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The Stability of a Class of Non-Newtonian Micropolar Fluid Equations with Unbounded Delays

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Abstract

We address the stability of stationary solutions to a class of 2D non-newtonian micropolar fluid equations, when the external force contains hereditary characteristics involving unbounded delays. Specifically, when the delay function is continuous with respect to time, the stability of the weak solution with respect to the non-trivial stationary solution is established by the definition of Lyapunov stability, and the asymptotic stability of the weak solution with respect to the trivial stationary solution is established by the method of constructing Lyapunov functional; when the delay function is only continuous with respect to time, the stability of the strong solution with respect to the non-trivial stationary solution is established by using Razumikhin technique; finally, the proportional time delay is introduced to establish the polynomial stability of the weak solution with respect to the trivial stationary solution.

Keywords

Non-Newtonian Micropolar Fluid, Unbounded Delays, Weak (Strong) Solutions, Stationary Solution, Asymptotic (Polynomial) Stability

1. Introduction

In fluid dynamics, fluids satisfying Newton's law of viscosity, namely those for which the stress tensor depends linearly on the rate-of-strain tensor, are generally referred to as Newtonian fluids. Typical examples include water, alcohol, and air. In contrast, fluids that do not obey this linear constitutive relation are called non-Newtonian fluids, such as molten plastics, synthetic fibers, paints, and greases. Micropolar fluids form a class of complex fluids that account for microstructural effects and non-symmetric stress tensors; examples include blood, polymer suspensions, and related materials. Non-Newtonian micropolar fluids combine the constitutive characteristics of non-Newtonian fluids with the microrotation effects of micropolar fluids. Therefore, the non-Newtonian micropolar fluid equations can be used to describe the motion of non-Newtonian fluids with microstructures, namely fluids composed of randomly oriented particles suspended in a viscous non-Newtonian medium, without considering the deformation of the particles, such as lubricants and other complex fluids [1].

At present, numerous results have been obtained on various mathematical aspects of the non-Newtonian micropolar fluid equations. For instance, Araújo[2] established the existence and uniqueness of weak solutions in a two-dimensional bounded domain and investigated the existence and upper semicontinuity of pullback attractors. Later, Ai and Tan [3] proved the existence of global and exponential attractors for the equations in a two-dimensional bounded domain and studied the existence of pullback exponential attractors. In another direction, Zhao, Zhang *et al.* [1] established the existence and degenerate regularity of statistical solutions for the equations in a two-dimensional bounded domain. Subsequently, Chen, Yang *et al.* [4] proved the existence and degenerate regularity of trajectory statistical solutions for the equations in a three-dimensional bounded domain.

Delay effects reveal the delayed reaction phenomenon of a system and have an important impact on the stability and evolution process of the system. In the real world, delay effects appear naturally, for example, in the motion of fluids in wind tunnel tests [5]. Currently, there have been studies on the long-term dynamical behavior of solutions to a class of non-Newtonian fluid and micropolar fluid equations with delays. For example, on the one hand, for a class of non-Newtonian fluids, when the bounded delay initial value space is $\mathcal{C}_H$, Zhao, Park, Lázaro *et al.* [6]-[9] established the well-posedness of weak solutions, the existence and properties of pullback attractors, Liu *et al.* [10] established the exponential stability of weak and strong solutions with respect to stationary solutions; when the bounded delay initial value space is $\mathcal{C}_\rho(H)$ ($\rho > 0$), Liu, Xu *et al.* [11] [12] established the existence and properties of pullback attractors for solutions in two-dimensional unbounded domains; when considering unbounded delays with initial value space $\mathcal{C}_\rho(H)$ ($\rho > 0$), Zhao *et al.* [13] established the well-posedness of weak solutions and the existence of pullback attractors in two-dimensional bounded domains; when $\rho = 0$, *i.e.*, considering the initial value space $\mathcal{C}_0(H)$, Liu *et al.* [14] established the stability of weak (strong) solutions with respect to stationary solutions for three special types of unbounded delay functions. On the other hand, for micropolar fluid equations, when the bounded delay initial value space is $L^2(-h,0;\hat{H})$, Sun and Liu [15] [16] established the global well-posedness of weak solutions and the existence and properties of pullback attractors in two-dimensional unbounded domains. On this basis, Sun [17] established the existence and uniqueness of stationary solutions and the exponential stability of weak solutions with respect to stationary solutions in two-dimensional unbounded domains. In the case of unbounded delays, with initial value space $\mathcal{C}_\rho(\hat{H})$ ($\rho > 0$), Zhao *et* *al.* [13] established the global well-posedness of weak solutions and the existence of pullback attractors in two-dimensional bounded domains. Furthermore, Zhou *et al.* [18] established the H^2 boundedness of pullback attractors for solutions in two-dimensional bounded domains. In contrast, research results on non-Newtonian micropolar fluid equations with delay effects are still relatively lacking.

