

Anisotropic Homogeneous Universe in $f(R, T)$ Gravity with Bulk Viscous Fluid

Rishi Kumar Tiwari¹, Aroon Beesham², Murleedhar Pal¹ 

¹Department of Mathematics, Government Model Science College, Rewa, India

²Department of Mathematical Sciences, University of Zululand, Kwa-Dlangezwa, South Africa

Email: rishitiwari59@rediffmail.com, abeesham@yahoo.com, murleedhar1994@gmail.com

How to cite this paper: Tiwari, R.K., Beesham, A. and Pal, M. (2026) Anisotropic Homogeneous Universe in $f(R, T)$ Gravity with Bulk Viscous Fluid. *Journal of Applied Mathematics and Physics*, 14, 2449-2469.
<https://doi.org/10.4236/jamp.2026.146121>

Received: May 9, 2026

Accepted: June 26, 2026

Published: June 29, 2026

Copyright © 2026 by author(s) and Scientific Research Publishing Inc. This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

In this work, we investigate a Bianchi type-I universe within the framework of $f(R, T) = f_1(R) + f_2(T)$ gravity, considering the specific case in the presence of bulk viscous matter. A time-dependent deceleration parameter is adopted to describe the transition from decelerated to accelerated cosmic expansion. Exact solutions of the field equations are derived by taking the relation between the deceleration parameter and the scale factor yields

$$a(t) = \left(\frac{e^{cnT} - d}{c} \right)^{\frac{1}{n}}. \text{ Where } c, d, \text{ and } n \text{ are model constants and } T \text{ represents}$$

cosmic time. The corresponding energy density and energy conditions (Weak, Dominant, and Strong) are analyzed graphically, indicating consistency with an accelerating universe. The impact of bulk viscosity on the future evolution of the universe is also examined. Furthermore, the cosmic jerk parameter is computed, and its behaviour is found to be qualitatively consistent with observational expectations, thereby consistent with the value predicted of the proposed model.

Keywords

$f(R, T)$ Gravity, Bulk Viscous Fluid, Bianchi Type-I Universe, Variable Deceleration Parameter, Accelerating Universe, Modified Gravity, Energy Conditions

1. Introduction

Over the past two decades, observational cosmology has made remarkable progress, significantly advancing our understanding of the universe. This progress has been driven by major surveys and telescope observations, including the Bar-

yon Oscillation Spectroscopic Survey [1], the Planck Collaboration [2], and the Atacama Cosmology Telescope Polarimeter [3]. One of the most important milestones in modern cosmology was the discovery of the universe's accelerated expansion, achieved through Type I a supernova observations conducted by the Supernova Cosmology Project and the High-Redshift Supernova Search Team [4]-[6], which provided the first direct evidence for cosmic acceleration. Further support came from measurements of the cosmic microwave background [7] [8], studies of large-scale cosmic structure [9] [10], and additional CMB analyses [11] [12]. Observations strongly indicate that the universe transitioned from an early decelerating phase to a late-time accelerating phase; although some recent studies have questioned the robustness of this interpretation [13]. According to the Planck results [2], the energy content of the universe consists of approximately 5% baryonic matter, 26% dark matter, and 69% dark energy. Despite extensive research, the physical nature of dark energy remains one of the biggest unresolved problems in cosmology. Two main approaches have been adopted to address this issue: introducing exotic matter components such as quintessence [14], Chaplygin gas [15] [16], polytropic fluids [17], phantom fields [18], tachyons [19], as well as the cosmological constant Λ , which suffers from fine-tuning and coincidence problems [20]. The second approach involves modifications of General Relativity, notably $f(R)$ gravity, where the Ricci scalar in the Einstein-Hilbert action is replaced by a general function [21] [22]. Further developments included coupling the matter Lagrangian with arbitrary $f(R)$ models [23], and Lorentz-violating extensions [24]. A major breakthrough came with $f(R, T)$ gravity, where the action depends on both the Ricci scalar and the trace of the energy-momentum tensor [25]. This theory enables matter-geometry coupling and provides a promising geometric explanation for late-time cosmic acceleration. Various cosmological investigations have supported the potential of $f(R, T)$ theory, including anisotropic and bulk viscous models [26]-[29], observational reconstruction methods [30], and recent studies focusing on dark-energy evolution and cosmic transit behaviour [7] [31] [32]. Section 1 introduces the basic ideas of gravity and reviews earlier research. In Section 2, the concept of $f(R, T)$ gravity has been introduced. In Section 3, the metric is defined, and the corresponding field equations are obtained. In Section 4, a time-varying deceleration parameter is considered and the field equation are solved. Important quantities such as energy density, viscous pressure, viscosity coefficient, matter trace, Ricci scalar, and energy conditions are derived and shown through graphs. In Section 5, the dynamical behaviour of the model is analyzed and physical parameters. While Section 6 a detailed discussion of the figures is presented. Finally, Section 7 presents the main conclusions of the study.

2. $f(R, T)$ Gravity Field Equations

Assuming that the matter Lagrangian density L_m depends explicitly on the metric tensor, the corresponding field equations in $f(R, T)$ gravity can be obtained through the application of the Hilbert-Einstein variational principle.

$$S = \int \left(\frac{1}{16\pi G} \sqrt{-g} f(R, T) d^4x + \sqrt{-g} L_m d^4x \right) \quad (1)$$

The standard matter Lagrangian density, denoted as L_m , corresponds to the matter source, while $f(R, T)$ represents a general function of the Ricci scalar R and the trace T of the energy-momentum tensor T_{ij} associated with the matter source. Additionally, g stands for the determinant of the metric tensor g_{ij} . The energy-momentum tensor derived from the matter Lagrangian is defined as follows:

$$T_{ij} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{ij}} \quad (2)$$

and its trace $T = g^{ij} T_{ij}$. It is assumed that the matter Lagrangian L_m depends solely on the metric components g_{ij} and not on their derivatives. Consequently, one obtains

$$T_{ij} = -\frac{\partial L_m}{\partial g^{ij}} + g_{ij} L_m \quad (3)$$

Varying the action S in Equation (1) with respect to g_{ij} the $f(R, T)$ metric tensor yields the field equations of gravity

$$\begin{aligned} f_R(R, T) R_{ij} - \frac{1}{2} f(R, T) g_{ij} + (g_{ij} \square - \nabla_i \nabla_j) f_R(R, T) \\ = 8T_{ij} - f_T(R, T) T_{ij} - f_T(R, T) \theta_{ij} \end{aligned} \quad (4)$$

where

$$\theta_{ij} = -2T_{ij} - 2g^{lm} \frac{\partial^2 L_m}{\partial g^{ij} \partial g^{lm}} + g_{ij} L_m \quad (5)$$

Here, $f_R(R, T) = \frac{\partial f_R(R, T)}{\partial R}$, $f_T(R, T) = \frac{\partial f_R(R, T)}{\partial T}$, $\square \equiv \nabla_i \nabla_j$ where ∇_i is the covariant Derivative.

$$(R, T) R - 2f(R, T) + f_T(R, T) \theta = (8\pi - f_T(R, T)) T - 3\square f_R(R, T) \quad (6)$$

where $\theta = g^{ij} \theta_{ij}$.

