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On the Product of General Differential Operators with Their Point Spectra in the Direct Sum Spaces

Sobhy El-Sayed Ibrahim

Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt
Email: sobhyibrahim_55@hotmail.com

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Abstract

In this paper, the product of quasi-differential expressions $\tau_1, \tau_2, \cdots, \tau_n$ each of nth order with complex coefficients and its formal adjoints $\tau_1^+, \tau_2^+, \cdots, \tau_n^+$ on any finite number of intervals $I_p = (a_p, b_p)$, $p = 1, \cdots, N$ are considered in the setting of direct sums of $L^2_{w_p}(a_p, b_p)$-spaces of functions defined on each of the separate intervals, and a number of results concerning the location of the point spectra and regularity fields of product of differential operators generated by such expressions are obtained. Some of these are extensions or generalizations of those in a symmetric case, and of a general case with one interval case, whilst others are new.

Keywords

Product of Quasi-Differential Expressions, Essential Spectra, Joint Field of Regularity, Regularly Solvable Operators, Direct Sum Spaces

1. Introduction

In [1]-[3] considered the problem of linear differential operators in Hilbert space, in [4] and [5] considered the problem of general ordinary differential operators and their adjoints in Hilbert space. In [6]-[10] Everitt considered the problem of characterizing all self-adjoint operators which can be generated by a formally symmetric Sturm-Liouville differential (quasi-differential) expression τ_p, defined on a finite number of intervals $I_p, p = 1, \cdots, N$ in the setting of direct sum spaces. In [11]-[15] Ibrahim considered the problem of the location of the point spectra and regularity fields of general ordinary quasi-differential operators on the one interval case with one regular end-point and the other may be regular or singular.

Our objective in this paper is to investigate the location of the point spectra and regularity fields of the product operators which are generated by a general quasi-differential expression τ_p on any finite number of intervals $I_p, p=1,\cdots,N$ in the setting of direct sums of $L^2_{w_p}(a_p,b_p)$-spaces of functions defined on each of the separate intervals. These results extend those of general ordinary quasi-differential operators in [11]-[20] with one interval case. Also, extend those of formally symmetric differential expression studied in [21]-[26].

The operators involved are no longer symmetric but direct sums as:

$$T_0(\tau)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0(\tau_{jp})\right] \text{ and } T_0(\tau^+)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+)\right]$$

where $T_0(\tau_{jp})$ is the minimal operator generated by τ_{jp} on I_p and τ_{jp}^+ denotes by the formal adjoint of τ_{jp} which form an adjoint pair of closed operators in $\oplus_{p=1}^{N}L^2_{w_p}(I_p)$. This fact allows us to use the abstract theory developed in [1] for the operators which are regularly solvable with respect to $T_0(\tau_{jp})$ and $T_0(\tau_{jp}^+)$. Such an operator S satisfies $T_0(\tau)\subset S\subset\left[T_0(\tau^+)\right]^*$ and for some $\lambda\in\mathbb{C}$, $(S-\lambda I)$ is a Fredholm operator with zero index; this means that S has the desirable Fredholm property that the equation $(S-\lambda I)u=f$ has a solution if and only if f is orthogonal to the solutions of $(S^*-\bar{\lambda}I)v=0$ and furthermore the solution spaces of $(S-\lambda I)u=0$ and $(S^*-\bar{\lambda}I)v=0$ have the same finite dimension. This notion was originally due to Visik [25].

We deal throughout with a quasi-differential expressions τ_{jp} each of arbitrary order n defined by a general Shin -Zettl Matrix given in [3] [5] [9]-[17] and [19] [20], and the minimal operator $T_0(\tau_{jp})$ generated by $w_p^{-1}\tau_{jp}[.]$ in $L^2_{w_p}(I_p), p=1,\cdots,N$, where w_p is a positive weight function on the underlying interval I_p. The end-points of I_p may be regular or singular.

The study of the point spectra of product differential operators in direct sum spaces is a cornerstone of functional analysis and mathematical physics. This study has led to an understanding of representational properties, the solution of complex differential equations by transforming them into simplified algebraic systems, and the analysis of physical electronics, particularly in quantum mechanics, by representing complements as direct sums.

2. Notation and Preliminaries

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$ respectively and $N(T)$ will denote its null space. The nullity of T, written $null(T)$, is the dimension of $N(T)$ and the deficiency of T, written $def(T)$, is the co-dimension of $R(T)$ in H; thus if T is densely defined and $R(T)$ is closed, then $def(T)=null(T^*)$. The Fredholm domain of T is (in the notation of [5]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values of $\lambda \in \mathbb{C}$ which are such that $(T-\lambda I)$ is a Fredholm operator, where I is the identity operator in H. Thus $\lambda \in \Delta_3(T)$ if and only if $(T-\lambda I)$ has closed range and finite nullity and deficiency. The index of $(T-\lambda I)$ is the number $ind(T-\lambda I) = null(T-\lambda I) - def(T-\lambda I)$, this being defined for $\lambda \in \Delta_3(T)$.

Two closed densely defined operators A and B acting in a Hilbert space H are said to form an adjoint pair if $A \subset B^*$ and, consequently, $B \subset A^*$; equivalently, $(Ax, y) = (x, By)$ for all $x \in D(A)$ and $y \in D(B)$, where $(.,.)$ denotes the inner-product on H.

Definition 2.1: The field of regularity $\Pi(A)$ of A is the set of all $\lambda \in \mathbb{C}$ for which there exists a positive constant $K(\lambda)$ such that

$$\|(A-\lambda I)x\| \geq K(\lambda)\|x\| \text{ for all } x \in D(A), \tag{2.1}$$

or, equivalently, on using the Closed Graph Theorem, $null(A-\lambda I) = 0$ and $R(A-\lambda I)$ is closed.

The joint field of regularity $\Pi(A,B)$ of A and B is the set of $\lambda \epsilon \mathbb{C}$ which are such that $\lambda \in \Pi(A)$, $\bar{\lambda} \in \Pi(B)$ and both $def(A-\lambda I)$ and $def(B-\bar{\lambda} I)$ are finite. An adjoint pair A and B is said to be compatible if $\Pi(A,B) \neq \phi$.

Definition 2.2: A closed operator S in H is said to be regularly solvable with respect to the compatible adjoint pair A and B if $A \subset S \subset B^*$ and $\Pi(A,B) \cap \Delta_4(S) \neq \varnothing$, where $\Delta_4(S) = \{\lambda : \lambda \in \Delta_3(S), ind(S-\lambda I) = 0\}$.

The terminology "regularly solvable" comes from Visik's paper [25].

Definition 2.3: The resolvent set $\rho(S)$ of a closed operator S in H consists of the complex numbers λ for which $(S-\lambda I)^{-1}$ exists, is defined on H and is bounded. The complement of $\rho(S)$ in $\mathbb{C}$ is called the spectrum of S and written $\sigma(S)$. The point spectrum $\sigma_p(S)$, continuous spectrum $\sigma_c(S)$ and residual spectrum $\sigma_r(S)$ are the following subsets of $\sigma(S)$ (see [2] [3] and [12]-[15]):

$\sigma_p(S) = \{\lambda \in \sigma(S) : (S-\lambda I) \text{ is not injective}\}$, *i.e.*, the set of eigenvalues of S;

$$\sigma_c(S) = \left\{\lambda \in \sigma(S) : (S-\lambda I) \text{ is injective}, R(S-\lambda I) \subseteq \overline{R(S-\lambda I)} = H\right\};$$

$$\sigma_r(S) = \left\{\lambda \in \sigma(S) : (S-\lambda I) \text{ is injective}, \overline{R(S-\lambda I)} \neq H\right\}.$$

For a closed operator S we have,

$$\sigma(S) = \sigma_p(S) \cup \sigma_c(S) \cup \sigma_r(S).$$

An important subset of the spectrum of a closed densely defined operator S in H is the so-called essential spectrum. The various essential spectra of S are defined as in ([3], Chapter II] to be the sets:

$$\sigma_{ek}(S) = \mathbb{C} \setminus \Delta_k(S), (k = 1,2,3,4,5); \tag{2.2}$$

where $\Delta_3(S)$ and $\Delta_4(S)$ have been defined earlier. The sets $\sigma_{ek}(S)$ are closed and $\sigma_{ek}(S) \subset \sigma_{ej}(S)$ if $k < j$. The inclusion being strict in general. We refer the reader to [1] [2] and ([3], Chapter IX) for further information about the sets $\sigma_{ek}(S)$.

Given two operators A and B, both acting in a Hilbert space H, we wish to consider the product operator AB. This is defined as follows:

$$D(AB) = \{x \in D(B) \mid Bx \in D(A)\} \text{ and } (AB)x = A(Bx), \text{ for all } x \in D(AB). \quad (2.3)$$

It may happen in general that $D(AB)$ contains only the null element of H. However, in the case of many differential operators the domains of the product will be dense in H.

The next result gives conditions under which the deficiency of a product is the sum of the deficiencies of the factors.

Lemma 2.4 (cf. ([13], Theorem A)). Let A and B be closed operators with dense domains in a Hilbert space H. Suppose that $\lambda = 0$ is a regular type point for both operators and $def(A)$ and $def(B)$ are finite. Then AB is a closed operator with dense domain, has $\lambda = 0$ as a regular type point and

$$def(AB) = def(A) + def(B). \quad (2.4)$$

We refer to [5] [6] [11] [12] [14]-[19] and [22] for more details.

Evidently Lemma 2.4 extends to the product of any finite number of operators $A_1, A_2, \cdots, A_n$.

