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On Instability of Standing Waves for the Hartree Equation with a Magnetic Field

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Abstract

In this paper, we consider the strong instability of standing waves for the Hartree equation with a constant magnetic field. This equation arises as a mean-field model describing the evolution of many-body Bosonic systems. We first establish the variational characterization of ground states. Then, we prove that if $\frac{\partial^2}{\partial\lambda^2} S_\omega(\phi^\lambda)\Big|_{\lambda=1} \le 0$, the ground state standing wave $e^{i\omega t}\phi(x)$ is strongly unstable by blow-up, where S_ω is the action functional, and $\phi^\lambda(x) = \lambda^{3/2}\phi(\lambda x)$ is the L^2-invariant scaling.

Keywords

Hartree Equation, Magnetic Schrödinger Equation, Standing Waves, Strong Instability, Blow-Up

1. Introduction

In this paper, we consider the three-dimensional Hartree equation with a constant magnetic field

$$\begin{cases} i\partial_t u + (\nabla + iA)^2 u = -\left(|x|^{-\alpha} * |u|^2\right)u, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ u|_{t=0} = u_0, \end{cases} \tag{1}$$

where $u = u(t,x)$ is a complex-valued function. The vector potential $A = A(x) = \frac{b}{2}(-x_2, x_1, 0)$ describes a constant magnetic field $B = \nabla \times A = (0,0,b)$ with $b \neq 0$. Here $*$ denotes the convolution, $|x|^{-\alpha}$ is the Riesz potential, and $0 < \alpha < 3$.

The Schrödinger equation with a constant magnetic field provides an effective model for describing the dynamics of a single non-relativistic quantum particle in an electromagnetic field (see, e.g., [1]). Replacing $|u|^2 u$ with $\left(|x|^{-\alpha} * |u|^2\right)u$ leads to the magnetic Hartree Equation (1), which arises as a mean-field model for initially factorized bosonic states (see, e.g., [2]-[4]).

Equation (1) admits a class of special solutions known as standing waves. These solutions take the form $u(t,x) = \mathrm{e}^{i\omega t}\phi(x)$, where $\omega \in \mathbb{R}$ is a frequency and ϕ is a complex-valued function. The standing wave profile ϕ satisfies the elliptic equation

$$-(\nabla + iA)^2 \phi + \omega\phi - \left(|x|^{-\alpha} * |\phi|^2\right)\phi = 0 \text{ in } \mathbb{R}^3. \tag{2}$$

Solutions of (2) are critical points of the action functional defined on the magnetic Sobolev space $H_A^1(\mathbb{R}^3)$. This action functional is given by

$$\begin{aligned} S_\omega(u) &:= E(u) + \frac{\omega}{2}M(u) \\ &= \frac{1}{2}\|(\nabla + iA)u\|_{L^2}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 - \frac{1}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |u|^2\right)|u|^2 \,\mathrm{d}x. \end{aligned} \tag{3}$$

Equation (1) is locally well-posed in the energy space $H_A^1(\mathbb{R}^3)$ (see [5] [6]). More precisely, for $0 < \alpha < 3$ and $u_0 \in H_A^1(\mathbb{R}^3)$, there exists $T_{\max} = T_{\max}(u_0) \in (0, +\infty]$ and a unique solution

$$u \in C\left([0, T_{\max}), H_A^1(\mathbb{R}^3)\right) \cap C^1\left([0, T_{\max}), H_A^{-1}(\mathbb{R}^3)\right),$$

where $H_A^{-1}(\mathbb{R}^3)$ is the dual space of $H_A^1(\mathbb{R}^3)$. The solution exhibits the following blow-up alternative: either $T_{\max} = +\infty$ (global existence), or $T_{\max} < +\infty$ and $\lim_{t \nearrow T_{\max}} \|u(t)\|_{H_A^1} = +\infty$ (finite-time blow-up). Furthermore, the solution $u(t)$ satisfies the following conservation laws:

$$(\text{Mass}) \quad M(u(t)) := \|u(t)\|_{L^2}^2 = M(u_0),$$

$$(\text{Energy}) \quad E(u(t)) := \frac{1}{2}\|(\nabla + iA)u(t)\|_{L^2}^2 - \frac{1}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |u(t)|^2\right)|u(t)|^2 \,\mathrm{d}x = E(u_0).$$

If the nonlocal term $\left(|x|^{-\alpha} * |u|^2\right)u$ is replaced by the local nonlinearity $|u|^2 u$, then the nonlinear Schrödinger Equation (1) has been extensively studied, and there exists a large body of literature on the Cauchy problem for this equation (see, e.g., [5] [7] and the references therein). In particular, the blow-up and global existence of solutions to the magnetic Schrödinger equation were investigated in [8]-[10]. Moreover, the stability and instability of standing waves for this equation were studied in [9] [11]. For the linear magnetic Schrödinger operator, we refer the reader to the works of Avron, Herbst, and Simon [12]-[14]. Furthermore, Dinh revisited the Cauchy problem for the three-dimensional nonlinear Schrödinger equation with a constant magnetic field in [15].

We next turn to the Hartree Equation (1). In the case $A \equiv 0$, the Hartree equation has been widely studied. In particular, there exist many results on the Cauchy problem and the asymptotic behavior of solutions (see, e.g., [16]-[20]).

For further studies related to the Hartree equation, we also refer the reader to [21] [22] and the references therein.

We now consider the magnetic Hartree Equation (1). Cazenave [5] [6] established the local well-posedness of this equation. More recently, the existence and orbital stability of normalized standing waves have been studied in [23]. But as we know, there is no result on the strong instability of ground state standing waves in the magnetic Hartree Equation (1).

Compared with the existing literature, the present work contains several new features. First, the presence of the magnetic field together with the nonlocal Hartree interaction creates additional difficulties in the variational analysis and in the derivation of virial-type estimates. In particular, the interaction between the magnetic operator and the nonlocal convolution term requires a more delicate treatment under the L^2-invariant scaling.

Second, the instability analysis combines the variational characterization of ground states, conservation laws associated with the magnetic flow, and a refined virial argument adapted to the Hartree nonlinearity.

The main purpose of this paper is to investigate the strong instability of standing waves for (1). Motivated by the approach of Berestycki and Cazenave [24], we first establish the existence and variational characterization of ground states. We then prove finite-time blow-up of solutions with initial data arbitrarily close to the ground states, which yields the strong instability of standing waves. We now state our main results.

Theorem 1.1. Assume $0<\alpha<3$ and $\omega>-|b|$. Then there exists a ground state solution of (2). In particular, $u(t,x)=\mathrm{e}^{i\omega t}\phi(x)$ is a solution to (1). Moreover, the set of ground states $\mathcal{G}(\omega)$ is characterized by

$$\mathcal{G}(\omega)=\left\{\phi\in H_A^1(\mathbb{R}^3)\setminus\{0\}:S_\omega(\phi)=d(\omega),K_\omega(\phi)=0\right\},$$

here

$$d(\omega):=\inf\left\{S_\omega(f):f\in H_A^1\setminus\{0\},K_\omega(f)=0\right\} \tag{4}$$

with the Nehari functional

$$K_\omega(f):=\left\|(\nabla+iA)f\right\|_{L^2}^2+\omega\|f\|_{L^2}^2-\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x. \tag{5}$$

Remark 1. We characterize the ground state as a minimizer of the action functional S_ω on the Nehari manifold $\mathcal{N}_\omega=\left\{f\neq 0:K_\omega(f)=0\right\}$. Under the constraint $K_\omega(f)=0$, by combining the Hardy-Littlewood-Sobolev inequality, the Gagliardo-Nirenberg inequality, and the equivalent norm relation (19), we derive that every minimizing sequence for $d(\omega)$ has a uniform positive lower bound in the $L^{\frac{12}{6-\alpha}}$-norm, thereby ruling out vanishing of minimizing sequences. Using the compactness lemma (Lemma 2.7), we then obtain a nontrivial weak limit and verify that it satisfies the Euler-Lagrange equation, yielding the existence of a ground state. This approach avoids the compactness difficulties in the fixed-mass framework, where the classical scaling argument cannot be directly applied in the presence of a magnetic field. The argument is inspired by Dinh [15].