Using Equations (4) and (6), the field equation of $f(R, T)$ gravity take the form:

$$\begin{aligned} f_T(R, T) \left(R_{ij} - \frac{1}{3} R g_{ij} \right) + f_T(R, T) \left(\theta_{ij} - \frac{1}{3} \theta g_{ij} \right) \\ = 8\pi - f_T(R, T) \left(T_{ij} - \frac{1}{3} T g_{ij} \right) + \nabla_i \nabla_j f_R(R, T) - \frac{1}{6} f(R, T) g_{ij} \end{aligned} \quad (7)$$

It is important to note that the physical properties of the matter field, through the pressure components, influence the structure of the field equations in $f(R, T)$ gravity. Based on the nature of the matter source, proposed three specific formulatis of $f(R, T)$ gravity, which are outlined below

The respective

$$f(R, T) = \begin{cases} R + 2f(T) \\ f(R, T) = f_1(R) + f_2(T) \\ f_1(R) + f_2(R) f_3(T) \end{cases} \quad (8)$$

field equation corresponding to each frame in $f(R, T)$ gravity are expressed as follow

$$f(R, T) = f_1(R) + f_2(T)$$

$$R_{ij} - \frac{1}{2} R g_{ij} = \left(\frac{8\pi + \mu}{\mu} \right) T_{ij} + \left(p + \frac{T}{2} \right) g_{ij} = \chi T_{ij} + \left(p + \frac{T}{2} \right) g_{ij} \quad (9)$$

In recent years, numerous researchers have developed cosmological models incorporating perfect fluid matter to investigate the universe's accelerated expansion. Observational evidence strongly confirms this acceleration, attributed to an unknown energy component known as dark energy, characterized by negative pressure. However, increasing effort has been devoted to constructing cosmological models that explain the acceleration without invoking dark energy or dark matter, instead relying on realistic matter components. One promising approach is cosmic bulk viscosity, which can mimic the dynamical behaviour of dark energy by generating negative effective pressure [33] [34], thereby driving late-time acceleration. Consequently, dissipative fluid matter is now considered a more realistic framework compared to perfect fluid or dust models for describing homogeneous and isotropic cosmology. It is widely believed that during the early universe, particularly near the neutrino decoupling epoch in the radiation-dominated era, the cosmic medium exhibited viscous behaviour [35]-[37]. Viscous cosmological models significantly modify singular behaviour and provide explanations for high entropy production, phase transitions, and baryogenesis. It has further been shown that a scalar field interacting with viscous fluid can account for cosmic acceleration more effectively than when coupled with a perfect fluid alone [38]. As a result, viscous fluid cosmologies have attracted considerable attention in early-universe research [19] [39] [40]. Observations also investigate the homogeneity and isotropy of the universe. Although the post-inflationary universe is modeled using the FLRW metric, anomalies in CMB data from the Planck mission suggest an early anisotropic stage [41]. This motivates the use of Bianchi-type models, particularly Bianchi Type-I, the simplest anisotropic generalization of FLRW. Studies indicate that viscosity influences singularity behaviour but does not completely remove the initial singularity [42]. More recent research has focused on bulk viscous matter in Bianchi Type-I geometry within modified gravity frameworks such as, and theories, demonstrating strong compatibility with observational constraints on late-time acceleration [16] [43]-[46]. The present work extends recent investigations on bulk viscous cosmology in modified gravity by introducing a variable deceleration parameter and studying the combined influence of anisotropy, viscosity, and matter-geometry coupling on the late-time accelerated expansion of the universe. Motivated by recent developments in modified gravity and viscous cosmology, in the present work we investigate an LRS Bianchi type-I cosmological model in the framework of $f(R, T)$ gravity with bulk viscous matter. Unlike several earlier studies based on perfect fluid cosmology or constant deceleration parameters, the present model employs a time-dependent

deceleration parameter capable of describing the transition from an early decelerating phase to the present accelerated phase of the universe. The combined effects of anisotropy, bulk viscosity, and matter-geometry coupling are analyzed through various cosmological and dynamical parameters, including the jerk and statefinder diagnostics.

3. Metric and Field Equations

We examine the spatially homogeneous LRS Bianchi type-I metric as follows

$$ds^2 = dt^2 - A^2(t) - B^2(t)(dy^2 + dz^2) \quad (10)$$

Here, $A(t)$ and $B(t)$ are assumed to be functions dependent only on the cosmic time t .

The energy-momentum tensor corresponding to a bulk viscous fluid is considered in the following form:

$$T_{ij} = (\rho + \bar{p})u_i u_j - \bar{p}g_{ij} \quad (11)$$

where $u^i = (0, 0, 0, 1)$ is the four velocity vector in co-moving coordinate system satisfyin

$$\begin{aligned} u^i u_j &= 1 \\ \bar{p} &= p - 3\xi H \end{aligned} \quad (12)$$

The bulk viscous pressure satisfies the linear equation of state $p = \gamma p$, where $0 \leq \gamma \leq 1$. Here, ξ is the bulk viscosity coefficient, H is the Hubble parameter, $\bar{p}$ denotes the effective bulk viscous pressure, p denotes the pressure, and ρ represents the energy density.

The trace of the energy-momentum tensor is given by:

$$T = \rho - 3\bar{p} \quad (13)$$

$$f(R, T) = f_1(R) + f_2(T)$$

Considering the function $f_1(R) = \mu T$ and $f_2(T) = \mu T$ (where μ is an arbitrary constant), the field Equation (9) for the metric (10) becomes:

$$-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} = \left(\chi + \frac{1}{2}\right)\bar{p} - \frac{1}{2}\rho \quad (14)$$

$$-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} = \left(\chi + \frac{1}{2}\right)\bar{p} - \frac{1}{2}\rho \quad (15)$$

$$-2\frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} = \left(\chi + \frac{1}{2}\right)\rho - \frac{1}{2}\bar{p} \quad (16)$$

where $\chi = \frac{8\pi + \mu}{\mu}$.

Here, $\dot{}$ denotes differentiation with respect to cosmic time t .

The mean scale factor a and the spatial volume V of LRS Bianchi type -I metric, are defined as

$$V = a^3 = AB^2 \quad (17)$$

and the mean Hubble parameter H may be given by

$$H = \frac{\dot{a}}{a} \quad (18)$$

from Equations (15)-(17) we have

$$\frac{\dot{A}}{A} = \frac{\dot{a}}{a} + \frac{2k}{3a^3} \quad (19)$$

and

$$\frac{\dot{B}}{B} = \frac{\dot{a}}{a} - \frac{k}{3a^3} \quad (20)$$

where K is a constant of integration.

4. Field Equation with a Varying Deceleration Parameter

The deceleration parameter (DP) is Defined as

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \quad (21)$$

The deceleration parameter is a crucial tool for understanding the universe's expansion [47]-[55]. Previous research indicates that this parameter may either remain constant or change over time. When the expansion rate is constant, the scale factor increases linearly with time, resulting in a deceleration parameter of zero. Conversely, if the Hubble parameter H remains constant, the deceleration parameter takes a constant value of -1 . Based on Equation (21), we get

$$\frac{\ddot{a}}{a} + q \frac{\dot{a}^2}{a^2} = 0 \quad (22)$$

On solving

$$\int e^{\int \frac{q}{a} da} da = t + t_0 = T \quad (23)$$

Here, t_0 is the constant of integration.

This solution suggests that the universe could initially experience a decelerating phase and later transition to an accelerating phase. Such behaviour is consistent with observations from Type Ia supernovae (SNeIa), which support the concept of an accelerating universe.

For a flat FRW universe, we assume:

$$e^{\int \frac{q}{a} da} da = f(a) = \frac{1}{ca + da^{-n+1}} \quad (24)$$

where c, d, n are positive constants. In the early universe (small a), the function $f(a)$ is non-zero. At late times $a \rightarrow \infty$, $f(a) \rightarrow 0$.

The chosen functional form of the deceleration parameter is motivated by the observational requirement that the universe should evolve from an early deceler-

ating phase to the present accelerated phase. The adopted parametrization provides a smooth cosmic transit behaviour and yields exact analytical solutions of the highly nonlinear modified field equations. Moreover, the model asymptotically approaches a de-Sitter type accelerating universe at late times.