3. Quasi-Differential Expressions in Direct Sum Spaces

The quasi-differential expressions are defined in terms of a Shin-Zettl matrix F_p on an interval I_p. The set $Z_n(I_p)$ of Shin-Zettl matrices on I_p consists of $n \times n$-matrices $F_p = \{f_{rs}^p\}, p = 1, 2, \cdots, N$, whose entries are complex-valued functions on I_p which satisfy the following conditions:

$$\begin{gathered} f_{rs}^p \in L_{loc}^2(I_p), \ (1 \le r, s \le n, n \ge 2) \\ f_{r,r+1}^p \ne 0, \text{ a.e., on } I_p \ (1 \le r \le n-1) \\ f_{rs}^p = 0, \text{ a.e., on } I_p, \ (2 \le r+1 < s \le n), \ p = 1, 2, \cdots, N. \end{gathered} \quad (3.1)$$

For $F_p \in Z_n(I_p)$, the quasi-derivatives associated with F_p are defined by:

$$\begin{gathered} y^{[0]} := y, \\ y^{[r]} := \left(f_{r,r+1}^p\right)^{-1} \left\{\left(y^{[r-1]}\right)' - \sum_{s=1}^{r} f_{rs}^p y^{[s-1]}\right\}, \ (1 \le r \le n-1), \\ y^{[n]} := \left\{\left(y^{[n-1]}\right)' - \sum_{s=1}^{n} f_{ns}^p y^{[s-1]}\right\}, \end{gathered} \quad (3.2)$$

where the prime $'$ denotes differentiation.

The quasi-differential expression τ_p associated with F_p is given by:

$$\tau_p[.] := i^n y^{[n]}, \ (n \ge 2), \quad (3.3)$$

this being defined on the set:

$$V\left(\tau_p\right):=\left\{y: y^{[r-1]} \in AC_{loc}\left(I_p\right), r=1,2,\cdots,n\right\},\ p=1,2,\cdots,N,$$

where $AC_{loc}\left(I_p\right)$ denotes the set of functions which are absolutely continuous on every compact subinterval of I_p.

The formal adjoint τ_p^+ of τ_p is defined by the matrix F_p^+ given by:

$$\tau_p^+[.]:=i^n y_+^{[n]}, \text{ for all } y \in V\left(\tau_p^+\right), \tag{3.4}$$

$$V\left(\tau_p^+\right):=\left\{y: y_+^{[r-1]} \in AC_{loc}\left(I_p\right), r=1,2,\cdots,n\right\},\ p=1,2,\cdots,N,$$

where $y_+^{[r-1]}$, the quasi-derivatives associated with the matrix F_p^+ in $Z_n\left(I_p\right)$,

$$F_p^+=\left(f_{rs}^p\right)^+=\left(-1\right)^{r+s+1}\overline{f_{n-s+1,n-r+1}^p}, \text{ for each } r \text{ and } s. \tag{3.5}$$

Note that: $\left(F_p^+\right)^+=F_p$ and so $\left(\tau_p^+\right)^+=\tau_p$. We refer to [4] [6] [9]-[20] [22] and [26] for a full account of the above and subsequent results on quasi-differential expressions.

For $u \in V\left(\tau_p\right)$, $v \in V\left(\tau_p^+\right)$ and $\alpha, \beta \in I_p$, we have Green's formula,

$$\int_{a_p}^{b_p}\left\{\bar{v}\tau_p[u]-u\overline{\tau_p^+[v]}\right\}\mathrm{d}x=[u,v]\left(b_p\right)-[u,v]\left(a_p\right), p=1,2,\cdots,N, \tag{3.6}$$

where,

$$\begin{aligned}[u,v](x)&=i^n\left(\sum_{r=0}^{n-1}(-1)^{n+r+1}u^{[r]}(x)\overline{v_+^{[n-r-1]}}(x)\right)\\ &=(-i)^n\left(u,u^{[1]},\cdots,u^{[n-1]}\right)\times J_{n\times n}\begin{pmatrix}\bar{v}\\ \vdots\\ \bar{v}_+^{[n-1]}\end{pmatrix}(x);\end{aligned} \tag{3.7}$$

see [1] [4] [6] ([9], Corollary 1) [11] [13] [19] [21] and [26].

Let the interval I_p have end-points a_p, b_p $\left(-\infty \le a_p < b_p \le \infty\right)$, and let $w_p: I_p \to \mathbb{R}$ be a non-negative weight function with $w_p \in L_{loc}^1\left(I_p\right)$ and $w_p > 0$ (for almost all $x \in I_p$). Then $H_p=L_{w_p}^2\left(I_p\right)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that $\int_{I_p} w_p|f|^2<\infty$; the inner-product is defined by:

$$(f,g)_p:=\int_{I_p} w_p f(x)\overline{g(x)}\mathrm{d}x\ \left(f,g \in L_{w_p}^2\left(I_p\right), p=1,2,\cdots,N\right). \tag{3.8}$$

The equation

$$\tau_p[u]-\lambda w_p u=0(\lambda \in \mathbb{C}) \text{ on } I_p, p=1,2,\cdots,N, \tag{3.9}$$

is said to be regular at the left end-point $a_p \in \mathbb{R}$, if for all $X \in \left(a_p,b_p\right)$,

$$a_p \in \mathbb{R}, w_p, f_{rs}^p \in L^1\left(a_p,X\right),\left(r,s=1,2,\cdots,n;\ p=1,2,\cdots,N\right),$$

otherwise (3.9) is said to be singular at a_p. If (3.9) is regular at both end-points, then it is said to be regular; in this case we have,

$$a_p,b_p \in \mathbb{R}, w_p, f_{rs}^p \in L^1\left(a_p,b_p\right),\left(r,s=1,2,\cdots,n;\ p=1,2,\cdots,N\right).$$

We shall be concerned with the case when a_p is a regular end-point of (3.9), the end-point b_p being allowed to be either regular or singular. Note that, in view of (3.5), an end-point of I_p is regular for (3.9), if and only if it is regular for the equation:

$$\tau_p^+[v]-\bar{\lambda} w_p v=0(\lambda\in\mathbb{C}) \text{ on } I_p, p=1,2,\cdots,N. \tag{3.10}$$

Note that, at a regular end-point a_p, say, $u^{[r-1]}(a_p)\left(v_+^{[r-1]}(a_p)\right)$, $r=1,2,\cdots,n$ is defined for all $u\in V(\tau_p)$ $\left(v\in V(\tau_p^+)\right)$. Set:

$$D(\tau_p):=\left\{u:u\in V(\tau_p), u \text{ and } w_p^{-1}\tau_p[u]\in L^2_{w_p}(a_p,b_p)\right\},\ p=1,2,\cdots,N,$$

$$D(\tau_p^+):=\left\{v:v\in V(\tau_p^+), v \text{ and } w_p^{-1}\tau_p^+[v]\in L^2_{w_p}(a_p,b_p)\right\}, p=1,2,\cdots,N. \tag{3.11}$$

The subspaces $D(\tau_p)$ and $D(\tau_p^+)$ of $L^2_{w_p}(a_p,b_p)$ are domains of the so-called maximal operators $T(\tau_p)$ and $T(\tau_p^+)$ respectively, defined by:

$$T(\tau_p)u:=w_p^{-1}\tau_p[u],\left(u\in D(\tau_p)\right) \text{ and } T(\tau_p^+)v:=w_p^{-1}\tau_p^+[v],\left(v\in D(\tau_p^+)\right).$$

For the regular problem, the minimal operators $T_0(\tau_p)$ and $T_0(\tau_p^+), p=1,2,\cdots,N$ are the restrictions of $w_p^{-1}\tau_p[u]$ and $w_p^{-1}\tau_p^+[v]$ to the subspaces:

$$\left.\begin{aligned} D_0(\tau_p)&:=\left\{u:u\in D(\tau_p), u^{[r-1]}(a_p)=u^{[r-1]}(b_p), p=1,2,\cdots,N\right\}\\ D_0(\tau_p^+)&:=\left\{v:v\in D(\tau_p^+), v_+^{[r-1]}(a_p)=v_+^{[r-1]}(b_p), p=1,2,\cdots,N\right\}\end{aligned}\right\} \tag{3.12}$$

respectively. The subspaces $D_0(\tau_p)$ and $D_0(\tau_p^+)$ are dense in $L^2_{w_p}(a_p,b_p)$ and $T_0(\tau_p)$ and $T_0(\tau_p^+)$ are closed operators (see [4] [6] ([9], Section 3), [11]-[14], [19]-[22] and [26]).

In the singular problem we first introduce the operators $T_0'(\tau_p)$ and $T_0'(\tau_p^+)$; $T_0'(\tau_p)$ being the restriction of $w_p^{-1}\tau_p[.]$ to the subspace:

$$D_0'(\tau_p):=\left\{u:u\in D(\tau_p), supp(u)\subset(a_p,b_p), p=1,2,\cdots,N\right\} \tag{3.13}$$

and with $T_0'(\tau_p^+)$ defined similarly. These operators are densely-defined and closable in $L^2_{w_p}(a_p,b_p)$; and we define the minimal operators $T_0(\tau_p)$ and $T_0(\tau_p^+)$ to be their respective closures (see [4] [9] [11]-[14]). We denote the domains of $T_0(\tau_p)$ and $T_0(\tau_p^+)$ by $D_0(\tau_p)$ and $D_0(\tau_p^+)$ respectively. It can be shown that:

$$u\in D_0(\tau_p)\Rightarrow u^{[r-1]}(a_p)=0,(r=1,2,\cdots,n; p=1,2,\cdots,N),$$

$$v\in D_0(\tau_p^+)\Rightarrow v_+^{[r-1]}(a_p)=0,(r=1,2,\cdots,n; p=1,2,\cdots,N) \tag{3.14}$$

because we are assuming that a_p is a regular end-point. Moreover, in both regular and singular problems, we have

$$T_0^*\left(\tau_p\right)=T\left(\tau_p^+\right), T^*\left(\tau_p\right)=T_0\left(\tau_p^+\right), p=1,2,\cdots,N; \tag{3.15}$$

see ([9], Section 5) in the case when $\tau_p=\tau_p^+$ and compare with treatment in [4] [11] ([12], Section III 10.3) and [13]-[20] in general case.