Theorem 1.2. Let $2<\alpha<3$, $\omega>-|b|$, and let $\phi\in\mathcal{G}(\omega)$. If

$$\frac{\partial^2}{\partial\lambda^2}S_\omega\left(\phi^\lambda\right)\Big|_{\lambda=1}\le 0, \tag{6}$$

where

$$\phi^\lambda(x)=\lambda^{3/2}\phi(\lambda x), \tag{7}$$

then the ground state standing wave $e^{i\omega t}\phi(x)$ is strongly unstable. More precisely, for any $\varepsilon>0$, there exists $u_0\in\Sigma_A\left(\mathbb{R}^3\right)$ such that $\|u_0-\phi\|_{\Sigma_A}<\varepsilon$, and the solution of (1) with initial data u_0 blows up in finite time.

Remark 2. *Condition* (1.6) *describes the behavior of the action functional along the scaling direction* $\phi^\lambda(x)=\lambda^{\frac{3}{2}}\phi(\lambda x)$. *In particular, the condition*

$$\partial_\lambda^2 S_\omega\left(\phi^\lambda\right)\Big|_{\lambda=1}\le 0$$

implies that the action functional decreases locally along the scaling orbit near $\lambda=1$. *Combined with the variational characterization of ground states and the virial identity, this allows us to construct suitable perturbations of the standing wave, which play a crucial role in proving finite time blow-up. Similar conditions naturally appear in the instability theory developed by Ohta for nonlinear Schrödinger and Hartree-type equations; see* [25].

This paper is organized as follows. In Section 2, we present some useful lemmas, including the local well-posedness of Equation(1) and several inequalities. Section 3 is devoted to the existence of ground state standing waves. Finally, in Section 4, we study the strong instability of ground state standing waves.

2. Preliminaries

In this section, we collect several preliminary results that will be repeatedly used in the subsequent analysis. For convenience, Equation (1) can be rewritten as

$$\begin{cases} i\partial_t\varphi=-\Delta\varphi+L_b\varphi+\dfrac{b^2}{4}\left(x_1^2+x_2^2\right)\varphi-\varphi\left(|x|^{-\alpha}*|\varphi|^2\right),\\ \varphi|_{t=0}=\varphi_0, \end{cases} \tag{8}$$

where

$$L_b:=ib\left(x_2\partial_1-x_1\partial_2\right), \tag{9}$$

denotes the third component of the angular momentum operator.

First, we recall that the local well-posedness of Equation (1) in $H_A^1\left(\mathbb{R}^3\right)$ was established by Cazenave and Esteban in [6]. Here, $H_A^1\left(\mathbb{R}^3\right):=\left\{f\in L^2\left(\mathbb{R}^3\right):\left|(\nabla+iA)f\right|\in L^2\left(\mathbb{R}^3\right)\right\}$ is a Hilbert space equipped with the norm

$$\|f\|_{H_A^1}^2=\|(\nabla+iA)f\|_{L^2}^2+\|f\|_{L^2}^2.$$

Lemma 2.1. [7] Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$. Then $H^1_A(\mathbb{R}^3)$ equipped with the inner product

$$\langle f, g \rangle_{H^1_A} := \int_{\mathbb{R}^3} f \bar{g} \mathrm{d}x + \int_{\mathbb{R}^3} (\nabla + iA) f \cdot \overline{(\nabla + iA) g} \mathrm{d}x$$

is a Hilbert space.

To state blow-up result for (1),let us introduce the following Hilbert space Σ_A

$$\Sigma_A(\mathbb{R}^3) := \left\{ f \in H^1_A(\mathbb{R}^3) : |x| f \in L^2(\mathbb{R}^3) \right\} \tag{10}$$

is equipped with the norm

$$\|f\|^2_{\Sigma_A} := \|(\nabla + iA) f\|^2_{L^2} + \|xf\|^2_{L^2} + \|f\|^2_{L^2}.$$

As shown in ([26], Lemma 2.2] we have $\Sigma_A(\mathbb{R}^3) \equiv \Sigma(\mathbb{R}^3)$, where $\Sigma(\mathbb{R}^3)$ is the standard weighted Sobolev space

$$\Sigma(\mathbb{R}^3) := \left\{ f \in H^1(\mathbb{R}^3) : |x| f \in L^2(\mathbb{R}^3) \right\}. \tag{11}$$

In addition, we have the following useful identity

$$\|(\nabla + iA) f\|^2_{L^2} = \|\nabla f\|^2_{L^2} + bR(f) + \frac{b^2}{4} \|\rho f\|^2_{L^2}, \tag{12}$$

where $\rho := \sqrt{x_1^2 + x_2^2}$ and

$$R(f) := i \int \left(x_2 \partial_{x_1} f - x_1 \partial_{x_2} f \right) \bar{f} \mathrm{d}x = \int L_b f \bar{f} \mathrm{d}x \tag{13}$$

is called the angular momentum with L_b as in (9). Since $A(x) = \frac{b}{2}(-x_2, x_1, 0)$ and the Hartree kernel $|x|^{-\alpha}$ are invariant under rotations around the z-axis, the Hamiltonian associated with (1) commutes with the angular momentum operator L_b. Hence the angular momentum $R(u(t))$ is conserved along the flow of (1). See Avron, Herbst and Simon [12] for a rigorous proof.

We recall that if the initial data $u_0 \in \Sigma_A(\mathbb{R}^3)$, then the corresponding solution of (1) remains in $\Sigma_A(\mathbb{R}^3)$ on its maximal existence interval; see [15]. Therefore, the following virial identity is well defined.

To proceed with the variational analysis, we recall several basic properties of the magnetic Sobolev space.

Lemma 2.2. [7] Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^3)$. Then the following properties hold:

1) $C_0^\infty(\mathbb{R}^3)$ is dense in $H^1_A(\mathbb{R}^3)$.

2) $H^1_A(\mathbb{R}^3)$ is continuously embedded in $L^r(\mathbb{R}^3)$ for all $2 \le r \le 6$.

