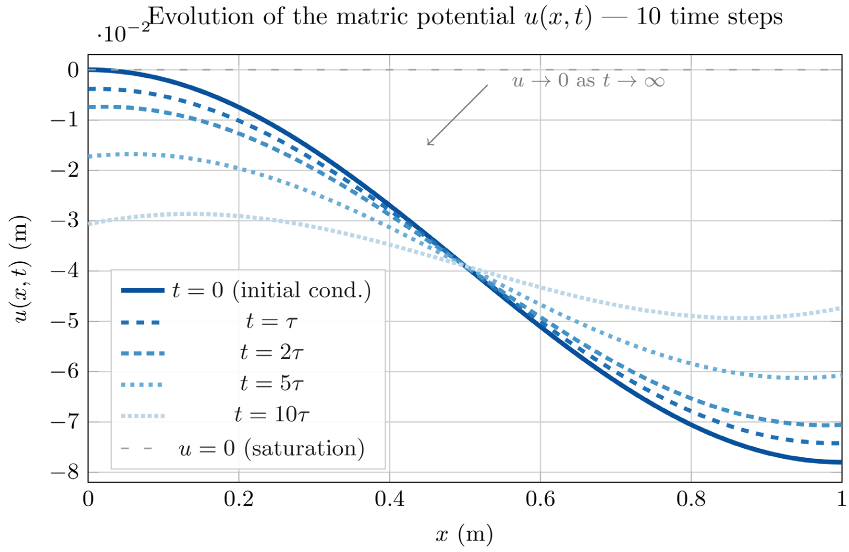

Figure 1. Evolution of the matric potential $u(x,t)$ over ten time steps ($\tau = 0.01\,\text{s}$). The blue curve is the initial condition; as $t \to \infty$ the profile tends uniformly to zero (fully saturated medium). Computed with Scilab on a mesh of $N = 100$ nodes.

The simulation exhibits three physically consistent features:

1) The initial profile (blue) displays a strong gradient near $x = 1$, consistent

with the prescribed base condition.

2) As t increases, u converges uniformly toward zero, reflecting the progressive saturation of the medium.

3) Convergence is rapid in the early steps and slows as the profile becomes quasi-uniform—a behaviour well-known from Richards equation theory.

Unconditional stability (Proposition 4.2) justifies the choice $\tau = h = 0.01$ independently of any CFL constraint.

6. Conclusions

We have established the well-posedness of the Richards-type elliptic-parabolic problem (1) in the renormalized sense for L^1 data, and designed a provably convergent and unconditionally stable finite-element scheme. The two-parameter approximation $\psi_{m,n}$ provides a flexible tool for handling the degeneracy of M and the lack of integrability of the data simultaneously.

Several extensions are natural:

- Multidimensional simulations on unstructured meshes (FEM or FVM).
- Extension to the fully degenerate case $m_0 = 0$ (Richards equation at full saturation) using time-space estimates of [10].
- Adaptive time-stepping guided by the Picard convergence rate.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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