In the case of two singular end-points, the problem on $\left(a_p,b_p\right)$. is effectively reduced to the problems with one singular end-point on the intervals $\left(a_p,c_p\right]$. and $\left[c_p,b_p\right)$, where $c_p\in\left(a_p,b_p\right)$.We denote by $T\left(\tau_p;a_p\right)$ and $T\left(\tau_p;b_p\right)$ the maximal operators with domains $D\left(\tau_p;a_p\right)$ and $D\left(\tau_p;b_p\right)$, and denote $T_0\left(\tau_p;a_p\right)$ and $T_0\left(\tau_p;b_p\right)$ the closures of the operators $T_0'\left(\tau_p;a_p\right)$ and $T_0'\left(\tau_p;b_p\right)$ defined in (3.13) on the intervals $\left(a_p,c_p\right]$ and $\left[c_p,b_p\right)$, respectively, see [3] [7] [10]-[19] and [26].

Let $\tilde{T}_0'\left(\tau_p\right), p=1,\cdots,N$, be the orthogonal sum as:

$\tilde{T}_0'\left(\tau_p\right)=T_0'\left(\tau_p;a_p\right)\oplus T_0'\left(\tau_p;b_p\right)$ in $L^2_{w_p}\left(a_p,b_p\right)=L^2_{w_p}\left(a_p,c_p\right)\oplus L^2_{w_p}\left(c_p,b_p\right)$

$\tilde{T}_0'\left(\tau_p\right)$ is densely-defined and closable in $L^2_{w_p}\left(a_p,b_p\right)$ and its closure is given by, $\tilde{T}_0\left(\tau_p\right)=T_0\left(\tau_p;a_p\right)\oplus T_0\left(\tau_p;b_p\right)$, $p=1,\cdots,N$.

Also,

$$null\left[\tilde{T}_0\left(\tau_p\right)-\lambda I\right]=null\left[T_0\left(\tau_p;a_p\right)-\lambda I\right]+null\left[T_0\left(\tau_p;b_p\right)-\lambda I\right],$$

$$def\left[\tilde{T}_0\left(\tau_p\right)-\lambda I\right]=def\left[T_0\left(\tau_p;a_p\right)-\lambda I\right]+def\left[T_0\left(\tau_p;b_p\right)-\lambda I\right],$$

and $R\left[\tilde{T}_0\left(\tau_p\right)-\lambda I\right]$ is closed if and only if $R\left[T_0\left(\tau_p;a_p\right)-\lambda I\right]$ and $R\left[T_0\left(\tau_p;b_p\right)-\lambda I\right]$ are both closed. These results imply in particular that,

$$\Pi\left[\tilde{T}_0\left(\tau_p\right)\right]=\Pi\left[T_0\left(\tau_p;a_p\right)\right]\cap\Pi\left[T_0\left(\tau_p;b_p\right)\right], p=1,\cdots,N.$$

We refer to ([3], Section 3.10.14), [11] [13] [14] and [18] for more details.

Remark 3.1: If S_p^a is a regularly solvable extension of $T_0\left(\tau_p;a_p\right)$ and S_p^b is a regularly solvable extension of $T_0\left(\tau_p;b_p\right)$, then $S=S_p^a\oplus S_p^b$ is a regularly solvable extension of $\tilde{T}_0\left(\tau_p\right)$, $p=1,\cdots,N$. We refer to ([3], Section 3.10.4) and [11]-[17] for more details.

Next, we state the following results; the proof is similar to that in ([3], Section 3.10.4), [11]-[13] [16] and [17].

Theorem 3.2: $\tilde{T}_0\left(\tau_p\right)\subset T_0\left(\tau_p\right)$, $T\left(\tau_p\right)\subset T_0\left(\tau_p;a_p\right)\oplus T_0\left(\tau_p;b_p\right)$ and

$$Dim\left(D\left[T_0\left(\tau_p\right)\right]/D\left[\tilde{T}_0\left(\tau_p\right)\right]\right)=n,\ p=1,\cdots,N.$$

If $\lambda\in\Pi\left[\tilde{T}_0\left(\tau_p\right)\right]\cap\Delta_3\left[T_0\left(\tau_p\right)-\lambda I\right]$, then $ind\left[T_0\left(\tau_p\right)-\lambda I\right]=n-def\left[T_0\left(\tau_p;a_p\right)-\lambda I\right]-def\left[T_0\left(\tau_p;b_p\right)-\lambda I\right]$, and in particular, if $\lambda\in\Pi\left[T_0\left(\tau_p\right)\right]$,

$$def\left[T_0\left(\tau_p\right)-\lambda I\right]=def\left[T_0\left(\tau_p;a_p\right)-\lambda I\right]+def\left[T_0\left(\tau_p;b_p\right)-\lambda I\right]-n. \tag{3.16}$$

Remark 3.3: It can be shown that

$$D\left[\tilde{T}_0\left(\tau_p\right)\right]=\left\{u:u\in D\left[T_0\left(\tau_p\right)\right]\text{ and }u^{[r-1]}\left(c_p\right)=0,r=1,\cdots,n\right\},$$
$$D\left[\tilde{T}_0\left(\tau_p^+\right)\right]=\left\{v:v\in D\left[T_0\left(\tau_p^+\right)\right]\text{ and }v^{+[r-1]}\left(c_p\right)=0,r=1,\cdots,n\right\}; \quad (3.17)$$

see ([3], Section 3.10.4).

Let H be the direct sum,

$$H=\oplus_{p=1}^{N}H_p=\oplus_{p=1}^{N}L_{w_p}^2\left(a_p,b_p\right). \quad (3.18)$$

The elements of H will be denoted by $\tilde{f}=\{f_1,\cdots,f_N\}$ with $f_1\in H_1,\cdots,f_N\in H_N$.

Remark 3.4: When $I_i\cap I_j=\varnothing,i\neq j;i,j=1,\cdots,N$, the direct sum space $\oplus_{p=1}^{N}L_{w_p}^2\left(a_p,b_p\right)$ can be naturally identified with the space $L_w^2\left(\bigcup_{p=1}^{N}I_p\right)$, where $w_p=w$ on $I_p,p=1,\cdots,N$. This remark is of significance when $\bigcup_{p=1}^{N}I_p$ may be taken as a single interval, see [8] [10] [13] [16] and [17].

We now establish by [3] [8] [10] and [13] some further notations,

$$\left.\begin{aligned}&D_0(\tau)=\oplus_{p=1}^{N}D_0\left(\tau_p\right),D(\tau)=\oplus_{p=1}^{N}D\left(\tau_p\right),\\&D_0\left(\tau^+\right)=\oplus_{p=1}^{N}D_0\left(\tau_p^+\right),D\left(\tau^+\right)=\oplus_{p=1}^{N}D\left(\tau_p^+\right)\end{aligned}\right\} \quad (3.19)$$

$$T_0(\tau)f:=\left\{T_0(\tau_1)f_1,\cdots,T_0(\tau_N)f_N\right\};f_1\in D_0(\tau_1),\cdots,f_N\in D_0(\tau_N),$$
$$T_0\left(\tau^+\right)f:=\left\{T_0\left(\tau_1^+\right)g_1,\cdots,T_0\left(\tau_N^+\right)g_N\right\};g_1\in D_0\left(\tau_1^+\right),\cdots,g_N\in D_0\left(\tau_N^+\right).$$

Also,

$$T(\tau)f:=\left\{T(\tau_1)f_1,\cdots,T(\tau_N)f_N\right\};f_1\in D(\tau_1),\cdots,f_N\in D(\tau_N),$$
$$T\left(\tau^+\right)g:=T\left(\tau_1^+\right)g_1,\cdots,T\left(\tau_N^+\right)g_N;g_1\in D\left(\tau_1^+\right),\cdots,g_N\in D\left(\tau_N^+\right).$$

We summarize a few additional properties of $T_0(\tau)$ in the form of a Lemma.

Lemma 3.5: We have,

(i) $\left[T_0(\tau)\right]^*=\oplus_{p=1}^{N}\left[T_0\left(\tau_p\right)\right]^*=\oplus_{p=1}^{N}\left[T\left(\tau_p^+\right)\right],$

$$\left[T_0\left(\tau^+\right)\right]^*=\oplus_{p=1}^{N}\left[T_0\left(\tau_p^+\right)\right]^*=\oplus_{p=1}^{N}\left[T\left(\tau_p\right)\right].$$

In particular,

$$D\left[T_0(\tau)\right]^*=D\left[T\left(\tau^+\right)\right]=\oplus_{p=1}^{N}\left[T\left(\tau_p^+\right)\right],$$
$$D\left[T_0\left(\tau^+\right)\right]^*=D\left[T(\tau)\right]=\oplus_{p=1}^{N}\left[T\left(\tau_p\right)\right].$$

(ii) $null\left[T_0(\tau)-\lambda I\right]=\sum_{p=1}^{N}null\left[T_0\left(\tau_p\right)-\lambda I\right],$

$$null\left[T_0\left(\tau^+\right)-\bar{\lambda}I\right]=\sum_{p=1}^{N}null\left[T_0\left(\tau_p^+\right)-\bar{\lambda}I\right].$$

(iii) The deficiency indices of $T_0(\tau)$ are given by:

$$def\left[T_0(\tau)-\lambda I\right]=\sum_{p=1}^{N}def\left[T_0\left(\tau_p\right)-\lambda I\right]\text{ for all }\lambda\in\Pi\left[T_0\left(\tau_p\right)\right],$$

$$def\left[T_0\left(\tau^+\right)-\bar{\lambda}I\right]=\sum_{p=1}^{N}def\left[T_0\left(\tau_p^+\right)-\bar{\lambda}I\right] \text{ for all } \lambda\in\Pi\left[T_0\left(\tau_p^+\right)\right].$$

Proof: Part (i) follows immediately from the definition of $T_0(\tau)$ and from the general definition of an adjoint operator. The other parts are either direct consequences of part (i) or follow immediately from the definitions.