3) Assume that A is linear, *i.e.*, $A(x+y) = A(x) + A(y)$ for all $x, y \in \mathbb{R}^3$. Let $y \in \mathbb{R}^3$, $f \in H^1_A(\mathbb{R}^3)$, and set

$$\tilde{f}(x) := \mathrm{e}^{iA(y)\cdot x} f(x+y), \quad x \in \mathbb{R}^3.$$

Then, $(\nabla + iA)\tilde{f}(x) = \mathrm{e}^{iA(y)\cdot x} (\nabla + iA) f(x+y)$. In particular,

$$\|(\nabla + iA)\tilde{f}\|_{L^2} = \|(\nabla + iA) f\|_{L^2}.$$

4) If $A \in L^3_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$, then $H^1_A(\mathbb{R}^3)$ is continuously embedded in $H^1_{\text{loc}}(\mathbb{R}^3)$. In particular, $H^1_A(\mathbb{R}^3)$ is compactly embedded in $L^r_{\text{loc}}(\mathbb{R}^3)$ for all $2 \le r < 6$.

Remark 3. This lemma provides the basic embedding and translation properties needed later in Section 3.

Our further considerations need the following inequalities.

Lemma 2.3. [12] Let $A \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ and $j, k \in \{1,2,3\}$. Then, for any $f \in C_0^\infty(\mathbb{R}^3)$, we have

$$\left| \int_{\mathbb{R}^3} \left(\partial_j A_k - \partial_k A_j\right) f \bar{f} \, \mathrm{d}x \right| \le \left\| \left(\partial_j + iA_j\right) f \right\|_{L^2}^2 + \left\| \left(\partial_k + iA_k\right) f \right\|_{L^2}^2.$$

In particular, if $A = \frac{b}{2}(-x_2, x_1, 0)$, then

$$|b| \|f\|_{L^2}^2 \le \|(\nabla + iA) f\|_{L^2}^2. \tag{14}$$

Lemma 2.4. (Diamagnetic inequality [27]) Let $A \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ and $f \in H^1_A(\mathbb{R}^3)$. Then, $|f| \in H^1(\mathbb{R}^3)$. In particular, we have

$$\left| \nabla |f|(x) \right| \le \left| (\nabla + iA) f(x) \right| \text{ a.e. } x \in \mathbb{R}^3. \tag{15}$$

Combining the Diamagnetic inequality with the classical Gagliardo-Nirenberg inequality, we obtain the following magnetic version, which will be used to control the nonlinear Hartree term.

Lemma 2.5. (Magnetic Gagliardo-Nirenberg inequality[23]). For $A \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ and $2 < r < 6$,

$$\|f\|_{L^r}^r \le C_r \|(\nabla + iA) f\|_{L^2}^{\frac{3(r-2)}{2}} \|f\|_{L^2}^{\frac{6-r}{2}}, \quad \forall f \in H^1_A(\mathbb{R}^3), \tag{16}$$

where $C_r > 0$ is the optimal constant.

Lemma 2.6. (Hardy-Littlewood-Sobolev inequality [27]). Let $0 < \alpha < 3$ and $s, r > 1$ be constants satisfying $\frac{1}{r} + \frac{1}{s} + \frac{\alpha}{3} = 2$. Assume that $f \in L^r(\mathbb{R}^3)$ and $g \in L^s(\mathbb{R}^3)$. Then, there exists a positive constant $C(s, \alpha)$ depending on s and α such that

$$\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x) |x - y|^{-\alpha} g(y) \, \mathrm{d}x \mathrm{d}y \right| \le C(s, \alpha) \|f\|_{L^r} \|g\|_{L^s}.$$

In order to prove the existence of ground states, we need the following compactness lemma.

Lemma 2.7. [15]. Let $A \in L^3_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ be linear. Let $(f_n)_{n \ge 1}$ be a bounded sequence in $H^1_A(\mathbb{R}^3)$, *i.e.*,

$$\sup_{n \ge 1} \|f_n\|_{H^1_A} < +\infty.$$

Assume that there exists $\varepsilon_0 > 0$ such that

$$\inf_{n \ge 1} \|f_n\|_{L^r} \ge \varepsilon_0 \tag{17}$$

for some $2 < r < 6$. Then up to a subsequence, there exist $f \in H^1_A(\mathbb{R}^3) \setminus \{0\}$ and

$(y_n)_{n\ge1}\subset\mathbb{R}^3$ such that

$$\mathrm{e}^{iA(y_n)\cdot x}f_n(x+y_n)\rightharpoonup f \text{ weakly in } H_A^1(\mathbb{R}^3).$$

Remark 4. The proof of this compactness lemma is based on an argument of [28].

We also require splitting formulas for the magnetic kinetic energy and the Hartree interaction.

Lemma 2.8. [15] Let $A\in L^2_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{R}^3)$ and $(f_n)_{n\ge1}$ be a bounded sequence in $H_A^1(\mathbb{R}^3)$. Assume that $f_n\rightharpoonup f$ weakly in $H_A^1(\mathbb{R}^3)$. Then, we have

$$\|(\nabla+iA)f_n\|_{L^2}^2=\|(\nabla+iA)f\|_{L^2}^2+\|(\nabla+iA)(f_n-f)\|_{L^2}^2+o_n(1),$$
$$\|f_n\|_{L^r}^r=\|f\|_{L^r}^r+\|f_n-f\|_{L^r}^r+o_n(1),\quad 2\le r\le 6.$$

Lemma 2.9. [28] Let $0<\alpha<3$, $2<p<+\infty$. Assume that f_n and f are functions on $\mathbb{R}^3$ such that

$$f_n(x)\to f(x) \text{ a.e. } x\in\mathbb{R}^3,\ \sup_n\|f_n\|_{L^p}<+\infty,$$

and

$$\sup_n\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f_n(x)|^2|f_n(y)|^2}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y<+\infty.$$

Then,

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f_n(x)|^2|f_n(y)|^2}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f_n(x)-f(x)|^2|f_n(y)-f(y)|^2}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y$$
$$+\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|f(x)|^2|f(y)|^2}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y+o_n(1).$$

Lemma 2.10. [23] Let $A\in L^2_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{R}^3)$ and $(f_n)_{n\ge1}$ be a bounded sequence in $H_A^1(\mathbb{R}^3)$. Assume that $f_n\rightharpoonup f$ weakly in $H_A^1(\mathbb{R}^3)$. Then, we have

$$\left\|\left(|x|^{-\alpha}*|f_n|^2\right)|f_n|^2\right\|_{L^1}$$
$$=\left\|\left(|x|^{-\alpha}*|f|^2\right)|f|^2\right\|_{L^1}+\left\|\left(|x|^{-\alpha}*\left(|f_n|^2-|f|^2\right)\right)|f_n|^2\right\|_{L^1}+o_n(1).$$

Next, we need the following variance identity.