From Equations (23) and (24), we get:

$$a(t) = \left(\frac{e^{cnT} - d}{c} \right)^{\frac{1}{n}} \quad (25)$$

The time varying deceleration parameter is defined from above Equation (21), we obtain

$$q = -1 + nde^{-cnT} \quad (26)$$

The Hubble parameter is defined as $H = \frac{\dot{a}}{a}$ and from the above equation we get

$$H = \frac{ce^{cnT}}{e^{cnT} - d} \quad (27)$$

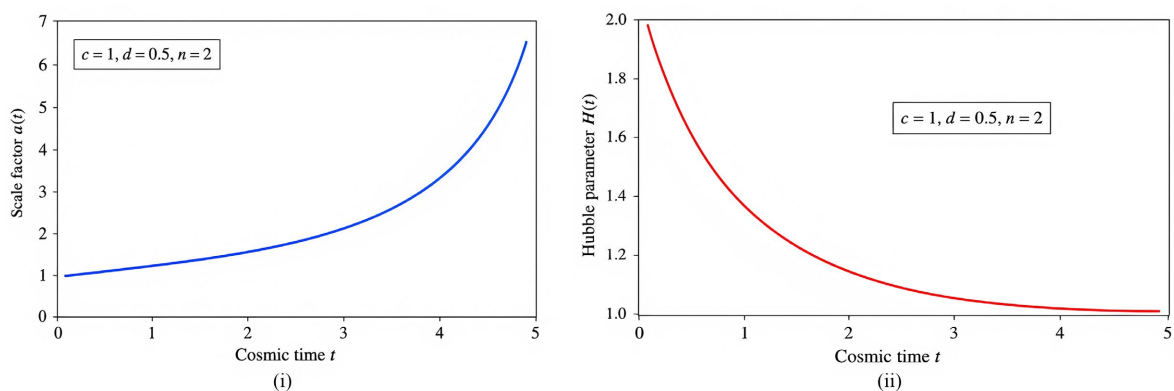


Figure 1. (i) Scale factor $a(t)$ versus Cosmic time t ; (ii) Hubble parameter $H(t)$ versus Cosmic time t .

Figure 1(i) and **Figure 1(ii)** show the combined behavior of the scale factor and the Hubble parameter. The scale factor increases continuously with time, indicating the expansion of the universe, while the Hubble parameter decreases gradually and approaches a constant value at late times. This demonstrates a transition from early rapid expansion to a stable accelerated phase.

Solving the field Equations (14)-(16) the value of ρ and $\bar{p}$ obtained as

$$\rho = \frac{1}{\chi(\chi+1)} \left(\frac{c^2 nde^{cnT}}{(e^{cnT} - d)^2} + \frac{(3\chi+1)c^2 de^{2ncT}}{(e^{cnT} - d)^2} - \frac{2Kc^{1+\frac{3}{n}}e^{cnT}}{3(e^{cnT} - d)^{1+\frac{3}{n}}} - \frac{K^2(3\chi+2)c^{\frac{6}{n}}}{9(e^{cnT} - d)^{\frac{6}{n}}} \right) \quad (28)$$

$$\bar{p} = \frac{1}{\chi(\chi+1)} \left[\frac{2\left(\chi + \frac{1}{2}\right)c^2 n d e^{cnT}}{(e^{cnT} - d)^2} + \frac{(1-\chi)c^2 d e^{cnT}}{(e^{cnT} - d)^2} - \frac{4K\left(\chi + \frac{1}{2}\right)c^{1+\frac{3}{n}}e^{cnT}}{3(e^{cnT} - d)^{1+\frac{3}{n}}} - \frac{K^2(\chi+2)c^{\frac{6}{n}}}{9(e^{cnT} - d)^{\frac{6}{n}}} \right] \quad (29)$$

The expressions for the coefficient of bulk viscosity ξ and the pressure are obtained as follows:

$$\frac{\Psi\rho - \bar{p}}{3H} = \frac{1}{\chi(\chi+1)} \frac{e^{cnT} - d}{3ce^{cnT^2}} \left[\Psi \left\{ \frac{c^2 n d e^{cnT}}{(e^{cnT} - d)^2} + \frac{(3\chi+1)c^2 d e^{2ncT}}{(e^{cnT} - d)^2} - \frac{2Kc^{1+\frac{3}{n}}e^{cnT}}{3(e^{cnT} - d)^{1+\frac{3}{n}}} - \frac{K^2(3\chi+2)c^{\frac{6}{n}}}{9(e^{cnT} - d)^{\frac{6}{n}}} \right\} - \left\{ \frac{2\left(\chi + \frac{1}{2}\right)c^2 n d e^{cnT}}{(e^{cnT} - d)^2} + \frac{(1-\chi)c^2 d e^{cnT}}{(e^{cnT} - d)^2} - \frac{4K\left(\chi + \frac{1}{2}\right)c^{1+\frac{3}{n}}e^{cnT}}{3(e^{cnT} - d)^{1+\frac{3}{n}}} - \frac{K^2(\chi+2)c^{\frac{6}{n}}}{9(e^{cnT} - d)^{\frac{6}{n}}} \right\} \right] \quad (30)$$

$$p = \Psi\rho$$

$$= \frac{\Psi}{\chi(\chi+1)} \left[\frac{c^2 n d e^{cnT}}{(e^{cnT} - d)^2} + \frac{(3\chi+1)c^2 d e^{2ncT}}{(e^{cnT} - d)^2} - \frac{2Kc^{1+\frac{3}{n}}e^{cnT}}{3(e^{cnT} - d)^{1+\frac{3}{n}}} - \frac{K^2(3\chi+2)c^{\frac{6}{n}}}{9(e^{cnT} - d)^{\frac{6}{n}}} \right] \quad (31)$$

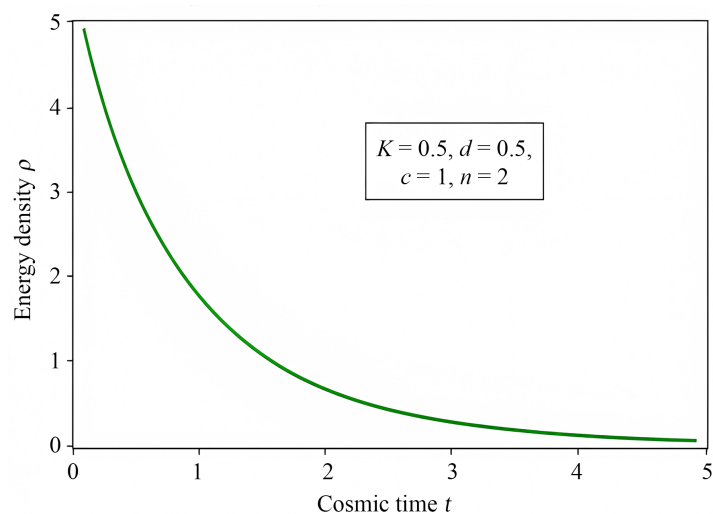


Figure 2. Energy density ρ versus cosmic time t .

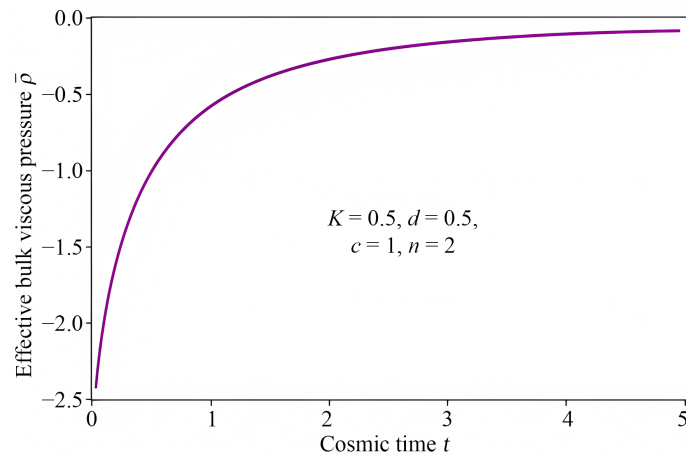


Figure 3. Effective bulk viscous pressure $\bar{p}$ versus cosmic time t .

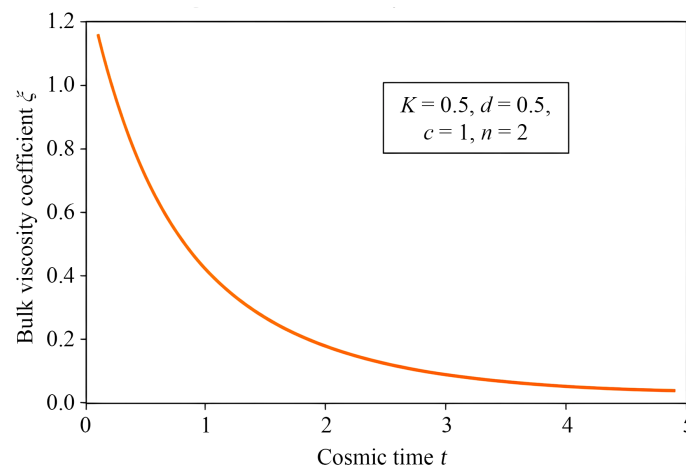


Figure 4. Bulk viscosity coefficient ξ versus cosmic time t .