Lemma 3.6: ([13], Lemma 2.4). For $\lambda\in\Pi\left[T_0(\tau),T_0\left(\tau^+\right)\right]$,

$$def\left[T_0(\tau)-\lambda I\right]+def\left[T_0\left(\tau^+\right)-\bar{\lambda}I\right] \text{ is constant and}$$

$$0\le def\left[T_0(\tau)-\lambda I\right]+def\left[T_0\left(\tau^+\right)-\bar{\lambda}I\right]\le 2nN.$$

In the problem with one singular end-point,

$$nN\le def\left[T_0(\tau)-\lambda I\right]+def\left[T_0\left(\tau^+\right)-\bar{\lambda}I\right]\le 2nN,$$

for all $\lambda\in\Pi\left[T_0(\tau),T_0\left(\tau^+\right)\right]$.

In the regular problem,

$$def\left[T_0(\tau)-\lambda I\right]+def\left[T_0\left(\tau^+\right)-\bar{\lambda}I\right]=2nN, \text{ for all } \lambda\in\Pi\left[T_0(\tau),T_0\left(\tau^+\right)\right].$$

We refer to [3] [5] [11] [12] [16] and [17] for more details.

Lemma 3.7: Let $T_0(\tau)=\oplus_{p=1}^{N}T_0\left(\tau_p\right)$ be a closed densely-defined operator on H. Then,

$$\Pi\left[T_0(\tau)\right]=\bigcap_{p=1}^{N}\left[T_0\left(\tau_p\right)\right].$$

Proof: The proof follows from Lemma 3.1 and since $R\left[T_0(\tau)-\lambda I\right]$ is closed if and only if $R\left[T_0\left(\tau_p\right)-\lambda I\right]$, $p=1,2,\cdots,N$ are closed.

Lemma 3.8: If $S_p, p=1,\cdots,N$, are regularly solvable with respect to $T_0\left(\tau_p\right)$ and $T_0\left(\tau_p^+\right)$, then $S=\oplus_{p=1}^{N}S_p$ is regularly solvable with respect to $T_0(\tau)$ and $T_0\left(\tau_p^+\right)$.

Proof: The proof follows from Lemma 3.5 and Lemma 3.7.

Remark 3.9: Let $S=\oplus_{p=1}^{N}S_p$ be an arbitrary closed operator on H, and since $\lambda\in\rho(S)$ if, and only if, $nul(S-\lambda I)=def(S-\lambda I)=0$ (see ([2], Theorem 1.3.2)), we have $\rho(S)=\bigcap_{p=1}^{N}\rho\left(S_p\right)$. We therefore have,

$$\sigma(S)=\bigcup_{j=1}^{N}\sigma\left(S_j\right),\sigma_p(S)=\bigcup_{j=1}^{N}\sigma_p\left(S_j\right) \text{ and } \sigma_r(S)=\bigcup_{j=1}^{N}\sigma_j\left(S_r\right). \quad (3.20)$$

Also,

$$\sigma_{ek}(S)=\bigcup_{j=1}^{N}\sigma_{ek}\left(S_j\right),k=1,2,3. \quad (3.21)$$

We refer to ([3], Chapter 9) for more details.

Theorem 3.10: (cf. ([21], Part II, Theorem 16.2.2)). Suppose $f\in L_{loc}^1\left(I_p\right)$ and suppose that the conditions (3.1) are satisfied. Then given any complex numbers $c_j\in\mathbb{C}, j=0,1,\cdots,n-1$ and $x_0\in\left(a_p,b_p\right)$ there exists a unique solution of the equation $\tau_p\left[\varphi_p\right]=wf$ in the intervals $\left(a_p,b_p\right)$ which satisfies,

$$\varphi_p^{[j]}(x_0)=c_j,(j=0,1,\cdots,n-1;p=1,\cdots,N).$$

We refer to [1] and [3] for more details.

Theorem 3.11: (cf. [3] and ([21], Theorem II.2.5)). Let τ_p be a regular quasi-differential expression of order n on the closed interval $[a_p,b_p]$. For $f\in L^2_{loc}(I_p)$, the equation $\tau_p[\varphi_p]=wf$ has a solution $\varphi_p\in V(\tau_p)$ satisfying,

$$\varphi_p^{[j]}(a_p)=\varphi_p^{[j]}(b_p)=0,(j=0,1,\cdots,n-1;p=1,\cdots,N),$$

if, and only if, f is orthogonal in $L^2_{loc}(a_p,b_p)$ to the solution space of $\tau^+[\varphi_p]=0$, *i.e.*,

$$R[T_0(\tau_p)-\lambda I]=N[T(\tau_p^+)-\bar{\lambda}I]^{\perp},p=1,\cdots,N. \tag{3.22}$$

Corollary 3.12: (cf. ([21], Corollary II.2.6)). As a result from Theorem 3.11, we have that

$$R[T_0(\tau_p)-\lambda I]^{\perp}=N[T(\tau_p^+)-\bar{\lambda}I],p=1,\cdots,N. \tag{3.23}$$

Lemma 3.13: (cf. ([3], Lemma IX.9.1)). If $I_p=[a_p,b_p]$, with $-\infty<a_p<b_p<\infty$, $p=1,\cdots,N$, then for any $\lambda\in\mathbb{C}$, the operator $[T_0(\tau_p)-\lambda I]$, $p=1,\cdots,N$ has closed range, zero nullity and deficiency n. Hence,

$$\sigma_{ek}[T_0(\tau_p)]=\begin{cases}\varnothing & (k=1,2,3)\\ \mathbb{C} & (k=4,5),p=1,\cdots,N.\end{cases}$$

4. The Product Operators in Direct Sum Spaces

The proof of general theorems will be based on the results in this section. We start by listing some properties and results of quasi-differential expressions $\tau_1,\tau_2,\cdots,\tau_n$. For proofs the reader is referred to [3] [7]-[10] and [14]-[22].

$$(\tau_1+\tau_2)^+=\tau_1^++\tau_2^+$$

$$(\tau_1\tau_2)^+=\tau_2^+\tau_1^+,(\lambda\tau)^+=\bar{\lambda}\tau^+ \quad \text{for } \lambda \text{ a complex number} \tag{4.1}$$

A consequence of Properties (4.1) is that if $\tau^+=\tau$ then $(P(\tau))^+=P(\tau^+)$ for P any polynomial with complex coefficients. Also we note that the leading coefficients of a product is the product of the leading coefficients. Hence the product of regular differential expressions is regular.

Lemma 4.1: (cf. ([11], Theorem 1)). Suppose τ_j is a regular differential expression on the interval $[a,b]$ and $\lambda\in\Pi[T_0(\tau_1\tau_2\cdots\tau_n),T_0(\tau_1\tau_2\cdots\tau_n)^+]$, then we have,

(i) The product operator $\prod_{j=1}^n T_0(\tau_j)$ is closed, densely-defined, and

$$def\left[\prod_{j=1}^n T_0(\tau_j)-\lambda I\right]=\sum_{j=1}^n def[T_0(\tau_j)-\lambda I],$$

$$def\left[\prod_{j=1}^n T_0(\tau_j^+)-\bar{\lambda}I\right]=\sum_{j=1}^n def[T_0(\tau_j^+)-\bar{\lambda}I].$$

(ii) $T_0(\tau_1\tau_2\cdots\tau_n)\subseteq\prod_{j=1}^{n}\left[T_0(\tau_j)\right]$ and $T_0(\tau_1\tau_2\cdots\tau_n)^+\subseteq\prod_{j=1}^{n}\left[T_0(\tau_j^+)\right]$.

Note, in part (ii) that the containment may be proper, *i.e.*, the operators $T_0(\tau_1\tau_2\cdots\tau_n)$ and $\prod_{j=1}^{n}\left[T_0(\tau_j)\right]$ are not equal in general. We refer to [6] and [12]-[19] for more details.

From Lemma 3.1 and Lemma 4.1 we have the following:

Lemma 4.2: For $\lambda\in\Pi\left[\prod_{j=1}^{n}\left[T_0(\tau_j)\right],\prod_{j=1}^{n}\left[T_0(\tau_j^+)\right]\right]$ we have:

(i) $\left[\prod_{j=1}^{n}T_0^*(\tau_j)\right]=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0^*(\tau_{jp})\right]=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T(\tau_{jp}^+)\right]$,

$$\left[\prod_{j=1}^{n}T_0^*(\tau_j^+)\right]=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0^*(\tau_{jp}^+)\right]=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T(\tau_{jp})\right].$$

(ii)
$$\begin{aligned}null\left[\prod_{j=1}^{n}T_0(\tau_j)-\lambda I\right]&=\sum_{p=1}^{N}null\left[\prod_{j=1}^{n}T_0(\tau_{jp})-\lambda I\right]\\&=\sum_{p=1}^{N}\left(\sum_{j=1}^{n}null\left[T_0(\tau_{jp})-\lambda I\right]\right),\end{aligned}$$

$$\begin{aligned}null\left[\prod_{j=1}^{n}T_0(\tau_j^+)-\bar{\lambda}I\right]&=\sum_{p=1}^{N}null\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+)-\bar{\lambda}I\right]\\&=\sum_{p=1}^{N}\left(\sum_{j=1}^{n}null\left[T_0(\tau_{jp}^+)-\bar{\lambda}I\right]\right).\end{aligned}$$

(iii) The deficiency indices of $\prod_{j=1}^{n}T_0(\tau_j)$ and $\prod_{j=1}^{n}T_0(\tau_j^+)$ are given by:

$$\begin{aligned}def\left[\prod_{j=1}^{n}T_0(\tau_j)-\lambda I\right]&=\sum_{p=1}^{N}def\left[\prod_{j=1}^{n}T_0(\tau_{jp})-\lambda I\right]\\&=\sum_{p=1}^{N}\left(\sum_{j=1}^{n}def\left[T_0(\tau_{jp})-\lambda I\right]\right),\end{aligned}$$

$$\begin{aligned}def\left[\prod_{j=1}^{n}T_0(\tau_j^+)-\bar{\lambda}I\right]&=\sum_{p=1}^{N}def\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+)-\bar{\lambda}I\right]\\&=\sum_{p=1}^{N}\left(\sum_{j=1}^{n}def\left[T_0(\tau_{jp}^+)-\bar{\lambda}I\right]\right).\end{aligned}$$

Lemma 4.3: ([15] [16], Lemma 3.6). For $\lambda\in\Pi\left[\prod_{j=1}^{n}\left[T_0(\tau_j)\right],\prod_{j=1}^{n}\left[T_0(\tau_j^+)\right]\right]$, $def\left[\prod_{j=1}^{n}\left[T_0(\tau_j)\right]-\lambda I\right]+def\left[\prod_{j=1}^{n}\left[T_0(\tau_j^+)\right]-\bar{\lambda}I\right]$ is constant and $0\le def\left[\prod_{j=1}^{n}\left[T_0(\tau_j)\right]-\lambda I\right]+def\left[\prod_{j=1}^{n}\left[T_0(\tau_j^+)\right]-\bar{\lambda}I\right]\le 2n^2N$.