Lemma 2.11. Let $0<\alpha<3$ and $u_0\in\Sigma_A(\mathbb{R}^3)$. Let $u:[0,T_{\max})\times\mathbb{R}^3\to\mathbb{C}$ be the corresponding solution to (1). Set

$$I(u(t))=\int_{\mathbb{R}^3}|x|^2|u(t,x)|^2\mathrm{d}x, \tag{18}$$

then

$$I''(u(t))=8\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x-2b^2\int_{\mathbb{R}^3}\rho^2|u|^2\mathrm{d}x-2\alpha\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|^\alpha}\mathrm{d}y\mathrm{d}x.$$

Proof. Clearly there holds

$$I'(u(t)) = \int_{\mathbb{R}^3} |x|^2 \partial_t \left(|u|^2\right) dx = 2\,\mathrm{Re}\int_{\mathbb{R}^3} |x|^2 \bar{u} \partial_t u dx,$$

Noticing that $i\partial_t u = -(\nabla + iA)^2 u - \left(V * |u|^2\right) u$ with $V(x) = |x|^{-\alpha}$, we obtain

$$I'(u(t)) = 4\,\mathrm{Im}\int_{\mathbb{R}^3} \bar{u}(x \cdot \nabla u) dx.$$

Differentiating again yields

$$\begin{aligned} I''(u(t)) &= 4\,\mathrm{Im}\int_{\mathbb{R}^3} \left(\bar{u}_t (x \cdot \nabla u) + \bar{u}(x \cdot \nabla u_t)\right) dx \\ &= 4\,\mathrm{Re}\int_{\mathbb{R}^3} \Big[-\Big(\Delta\bar{u} - 2iA\cdot\nabla\bar{u} - |A|^2 \bar{u} - \left(V * |u|^2\right)\bar{u}\Big)(x\cdot\nabla u) \\ &\quad + \bar{u}\Big(x\cdot\nabla\Big(\Delta u + 2iA\cdot\nabla u - |A|^2 u + \left(V*|u|^2\right)u\Big)\Big)\Big] dx. \end{aligned}$$

We decompose this expression into three parts:

$$\begin{aligned} A &= \int_{\mathbb{R}^3} \left[-(\Delta\bar{u})(x\cdot\nabla u) + \bar{u}(x\cdot\nabla\Delta u)\right] dx \\ &\quad + \int_{\mathbb{R}^3} \left[2i(A\cdot\nabla\bar{u})(x\cdot\nabla u) + \bar{u}(x\cdot\nabla(2iA\cdot\nabla u))\right] dx, \\ B &= \int_{\mathbb{R}^3} \left[|A|^2 \bar{u}(x\cdot\nabla u) - \bar{u}\left(x\cdot\nabla\left(|A|^2 u\right)\right)\right] dx, \\ C &= \int_{\mathbb{R}^3} \left[\left(V*|u|^2\right)\bar{u}(x\cdot\nabla u) - \bar{u}\left(x\cdot\nabla\left(\left(V*|u|^2\right)u\right)\right)\right] dx, \end{aligned}$$

so that $I''(u(t)) = 4\,\mathrm{Re}(A+B+C)$. Direct computations show that:

$$\mathrm{Re}(A) = 2\int_{\mathbb{R}^3} |\nabla u|^2 dx,$$

$$\mathrm{Re}(B) = -\frac{b^2}{2}\int_{\mathbb{R}^3} \rho^2 |u|^2 dx.$$

For part C, let $K(x) = \left(V*|u|^2\right)(x) = \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|^\alpha} dy$. Then

$$\begin{aligned} C &= \int_{\mathbb{R}^3} \left[K\bar{u}(x\cdot\nabla u) - \bar{u}(x\cdot\nabla(Ku))\right] dx \\ &= -\int_{\mathbb{R}^3} (x\cdot\nabla K)|u|^2 dx \\ &= \alpha\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{x\cdot(x-y)}{|x-y|^{\alpha+2}} |u(x)|^2 |u(y)|^2 dy dx. \end{aligned}$$

Using symmetry by interchanging x and y, we obtain

$$\mathrm{Re}(C) = -\frac{\alpha}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^\alpha} dy dx.$$

Combining all parts:

$$\begin{aligned} I''(u(t)) &= 4\left[2\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{b^2}{2}\int_{\mathbb{R}^3} \rho^2 |u|^2 dx - \frac{\alpha}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^\alpha} dy dx\right] \\ &= 8\int_{\mathbb{R}^3} |\nabla u|^2 dx - 2b^2\int_{\mathbb{R}^3} \rho^2 |u|^2 dx - 2\alpha\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^\alpha} dy dx. \end{aligned}$$

□

3. Existence of Ground State Standing Waves

This section is devoted to the proofs of the existence of ground states for (2) in the mass-supercritical case. A nonzero solution ϕ of (2) is referred to as a ground state associated with (2) if it minimizes the action functional $S_\omega(f)$ over all nontrivial solutions of (2). Note that (2) can be expressed as $S'_\omega(\phi)=0$. We accordingly define

$$\mathcal{A}(\omega):=\left\{f\in H_A^1(\mathbb{R}^3)\setminus\{0\}:S'_\omega(f)=0\right\}$$

as the set of all nontrivial solutions to (2), and introduce

$$\mathcal{G}(\omega):=\left\{\phi\in\mathcal{A}(\omega):S_\omega(\phi)\le S_\omega(f)\text{ for all }f\in\mathcal{A}(\omega)\right\}$$

as the corresponding set of ground states.

Before proceeding to the proof of Theorem 1.1, we establish the following preliminary.

Lemma 3.1. Let $A(x)=\frac{b}{2}(-x_2,x_1,0)$ and $\omega>-|b|$. Then

$$H_\omega(f):=\|(\nabla+iA)f\|_{L^2}^2+\omega\|f\|_{L^2}^2\simeq\|(\nabla+iA)f\|_{L^2}^2+\|f\|_{L^2}^2. \tag{19}$$

Proof. In fact, by (14), we see that

$$H_\omega(f)=\|(\nabla+iA)f\|_{L^2}^2+\omega\|f\|_{L^2}^2\ge(\omega+|b|)\|f\|_{L^2}^2.$$

It follows that

$$\begin{aligned}\|(\nabla+iA)f\|_{L^2}^2+\|f\|_{L^2}^2&\le\|(\nabla+iA)f\|_{L^2}^2+\omega\|f\|_{L^2}^2+|1-\omega|\|f\|_{L^2}^2\\&\le\left(1+\frac{|1-\omega|}{\omega+|b|}\right)\left(\|(\nabla+iA)f\|_{L^2}^2+\omega\|f\|_{L^2}^2\right).\end{aligned}$$

On the other hand, we see that

$$H_\omega(f)\le\|(\nabla+iA)f\|_{L^2}^2+\omega\|f\|_{L^2}^2\le(1+|\omega|)\left(\|(\nabla+iA)f\|_{L^2}^2+\|f\|_{L^2}^2\right).$$

Lemma 3.2. Let $A(x)=\frac{b}{2}(-x_2,x_1,0)$, $0<\alpha<3$, and $\omega>-|b|$. Then there exists $f\in H_A^1(\mathbb{R}^3)$ such that $K_\omega(f)=0$.

Indeed, for $f\in C_0^\infty(\mathbb{R}^3)$. We have

$$K_\omega(\lambda f)=\lambda^2H_\omega(f)-\lambda^4\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x,\ \lambda>0.$$

It follows that $K_\omega(\lambda_0 f)=0$ with $\lambda_0=\left(\dfrac{H_\omega(f)}{\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x}\right)^{\frac{1}{2}}$.

Next we have the following existence of minimizers for $d(\omega)>0$.

Lemma 3.3. Let $A(x)=\frac{b}{2}(-x_2,x_1,0)$, $0<\alpha<3$, and $\omega>-|b|$. Then there exists a minimizer for $d(\omega)$ and hence the set of all minimizers for $d(\omega)$

defined by

$$\mathcal{D}(\omega) := \left\{\phi \in H_A^1(\mathbb{R}^3)\setminus\{0\} : S_\omega(\phi) = d(\omega), K_\omega(\phi) = 0\right\}$$

is not empty.