In our model, **Figure 2** & **Figure 3** shows that the energy density remains positive throughout the evolution of the universe. It gradually decreases over time, starting from a high value in the early universe and approaching zero as $t \rightarrow \infty$. Conversely, the bulk viscous pressure $\bar{p}$ shows an opposite behavior: it begins with a significantly negative value and steadily increases, moving toward zero in the current epoch. This negative pressure is consistent with the influence of dark energy in driving the universe's accelerated expansion. Consequently, the behavior of the bulk viscous pressure in our model aligns well with current observations.

Figure 4 demonstrates that the bulk viscous coefficient ξ stays positive during cosmic evolution and reaches a finite limit as $t \rightarrow \infty$. To better understand the universe's matter content, we also examine the energy conditions. These conditions play a vital role in general relativity, helping to establish important results about singularities and black holes, [56] and serve as tools to test the physical soundness of cosmological models. Our study focuses on three widely used energy conditions the Weak Energy Condition, the Dominant Energy Condition, and the Strong Energy Condition and evaluates their applicability to our model.

$$\rho > 0, \rho - p \geq 0 \text{ (WEC)} \quad (32)$$

$$\rho + p \geq 0 \text{ (DEC)} \quad (33)$$

$$\rho + 3p \geq 0 \text{ (SEC)} \quad (34)$$

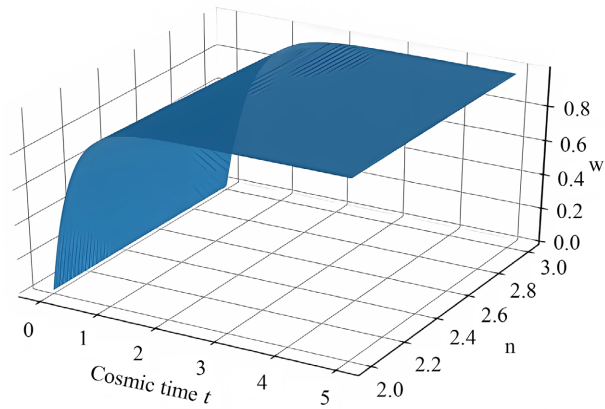


Figure 5. Behaviour of WEC versus Cosmic time t and n .

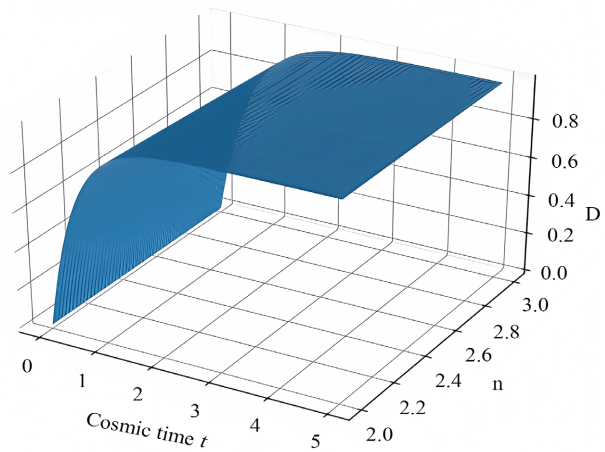


Figure 6. Behaviour of DEC versus Cosmic time t and n .

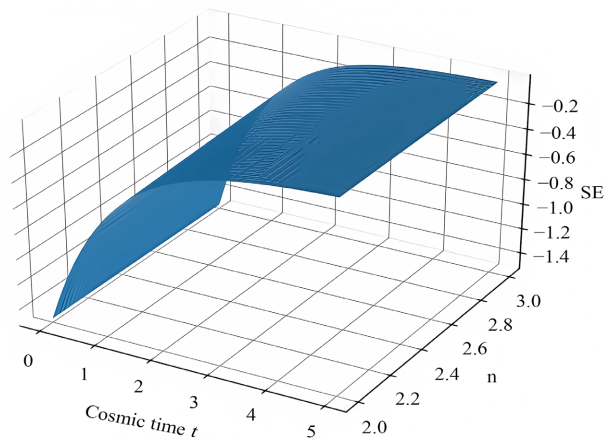


Figure 7. Behaviour of SEC versus Cosmic time t .

Energy conditions are essential because each provides unique physical insights. For instance, the Dominant Energy Condition (DEC) helps assess the stability of the matter source and restricts the equation of state parameter to $\omega \geq -1$. Violating this bound can lead to a future singularity known as the Big Rip [57]. The Strong Energy Condition (SEC) is often violated in the presence of a positive cosmological constant Λ [58]. Meanwhile, the Weak Energy Condition (WEC) ensures that the matter-energy density remains non-negative in all situations. In gravity theories, energy conditions have been extensively used to investigate the matter supporting wormhole geometries. Similarly, for the Friedmann-Robertson-Walker (FRW) model with perfect fluid matter, energy conditions have been studied to understand the behaviour of solutions [59] [60]. Accordingly, prior research demonstrates that energy conditions are valuable tools for analyzing the evolution of cosmological solutions throughout the universe. In this study, we consider three key energy conditions—the WEC, DEC, and SEC to evaluate our solutions under various scenarios. The behavior of these conditions is illustrated in **Figures 5-7** using appropriate parameter choices. **Figures 5-7** demonstrate that all the energy conditions are satisfied in the proposed model. The expressions for the Ricci scalar R and the trace T of the energy-momentum tensor (arising from the matter source) are provided below:

$$R = - \left[\frac{2\ddot{A}}{A} + \frac{4\ddot{B}}{B} + \frac{4\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} \right] - \left(\frac{11c^2e^{2ncT} - 6c^2nde^{cnT}}{(e^{cnT} - d)^2} - \frac{2ke^{cnT}c^{1+\frac{3}{n}}}{3(e^{cnT} - d)^{1+\frac{3}{n}}} - \frac{5K^2c^{\frac{6}{n}}}{9(e^{cnT} - d)^{\frac{6}{n}}} \right) \quad (35)$$

$$T = \rho - 3\bar{p} = \frac{1}{X(X+1)} \left(\frac{c^2nde^{cnT(-6X-2)}}{(e^{cnT} - d)^2} + \frac{(3X+1)c^2de^{2ncT(6X-2)}}{(e^{cnT} - d)^2} + \frac{Kc^{1+\frac{3}{n}}e^{cnT}4\left(X+\frac{1}{3}\right)}{(e^{cnT} - d)^{1+\frac{3}{n}}} + \frac{4K^2c^{\frac{6}{n}}}{9(e^{cnT} - d)^{\frac{6}{n}}} \right) \quad (36)$$

By utilizing the above equation, the functional form of $f(R, T)$ is derived as follows

$$f(R, T) = \mu \left(- \frac{11c^2e^{2ncT} - 6c^2nde^{cnT}}{(e^{cnT} - d)^2} + \frac{2ke^{cnT}c^{1+\frac{3}{n}}}{3(e^{cnT} - d)^{1+\frac{3}{n}}} + \frac{5K^2c^{\frac{6}{n}}}{9(e^{cnT} - d)^{\frac{6}{n}}} \right) + \frac{1}{X(X+1)} \left(\frac{c^2nde^{cnT(-6X-2)}}{(e^{cnT} - d)^2} + \frac{(3X+1)c^2de^{2ncT(6X-2)}}{(e^{cnT} - d)^2} + \frac{Kc^{1+\frac{3}{n}}e^{cnT}4\left(X+\frac{1}{3}\right)}{(e^{cnT} - d)^{1+\frac{3}{n}}} + \frac{4K^2c^{\frac{6}{n}}}{9(e^{cnT} - d)^{\frac{6}{n}}} \right) \quad (37)$$

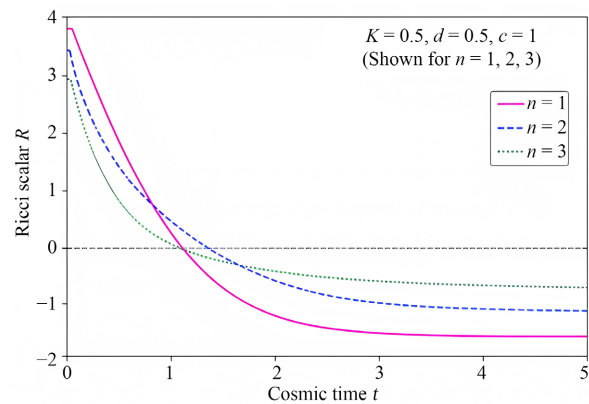


Figure 8. Ricci scalar R versus Cosmic time t .