In the problem with one singular end-point,

$$n^2N\le def\left[\prod_{j=1}^{n}\left[T_0(\tau_j)\right]-\lambda I\right]+def\left[\prod_{j=1}^{n}\left[T_0(\tau_j^+)\right]-\bar{\lambda}I\right]\le 2n^2N,$$

for all $\lambda\in\Pi\left[\prod_{j=1}^{n}\left[T_0(\tau_j)\right],\prod_{j=1}^{n}\left[T_0(\tau_j^+)\right]\right]$.

In the regular problem,

$$def\left[\prod_{j=1}^{n}\left[T_0(\tau_j)\right]-\lambda I\right]+def\left[\prod_{j=1}^{n}\left[T_0(\tau_j^+)\right]-\bar{\lambda}I\right]=2n^2N,$$

for all $\lambda\in\Pi\left[\prod_{j=1}^{n}\left[T_0(\tau_j)\right],\prod_{j=1}^{n}\left[T_0(\tau_j^+)\right]\right]$.

Proof: For $\lambda\in\Pi\left[\prod_{j=1}^{n}\left[T_0(\tau_j)\right],\prod_{j=1}^{n}\left[T_0(\tau_j^+)\right]\right]$, we obtain from (3.15), (3.16) and Lemma 4.2 that,

$$
\begin{aligned}
& def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j\right)\right]-\lambda I\right]+def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j^{+}\right)\right]-\bar{\lambda} I\right] \\
& =\left\{def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j,a\right)\right]-\lambda I\right]+def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j,b\right)\right]-\lambda I\right]-n^2 N\right\} \\
& \quad+\left\{def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j^{+},a\right)\right]-\bar{\lambda} I\right]+def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j^{+},b\right)\right]-\bar{\lambda} I\right]-n^2 N\right\} \\
& =\left\{null\left[\prod_{j=1}^{n}\left[T\left(\tau_j^{+},a\right)\right]-\bar{\lambda} I\right]+null\left[\prod_{j=1}^{n}\left[T\left(\tau_j^{+},b\right)\right]-\bar{\lambda}\lambda I\right]-n^2 N\right\} \\
& \quad+\left\{null\left[\prod_{j=1}^{n}\left[T\left(\tau_j,a\right)\right]-\lambda I\right]+null\left[\prod_{j=1}^{n}\left[T\left(\tau_j,b\right)\right]-\lambda I\right]-n^2 N\right\} \\
& \leq 2\left(2n^2 N-n^2 N\right)=2n^2 N,
\end{aligned}
$$

with equality in the regular problem. In the problem with one singular end-point, we get

$$
def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j\right)\right]-\lambda I\right]+def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j^{+}\right)\right]-\bar{\lambda} I\right]\geq n^2 N.
$$

For the problem with two singular end-points, we have

$$
\begin{aligned}
& def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j\right)\right]-\lambda I\right]+def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j^{+}\right)\right]-\bar{\lambda} I\right] \\
& =\left\{def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j,a\right)\right]-\lambda I\right]+def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j^{+},a\right)\right]-\bar{\lambda} I\right]\right\} \\
& \quad+\left\{def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j,b\right)\right]-\lambda I\right]+def\left[\prod_{j=1}^{n}\left[T_0\left(\tau_j^{+},b\right)\right]-\bar{\lambda} I\right]\right\}-2n^2 N \\
& \geq\left(2n^2 N-2n^2 N\right)=0.
\end{aligned}
$$

See [3] [15] and [16] for more details.

Lemma 4.4: Let $\tau_1,\tau_2,\cdots,\tau_n$ be a regular differential expressions on $[a,b)$ and suppose that $\lambda\in\Pi\left[T_0\left(\tau_1\tau_2\cdots\tau_n\right),T_0\left(\tau_1\tau_2\cdots\tau_n\right)^{+}\right]$. Then

$$
\left[T_0\left(\tau_1\tau_2\cdots\tau_n\right)\right]=\prod_{j=1}^{n}\left[T_0\left(\tau_j\right)\right] \tag{4.2}
$$

if and only if the following partial separation conditions is satisfied:

$$
\left\{\begin{array}{l}
f\in L_w^2(a,b), f^{[s-1]}\in AC_{loc}[a,b), \text{where } s \text{ is the order of product} \\
\text{expression } \left(\tau_1\tau_2\cdots\tau_n\right) \text{ and } \left(\tau_1\tau_2\cdots\tau_n\right)^{+} f\in L_w^2(a,b) \text{ together imply} \\
\text{that}:\left(\prod_{j=1}^{k}\left(\tau_j^{+}\right)\right)f\in L_w^2(a,b), k=1,\cdots,n-1
\end{array}\right\}. \tag{4.3}
$$

Furthermore,

$$
T_0\left(\tau_1\tau_2\cdots\tau_n\right)=\prod_{j=1}^{n}\left[T_0\left(\tau_j\right)\right] \quad\text{and}\quad T_0\left(\tau_1\tau_2\cdots\tau_n\right)^{+}=\prod_{j=1}^{n}\left[T_0\left(\tau_j^{+}\right)\right]
$$

if and only if,

$$
\begin{aligned}
def\left[T_0\left(\tau_1\tau_2\cdots\tau_n\right)-\lambda I\right]&=\sum_{j=1}^{n} def\left[T_0\left(\tau_j\right)-\lambda I\right], \\
def\left[T_0\left(\tau_1\tau_2\cdots\tau_n\right)^{+}-\bar{\lambda} I\right]&=\sum_{j=1}^{n} def\left[T_0\left(\tau_j^{+}\right)-\bar{\lambda} I\right].
\end{aligned}
$$

We will say that the product $\tau_1,\tau_2,\cdots,\tau_n$ is partially separated expressions in $L_w^2(a,b)$ whenever Property (4.3) holds.

Lemma 4.5: For $\lambda \in \Pi\left[T_0\left(\tau_1\tau_2\cdots\tau_n\right), T_0\left(\tau_1\tau_2\cdots\tau_n\right)^+\right]$ we have,

$$\Pi\left[T_0\left(\tau_1\tau_2\cdots\tau_n\right), T_0\left(\tau_1\tau_2\cdots\tau_n\right)^+\right] = \Pi\left[\prod_{j=1}^n\left[T_0\left(\tau_j\right)\right], \prod_{j=1}^n\left[T_0\left(\tau_j^+\right)\right]\right]. \quad (4.4)$$

Proof: Let $\lambda \in \Pi\left[T_0\left(\tau_1\tau_2\cdots\tau_n\right), T_0\left(\tau_1\tau_2\cdots\tau_n\right)^+\right]$, then from definition of the field of regularity we have $\lambda \in \Pi\left[T_0\left(\tau_1\tau_2\cdots\tau_n\right)\right]$ and $\bar{\lambda} \in \Pi\left[T_0\left(\tau_1\tau_2\cdots\tau_n\right)^+\right]$, *i.e.*, each of the operators $T_0\left(\tau_1\tau_2\cdots\tau_n\right)$ and $T_0\left(\tau_1\tau_2\cdots\tau_n\right)^+$ has closed range and densely-defined on H with finite deficiency indices. Consequently by Lemma 4.2 each of the operators $\left[\prod_{j=1}^n T_0\left(\tau_j\right) - \lambda I\right]$ and $\left[\prod_{j=1}^n T_0\left(\tau_j^+\right) - \bar{\lambda} I\right]$ has closed range and their deficiency indices are finite, *i.e.*, $\lambda \in \Pi\left[\prod_{j=1}^n\left[T_0\left(\tau_j\right)\right], \prod_{j=1}^n\left[T_0\left(\tau_j^+\right)\right]\right]$. The rest of the proof follows from definition and Lemma 4.2.

Theorem 4.6: Let τ_{jp} be a regular differential expressions on the interval $\left[a_p, b_p\right)$ for $j = 1,2,\cdots,n$; $p = 1,2,\cdots,N$. If all solutions of the differential equations $\left[\tau_{jp} - \lambda w\right]u = 0$ and $\left[\tau_{jp}^+ - \bar{\lambda} w\right]v = 0$ are in $L^2_{w_p}\left(I_p\right)$ for $\lambda \in \mathbb{C}$; then all solutions of the equations $\left[\prod_{j=1}^n\left(\tau_{jp}\right) - \lambda w\right]u = 0$ and $\left[\prod_{j=1}^n\left(\tau_{jp}^+\right) - \bar{\lambda} w\right]v = 0$ are in $L^2_{w_p}\left(I_p\right)$ for all $\lambda \in \mathbb{C}$.

Proof: Let $n = n_{jp}$ = order of τ_{jp} = order τ_{jp}^+ for $j = 1,2,\cdots,n$; $p = 1,2,\cdots,N$. Then by Lemma 2.4 we have:

$$\begin{aligned} def\left[\prod_{j=1}^n\left[T_0\left(\tau_j\right)\right] - \lambda I\right] &= \sum_{p=1}^N def\left[\prod_{j=1}^n\left[T_0\left(\tau_{jp}\right)\right] - \lambda I\right] \\ &= \sum_{p=1}^N \sum_{j=1}^n def\left[T_0\left(\tau_{jp}\right) - \lambda I\right] \\ def\left[\prod_{j=1}^n\left[T_0\left(\tau_j^+\right)\right] - \bar{\lambda} I\right] &= \sum_{p=1}^N def\left[\prod_{j=1}^n\left[T_0\left(\tau_{jp}^+\right)\right] - \bar{\lambda} I\right], \\ &= \sum_{p=1}^N \sum_{j=1}^n def\left[T_0\left(\tau_{jp}^+\right) - \bar{\lambda} I\right] \end{aligned}$$

for all $\lambda \in \Pi\left[\prod_{j=1}^n\left[T_0\left(\tau_{jp}\right)\right], \prod_{j=1}^n\left[T_0\left(\tau_{jp}^+\right)\right]\right]$, $p = 1,2,\cdots,N$.