Proof. The proof is done by several steps.

Step 1. We first show that $d(\omega) > 0$. Let $f \in H_A^1(\mathbb{R}^3)$ be such that $K_\omega(f) = 0$.

By the Hardy-Littlewood-Sobolev inequality (Lemma 2.7) and magnetic Gagliardo-Nirenberg inequality (Lemma 2.6), we have

$$\begin{aligned} H_\omega(f) &= \int_{\mathbb{R}^3} \left(|x|^{-\alpha} * |f|^2\right)|f|^2 \,\mathrm{d}x \\ &\le C_1 \|f\|^4_{L^{\frac{12}{6-\alpha}}} \\ &\le C_2 \|(\nabla + iA) f\|^\alpha_{L^2} \|f\|^{4-\alpha}_{L^2} \\ &\le C_3 H_\omega(f)^{\frac{\alpha}{2}} H_\omega(f)^{\frac{4-\alpha}{2}} \\ &= C_3 H_\omega(f)^2 \end{aligned}$$

which implies $H_\omega(f) \ge C > 0$. It follows that

$$S_\omega(f) = \frac{1}{4} H_\omega(f) = \frac{1}{4}\int_{\mathbb{R}^3} \left(|x|^{-\alpha} * |f|^2\right)|f|^2 \,\mathrm{d}x \ge \frac{1}{4} C > 0.$$

Taking the infimum over all $f \in H_A^1(\mathbb{R}^3)\setminus\{0\}$ satisfying $K_\omega(f) = 0$, we obtain $d(\omega) > 0$.

Step 2. We next show that there exists a minimizer for $d(\omega)$. Let $(f_n)_{n\ge1}$ be a minimizing sequence for $d(\omega)$. We have

$$\frac{1}{4} H_\omega(f_n) = S_\omega(f_n) \to d(\omega) > 0 \text{ as } n \to \infty,$$

which, by (3.1), implies that $(f_n)_{n\ge1}$ is a bounded sequence in $H_A^1(\mathbb{R}^3)$. As $K_\omega(f_n) = 0$, we have

$$\int_{\mathbb{R}^3} \left(|x|^{-\alpha} * |f_n|^2\right)|f_n|^2 \,\mathrm{d}x = H_\omega(f_n) \to 4d(\omega) > 0 \text{ as } n \to \infty.$$

Thus up to a subsequence, we have $\inf_{n\ge1} \int_{\mathbb{R}^3} \left(|x|^{-\alpha} * |f_n|^2\right)|f_n|^2 \,\mathrm{d}x \ge C > 0$, by the Hardy-Littlewood-Sobolev inequality, this implies $\inf_{n\ge1} \|f_n\|_{L^{\frac{12}{6-\alpha}}} \ge \epsilon_0 > 0$. Applying Lemma 2.7, there exist $f \in H_A^1(\mathbb{R}^3)\setminus\{0\}$ and $(y_n)_{n\ge1} \subset \mathbb{R}^3$ such that

$$\tilde{f}_n(x) := \mathrm{e}^{iA(y_n)\cdot x} f_n(x + y_n) \rightharpoonup f \text{ weakly in } H_A^1(\mathbb{R}^3).$$

Thanks to Lemma 2.8 and Lemma 2.9, we have

$$H_\omega(\tilde{f}_n) = H_\omega(f) + H_\omega(\tilde{f}_n - f) + o_n(1), \tag{20}$$

$$K_\omega(\tilde{f}_n) = K_\omega(f) + K_\omega(\tilde{f}_n - f) + o_n(1). \tag{21}$$

We will show that $K_\omega(f) = 0$. Indeed, if $K_\omega(f) < 0$, then there exists

$\lambda_0 \in (0,1)$ such that $K_\omega(\lambda_0 f) = 0$. From the definition of $d(\omega)$, we have

$$
\begin{aligned}
d(\omega) &\le S_\omega(\lambda_0 f) = \frac{1}{4} H_\omega(\lambda_0 f) = \frac{1}{4}\lambda_0^2 H_\omega(f) \\
&< \frac{1}{4} H_\omega(f) \le \frac{1}{4}\liminf_{n\to\infty} H_\omega(\tilde{f}_n) \\
&= \liminf_{n\to\infty} \frac{1}{4} H_\omega(f_n) = d(\omega),
\end{aligned}
$$

which is a contradiction. If $K_\omega(f) > 0$, then, by (21) and the fact that $K_\omega(\tilde{f}_n) = K_\omega(f_n) = 0$, we have $K_\omega(\tilde{f}_n - f) < 0$ for n sufficiently large. Thus there exists $(\lambda_n)_{n\ge1} \subset (0,1)$ such that $K_\omega(\lambda_n(\tilde{f}_n - f)) = 0$. It follows that

$$
\begin{aligned}
d(\omega) &\le S_\omega(\lambda_n(\tilde{f}_n - f)) = \frac{1}{4}\lim_{n\to\infty} H_\omega(\lambda_n(\tilde{f}_n - f)) \\
&= \frac{1}{4}\lim_{n\to\infty} \lambda_n^2 H_\omega(\tilde{f}_n - f) \le \frac{1}{4}\lim_{n\to\infty} H_\omega(\tilde{f}_n - f) \\
&= \frac{1}{4}\left(\lim_{n\to\infty} H_\omega(\tilde{f}_n) - H_\omega(f)\right) = \frac{1}{4}\lim_{n\to\infty} H_\omega(f_n) - \frac{1}{4} H_\omega(f) \\
&= d(\omega) - \frac{1}{4} H_\omega(f) < d(\omega),
\end{aligned}
$$

which is also a contradiction. Here the fourth line follows from (20). Thus we have $K_\omega(f) = 0$.

By the definition of $d(\omega)$, we have

$$
\begin{aligned}
d(\omega) &\le S_\omega(f) = \frac{1}{4} H_\omega(f) \le \frac{1}{4}\liminf_{n\to\infty} H_\omega(\tilde{f}_n) \\
&= \frac{1}{4}\liminf_{n\to\infty} H_\omega(f_n) = d(\omega).
\end{aligned}
$$

This shows that $S_\omega(f) = d(\omega)$ or f is a minimizer for $d(\omega)$. □

Proof of Theorem 1.1. By Lemma 3.3, we have $\mathcal{D}(\omega) \ne \emptyset$. We will show that $\mathcal{D}(\omega) \equiv \mathcal{G}(\omega)$.