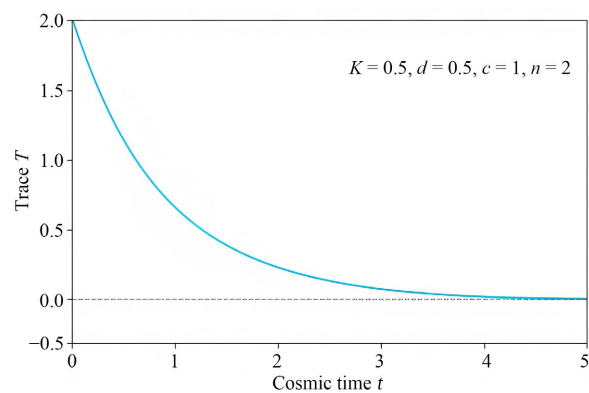


Figure 9. Trace T versus Cosmic time t .

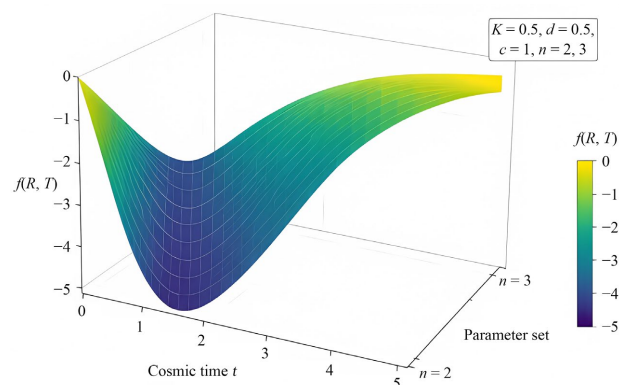


Figure 10. Shows the behaviour of the function $f(R, T)$ for this model.

The variation of the Ricci scalar R with cosmic time is depicted in **Figure 8**. It is observed that the Ricci scalar remains finite throughout the cosmic evolution and gradually approaches a constant value at late times. This behaviour indicates that the space-time geometry evolves smoothly and supports the late-time accelerated expansion of the universe.

The evolution of the trace of the energy-momentum tensor T is illustrated in **Figure 9**. The quantity T decreases with cosmic time and reflects the dynamical

influence of bulk viscous matter. The behaviour confirms that matter contributions become less significant during the late stages of cosmic evolution.

The functional behaviour of $f(R, T)$ is shown in **Figure 10**. The figure demonstrates the combined influence of curvature and matter-geometry coupling on the evolution of the model. The obtained behaviour remains regular throughout the cosmic history and is consistent with the accelerated expansion scenario.

The deceleration parameter can also be expressed as a function of redshift z . To achieve this, we begin with the relation

$$a = \frac{a_0}{1+z} \quad (38)$$

and, utilizing Equation (26) with the normalized scale factor $a_0 = 1$, we derive the corresponding t - z relation. T_0 is the present cosmic time, defined so that redshift $z = 0$.

$$t = \frac{1}{cn} \ln \left(d + \frac{e^{cnT_0} - d}{(1+z)^n} \right) \quad (39)$$

Substituting this relation into Equation (27) yields the expression for the deceleration parameter in terms of redshift.

$$q(z) = -1 + \frac{nd(1+z)^n}{d(1+z)^n + (e^{cnT_0} - d)} \quad (40)$$

The deceleration parameter as a function of redshift is presented in **Figure 11**. The figure clearly exhibits the transition from an early decelerating phase $q > 0$ to a late-time accelerating phase $q < 0$. Such a cosmic transit behaviour is in agreement with current observational evidence.

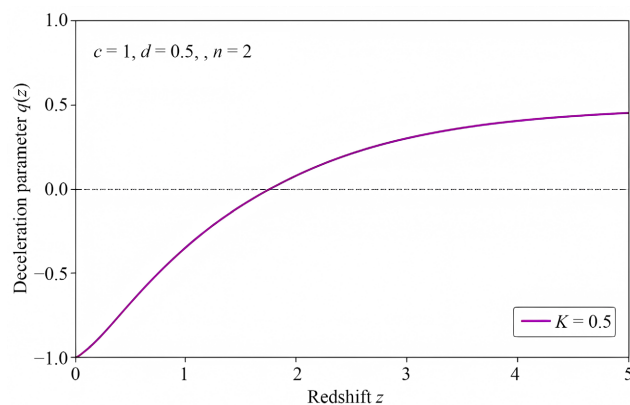


Figure 11. Deceleration Parameter q versus z redshift with n .

5. Dynamical Behaviour of the Model

The Hubble's parameter H , expansion scalar θ , shear scalar σ^2 , anisotropy parameter Δ , Jerk Parameter j , and Statefinder Parameters (r - s) become

$$H = \frac{1}{3} (H_1 + 2H_2) = \frac{ce^{cnT}}{e^{cnT} - d} \quad (41)$$

$$\theta = 3H = \frac{3ce^{cnT}}{e^{cnT} - d} \quad (42)$$

$$\sigma^2 = \frac{1}{3} \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right)^2 = \frac{k^2}{3a^6} = \frac{k^2 C_n^{\frac{6}{n}}}{3(e^{cnT} - d)^{\frac{6}{n}}} \quad (43)$$

$$\Delta = \frac{1}{3} = 6 \left(\frac{\sigma}{\theta} \right)^2 = \frac{2K^2 C_n^{\frac{6}{n}-2} (e^{cnT} - d)^{2-\frac{6}{n}}}{9e^{2cnT}} \quad (44)$$

$$j = \frac{a^2}{\dot{a}^3} \frac{d^3 a}{dt^3} = q + 2q^2 - \frac{\dot{q}}{H} = 1 + \frac{(n^2 - 3n)d}{e^{cnT}} + \frac{n^2 d^2}{e^{2cnT}} \quad (45)$$

$$r = \frac{\ddot{a}}{aH^3} = 1 + \frac{(n^2 - 3n)d}{e^{cnT}} + \frac{n^2 d^2}{e^{2cnT}} \quad (46)$$

$$s = \frac{r-1}{3\left(q - \frac{1}{2}\right)} = \frac{-2\left((n^2 - 3n)d e^{cnT} + n^2 d^2\right)}{e^{cnT} (9e^{cnT} - 6nd)} \quad (47)$$