This paper mainly studies the following class of non-Newtonian micropolar fluid equations with bounded delay:

$$\frac{\partial \mathbf{u}}{\partial t}-\nabla\cdot\left[\mu(\mathbf{u})e(\mathbf{u})-2\mu_1\Delta e(\mathbf{u})\right]+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p-2\nu_r\nabla\times\omega=\mathbf{f}+\mathbf{g}(t,\mathbf{u}_t), \tag{1}$$

$$\nabla\cdot\mathbf{u}=0, \tag{2}$$

$$\frac{\partial \omega}{\partial t}-\nu_1\Delta\omega+(\mathbf{u}\cdot\nabla)\omega+4\nu_r\omega-2\nu_r\nabla\times\mathbf{u}=\tilde{f}+\tilde{g}(t,\omega_t), \tag{3}$$

where $x\in\Omega\subset\mathbb{R}^2$, $t\in\mathbb{R}^+$, $\mathbf{u}=(u_1,u_2)$ denotes the velocity field of the fluid, ω denotes the angular velocity of particle rotation, $p=p(x,t)$ denotes the pressure of the fluid, $\mathbf{f}=(f_1,f_2)$ and $\tilde{f}$ denote the external force and torque respectively, the positive constants ν_1,ν_r denote viscosity coefficients, ∇ denotes the gradient operator, Δ denotes the Laplace operator,

$$\nabla\times\omega=\left(\frac{\partial\omega}{\partial x_2},-\frac{\partial\omega}{\partial x_1}\right),\ \nabla\times\mathbf{u}=\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2},$$

$\mu(\mathbf{u}):=2\mu_0\left(\varepsilon+|e(\mathbf{u})|^2\right)^{-\alpha/2}$, denotes the rate-of-strain tensor, where

$$e_{ij}(\mathbf{u})=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\ i,j=1,2,\cdots,n.$$

$\nabla\cdot\mathbf{u}=0$ describes the incompressibility of the fluid.
$\mathbf{g}(t,\mathbf{u}_t)=(g_1(t,\mathbf{u}_t),g_2(t,\mathbf{u}_t))$, $\tilde{g}(t,\omega_t)$ denote the external force terms with delays that have some memory or hereditary characteristics. $\mathbf{u}_t,\omega_t$ are the delay functions defined as

$$\mathbf{u}_t(\theta)=\mathbf{u}(t+\theta),\ \omega_t(\theta)=\omega(t+\theta),\ \theta\in(-\infty,0].$$

Let $\partial\Omega$ be the boundary of Ω and be sufficiently smooth. The initial-boundary value conditions for Equations (1)-(3) are given as

$$\mathbf{u}(\theta,x)=\phi(\theta,x),\ \omega(\theta,x)=\phi_3(\theta,x), \tag{4}$$

$$\mathbf{u}\big|_{(0,T)\times\partial\Omega}=0,\ \omega\big|_{(0,T)\times\partial\Omega}=0,\ \mathbb{T}^{\mathbf{u}}_{ijk}\mathbf{n}_j\mathbf{n}_k\big|_{(0,T)\times\partial\Omega}=0, \tag{5}$$

where $\boldsymbol{\phi}=(\phi_1,\phi_2)$, ϕ_3 are initial velocities defined on the delay interval, $\mathbb{T}^{\mathbf{u}}_{ijk}:=2\mu_1\frac{\partial e_{ij}(\mathbf{u})}{\partial x_k}$, $i,j,k=1,2$, $(\mathbf{n}_1,\mathbf{n}_2)$ denotes the unit outward normal vector on $\partial\Omega$. In the boundary conditions (5), the first two conditions indicate no-slip phenomenon of the fluid on the boundary, and the third condition indicates that the traction force of the viscous fluid vanishes on the boundary. For the specific physical background, references [19]-[25] can be consulted.