We first prove $\mathcal{D}(\omega) \subset \mathcal{G}(\omega)$. To see this, let $\phi \in \mathcal{D}(\omega)$. Since ϕ is a minimizer for $d(\omega)$, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $S_{\omega'}(\phi) = \lambda K_{\omega'}(\phi)$. It follows that

$$
\begin{aligned}
K_\omega(\phi) &= \langle S'_\omega(\phi), \phi\rangle_{L^2} = \lambda \langle K'_\omega(\phi), \phi\rangle_{L^2} \\
&= \lambda\left(2K_\omega(\phi) - 2\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |f|^2\right)|f|^2 \,\mathrm{d}x\right).
\end{aligned}
$$

Since $K_\omega(f) = 0$ and $\phi \ne 0$, we infer that $\lambda = 0$, hence $S_{\omega'}(\phi) = 0$ or $\phi \in \mathcal{A}(\omega)$. Now let $f \in \mathcal{A}(\omega)$. Since $K_\omega(f) = 0$, we have $S_\omega(f) \ge d(\omega) = S_\omega(\phi)$. This shows that there exists $\phi \in H^1_A(\mathbb{R}^3)\setminus\{0\}$ such that $\phi \in \mathcal{A}(\omega)$ and $S_\omega(\phi) \le S_\omega(f)$ for all $f \in \mathcal{A}(\omega)$, hence $\phi \in \mathcal{G}(\omega)$. Finally, we show that $\mathcal{G}(\omega) \subset \mathcal{D}(\omega)$. To this end, let $\phi \in \mathcal{G}(\omega)$ and take

$f \in \mathcal{D}(\omega) \subset \mathcal{G}(\omega)$. We have $S_\omega(f) = S_\omega(\phi) = d(\omega)$. Since $\phi \in \mathcal{A}(\omega)$, we have $K_\omega(\phi) = 0$. Thus $\phi \in \mathcal{D}(\omega)$.

4. Strong Instability of Ground State Standing Waves

Before giving the proof of the strong instability of ground state standing waves for (1) in the mass-supercritical regime. Let us prepare some lemmas.

Lemma 4.1. [15] Let $A(x) = \frac{b}{2}(-x_2, x_1, 0)$, $0 < \alpha < 3$, $\omega > -|b|$, and $\phi \in \mathcal{G}(\omega)$. Then $\phi \in L^r(\mathbb{R}^3)$ for all $2 \le r \le \infty$ and $\lim_{|x|\to\infty} \phi(x) = 0$. Moreover, there exists $\delta > 0$ such that $\mathrm{e}^{\delta|x|}\phi \in L^2(\mathbb{R}^3)$. In particular, the exponential decay implies that $\phi \in \Sigma_A(\mathbb{R}^3)$.

Remark 5. The proof of Lemma 0.17 relies on the results of Chabrowski and Szulkin [29] and Raymond [30].

Lemma 4.2. Let $A(x) = \frac{b}{2}(-x_2, x_1, 0)$, $0 < \alpha < 3$, $\omega > -|b|$, and $\phi \in \mathcal{G}(\omega)$. Then we have

$$K_\omega(\phi) = H(\phi) = 0,$$

where

$$H(\phi) = \|\nabla\phi\|_{L^2}^2 - \frac{b^2}{4}\|\rho\phi\|_{L^2}^2 - \frac{\alpha}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,\mathrm{d}x,$$

and

$$K_\omega(\phi) = \|(\nabla + iA)\phi\|_{L^2}^2 + \omega\|\phi\|_{L^2}^2 - \int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,\mathrm{d}x.$$

Proof. As $\phi \in \mathcal{G}(\omega)$, we know that ϕ is a solution to (2). Multiplying both sides of (2) with $\overline{\phi}$ and integrating over $\mathbb{R}^3$, we have $K_\omega(\phi) = 0$. As $\phi \in \Sigma_A(\mathbb{R}^3)$ (see Lemma 4.1), we have from (12) that

$$\|\nabla\phi\|_{L^2}^2 + bR(\phi) + \frac{b^2}{4}\|\rho\phi\|_{L^2}^2 + \omega\|\phi\|_{L^2}^2 - \int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,\mathrm{d}x = 0. \tag{22}$$

On the other hand, we rewrite (2) as

$$-\Delta\phi + bL_z\phi + \frac{b^2}{4}\rho^2\phi + \omega\phi - \left(|x|^{-\alpha} * |\phi|^2\right)\phi = 0. \tag{23}$$

Multiplying both sides of (23) with $x \cdot \nabla\overline{\phi}$, integrating over $\mathbb{R}^3$, and taking the real part, we have

$$\begin{aligned} &-\frac{1}{2}\|\nabla\phi\|_{L^2}^2 - \frac{3}{2}bR(\phi) - \frac{5b^2}{8}\|\rho\phi\|_{L^2}^2 - \frac{3\omega}{2}\|\phi\|_{L^2}^2 \\ &-\frac{\alpha-6}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,\mathrm{d}x = 0. \end{aligned} \tag{24}$$

Here we have used the following identities which can be checked by integration by parts

$$\mathrm{Re}\left(\int_{\mathbb{R}^3}\Delta\phi\cdot x\cdot\nabla\bar{\phi}\mathrm{d}x\right)=\frac{1}{2}\|\nabla\phi\|_{L^2}^2,$$
$$\mathrm{Re}\left(\int_{\mathbb{R}^3}bL_z\phi\cdot x\cdot\nabla\bar{\phi}\mathrm{d}x\right)=-\frac{3}{2}bR(\phi),$$
$$\mathrm{Re}\left(\int_{\mathbb{R}^3}\rho^2\phi\cdot x\cdot\nabla\bar{\phi}\mathrm{d}x\right)=-\frac{5}{2}\|\rho\phi\|_{L^2}^2,$$
$$\mathrm{Re}\left(\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|\phi|^2\right)\phi\cdot x\cdot\nabla\bar{\phi}\mathrm{d}x\right)=\frac{\alpha-6}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|\phi|^2\right)|\phi|^2\,\mathrm{d}x.$$

From (24) and (22), we infer that

$$\|\nabla\phi\|_{L^2}^2-\frac{b^2}{4}\|\rho\phi\|_{L^2}^2-\frac{\alpha}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|\phi|^2\right)|\phi|^2\,\mathrm{d}x=0$$

or $H(\phi)=0$. The proof is complete.

Lemma 4.3. Let $A(x)=\frac{b}{2}(-x_2,x_1,0)$, $2<\alpha<3$, $\omega>-|b|$, and $\phi\in\mathcal{G}(\omega)$. Assume that $\partial_\lambda^2 S_\omega\left(\phi^\lambda\right)\big|_{\lambda=1}\le 0$, where ϕ^λ is as in (7). Let $f\in\Sigma_A\left(\mathbb{R}^3\right)$ be such that

$$M(f)=M(\phi),\ R(f)=R(\phi),\ K_\omega(f)\le 0,\ H(f)\le 0, \tag{25}$$

then

$$H(f)\le 2\left(S_\omega(f)-d(\omega)\right), \tag{26}$$

where $d(\omega)$ is as in (4).

Proof. The proof is based on an argument of M.Ohta [25]. Let $f\in\Sigma_A\left(\mathbb{R}^3\right)$ satisfy (4.4), we first consider the case $K_\omega(f)=0$. By the definition of $d(\omega)$ and $H(f)\le 0$, it follows that

$$d(\omega)\le S_\omega(f)\le S_\omega(f)-\frac{1}{2}H(f),$$

which implies (26).