The statefinder pair $(r,s) \rightarrow (1,0)$ at late times, which corresponds to the standard Λ CDM cosmology. This confirms that the present model behaves consistently with the observed late-time universe. The physical behaviour of the universe can be understood by studying the kinematical parameters of the model. For the

chosen mean scale factor $a = \left(\frac{e^{cnT} - d}{c} \right)^{\frac{1}{n}}$ the Hubble parameter is obtained as

$$H = \frac{ce^{cnT}}{e^{cnT} - d} \quad \text{while the, corresponding expansion scalar is } \theta = 3H. \text{ These quanti-}$$

ties are positive and monotonically decreasing functions of cosmic time, tending to constant values in the late universe, which indicates that the model asymptotically approaches a de-Sitter phase of accelerated expansion. The deceleration parameter evolves according to $q = -1 + nde^{-cnT}$. At early times, the second term dominates and $q > 0$ ($e^{-cnT} \approx 1$, $nd > 1$), so the universe experiences deceleration, whereas at late times $q \rightarrow -1$, corresponding to accelerated expansion. Thus, the model naturally provides a transition from a decelerated phase to the current acceleration. The anisotropic nature of the space-time is described through the shear scalar (σ^2) the mean anisotropy parameter (Δ). Both of these quantities decrease as time increases, which shows that anisotropy is gradually suppressed and the universe evolves towards isotropy at late times. The jerk parameter, which is a higher-order diagnostic of cosmic acceleration, is obtained as towards isotropy at late times. The jerk parameter, which is a higher-order diagnostic of cosmic acceleration, is obtained as $j(T)$ For large values of T , the exponential terms vanish and $j \rightarrow 1$, which is exactly the value predicted by the concordance Λ CDM model. This behaviour is also confirmed through the statefinder analysis: The pair (r,s) tends to the fixed point $(1,0)$ as $T \rightarrow \infty$. From the above analysis it is evident that the model describes a realistic cosmic evolution. The universe begins in a deceler-

ating regime, undergoes a smooth transition to acceleration, and asymptotically approaches a Λ CDM-like stage. The anisotropy decays with time, the expansion rate stabilizes, and the higher-order kinematical parameters converge to their standard values, confirming the physical viability of the model.

	Parameters (c, d, n)	Transition time $T_r \frac{1}{cn} \ln(nd)$, ($nd > 1$) ($q = 0$)	Deceleration $q(T \rightarrow \infty)$	Jerk $j(T \rightarrow \infty)$	Statefinder (r, s) at late times
1	$c = 1, d = 0.5; n = 2$	=0	-1	1	(1,0)
2	$c = 1, d = 0.8; n = 2$	≈ 0.235	-1	1	(1,0)
3	$c = 0.8, d = 0.5; n = 3$	≈ 0.169	-1	1	(1,0)

Today, scientists have many different cosmological models to explain dark energy. Current observations cannot rule out most of these models. A few years ago, researchers introduced two new cosmological parameters [61] [62] to help compare and separate these models. These parameters are called statefinder parameters. They are written using the scale factor and its time derivatives, up to the third order. The statefinder parameters are defined for late cosmic times ($T \rightarrow \infty$), the statefinder pair becomes $(r, s) = (1, 0)$. The model show a singularity at $T = \frac{1}{na} \log b$, which represents the beginning of the universe with a Big Bang. We observe that the expansion θ of the universe decrease with time and slowly approaches zero. The scale factor a increases with time. The expansion rate become finite for large value of T . At $T = \frac{1}{na} \log b$, the deceleration parameter is $q = n - 1 > 0$, which shows decelerated expansion. For large time ($T \rightarrow \infty$), we get $q = -1$, showing accelerated expansion.

The variation of the jerk parameter for different values of n is displayed in **Figure 12**. It is observed that the jerk parameter approaches unity at late times, which is a characteristic feature of the standard Λ CDM cosmological model. This indicates the observational consistency of the proposed model.

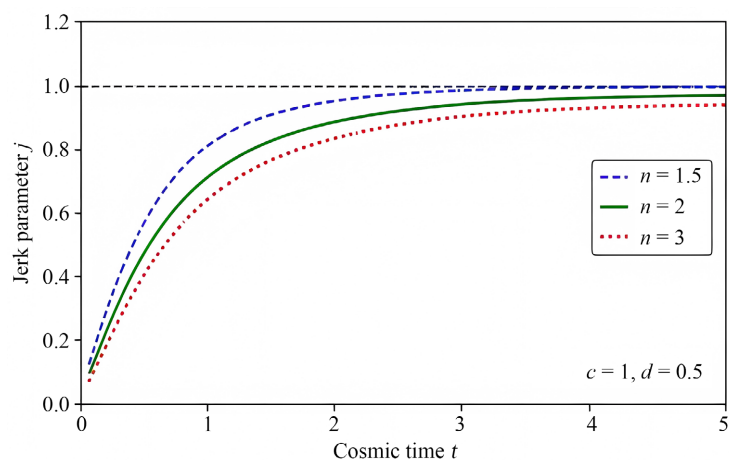


Figure 12. Behaviour of Jerk parameter versus cosmic time t with different n .

The statefinder trajectory (r,s) is illustrated in **Figure 13**. The trajectory converges towards the fixed point $(1,0)$ corresponding to the Λ CDM model. Therefore, the statefinder diagnostic confirms that the present model successfully reproduces the observed late-time behaviour of the universe.

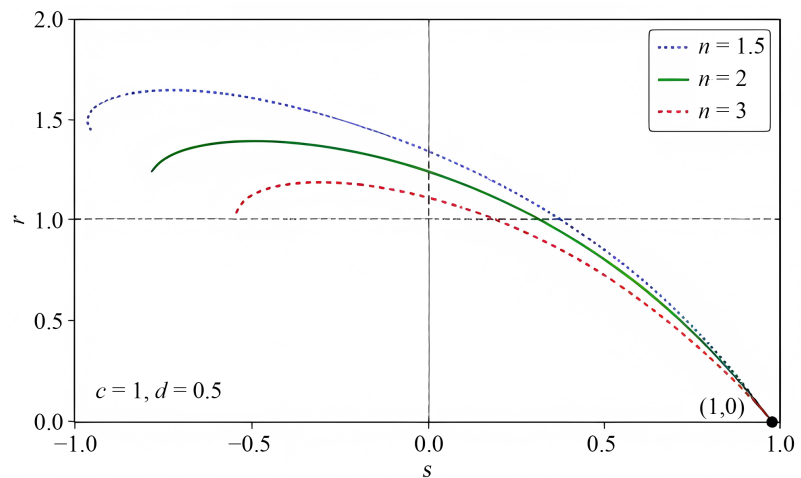


Figure 13. Statefinder Pair r versus s with different n .

6. Discussion

In this model we studied a Bianchi type-I universe in $f(R, T)$ gravity with bulk viscous matter. The results indicate that the energy density remains positive throughout cosmic evolution (**Figure 2**) and decreases with time, starting from a large value in the early universe and approaching zero in the distant future. The effective bulk viscous pressure is negative (**Figure 3**), and although its magnitude decreases gradually, it is this negative pressure that drives the accelerated expansion of the universe. The bulk viscosity coefficient (**Figure 4**) stays positive during the whole evolution and gradually approaches a constant value at late times, which indicates a stabilizing role of viscosity in cosmic dynamics. The physical acceptability of the model is supported by the fact that the weak, dominant, and strong energy conditions (**Figures 5-7**) are satisfied. The deceleration parameter (**Figure 11**) clearly shows a transition from a positive value in the early phase (deceleration) to a negative value in the later phase (acceleration), in qualitatively agrees with observational cosmology. The higher-order diagnostics also confirm this behaviour the jerk parameter (**Figure 12**) approaches unity and the statefinder pair (**Figure 13**) tends to the fixed point $(1,0)$, which is exactly the prediction of the standard Λ CDM model. Thus, the present model provides a smooth description of cosmic evolution, beginning with a decelerating expansion, undergoing a transition, and finally approaching an accelerated de Sitter-like phase driven by bulk viscosity in $f(R, T)$ gravity.