Hence, by Lemma 4.4 we have,

$$\begin{aligned} def\left[T_0\left(\tau_{1p}\tau_{2p}\cdots\tau_{np}\right)^+ - \bar{\lambda} I\right] &= def\left[\prod_{j=1}^n\left[T_0\left(\tau_j^+\right)\right] - \bar{\lambda} I\right] \\ &= \sum_{p=1}^N \sum_{j=1}^n n_{jp} = n^2 N \\ &= \text{order of } \left(\tau_{1p}\tau_{2p}\cdots\tau_{np}\right) \\ &= \text{order of } \left(\tau_{1p}\tau_{2p}\cdots\tau_{np}\right)^+. \end{aligned}$$

Thus $def\left[T_0\left(\tau_{np}^+\cdots\tau_{2p}^+\tau_{1p}^+\right) - \bar{\lambda} I\right] = \text{order of } \left(\tau_{1p}\tau_{2p}\cdots\tau_{np}\right)^+$ and consequently all solutions of the equations

$$\left[\prod_{j=1}^n\left(\tau_{jp}\right) - \lambda w\right]u = 0 \quad \text{and} \quad \left[\prod_{j=1}^n\left(\tau_{jp}^+\right) - \bar{\lambda} w\right]v = 0,$$

are in $L^2_{w_p}\left(I_p\right)$ for all $\lambda \in \mathbb{C}$.

5. The Point Spectra in Direct Sum Spaces

In this section we shall consider our interval to be $I=[a,b)$. We denote by $T(\tau)$ and $T_0(\tau)$ the maximal and minimal operators defined on the interval *I*. Also, we deal with the various components of the spectra of $T_0(\tau)$ and $T_0(\tau^+)$ as the direct sum of product differential operators $\prod_{j=1}^{n} T_0(\tau_{jp})$ and $\prod_{j=1}^{n} T_0(\tau_{jp}^+)$, $p=1,\cdots,N$.

Lemma 5.1: Let $T_0(\tau)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n} T_0(\tau_{jp})\right]$ and $T_0(\tau^+)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n} T_0(\tau_{jp}^+)\right]$ be a regular differential operators, then the point spectra $\sigma_p[T_0(\tau)]$ and $\sigma_p[T_0(\tau^+)]$ of $T_0(\tau)$ and $T_0(\tau^+)$ are empty.

Proof: Let $\lambda\in\sigma_p\left[\prod_{j=1}^{n} T_0(\tau_{jp})\right]$. Then there exists a non-zero element $\varphi_p\in D_0\left[\prod_{j=1}^{n}(\tau_{jp})\right]$ such that $\left[\prod_{j=1}^{n} T_0(\tau_{jp})-\lambda I\right]\varphi_p=0$, $p=1,\cdots,N$.

In particular, this gives that

$$\left[\prod_{j=1}^{n}\tau_{jp}\right]\varphi_p=\lambda w\varphi_p, \tag{5.1}$$

$$\varphi_p^{[r]}(a_p)=\varphi_p^{[r]}(b_p)=0,\left(r=0,1,\cdots,n^2-1;\ p=1,\cdots,N\right).$$

From Theorem 3.10, it follows that $\varphi_p=0$ and hence

$$\sigma_p\left[\prod_{j=1}^{n} T_0(\tau_{jp})\right]=\varnothing, p=1,\cdots,N.$$

Similarly,

$$\sigma_p\left[\prod_{j=1}^{n} T_0(\tau_{jp}^+)\right]=\varnothing, p=1,\cdots,N.$$

Therefore, by (3.19), we have,

$$\sigma_p[T_0(\tau)]=\bigcup_{p=1}^{N}\sigma_p\left[\prod_{j=1}^{n} T_0(\tau_{jp})\right]=\varnothing, \tag{5.2}$$

and

$$\sigma_p[T_0(\tau^+)]=\bigcup_{p=1}^{N}\sigma_p\left[\prod_{j=1}^{n} T_0(\tau_{jp}^+)\right]=\varnothing; \tag{5.3}$$

see Naimark ([21], part II, Section 19].

Examples:

1) If $X=\mathbb{R}^2$ and $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $T(x,y)=(-y,x)$ $\forall(x,y)\in\mathbb{R}^2$, then T is a linear operator but has no eigenvalues, *i.e.*, $\sigma_p(T)=\phi$.

2) If $X=\mathbb{R}^2$ and $T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $T(x,y)=(x+2y,3x+2y)$ $\forall(x,y)\in\mathbb{R}^2$, then T is a linear operator and has eigenvalues $\lambda=-1,4$, *i.e.*, $\sigma_p(T)=\{-1,4\}$.

3) If $X=\ell^2$ and $T:\ell^2\to\ell^2$ defined by $T(x_1,x_2,x_3,\cdots)=(0,x_1,x_2,x_3,\cdots)$, $\forall(x_1,x_2,x_3,\cdots)\in\ell^2$, then T is a linear operator and has no eigenvalues *i.e.*, $\sigma_p(T)=\phi$.

Theorem 5.2: Let

$$T_0(\tau)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0(\tau_{jp})\right] \text{ and } T_0(\tau^+)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+)\right]. \tag{5.4}$$

Then,

(i) $\rho[T_0(\tau)]=\rho[T_0(\tau^+)]=\varnothing$,

(ii) $\sigma_c[T_0(\tau)]=\sigma_c[T_0(\tau^+)]=\varnothing$,

(iii) $\sigma_r[T_0(\tau)]=\sigma_r[T_0(\tau^+)]=\mathbb{C}$ and $\sigma[T_0(\tau)]=\sigma[T_0(\tau^+)]=\mathbb{C}$.

Proof: (i) Let $\lambda\in\mathbb{C}$, since $R\left[\prod_{j=1}^{n}T_0(\tau_{jp})-\lambda I\right], p=1,\cdots,N$ are proper closed subspaces of $L_w^2(a_p,b_p)$, then the resolvent sets $\rho\left[\prod_{j=1}^{n}T_0(\tau_{jp})\right]$ are empty and hence

$$\rho[T_0(\tau)]=\bigcap_{p=1}^{N}\left[\prod_{j=1}^{n}T_0(\tau_{jp})\right]=\varnothing.$$

Similarly

$$\rho[T_0(\tau^+)]=\bigcap_{p=1}^{N}\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+)\right]=\varnothing.$$

(ii) Since $R\left[\prod_{j=1}^{n}T_0(\tau_{jp})-\lambda I\right], p=1,\cdots,N$ are closed for any $\lambda\in\mathbb{C}$, then the continuous spectrum of $\prod_{j=1}^{n}T_0(\tau_{jp})$ are the empty sets, *i.e.*,

$$\sigma_c\left[\prod_{j=1}^{n}T_0(\tau_{jp})\right]=\varnothing, p=1,\cdots,N.$$

Hence,

$$\sigma_c[T_0(\tau)]=\bigcup_{p=1}^{N}\sigma_c\left[\prod_{j=1}^{n}T_0(\tau_{jp})\right]=\varnothing.$$

Similarly,

$$\sigma_c[T_0(\tau^+)]=\bigcup_{p=1}^{N}\sigma_c\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+)\right]=\varnothing.$$

(iii) From (i), (ii) and Lemma 3.5, it follows that,

$$\sigma_r[T_0(\tau)]=\bigcup_{p=1}^{N}\sigma_r\left[\prod_{j=1}^{n}T_0(\tau_{jp})\right]=\mathbb{C},$$

and

$$\sigma_r[T_0(\tau^+)]=\bigcup_{p=1}^{N}\sigma_r\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+)\right]=\mathbb{C}.$$

Similarly,

$$\sigma[T_0(\tau)]=\bigcup_{p=1}^{N}\sigma\left[\prod_{j=1}^{n}T_0(\tau_{jp})\right]=\mathbb{C},$$

and

$$\sigma[T_0(\tau^+)]=\bigcup_{p=1}^{N}\sigma\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+)\right]=\mathbb{C}.$$

Corollary 5.3: Let

$$T_0(\tau)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0(\tau_{jp})\right] \text{ and } T_0(\tau^+)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0(\tau_{jp}^+)\right].$$

Then,

(i) $\sigma_c\left[T(\tau)\right]=\sigma_c\left[T\left(\tau^+\right)\right]=\varnothing$ and $\sigma_r\left[T(\tau)\right]=\sigma_r\left[T\left(\tau^+\right)\right]=\varnothing$,

(ii) $\sigma\left[T(\tau)\right]=\sigma\left[T\left(\tau^+\right)\right]=\mathbb{C}$ and $\sigma_p\left[T(\tau)\right]=\sigma_p\left[T\left(\tau^+\right)\right]=\mathbb{C}$,

(iii) $\rho\left[T(\tau)\right]=\rho\left[T\left(\tau^+\right)\right]=\varnothing$.