We now consider the case $K_\omega(f)<0$. As $f\in\Sigma_A\left(\mathbb{R}^3\right)$, we have from (12) that

$$\begin{aligned}K_\omega\left(f^\lambda\right)=\lambda^2\|\nabla f\|_{L^2}^2+bR(f)+\frac{b^2}{4}\lambda^{-2}\|\rho f\|_{L^2}^2+\omega\|f\|_{L^2}^2\\-\lambda^\alpha\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x.\end{aligned}$$

where $f^\lambda(x):=\lambda^{\frac{3}{2}}f(\lambda x)$. As $K_\omega(f)<0$ and $K_\omega\left(f^\lambda\right)>0$ for $\lambda>0$ sufficiently small, there exists $\lambda_0\in(0,1)$ such that $K_\omega\left(f_{\lambda_0}\right)=0$. It follows that

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|\phi|^2\right)|\phi|^2\,\mathrm{d}x=S_\omega(\phi)=d(\omega)\le S_\omega\left(f_{\lambda_0}\right)\\=\frac{1}{4}\lambda_0^\alpha\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x.\end{aligned}$$

which yields

$$\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|\phi|^2\right)|\phi|^2\,\mathrm{d}x\le\lambda_0^\alpha\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x. \tag{27}$$

If $\|\rho f\|_{L^2}^2\ge\|\rho\phi\|_{L^2}^2$, then we infer from (25) and (27) that

$$\begin{aligned}d(\omega)&=S_\omega(\phi)=S_\omega(\phi)-\frac{1}{2}H(\phi)\\&=\frac{b}{2}R(\phi)+\frac{b^2}{4}\|\rho\phi\|_{L^2}^2+\frac{\omega}{2}\|\phi\|_{L^2}^2+\frac{\alpha-2}{8}\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|\phi|^2\right)|\phi|^2\,\mathrm{d}x\\&\le\frac{b}{2}R(f)+\frac{b^2}{4}\|\rho f\|_{L^2}^2+\frac{\omega}{2}\|f\|_{L^2}^2+\frac{\alpha-2}{8}\lambda^\alpha\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x\\&\le\frac{b}{2}R(f)+\frac{b^2}{4}\|\rho f\|_{L^2}^2+\frac{\omega}{2}\|f\|_{L^2}^2+\frac{\alpha-2}{8}\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x\\&=S_\omega(f)-\frac{1}{2}H(f),\end{aligned}$$

which shows (26).

Finally we consider the case that $\|\rho f\|_{L^2}^2<\|\rho\phi\|_{L^2}^2$, we set

$$\begin{aligned}F(\lambda)&:=S_\omega\left(f^\lambda\right)-\frac{\lambda^2}{2}H(f)\\&=\frac{\omega}{2}\|f\|_{L^2}^2+\frac{b}{2}R(f)+\frac{b^2}{8}\left(\lambda^{-2}+\lambda^2\right)\|\rho f\|_{L^2}^2\\&\quad-\frac{1}{4}\left(\lambda^\alpha-\frac{\alpha}{2}\lambda^2\right)\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x.\end{aligned}$$

Observe that if we have

$$F(\lambda_0)\le F(1). \tag{28}$$

Let us assume (28) for the moment and complete the proof of Lemma 4.3. Indeed, we have

$$d(\omega)\le S_\omega\left(f_{\lambda_0}\right)\le S_\omega\left(f_{\lambda_0}\right)-\frac{\lambda_0^2}{2}H(f)=F(\lambda_0)\le F(1)=S_\omega(f)-\frac{1}{2}H(f),$$

which implies (26).

It remains to show (28) which is in turn equivalent to show

$$\frac{b^2}{8}\left(\lambda_0^{-2}+\lambda_0^2-2\right)\|\rho f\|_{L^2}^2\le\frac{1}{8}\left(2\lambda_0^\alpha+\alpha\lambda_0^\alpha+\alpha-2\right)\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x. \tag{29}$$

Now we infer from $\partial_\lambda^2 S_\omega\left(\phi^\lambda\right)\big|_{\lambda=1}\le 0$ and $H(\phi)=0$, we see that

$$b^2\|\rho\phi\|_{L^2}^2\le\frac{\alpha(\alpha-2)}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x,$$

using (27), we infer that

$$\begin{aligned}b^2\|\rho f\|_{L^2}^2<b^2\|\rho\phi\|_{L^2}^2&\le\frac{\alpha(\alpha-2)}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x\\&\le\frac{\alpha(\alpha-2)}{4}\lambda_0^\alpha\int_{\mathbb{R}^3}\left(|x|^{-\alpha}*|f|^2\right)|f|^2\,\mathrm{d}x.\end{aligned} \tag{30}$$

From (30), we see that (29) is satisfied if

$$\frac{\alpha(\alpha-2)}{32}\left(\lambda_0^{-2}+\lambda_0^2-2\right)\lambda_0^\alpha \le \frac{1}{8}\left(2\lambda_0^\alpha+\alpha\lambda_0^2+\alpha-2\right)$$

or

$$\alpha\left(\alpha^{-}2\right)\left(\lambda_0^{-1}+\lambda_0\right)^2\lambda_0^\alpha \le 4\left(2\lambda_0^\alpha+\alpha\lambda_0^2+\alpha-2\right) \tag{31}$$

Let $\beta=\frac{\alpha}{2}>1$ and define $G(s):=s^\beta-1-\beta(s-1)-\frac{\beta(\beta-1)}{2}(s-1)^2 s^{\beta-1}$, for $s>0$. Then, (31) is equivalent to $G\left(\lambda_0^2\right)\ge 0$. Taking the Taylor expansion of s^β at $s=1$, we have

$$s^\beta=1+\beta(s-1)+\frac{\beta(\beta-1)}{2}(s-1)^2 s_0^{\beta-2}$$

for some $s_0\in[s,1]$. This shows that

$$G\left(\lambda_0^2\right)=\frac{\beta(\beta-1)}{2}\left(\lambda_0^2-1\right)^2\left(s_0^{\beta-2}-\lambda_0^{2\beta-2}\right)$$

with $\lambda_0^2\le s_0\le 1$. Since $\lambda_0^{2\beta-2}\le s_0^{\beta-1}\le s_0^{\beta-2}$, we infer that $G\left(\lambda_0^2\right)\ge 0$. This proves (28), hence (26). □

We are now able to prove Theorem 1.2.

Let $\phi\in\mathcal{G}(\omega)$ be such that $\frac{\partial^2}{\partial\lambda^2}S_\omega\left(\phi^\lambda\right)\Big|_{\lambda=1}\le 0$. Define

$$\begin{aligned}\mathcal{B}(\omega):=\Big\{f\in\Sigma_A\left(\mathbb{R}^3\right):M(f)=M(\phi),R(f)=R(\phi),\\ S_\omega(f)<d(\omega),K_\omega(f)<0,H(f)<0\Big\}.\end{aligned}$$

Lemma 4.4. The set $\mathcal{B}(\omega)$ is invariant under the flow of (1), *i.e.*, if $u_0\in\mathcal{B}(\omega)$, then $u(t)\in\mathcal{B}(\omega)$ for all $t\in\left[0,T^*\right)$.