7. Conclusion

In this study, we investigated an LRS Bianchi type-I cosmological model with bulk

viscosity in the framework of gravity. Exact analytical solutions of the modified field equations were obtained by assuming $e^{\int_a^q da}$ is a suitable functional relationship between the expansion scalar and the scale factor. Furthermore, by considering an alternative choice of the function and employing a specific form of the deceleration parameter, we derived an additional exact solution. The resulting model represents an accelerating universe undergoing exponential expansion. The energy density and the bulk viscosity coefficient are positive and decrease monotonically with cosmic time, approaching as $\rho \rightarrow 0$ as $t \rightarrow \infty$. The effective bulk viscous pressure is negative, thereby supporting accelerated cosmic expansion. In addition, all major energy conditions strong (SEC), weak (WEC), and dominant (DEC) are satisfied, ensuring the physical viability of the model. A detailed observational constraint analysis using recent cosmological datasets such as SNe Ia, BAO, and CMB remains an important direction for future investigation. Therefore, the obtained model successfully describes a realistic transition from an early decelerating anisotropic universe to the present accelerating epoch and asymptotically approaches the standard Λ CDM behaviour at late times.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

- [1] Allam, S., Ata, M., Bailey, S., et al. (2016) The Clustering of Galaxies in the Completed SDSS-III Baryon Oscillation Spectroscopic Survey: Cosmological Analysis of the DR12 Galaxy Sample. arXiv:1607.03155.
- [2] Ade, P.A.R., Aghanim, N., Arnaud, M., et al. (2016) Planck 2015 Results XIII: Cosmological Parameters. *Astronomy & Astrophysics*, **594**, A13.
- [3] Naess, S., Hasselfield, M., McMahon, J., Niemack, M.D., Addison, G.E., Ade, P.A.R., et al. (2014) The Atacama Cosmology Telescope: CMB Polarization at $200 < \ell < 9000$. *Journal of Cosmology and Astroparticle Physics*, **10**, Article 007. <https://doi.org/10.1088/1475-7516/2014/10/007>
- [4] Nielsen, J.T., Guffanti, A. and Sarkar, S. (2016) Marginal Evidence for Cosmic Acceleration from Type Ia Supernovae. *Scientific Reports*, **6**, Article No. 35596. <https://doi.org/10.1038/srep35596>
- [5] Myrzakulov, N., et al. (2015) Inflationary Dynamics in Modified $f(R, G)$ Gravity. *European Physical Journal C*, **75**, 111.
- [6] Sebastiani, L. and Myrzakulov, R. (2015) $F(R)$ -Gravity and Inflation. *International Journal of Geometric Methods in Modern Physics*, **12**, Article 1530003. <https://doi.org/10.1142/s0219887815300032>
- [7] Houndjo, M.J.S. (2012) Reconstruction of $f(R, T)$ Gravity Describing Matter Dominated and Accelerated Phases. *International Journal of Modern Physics D*, **21**, Article 1250003. <https://doi.org/10.1142/s0218271812500034>
- [8] Sahoo, P.K., Mishra, B. and Chakradhar Reddy, G. (2014) Axially Symmetric Cosmological Model in $f(R, T)$ Gravity. *The European Physical Journal Plus*, **129**, Article No. 49. <https://doi.org/10.1140/epjp/i2014-14049-7>
- [9] Sahoo, P.K. and Mishra, B. (2014) Kaluza-Klein Dark Energy Model in the Form of

- Wet Dark Fluid in $f(R, T)$ Gravity. *Canadian Journal of Physics*, **92**, 931-936.
<https://doi.org/10.1139/cjp-2014-0235>
- [10] Moraes, P.H.R.S. and Correa, R.A.C. (2019) The Trace of the Trace of the Energy-Momentum Tensor-Dependent Einstein's Field Equations. *The European Physical Journal C*, **79**, Article No. 674. <https://doi.org/10.1140/epjc/s10052-019-7195-4>
- [11] Koussour, M., Altaibayeva, A., Bekov, S., Donmez, O., Muminov, S. and Rayimbaev, J. (2024) Observational Constraints on the Equation of State of Viscous Fluid in $f(R, T)$ Gravity. *Physics of the Dark Universe*, **46**, Article 101577.
<https://doi.org/10.1016/j.dark.2024.101577>
- [12] Beesham, A., Singh, V. and Jokweni, S. (2021) Bulk-Viscous Bianchi Type-I Cosmology in $f(R, T)$ Gravity. *International Journal of Geometric Methods in Modern Physics*, **18**, Article 2150064.
- [13] Maurya, D.C., et al. (2024) Cosmological Evolution with Bulk Viscosity and Cosmic Transit in $f(R, T)$ Gravity. *New Astronomy*, **104**, Article 102085.
- [14] Fortunato, J.A.S., et al. (2024) Reconstruction of Cosmic Acceleration in $f(R, T)$ Gravity. *Physica Scripta*, **99**, Article 035002.
- [15] Jeakel, A.P. (2023) Dark Energy Behaviour in $f(R, T)$ Cosmology. *Annals of Physics*, **455**, Article 169373.
- [16] Debnath, P.S. (2019) Bulk Viscous Cosmological Model in $f(R, T)$ Theory of Gravity. *International Journal of Geometric Methods in Modern Physics*, **16**, Article 1950005.
<https://doi.org/10.1142/s0219887819500051>
- [17] Arora, S., Bhattacharjee, S. and Sahoo, P.K. (2021) Late-time Viscous Cosmology in $f(R, T)$ Gravity. *New Astronomy*, **82**, Article 101452.
<https://doi.org/10.1016/j.newast.2020.101452>
- [18] Brahma, B.P. and Dewri, M. (2022) Bulk Viscous Bianchi Type-V Cosmological Model in $f(R, T)$ Theory of Gravity. *Frontiers in Astronomy and Space Sciences*, **9**, Article 831431. <https://doi.org/10.3389/fspas.2022.831431>
- [19] Yang, J., Lin, R.H. and Zhai, X.H. (2022) Viscous Cosmology in $f(T)$ Gravity. *The European Physical Journal C*, **82**, Article No. 1039.
<https://doi.org/10.1140/epjc/s10052-022-11008-2>
- [20] Arora, S., Pacif, S.K.J., Parida, A. and Sahoo, P.K. (2022) Bulk Viscous Matter and the Cosmic Acceleration of the Universe in $f(Q, T)$ Gravity. *Journal of High Energy Astrophysics*, **33**, 1-9. <https://doi.org/10.1016/j.jheap.2021.10.001>
- [21] Joyce, A., Lombriser, L. and Schmidt, F. (2016) Dark Energy versus Modified Gravity. *Annual Review of Nuclear and Particle Science*, **66**, 95-122.
<https://doi.org/10.1146/annurev-nucl-102115-044553>
- [22] Joyce, A., Jain, B., Khoury, J. and Trodden, M. (2015) Beyond the Cosmological Standard Model. *Physics Reports*, **568**, 1-98.
<https://doi.org/10.1016/j.physrep.2014.12.002>
- [23] Smeenk, C. and Weatherall, J.O. (2023) Dark Energy or Modified Gravity? *Philosophy of Science*, **91**, 1232-1241. <https://doi.org/10.1017/psa.2023.143>
- [24] Santhi, M.V., Rao, V.U.M. and Aditya, Y. (2018) Bulk Viscous String Cosmological Models in $f(R)$ Gravity. *Canadian Journal of Physics*, **96**, 55-61.
<https://doi.org/10.1139/cjp-2017-0256>
- [25] Abbott, T.M.C., Abdalla, F.B., Avila, S., Banerji, M., Baxter, E., Bechtol, K., et al. (2019) Dark Energy Survey Year 1 Results: Constraints on Extended Cosmological Models from Galaxy Clustering and Weak Lensing. *Physical Review D*, **99**, Article 123505.
<https://doi.org/10.1103/physrevd.99.123505>