Throughout this paper, we investigate three special classes of unbounded delay functions and studies the stability of weak and strong solutions with respect to stationary solutions by combining different analytical techniques with stability theory. Specifically, when the delay function is continuous in time, the Lyapunov stability of weak solutions with respect to nontrivial stationary solutions is established via the definition of Lyapunov stability, while the asymptotic stability of weak solutions with respect to the trivial stationary solution is obtained by constructing a suitable Lyapunov functional. When the delay function is only piecewise continuous in time, the stability of strong solutions with respect to nontrivial stationary solutions is proved by means of the Razumikhin technique. Compared with the case of bounded delay [10] [17], the unbounded delay considered in this paper does not yield exponential stability of weak or strong solutions with respect to the steady-state solution. Therefore, we introduce a proportional delay and establish the polynomial stability of weak solutions with respect to the trivial steady-state solution. The innovations of this paper are threefold. Firstly, while existing studies focus only on non-Newtonian micropolar fluid equations without delay, this paper considers the delayed case, which is theoretically significant because delay effects strongly influence well-posedness and solution dynamics. Secondly, new phase spaces are introduced for bounded and unbounded delays, respectively, overcoming the analytical difficulties caused by delay terms. Besides, the stability of weak and strong solutions with respect to steady-state solutions is established, enriching the study of the system's long-time dynamical behavior. The paper is organized as follows. Section 1 presents the preliminaries. By introducing suitable abstract operators, the initial-boundary value problem (1)-(5) is reformulated as an initial value problem for a functional differential equation. The definitions of weak and strong solutions are also given in this section. Section 2 proves the well-posedness of weak solutions and establishes the existence, uniqueness, and regularity of stationary solutions to problem (1)-(5). Section 3 investigates the exponential stability of weak and strong solutions with respect to stationary solutions for problem (1)-(5).

2. Preliminaries

In this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ denotes the set of integers. c, c_i, C denote constants, which may take different values in different places. $\to$ denotes strong convergence or convergence in norm, $\rightharpoonup$ denotes weak convergence, $\rightharpoonup^*$ denotes weak-star convergence. $\hookrightarrow\hookrightarrow$ denotes a compact embedding between two spaces. Let $L^p(\Omega)$ and $W^{m,p}(\Omega)$ denote the classical Lebesgue space and Sobolev space, respectively, with norms

$$\|\mathbf{u}\|_p := \left(\int_\Omega |\mathbf{u}|^p \,\mathrm{d}x\right)^{1/p}, \quad \|\mathbf{u}\|_{m,p} := \left(\sum_{|\beta|\le m} \int_\Omega \left|D^\beta \mathbf{u}\right|^p \,\mathrm{d}x\right)^{1/p}.$$

In particular, denote $\|\cdot\| := \|\cdot\|_2$, $H^m(\Omega) := W^{m,2}(\Omega)$, the dual space of $H^m(\Omega)$ is $H^{-m}(\Omega)$, $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. On this basis, introduce vector function spaces:

$$\mathcal{V} := \left\{ \mathbf{u} \in C_0^\infty(\Omega) \times C_0^\infty(\Omega) \mid \mathbf{u} = (u_1, u_2), \nabla \cdot \mathbf{u} = 0 \right\},$$

$$H := \text{closure of } \mathcal{V} \text{ in } \left(L^2(\Omega) \right)^2, \text{ norm } \|\cdot\|_H \text{ and dual space } H' = H,$$

$$V := \text{closure of } \mathcal{V} \text{ in } \left(H^2(\Omega) \right)^2, \text{ norm } \|\cdot\|_V \text{ and dual space } V',$$

$$\hat{H} := H \times L^2(\Omega), \text{ norm } \|\cdot\|_{\hat{H}}, \text{ dual space } \hat{H}',$$

$$\hat{V} := V \times H_0^1(\Omega), \text{ norm } \|\cdot\|_{\hat{V}}, \text{ dual space } \hat{V}'.$$