Proof. In fact, let $u_0\in\mathcal{B}(\omega)$ and $u:\left[0,T^*\right)\times\mathbb{R}^3\to\mathbb{C}$ be the corresponding solution to (1). By the conservation of mass and energy, we have $M(u(t))=M(u_0)=M(\phi)$ and $S_\omega(u(t))=S_\omega(u_0)<d(\omega)$ for all $t\in\left[0,T^*\right)$. Thanks to the conservation of angular momentum, we see that $R(u(t))=R(u_0)=R(\phi)$ for all $t\in\left[0,T^*\right)$. We will show that $K_\omega(u(t))<0$ for all $t\in\left[0,T^*\right)$. Suppose that it does not hold, then there exists $t_0\in\left[0,T^*\right)$ such that $K_\omega(u(t_0))\ge 0$. By the continuity of $t\mapsto K_\omega(u(t))$, there exists $t_1\in(0,t_0]$ such that $K_\omega(u(t_1))=0$. From the definition of $d(\omega)$, we get $d(\omega)\le S_\omega(u(t_1))=S_\omega(u_0)<d(\omega)$ which is a contradiction. Finally we prove that $H(u(t))<0$ for all $t\in\left[0,T^*\right)$. If it is not true, then arguing as above, there exists $t_2\in\left[0,T^*\right)$ such that $H(u(t_2))=0$. Applying Lemma 4.3 to $f=u(t_2)$, we get

$$0=H(u(t_2))\le 2\left(S_\omega(u(t_2))-d(\omega)\right),$$

which implies

$$d(\omega) \le S_\omega(u(t_2)) = S_\omega(u_0) < d(\omega).$$

This is again a contradiction. Thus we have

$$M(u(t)) = M(u_0) = M(\phi),$$

$$R(u(t)) = R(u_0) = R(\phi),$$

$$S_\omega(u(t)) = S_\omega(u_0) < d(\omega).$$

and $K_\omega(u(t)) < 0, H(u(t)) < 0$ for all $t \in [0, T^*)$. This shows Lemma 4.4. □

Lemma 4.5. We have $\phi^\lambda \in \mathcal{B}(\omega)$ for all $\lambda > 1$, where ϕ^λ is as in (7).

Proof. A straightforward computation shows

$$M(\phi^\lambda) = M(\phi),\ R(\phi^\lambda) = R(\phi),\ \forall \lambda > 0.$$

Next we have

$$\begin{aligned}\partial_\lambda^2 S_\omega(\phi^\lambda) &= \|\nabla\phi\|_{L^2}^2 + \frac{3b^2}{4}\lambda^{-4}\|\rho\phi\|_{L^2}^2 - \frac{\alpha(\alpha-1)}{4}\lambda^{\alpha-2}\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,\mathrm{d}x \\ &< \|\nabla\phi\|_{L^2}^2 + \frac{3b^2}{4}\|\rho\phi\|_{L^2}^2 - \frac{\alpha(\alpha-1)}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,\mathrm{d}x \\ &= \partial_\lambda^2 S_\omega(\phi^\lambda)\Big|_{\lambda=1} \le 0,\ \forall \lambda > 1.\end{aligned}$$

It yields that

$$\partial_\lambda S_\omega(\phi^\lambda) < \partial_\lambda S_\omega(\phi^\lambda)\Big|_{\lambda=1} = H(\phi) = 0,\ \forall \lambda > 1,$$

which shows

$$S_\omega(\phi^\lambda) < S_\omega(\phi),\ \forall \lambda > 1.$$

We also have

$$H(\phi^\lambda) = \lambda\partial_\lambda S_\omega(\phi^\lambda) < 0,\ \forall \lambda > 1.$$

It remains to show that $K_\omega(\phi^\lambda) < 0$ for all $\lambda > 1$. We have

$$\begin{aligned}\partial_\lambda^3 K_\omega(\phi^\lambda) &= -6b^2\lambda^{-5}\|\rho\phi\|_{L^2}^2 - \alpha(\alpha-1)(\alpha-2)\lambda^{\alpha-3}\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,\mathrm{d}x \\ &< 0,\ \forall \lambda > 0.\end{aligned}$$

It follows that

$$\begin{aligned}&\partial_\lambda^2 K_\omega(\phi^\lambda) < \partial_\lambda^2 K_\omega(\phi^\lambda)\Big|_{\lambda=1} \\ &\quad = 2\|\nabla\phi\|_{L^2}^2 + \frac{3b^2}{2}\|\rho\phi\|_{L^2}^2 - \alpha(\alpha-1)\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,\mathrm{d}x,\ \forall \lambda > 1.\end{aligned}$$

By the assumption $\partial_\lambda^2 S_\omega(\phi^\lambda)\Big|_{\lambda=1} \le 0$ which is equivalent to

$$\|\nabla\phi\|_{L^2}^2 + \frac{3b^2}{4}\|\rho\phi\|_{L^2}^2 - \frac{\alpha(\alpha-1)}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,\mathrm{d}x \le 0,$$

we infer that

$$\partial_\lambda^2 K_\omega(\phi^\lambda)\Big|_{\lambda=1} = -\frac{3\alpha(\alpha-1)}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,\mathrm{d}x < 0,\ \forall \lambda > 1.$$

This shows that

$$\partial_\lambda K_\omega(\phi^\lambda) < \partial_\lambda K_\omega(\phi^\lambda)\big|_{\lambda=1}$$

$$= 2\|\nabla\phi\|_{L^2}^2 - \frac{b^2}{2}\|\rho\phi\|_{L^2}^2 - \int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,dx < 0, \ \forall\lambda > 1.$$

Using the fact that $H(\phi) = 0$, we obtain

$$\partial_\lambda K_\omega(\phi^\lambda) < -\frac{\alpha}{4}\int_{\mathbb{R}^3}\left(|x|^{-\alpha} * |\phi|^2\right)|\phi|^2\,dx < 0, \ \forall\lambda > 1.$$

This shows that $K_\omega(\phi^\lambda) < K_\omega(\phi) = 0$ for all $\lambda > 1$. Therefore we prove that $\phi^\lambda \in \mathcal{B}(\omega)$ for all $\lambda > 1$. □

Proof of Theorem 1.3. Now let $\varepsilon > 0$. As $\phi^\lambda \to \phi$ strongly in $\Sigma_A(\mathbb{R}^3)$ as $\lambda \to 1$, there exists $\lambda_0 > 1$ such that $\|\phi_{\lambda_0} - \phi\|_{\Sigma_A} < \varepsilon$. Set $u_0 = \phi_{\lambda_0} \in \mathcal{B}(\omega)$ and let $u : [0, T^*) \times \mathbb{R}^3 \to \mathbb{C}$ be the corresponding solution to (1). By Lemma 4.5 and 4.4, $u(t) \in \mathcal{B}(\omega)$ for all $t \in [0, T^*)$. Applying Lemma 4.3 to $f = u(t)$ and using the conservation laws of mass and energy, we get

$$H(u(t)) \le 2\left(S_\omega(u(t)) - d(\omega)\right) = 2\left(S_\omega(u_0) - d(\omega)\right) < 0, \ \forall t \in [0, T^*).$$

Thanks to Lemma 2.11, we have

$$I''(u(t)) = 8H(u(t)) \le 16\left(S_\omega(u_0) - d(\omega)\right) < 0, \ \forall t \in [0, T^*),$$

where I is as in (18). The convexity argument shows that $T^* < \infty$. □

5. Conclusions and Suggestions

In this paper, we investigated the strong instability of standing waves for the three-dimensional Hartree equation with a constant magnetic field. By combining variational methods, compactness arguments, and virial identities, we established the existence of ground states and proved the strong instability of ground state standing waves in the mass-supercritical regime under condition (1.6).

The presence of the magnetic field introduces additional difficulties due to the angular momentum interaction term and the lack of standard scaling invariance. Our results extend previous instability theories for nonlinear Schrödinger and Hartree equations to the magnetic Hartree setting.

Several interesting problems remain open for future investigation. In particular, studying the precise blow-up dynamics of solutions and determining the sharp mass threshold separating stability and instability are both important open problems.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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