- [26] Devi, L.A., Singh, S.S. and Alam, M.K. (2024) Phase Transition of Bianchi-Type I Cosmological Model in $f(T)$ Gravity. *New Astronomy*, **107**, Article 102156. <https://doi.org/10.1016/j.newast.2023.102156>
- [27] Myrzakulov, N., Shekh, S.H., Pradhan, A. and Dixit, A. (2025) Dark Energy and Cosmic Evolution: A Study in $f(R, T)$ Gravity. *Journal of High Energy Astrophysics*, **47**, Article 100374. <https://doi.org/10.1016/j.jheap.2025.100374>
- [28] Solanke, Y.S., Mhaske, S., Dagwal, V.J. and Pawar, D.D. (2025) LRS Bianchi Type-V Cosmological Model in $f(Q, T)$ Theory of Gravity with Cold Matter and Holographic Dark Energy. *Astronomy and Computing*, **52**, Article ID: 100961. <https://doi.org/10.1016/j.ascom.2025.100961>
- [29] Gadbail, G.N., Arora, S. and Sahoo, P.K. (2021) Viscous Cosmology in the Weyl-Type $f(Q, T)$ Gravity. *The European Physical Journal C*, **81**, Article No. 1088. <https://doi.org/10.1140/epjc/s10052-021-09889-w>
- [30] Navas, S., Amsler, C., Gutsche, T., et al. (Particle Data Group) (2024) Review of Particle Physics. *Physical Review D*, **110**, Article ID: 030001. <https://doi.org/10.1103/PhysRevD.110.030001>
- [31] Harko, T., Lobo, F.S.N., Nojiri, S. and Odintsov, S.D. (2011) $f(R, T)$ Gravity. *Physical Review D*, **84**, Article 024020. <https://doi.org/10.1103/physrevd.84.024020>
- [32] Zia, R., Kumar, D. and Pradhan, A. (2015) Bianchi Type-I Cosmological Model in $f(R, T)$ Theory of Gravity with Bulk Viscosity. *Astrophysics and Space Science*, **358**, 20.
- [33] Mohan, N.D.J., Sasidharan, A. and Mathew, T.K. (2017) Bulk Viscous Matter and Recent Acceleration of the Universe Based on Causal Viscous Theory. *The European Physical Journal C*, **77**, Article No. 508. <https://doi.org/10.1140/epjc/s10052-017-5428-y>
- [34] Brevik, I., Grøn, Ø., de Haro, J., Odintsov, S.D. and Saridakis, E.N. (2017) Viscous Cosmology for Early- and Late-Time Universe. *International Journal of Modern Physics D*, **26**, Article 1730024. <https://doi.org/10.1142/s0218271817300245>
- [35] Maartens, R. (1995) Dissipative Cosmology. *Classical and Quantum Gravity*, **12**, Article 1455.
- [36] Singh, C.P. (2011) Bulk Viscous Cosmology. *International Journal of Modern Physics D*, **20**, 2515.
- [37] Zimdahl, W. and Pavón, D. (2000) Expanding Universe with Positive Bulk Viscous Pressures? *Physical Review D*, **61**, Article 108301. <https://doi.org/10.1103/physrevd.61.108301>
- [38] Pouliaxis, C., et al. (2021) Viscous Cosmology with Scalar Field. *European Physical Journal C*, **81**, 1036.
- [39] Pradhan, A., Dixit, A. and Maurya, D.C. (2022) Quintessence Behavior of an Anisotropic Bulk Viscous Cosmological Model in Modified $f(Q)$ -Gravity. *Symmetry*, **14**, Article 2630. <https://doi.org/10.3390/sym14122630>
- [40] Solanki, R., et al. (2022) Bulk Viscosity in Symmetric Teleparallel Gravity. arXiv: 2205.04462.
- [41] Planck Collaboration (2018) Planck 2018 Results: Cosmological Parameters.
- [42] Barrow, J.D., et al. (2011) Anisotropic Bianchi Cosmology. *Physical Review D*, **83**, Article 043515.
- [43] Solanki, R., et al. (2023) Anisotropic Bulk Viscous Model in $f(R, LM)$ Gravity. arXiv: 2305.07683.

- [44] Sahlu, S., et al. (2025) Viscous-Fluid Constraints in $f(Q)$ Gravity. *European Physical Journal C*, **85**, 746.
- [45] Agrawal, P.R. and Nile, A.P. (2025) Exploration of Bulk Viscous Bianchi Type Cosmological Model in $f(T)$ Theory of Gravity. *New Astronomy*, **114**, Article 102300. <https://doi.org/10.1016/j.newast.2024.102300>
- [46] Thakre, M., Dhankar, P.K., Pourhassan, B. and Islam, S. (2025) Viscous Fluid Models Using MCMC Constraints. arXiv:2511.03258.
- [47] Tiwari, R.K., Beesham, A. and Shukla, B. (2018) Cosmological Model with Variable Deceleration Parameter in $f(R, T)$ Modified Gravity. *International Journal of Geometric Methods in Modern Physics*, **15**, Article 1850115. <https://doi.org/10.1142/s0219887818501153>
- [48] Akarsu, Ö. and Dereli, T. (2011) Cosmological Models with Linearly Varying Deceleration Parameter. *International Journal of Theoretical Physics*, **51**, 612-621. <https://doi.org/10.1007/s10773-011-0941-5>
- [49] Tiwari, R.K., Singh, R. and Shukla, B.K. (2015) A Cosmological Model with Variable Deceleration Parameter. *African Review of Physics*, **10**, 395-402.
- [50] Aygün, S., Aktaş, C. and Yılmaz, İ. (2016) Strange Quark Matter Solutions for Marder's Universe in $f(R, T)$ Gravity with Λ . *Astrophysics and Space Science*, **361**, Article No. 380. <https://doi.org/10.1007/s10509-016-2956-0>
- [51] Tiwari, R.K., Beesham, A. and Shukla, B.K. (2016) Behaviour of the Cosmological Model with Variable Deceleration Parameter. *The European Physical Journal Plus*, **131**, Article No. 447. <https://doi.org/10.1140/epjp/i2016-16447-1>
- [52] Tiwari, R.K., Beesham, A. and Shukla, B.K. (2017) Cosmological Models with Viscous Fluid and Variable Deceleration Parameter. *The European Physical Journal Plus*, **132**, Article No. 126. <https://doi.org/10.1140/epjp/i2017-11289-y>
- [53] Tiwari, R.K., Beesham, A. and Shukla, B.K. (2017) Scenario of a Two-Fluid FRW Cosmological Model with Dark Energy. *The European Physical Journal Plus*, **132**, Article No. 126. <https://doi.org/10.1140/epjp/i2017-11409-9>
- [54] Tiwari, R.K., Beesham, A. and Shukla, B.K. (2018) Scenario of Two-Fluid Dark Energy Models in Bianchi Type-III Universe. *International Journal of Geometric Methods in Modern Physics*, **15**, Article 1850189. <https://doi.org/10.1142/s021988781850189x>
- [55] Tiwari, R.K. and Sofuoğlu, D. (2020) Quadratically Varying Deceleration Parameter in $f(R, T)$ Gravity. *International Journal of Geometric Methods in Modern Physics*, **17**, Article 2030003. <https://doi.org/10.1142/s0219887820300032>
- [56] Wald, R.M. (1984) General Relativity. University of Chicago Press. <https://doi.org/10.7208/chicago/9780226870373.001.0001>
- [57] Carroll, S.M., Hoffman, M. and Trodden, M. (2003) Can the Dark Energy Equation-Of-State Parameter w Be Less Than -1? *Physical Review D*, **68**, Article 023509. <https://doi.org/10.1103/physrevd.68.023509>
- [58] Lake, K. (2004) Sudden Future Singularities in FLRW Cosmologies. *Classical and Quantum Gravity*, **21**, L129-L132. <https://doi.org/10.1088/0264-9381/21/21/L01>
- [59] Zubair, M., Waheed, S. and Ahmad, Y. (2016) Static Spherically Symmetric Wormholes in $f(R, T)$ Gravity. *The European Physical Journal C*, **76**, Article No. 444. <https://doi.org/10.1140/epjc/s10052-016-4288-1>
- [60] Sharif, M., Rani, S. and Myrzakulov, R. (2013) Analysis of $f(R, T)$ Gravity Models through Energy Conditions. *The European Physical Journal Plus*, **128**, Article No. 123. <https://doi.org/10.1140/epjp/i2013-13123-0>

- [61] Sahni, V., Saini, T.D., Starobinsky, A.A. and Alam, U. (2003) Statefinder—A New Geometrical Diagnostic of Dark Energy. *Journal of Experimental and Theoretical Physics Letters*, **77**, 201-206. <https://doi.org/10.1134/1.1574831>
- [62] Alam, U., Sahni, V., Deep Saini, T. and Starobinsky, A.A. (2003) Exploring the Expanding Universe and Dark Energy Using the Statefinder Diagnostic. *Monthly Notices of the Royal Astronomical Society*, **344**, 1057-1074. <https://doi.org/10.1046/j.1365-8711.2003.06871.x>