Denote $(\cdot,\cdot)$ as the inner product in $L^2(\Omega)$, H or $\hat{H}$, and $\langle \cdot,\cdot \rangle$ as the duality product between V and V' or $\hat{V}$ and $\hat{V}'$. When no confusion arises, the norms $\|\cdot\|_H$ and $\|\cdot\|_{\hat{H}}$ may be abbreviated as $\|\cdot\|$. Furthermore, let $L^p(I;X)$ be the set of all p-integrable functions defined on interval I with values in Banach space X, with norm

$$\|\mathbf{u}\|_{L^p(I,X)} := \left(\int_I \|\mathbf{u}\|_X^p \, \mathrm{d}x \right)^{\frac{1}{p}}, \ 1 \le p < \infty;$$

$\mathcal{C}(I;X)$ denotes the set of continuous functions defined on interval I with values in Banach space X, with norm $\|\cdot\|_{\mathcal{C}(I;X)}$.

To obtain the weak form of the non-Newtonian micropolar fluid equations, we introduce four abstract operators.

(i) Define operator A: for any $\mathbf{v} = (\mathbf{u}, \omega)$, $\boldsymbol{\Psi} = (\boldsymbol{\psi}, \psi_3) \in \hat{V}$, where $\mathbf{u} = (u_1, u_2)$, $\boldsymbol{\psi} = (\psi_1, \psi_2) \in V$,

$$\begin{aligned}(A\mathbf{v}, \boldsymbol{\Psi}) &= 2\mu_1 \langle A_1 \mathbf{u}, \boldsymbol{\psi} \rangle + \nu_1 \langle A_2 \omega, \psi_3 \rangle \\ &= 2\mu_1 \sum_{i,j,k=1}^{2} \int_\Omega \frac{\partial e_{ij}(\mathbf{u})}{\partial x_k} \frac{\partial e_{ij}(\boldsymbol{\psi})}{\partial x_k} \mathrm{d}x + \nu_1 \sum_{k=1}^{2} \int_\Omega \frac{\partial \omega}{\partial x_k} \frac{\partial \psi_3}{\partial x_k} \mathrm{d}x.\end{aligned}$$

Remark 1. $A_1 = \mathbf{P}\Delta^2$, where $\mathbf{P}$ is the Leray projection operator from $\left(L^2(\Omega) \right)^2$ to H, and $D(A_1) = \{ \mathbf{u} \in V : A_1 \mathbf{u} \in H \} = \left(H^4(\Omega) \right)^2 \cap V$ is a Hilbert space compactly embedded into V. Moreover, the operator A_1 satisfies ([13]): there exists a positive constant c_1 such that for any $\mathbf{u} \in V$,

$$c_1 \|\mathbf{u}\|_V^2 \le \langle A_1 \mathbf{u}, \mathbf{u} \rangle \le \|\mathbf{u}\|_V^2.$$

Remark 2. According to the definition of operators A_1, A_2 and the spectral theory of elliptic operators [25] [26], there exists a sequence $\{ \lambda_n, n = 1, 2, \cdots \}$ consisting of eigenvalues of A_1 and a sequence $\{ v_n \}_{n=1}^\infty \subset D(A_1)$ which is an orthonormal basis of H, such that $\operatorname{span}\{ v_1, v_2, \cdots, v_n, \cdots \}$ is dense in V. There exists a sequence $\{ \lambda_n^*, n = 1, 2, \cdots \}$ consisting of eigenvalues of A_2 and a sequence $\{ v_n^* \}_{n=1}^\infty \subset D(A_2)$ which is an orthonormal basis of H, such that $\operatorname{span}\{ v_1^*, v_2^*, \cdots, v_n^*, \cdots \}$ is dense in $H_0^1(\Omega)$. Moreover, for any $n \in \mathbb{N}$,

$$A_1 \upsilon_n = \lambda_n \upsilon_n, \ 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n \le \cdots, \ n \to \infty, \ \lambda_n \to \infty,$$

$$A_2\upsilon_n^* = \lambda_n^*\upsilon_n^*,\ 0<\lambda_1^*\le\lambda_2^*\le\lambda_3^*\le\cdots\le\lambda_n^*\le\cdots,\ n\to\infty,\ \lambda_n^*\to\infty.$$