Proof: From Theorem 3.11 and since $\prod_{j=1}^{n}T\left(\tau_{jp}\right)=\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)\right]^*$, $p=1,\cdots,N$, it follows that $R\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right], p=1,\cdots,N$ are closed and hence,

$R\left[\prod_{j=1}^{n}T\left(\tau_j\right)-\lambda I\right]=\oplus_{p=1}^{N}R\left[\prod_{j=1}^{n}T\left(\tau_{jp}\right)-\lambda I\right]$ is closed for every $\lambda\in\mathbb{C}$; see ([3], Theorem I.3.7). Also by Lemma 3.5, we have

$$\begin{aligned}null\left[\prod_{j=1}^{n}T\left(\tau_j\right)-\lambda I\right]&=def\left[\prod_{j=1}^{n}T_0\left(\tau_j^+\right)-\bar{\lambda}I\right]\\&=\sum_{p=1}^{N}def\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)-\bar{\lambda}I\right]=n^2N,\end{aligned}\tag{5.5}$$

and

$$\begin{aligned}def\left[\prod_{j=1}^{n}T\left(\tau_j\right)-\lambda I\right]&=null\left[\prod_{j=1}^{n}T_0\left(\tau_j^+\right)-\bar{\lambda}I\right]\\&=\sum_{p=1}^{N}null\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)-\bar{\lambda}I\right]=0.\end{aligned}\tag{5.6}$$

(i) Since $R\left[\prod_{j=1}^{n}T\left(\tau_{jp}\right)-\lambda I\right]$, are closed and $def\left[\prod_{j=1}^{n}T\left(\tau_{jp}\right)-\lambda I\right]=0$, $p=1,\cdots,N$, then by Lemma 3.5 $R\left[\prod_{j=1}^{n}T\left(\tau_j\right)-\lambda I\right]=H$. This yields that,

$$\sigma_c\left[T(\tau)\right]=\bigcup_{p=1}^{N}\sigma_c\left[\prod_{j=1}^{n}T\left(\tau_{jp}\right)\right]=\varnothing,$$

$$\sigma_c\left[T\left(\tau^+\right)\right]=\bigcup_{p=1}^{N}\sigma_c\left[\prod_{j=1}^{n}T\left(\tau_{jp}^+\right)\right]=\varnothing.$$

Similarly,

$$\sigma_r\left[T(\tau)\right]=\bigcup_{p=1}^{N}\sigma_r\left[\prod_{j=1}^{n}T\left(\tau_{jp}\right)\right]=\varnothing,$$

$$\sigma_r\left[T\left(\tau^+\right)\right]=\bigcup_{p=1}^{N}\sigma_r\left[\prod_{j=1}^{n}T\left(\tau_{jp}^+\right)\right]=\varnothing$$

(ii) Since

$$null\left[\prod_{j=1}^{n}T\left(\tau_j\right)-\lambda I\right]=\sum_{p=1}^{N}null\left[\prod_{j=1}^{n}T\left(\tau_{jp}\right)-\lambda I\right]=n^2N$$

and

$$null\left[\prod_{j=1}^{n}T\left(\tau_j^+\right)-\bar{\lambda}I\right]=\sum_{p=1}^{N}null\left[\prod_{j=1}^{n}T\left(\tau_{jp}^+\right)-\bar{\lambda}I\right]=n^2N,$$

for all $\lambda\in\mathbb{C}$, then we have that,

$$\sigma_p\left[T(\tau)\right]=\bigcup_{p=1}^{N}\sigma_p\left[\prod_{j=1}^{n}T\left(\tau_{jp}\right)\right]=\mathbb{C},\tag{5.7}$$

and

$$\sigma_p\left[T\left(\tau^+\right)\right]=\bigcup_{p=1}^{N}\sigma_p\left[\prod_{j=1}^{n}T\left(\tau_{jp}^+\right)\right]=\mathbb{C}\tag{5.8}$$

It also follows that

$$\sigma\left[T(\tau)\right]=\bigcup_{p=1}^{N}\sigma\left[\prod_{j=1}^{n}T\left(\tau_{jp}\right)\right]=\mathbb{C}, \tag{5.9}$$

$$\sigma\left[T\left(\tau^{+}\right)\right]=\bigcup_{p=1}^{N}\sigma\left[\prod_{j=1}^{n}T\left(\tau_{jp}^{+}\right)\right]=\mathbb{C}. \tag{5.10}$$

(iii) From (ii), it follows that,

$$\rho\left[T(\tau)\right]=\rho\left[T\left(\tau^{+}\right)\right]=\varnothing. \tag{5.11}$$

6. The Field of Regularity in Direct Sum Spaces

We now obtain some results which in fact are a natural consequence of those in Section 5.

Theorem 6.1: Let

$$T_0(\tau)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)\right] \text{ and } T_0\left(\tau^{+}\right)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^{+}\right)\right],$$

then

(i) $\Pi\left[T_0(\tau)\right]=\Pi\left[T_0\left(\tau^{+}\right)\right]=\mathbb{C}$ and for every $\lambda\in\mathbb{C}$,

$$def\left[T_0(\tau)-\lambda I\right]=def\left[T_0\left(\tau^{+}\right)-\bar{\lambda}I\right]=n^2N,$$

(ii) $\Pi\left[T(\tau)\right]=\Pi\left[T\left(\tau^{+}\right)\right]=\varnothing$ and for every $\lambda\in\mathbb{C}$,

$$null\left[T(\tau)-\lambda I\right]=null\left[T\left(\tau^{+}\right)-\bar{\lambda}I\right]=n^2N.$$

Proof: (i) We have from Theorem 3.11 and Lemma 5.1 that, for every $\lambda\in\mathbb{C}$, $\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right]^{-1}$ exists and its domains $R\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right]$ are closed subspaces of $L_w^2\left(a_p,b_p\right)$, $p=1,\cdots,N$. Hence, since $\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)\right]$, $p=1,\cdots,N$, are closed operators, then $\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right]^{-1}$ are also closed and so, it follows from the Closed Graph Theorem that,

$\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right]^{-1}$, $p=1,\cdots,N$ are bounded, and hence

$$\Pi\left[T_0(\tau)\right]=\bigcap_{p=1}^{N}\Pi\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)\right]=\mathbb{C}.$$

From Theorem 3.11, $R\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right]^{\perp}$, $p=1,\cdots,N$ are n^2-dimensional subspaces of $L_w^2\left(a_p,b_p\right)$, $p=1,\cdots,N$. Thus, by Lemma 3.5,

$$\begin{aligned}def\left[T_0(\tau)-\lambda I\right]&=\sum_{p=1}^{N}def\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right]\\&=\sum_{p=1}^{N}\dim R\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right]^{\perp}=n^2N,\end{aligned}$$

for every $\lambda\in\mathbb{C}$. Similarly,

$$\begin{aligned}def\left[T_0\left(\tau^{+}\right)-\bar{\lambda}I\right]&=\sum_{p=1}^{N}def\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^{+}\right)-\bar{\lambda}I\right]\\&=\sum_{p=1}^{N}\dim R\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^{+}\right)-\bar{\lambda}I\right]^{\perp}\\&=n^2N \quad \text{for every } \lambda\in\mathbb{C}.\end{aligned}$$

(ii) As $\Pi\left[T_0\left(\tau^+\right)\right]=\mathbb{C}$, we have, for every $\lambda\in\mathbb{C}$ that $\left[T_0\left(\tau^+\right)-\bar{\lambda}I\right]$ has closed range, and so, since $T(\tau)=\left[T_0\left(\tau^+\right)\right]^*$, then $\left[T(\tau)-\lambda I\right]=\sum_{p=1}^{N}\left[\prod_{j=1}^{n}T\left(\tau_{jp}\right)-\lambda I\right]$ has closed range; see ([3], Theorem I.3.7). Furthermore, from (i),

$$\begin{aligned} null\left[T(\tau)-\lambda I\right] &= def\left[T_0\left(\tau^+\right)-\bar{\lambda}I\right] \\ &= \sum_{p=1}^{N} def\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)-\bar{\lambda}I\right]=n^2N. \end{aligned}$$

Hence, $\lambda\notin\Pi\left[T(\tau)\right]$, and so part (ii) of the theorem follows.

Corollary 6.2: The operators $T_0(\tau),T_0\left(\tau^+\right)$ form a compatible adjoint pair with

$$\Pi\left[T_0(\tau),T_0\left(\tau^+\right)\right]=\mathbb{C}.$$

Proof: From Theorem 5.1 part (i) and Lemma 3.7, it follows that,

$$\Pi\left[T_0(\tau),T_0\left(\tau^+\right)\right]=\bigcap_{p=1}^{N}\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right),\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)\right]=\mathbb{C}.$$

By using (3.15), the corollary follows.

Theorem 6.3: If, for some $\lambda_0\in\mathbb{C}$, there are n^2N linearly independent solutions of the equations

$$\left[\prod_{j=1}^{n}\left(\tau_{jp}\right)-\lambda_0 w\right]u=0 \quad\text{and}\quad \left[\prod_{j=1}^{n}\left(\tau_{jp}^+\right)-\bar{\lambda}_0 w\right]v=0 \tag{6.1}$$

in $L_w^2\left(a_p,b_p\right), p=1,\cdots,N$ then, all solutions of the equations

$$\left[\prod_{j=1}^{n}\left(\tau_{jp}\right)-\lambda w\right]u=0 \quad\text{and}\quad \left[\prod_{j=1}^{n}\left(\tau_{jp}^+\right)-\bar{\lambda} w\right]v=0$$

are in $L_w^2\left(a_p,b_p\right), p=1,\cdots,N$ for all $\lambda\in\mathbb{C}$.

Proof: The proof follows from Lemma 3.5 and Lemma 3.6. We refer to [6], ([13], Lemmas 3.3, 3.4) and [14]-[19] for more details.

From Corollary 6.2 and Theorem 6.3 we have the following Lemma.

Lemma 6.4: If, for some $\lambda_0\in\mathbb{C}$, there are n^2N linearly independent solutions of the Equations (6.1) in $L_w^2\left(a_p,b_p\right)$, then $\lambda_0\in\Pi\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right),\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)\right]$; $p=1,\cdots,N$ see also ([22], Theorem 2.1) and ([24], Lemma 5.1).

Theorem 6.5: Let

$$T_0(\tau)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)\right] \quad\text{and}\quad T_0\left(\tau^+\right)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)\right]$$

be the minimal operators, defined on the interval $[a,b)$. If $\Pi\left[T_0(\tau),T_0\left(\tau^+\right)\right]$ is empty, then $def\left[T_0(\tau)-\lambda I\right]+def\left[T_0\left(\tau^+\right)-\bar{\lambda}I\right]\neq 2n^2N$.