Thus the following Poincaré inequalities hold:

$$\lambda_1\|\mathbf{u}\|^2\le\|\mathbf{u}\|_V^2,\ \lambda_1^*\|\omega\|^2\le\|\nabla\omega\|^2,\ \forall\mathbf{u}\in V,\omega\in H_0^1(\Omega). \tag{6}$$

From (6), for any $\mathbf{v}=(\mathbf{u},\omega)\in\hat{V}$,

$$\|\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\omega\|^2\le\lambda_1^{-1}\|\mathbf{u}\|_V^2+\lambda_1^{*-1}\|\nabla\omega\|^2\le\gamma^{-1}\|\mathbf{v}\|_{\hat{V}}^2, \tag{7}$$

where $\gamma=\min\{\lambda_1,\lambda_1^*\}$.

(ii) Define operator B: for any $\mathbf{u}=(u_1,u_2)\in V$, $\mathbf{\Psi}=(\psi_1,\psi_2,\psi_3)$, $\mathbf{\Phi}=(\phi_1,\phi_2,\phi_3)\in\hat{V}$,

$$\langle B(\mathbf{u},\mathbf{\Psi}),\mathbf{\Phi}\rangle=b(\mathbf{u},\mathbf{\Psi},\mathbf{\Phi})=\sum_{j=1}^{3}\sum_{i=1}^{2}\int_\Omega u_i\frac{\partial\psi_j}{\partial x_i}\phi_j\mathrm{d}x.$$

(iii) Define operator N:

$$N(\mathbf{v})=(N_1(\mathbf{u}),0),\ \forall\mathbf{v}=(\mathbf{u},\omega).$$

Here N_1 is defined as: for any $\mathbf{u}=(u_1,u_2)$, $\boldsymbol{\psi}=(\psi_1,\psi_2)\in V$,

$$\langle N_1(\mathbf{u}),\boldsymbol{\psi}\rangle=n_1(\mathbf{u},\mathbf{u},\boldsymbol{\psi})=\sum_{i,j=1}^{2}\int_\Omega 2\mu_0\left(\varepsilon+|e(\mathbf{u})|^2\right)^{-\alpha/2}e_{ij}(\mathbf{u})e_{ij}(\boldsymbol{\psi})\mathrm{d}x.$$

If we define for any $\mathbf{\Psi}=(\boldsymbol{\psi},\psi_3)\in\hat{V}$, $n(\mathbf{v},\mathbf{v},\mathbf{\Psi})=\langle N(\mathbf{v}),\mathbf{\Psi}\rangle$, then $n(\mathbf{v},\mathbf{v},\mathbf{\Psi})=n_1(\mathbf{u},\mathbf{u},\boldsymbol{\psi})$.

(iv) Define operator J: for any $\mathbf{v}=(\mathbf{u},\omega)\in\hat{V}$,

$$J(\mathbf{v})=(-2\nu_r\nabla\times\omega,-2\nu_r\nabla\times\mathbf{u}+4\nu_r\omega).$$

Lemma 1. The following are some properties of operators A,B,N,J ([4] [13]):

(i) $A:\hat{V}\to\hat{V}'$ and $D(A)\to\hat{H}$ is a continuous linear operator; $N(\cdot):\hat{V}\to\hat{V}'$ is a continuous nonlinear operator; $J(\cdot):\hat{V}\to\hat{H}$ is a continuous linear operator.

(ii) $B(\cdot,\cdot):V\times\hat{V}\to\hat{V}'$ is a continuous nonlinear operator, and

$$\langle B(\mathbf{u},\mathbf{\Psi}),\mathbf{\Psi}\rangle=0,\ \langle B(\mathbf{u},\mathbf{\Psi}),\mathbf{\Phi}\rangle=-\langle B(\mathbf{u},\mathbf{\Phi}),\mathbf{\Psi}\rangle. \tag{8}$$

(iii) For any $\mathbf{v}=(\mathbf{u},\omega)$, $\mathbf{\Psi}=(\boldsymbol{\psi},\psi_3)\in\hat{V}$, $\mathbf{u}=(u_1,u_2)$, $\boldsymbol{\psi}=(\psi_1,\psi_2)\in V$, there exists a positive constant c_2 such that

$$|\langle A\mathbf{v},\mathbf{\Psi}\rangle|\le c_2\|\mathbf{v}\|_{\hat{V}}\|\mathbf{\Psi}\|_{\hat{V}}, \tag{9}$$

and

$$\langle A\mathbf{\Psi},\mathbf{\Psi}\rangle+\langle J(\mathbf{\Psi}),\mathbf{\Psi}\rangle\ge\delta_1\|\mathbf{\Psi}\|_{\hat{V}}^2, \tag{10}$$

where $\delta_1=\min\{2c_1\mu_1-\nu_r,\nu_1\}$.

(iv) There exist positive constants $c_i, i=3,4,5$, such that for any $\mathbf{v}=(\mathbf{u},\omega)$, $\mathbf{\Phi}=(\boldsymbol{\psi},\phi_3)$, $\mathbf{\Psi}=(\boldsymbol{\psi},\psi_3)\in\hat{V}$ with $\mathbf{u}=(u_1,u_2)$, $\boldsymbol{\psi}=(\psi_1,\psi_2)$, $\boldsymbol{\phi}=(\phi_1,\phi_2)\in V$,

$$\left|\left\langle B(\mathbf{u},\mathbf{\Phi}),\mathbf{\Psi}\right\rangle\right| \le c_3 \|\mathbf{u}\|^{\frac{1}{2}} \|\mathbf{u}\|_{\hat{V}}^{\frac{1}{2}} \|\mathbf{\Phi}\|_{\hat{V}} \|\mathbf{\Psi}\|^{\frac{1}{2}} \|\mathbf{\Psi}\|_{\hat{V}}^{\frac{1}{2}}. \tag{11}$$

$$\left|\left\langle J(\mathbf{v}),\mathbf{\Psi}\right\rangle\right| \le c\|\mathbf{v}\|_{\hat{V}} \|\mathbf{\Psi}\| \le c_4 \|\mathbf{v}\|_{\hat{V}} \|\mathbf{\Psi}\|_{\hat{V}}. \tag{12}$$

$$\left|\left\langle N(\mathbf{v}),\mathbf{\Psi}\right\rangle\right| \le c\|\mathbf{u}\|_{V} \|\mathbf{\Psi}\|_{V} \le c_5 \|\mathbf{v}\|_{\hat{V}} \|\mathbf{\Psi}\|_{\hat{V}}. \tag{13}$$

Based on the above operators, in the divergence-free field, the weak form of the initial-boundary value problem (1)-(5) in the sense of $\mathcal{D}'(0,T)$ can be written as:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + A\mathbf{v} + B(\mathbf{u},\mathbf{v}) + N(\mathbf{v}) + J(\mathbf{v}) = \mathbf{F}(t) + \mathbf{G}(t,\mathbf{v}_t), \tag{14}$$

$$\mathbf{v}(\theta) = \mathbf{\Phi}(\theta),\ \theta \in (-\infty,0]. \tag{15}$$

where $\mathbf{v} = (\mathbf{u},\omega)$, $\mathbf{\Phi} = (\phi,\phi_3)$, $\phi = (\phi_1,\phi_2)$, $\mathbf{F} = (\mathbf{f},\tilde{f})$, $\mathbf{G}(t,\mathbf{v}_t) = (\mathbf{g}(t,\mathbf{u}_t),\tilde{g}(t,\omega_t))$. The variables in this equation have the same physical meanings as those in Equations (1)-(5). To deal with unbounded delays, we consider the initial value space $\mathcal{C}_0(\hat{H})$

$$\mathcal{C}_0(\hat{H}) = \left\{\mathbf{\Phi} \in \mathcal{C}\left((-\infty,0];\hat{H}\right) : \exists \lim_{s\to-\infty} \mathbf{\Phi}(s) \in \hat{H}\right\},$$

which is a Banach space with norm

$$\|\mathbf{\Phi}\|_{\mathcal{C}_0(\hat{H})} = \sup_{s\in(-\infty,0]} \|\mathbf{\Phi}(s)\|.$$

Now we give the definitions of weak and strong solutions.