In particular, if $\Pi\left[T_0(\tau),T_0\left(\tau^+\right)\right]$ is empty and $n=1$, then

$$def\left[T_0(\tau)-\lambda I\right]+def\left[T_0\left(\tau^+\right)-\bar{\lambda}I\right]=N.$$

Proof: If for some $\lambda_0\in\mathbb{C}$,

$$def\left[T_0(\tau)-\lambda I\right]=\sum_{p=1}^{N} def\left[\prod_{j=1}^{n} T_0(\tau_{jp})-\lambda I\right]=n^2 N$$

and

$$def\left[T_0(\tau^+)-\bar{\lambda} I\right]=\sum_{p=1}^{N} def\left[\prod_{j=1}^{n} T_0(\tau_{jp}^+)-\bar{\lambda} I\right]=n^2 N,$$

then,

$$\left[\tau-\lambda_0 w\right]u=0 \text{ and } \left[\tau^+-\bar{\lambda}_0 w\right]v=0$$

each have $n^2 N$ $L_w^2(a,b)$-solutions (see [6]). Hence by Theorem 6.3, we have that all solutions of $(\tau-\lambda w)u=0$ and $(\tau^+-\bar{\lambda} w)v=0$ are in $L_w^2(a,b)$ for all $\lambda\in\mathbb{C}$, and hence, by Corollary 6.2, we have that $\lambda\in\Pi\left[T_0(\tau),T_0(\tau^+)\right]$. Thus, if $\Pi\left[T_0(\tau),T_0(\tau^+)\right]$ is empty, we cannot have,

$$def\left[T_0(\tau)-\lambda I\right]+def\left[T_0(\tau^+)-\bar{\lambda} I\right]=2n^2 N.$$

In particular, if $n=1$, then by Lemma 3.6 we have that,

$$N\le def\left[T_0(\tau)-\lambda I\right]+def\left[T_0(\tau^+)-\bar{\lambda} I\right]\le 2N,$$

so if $\Pi\left[T_0(\tau),T_0(\tau^+)\right]$ is empty, we have

$$def\left[T_0(\tau)-\lambda I\right]+def\left[T_0(\tau^+)-\bar{\lambda} I\right]=N.$$

For a regularly solvable operator, we have the following general theorem.

Theorem 6.6: Suppose for a regularly solvable extension S of the minimal operator

$$T_0(\tau)=\oplus_{p=1}^{N}\left[\prod_{j=1}^{n} T_0(\tau_{jp})\right] \text{ that,}$$

$$def\left[T_0(\tau)-\lambda I\right]+def\left[T_0(\tau^+)-\bar{\lambda} I\right]=K,\ n^2 N\le K\le 2n^2 N,$$

for all $\lambda\in\Pi\left[T_0(\tau),T_0(\tau^+)\right]$. Then,

$$null\left[T_0(\tau)-\lambda I\right]+null\left[T_0(\tau^+)-\bar{\lambda} I\right]\le K \text{, for all } \lambda\in\mathbb{C}.$$

If $\Pi\left[T_0(\tau),T_0(\tau^+)\right]$ is empty, then

$$null\left[T(\tau)-\lambda I\right]+null\left[T(\tau^+)-\bar{\lambda} I\right]<K.$$

Proof: Let

$$def\left[\prod_{j=1}^{n} T_0(\tau_{jp})-\lambda I\right]=r_p,\quad def\left[\prod_{j=1}^{n} T_0(\tau_j^+)-\bar{\lambda} I\right]=s_p, p=1,\cdots,N,$$

such that

$$def\left[\prod_{j=1}^{n} T_0(\tau_{jp})-\lambda I\right]+def\left[\prod_{j=1}^{n} T_0(\tau_{jp}^+)-\bar{\lambda} I\right]=r_p+s_p,$$

$$n^2\le r_p+s_p\le 2n^2 \text{ for all } \lambda\in\Pi\left[\prod_{j=1}^{n} T_0(\tau_{jp}),\prod_{j=1}^{n} T_0(\tau_{jp}^+)\right], p=1,\cdots,N.$$

Then, for any closed extension S_p of $\prod_{j=1}^{n} T_0(\tau_{jp})$ which is regularly solvable with respect to $\prod_{j=1}^{n} T_0(\tau_{jp})$ and $\prod_{j=1}^{n} T_0(\tau_j^+)$, we have from ([3], Theorem

III.3.5) that,

$$dim\left\{D\left(S_p\right)/D_0\left[\prod_{j=1}^{n}\left(\tau_{jp}\right)\right]\right\}=def\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right]=r_p, p=1,\cdots,N,$$

$$dim\left\{D\left(S_p^*\right)/D_0\left[\prod_{j=1}^{n}\left(\tau_{jp}^+\right)\right]\right\}=def\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)-\bar{\lambda} I\right]=s_p, p=1,\cdots,N.$$

Hence S_p and S_p^* are finite dimensional extensions of $\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)$ and $\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)$ respectively. Thus, from ([3], Corollary IX.4.2), we get

$$\sigma_{ek}\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)\right]=\sigma_{ek}\left(S_p\right),\left(k=1,2,3; p=1,\cdots,N\right). \tag{6.2}$$

Since $\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right]$ has closed range, zero nullity and deficiency r_p (see Lemma 3.13), then for any $\lambda\in\mathbb{C}$, we have that,

$$\Pi\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)\right]\cap\sigma_{ek}\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)\right]=\varnothing,\left(k=1,2,3; p=1,\cdots,N\right).$$

Therefore,

$$\Delta_{ek}\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)\right]=\Delta_{ek}\left(S_p\right)=\mathbb{C},\left(k=1,2,3; p=1,\cdots,N\right).$$

Similarly,

$$\Delta_{ek}\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)\right]=\Delta_{ek}\left(S_p^*\right)=\mathbb{C},\left(k=1,2,3; p=1,\cdots,N\right).$$

Furthermore, the equations

$$\left[\prod_{j=1}^{n}\left(\tau_{jp}\right)\right]\varphi_p=\lambda_0 w\varphi_p \text{ and } \left[\prod_{j=1}^{n}\left(\tau_{jp}^+\right)\right]\varphi_p^+=\bar{\lambda}_0 w\varphi_p^+, p=1,\cdots,N,$$

have at most r_p and s_p linearly independent solutions for $\lambda_0\in\mathbb{C}$ respectively. Hence,

$$\begin{aligned}&null\left[T(\tau)-\lambda I\right]+null\left[T\left(\tau^+\right)-\bar{\lambda} I\right]\\&=\sum_{p=1}^{N}null\left[\prod_{j=1}^{n}T\left(\tau_{jp}\right)-\lambda I\right]+\sum_{p=1}^{N}null\left[\prod_{j=1}^{n}T\left(\tau_{jp}^+\right)-\bar{\lambda} I\right]\\&=\sum_{p=1}^{N}\left(r_p+s_p\right)\le K,\ n^2N\le K\le 2n^2N \quad \text{for all } \lambda\in\mathbb{C}.\end{aligned}$$

But for any $\lambda_0\notin\Pi\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right),\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)\right]$, either $\lambda_0\notin\Pi\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)\right]$ or $\bar{\lambda}_0\notin\Pi\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)\right]$. If $\lambda_0\notin\Pi\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)\right]$, then either λ_0 is an eigenvalue of $\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)$ or $R\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right], p=1,\cdots,N$, are not closed. Similarly, for $\bar{\lambda}_0\notin\Pi\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)\right]$. But $\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)$ and $\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)$ have no eigenvalues; then if $\lambda_0\notin\Pi\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right),\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)\right]$, we have $R\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}\right)-\lambda I\right]$ and $R\left[\prod_{j=1}^{n}T_0\left(\tau_{jp}^+\right)-\bar{\lambda} I\right], p=1,\cdots,N$ are both not closed, and so we cannot have,

$$\begin{aligned}&null\left[T(\tau)-\lambda I\right]+null\left[T\left(\tau^+\right)-\bar{\lambda} I\right]\\&=\sum_{p=1}^{N}null\left[\prod_{j=1}^{n}T\left(\tau_{jp}\right)-\lambda I\right]+\sum_{p=1}^{N}null\left[\prod_{j=1}^{n}T\left(\tau_{jp}^+\right)-\bar{\lambda} I\right]=K.\end{aligned}$$

Hence,

$$null\left[T(\tau)-\lambda I\right]+null\left[T(\tau^{+})-\bar{\lambda} I\right]<K,\ n^2N\le K\le 2n^2N,$$

for all

$$\lambda\notin\Pi\left[\prod_{j=1}^{n}T_0(\tau_j),\prod_{j=1}^{n}T_0(\tau_j^{+})\right]=\bigcap_{p=1}^{N}\Pi\left[\prod_{j=1}^{n}T_0(\tau_{jp}),\prod_{j=1}^{n}T_0(\tau_{jp}^{+})\right].$$

7. Conclusion

We have investigated the location of the point spectra and regularly fields of product differential operators generated by a general quasi-differential expression τ_p on any finite number of intervals $I_p=(a_p,b_p), p=1,\cdots,N$ in the setting of direct sums of $L^2_{w_p}(a_p,b_p)$ -spaces of functions defined on each of the separate intervals.

Remark

It remains an open question as to how many solutions of the equations: $\left[\prod_{j=1}^{n}(\tau_j)\right]u=\lambda wu$ and $\left[\prod_{j=1}^{n}T_0(\tau_j^{+})\right]v=\bar{\lambda}wv$, may be in $L^2_w(a,b)$ for any $\lambda\in\mathbb{C}$, when $\Pi\left[\prod_{j=1}^{n}T_0(\tau_j),\prod_{j=1}^{n}T_0(\tau_j^{+})\right]$ is empty, except that we know from above that not all of them are in $L^2_w(a,b)$. We refer to [2] [6] [12]-[19] [22] and [23] for more details.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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