Definition 1. Given $T > 0$ and $\mathbf{v}_0 = \mathbf{\Phi} \in \mathcal{C}_0(\hat{H})$, $\mathbf{\Psi} \in \hat{V}$. A function

$$\mathbf{v} \in \mathcal{C}\left((-\infty,T];\hat{H}\right) \cap L^2\left(0,T;\hat{V}\right)$$

is called a weak solution of the initial value problem (14)-(15) if it satisfies in the sense of $\mathcal{D}'(0,T)$:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t),\mathbf{\Psi}) + \langle A\mathbf{v},\mathbf{\Psi}\rangle + \langle B(u(t),\mathbf{v}(t)),\mathbf{\Psi}\rangle + \langle N(\mathbf{v}(t)),\mathbf{\Psi}\rangle + \langle J(\mathbf{v}(t)),\mathbf{\Psi}\rangle \\ &= \langle \mathbf{F}(t),\mathbf{\Psi}\rangle + (\mathbf{G}(t,\mathbf{v}_t),\mathbf{\Psi}).\end{aligned}$$

If a weak solution satisfies $\mathbf{v} \in \mathcal{C}\left((-\infty,0];\hat{V}\right) \cap L^2\left(0,T;D(A)\right)$, then it is called a strong solution.

3. Existence and Uniqueness of Weak Solutions and Stationary Solutions

In this section, we prove the global well-posedness of weak solutions to the initial value problem (14)-(15). For that purpose, the following assumption on the delay external force term $\mathbf{G}(t,\mathbf{v}_t)$ is needed.

Assumption 1. Let $\mathbf{G} : [0,T] \times \mathcal{C}_0(\hat{H}) \to \hat{H}$ satisfy:

(i) For any $\xi \in \mathcal{C}_0(\hat{H})$, the map $[0,T] \ni t \mapsto \mathbf{G}(t,\xi) \in \hat{H}$ is measurable.

(ii) $\mathbf{G}(\cdot,0)=0$.

(iii) There exists a constant $L_{\mathbf{G}}>0$ such that for any $t\in[0,T]$, $\xi,\eta\in\mathcal{C}_0(\hat{H})$,

$$\|\mathbf{G}(t,\xi)-\mathbf{G}(t,\eta)\|\le L_{\mathbf{G}}\|\xi-\eta\|_{\mathcal{C}_0(\hat{H})}.$$

Remark 3. From (ii) and (iii),

$$\|\mathbf{G}(t,\xi)\|\le L_{\mathbf{G}}\|\xi\|_{\mathcal{C}_0(\hat{H})},\ \forall\xi\in\mathcal{C}_0(\hat{H}). \tag{16}$$

Theorem 1. For any $T>0$, $\mathbf{F}(t,x)\in L^2(0,T;\hat{V})$, $\mathbf{\Phi}\in\mathcal{C}_0(\hat{H})$, and $\mathbf{G}(t,\mathbf{v}_t)$ satisfying Assumption 1, the initial value problem (14)-(15) has a unique weak solution $\mathbf{v}=(\mathbf{u},\omega)$ on the interval $(-\infty,T]$, and this weak solution depends continuously on the initial value. If $\mathbf{F}(t,x)\in L^2(0,T;\hat{H})$, $\mathbf{\Phi}\in\mathcal{C}_0(\hat{H})$ and $\Phi(0)\in\hat{V}$, then the weak solution $\mathbf{v}=(\mathbf{u},\omega)$ is actually a strong solution of (14)-(15).

Proof. The proof uses the classical Faedo-Galerkin approximation method and the energy method, it consists of three steps: constructing Galerkin approximate solutions, deriving a priori estimates, and passing to the limit, which is similar to [7] [13], only minor adjustments are neededthe detailed proof is omitted here. □

The stationary equation corresponding to the initial value problem (14)-(15) is

$$A\mathbf{v}^*+B(\mathbf{u}^*,\mathbf{v}^*)+N(\mathbf{v}^*)+J(\mathbf{v}^*)=\mathbf{F}+\mathbf{G}(\mathbf{v}^*). \tag{17}$$

Definition 2. A function $\mathbf{v}^*=(\mathbf{u}^*,\omega^*)\in\hat{V}$ is called a stationary solution of the initial value problem (14)-(15) if it satisfies

$$\begin{aligned}&\langle A\mathbf{v}^*,\mathbf{\Psi}\rangle+\langle B(u^*,\mathbf{v}^*),\mathbf{\Psi}\rangle+\langle N(\mathbf{v}^*),\mathbf{\Psi}\rangle+\langle J(\mathbf{v}^*),\mathbf{\Psi}\rangle\\&=\langle\mathbf{F},\mathbf{\Psi}\rangle+(\mathbf{G}(\mathbf{v}^*),\mathbf{\Psi}),\ \forall\mathbf{\Psi}\in\hat{V}.\end{aligned}$$

Theorem 2. Assume that G satisfies Assumption 1 and $\delta_1>\gamma^{-1}L_G$. Then,

(a) For all $\mathbf{F}\in\hat{V}'$, Equation (17) has at least one solution.

(b) If $(\delta_1-\gamma^{-1}L_G)^2>c_3\gamma^{-\frac{1}{2}}\|\mathbf{F}\|_{\hat{V}'}$, then Equation (17) has a unique solution.

(c) If $\mathbf{F}\in\hat{H}$, then the solution $\mathbf{v}^*\in D(A)$.

Proof. The proof mainly relies on the Lax–Milgram theorem and the Schauder fixed-point theorem, which is similar to references [17] [27], the detailed steps are omitted here. □

4. Stability of Weak and Strong Solutions with Respect to Stationary Solutions

This section considers several special forms of unbounded delay functions and further establishes the stability of stationary solutions to the initial value problems (14)-(15). First, when the unbounded delay function is continuously differentiable with respect to time, the stability of weak solutions with respect to nontrivial stationary solutions and the asymptotic stability of weak solutions with respect to trivial stationary solutions are established by applying the definition of Lyapunov stability and by constructing a Lyapunov functional, respectively. Next, when the unbounded delay function is only continuous with respect to time, the stability of strong solutions with respect to nontrivial stationary solutions is proved by means of the Razumikhin technique. Finally, when the unbounded delay function is a proportional delay, the polynomial stability of weak solutions with respect to trivial stationary solutions is obtained. The definitions of stability of weak and strong solutions with respect to stationary solutions are given below.

Definition 3. ([28]) Let $\mathbf{v}(t)$ and $\mathbf{v}^*$ be a weak (strong) solution and a stationary solution of the initial value problem (14)-(15), respectively.

(i) If for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever the initial value $\mathbf{\Phi} \in \mathcal{C}_0(\hat{H})$ satisfies $\|\mathbf{\Phi} - \mathbf{v}^*\|_{\mathcal{C}_0(\hat{H})} < \delta$, we have

$$\|\mathbf{v}(t) - \mathbf{v}^*\| < \varepsilon, \ \forall t > 0,$$

then $\mathbf{v}(t)$ is said to be stable with respect to the stationary solution $\mathbf{v}^*$.

(ii) If there exists $\tilde{\delta} > 0$ such that whenever $\mathbf{\Phi} \in \mathcal{C}_0(\hat{H})$ satisfies $\|\mathbf{\Phi} - \mathbf{v}^*\|_{\mathcal{C}_0(\hat{H})} < \tilde{\delta}$, we have

$$\lim_{t \to \infty} \|\mathbf{v}(t) - \mathbf{v}^*\| = 0,$$

then $\mathbf{v}(t)$ is said to be attractive with respect to $\mathbf{v}^*$.

(iii) If $\mathbf{v}(t)$ is both stable and attractive with respect to $\mathbf{v}^*$, then $\mathbf{v}(t)$ is said to be asymptotically stable with respect to $\mathbf{v}^*$.

4.1. Stability and Asymptotic Stability: The Unbounded Variable Delay Driven by a Continuously Differential Function

In this subsection, we consider delay functions satisfying the following condition.

Condition 1. Let

$$\mathbf{G}(t, \xi) = \hat{\mathbf{G}}(\xi(-\rho(t))),$$

where $(t, \xi) \in \mathbb{R}^+ \times \mathcal{C}_0(\hat{H})$, $\hat{G}: \hat{H} \to \hat{H}$ is a Lipschitz map with Lipschitz constant $L_{\hat{\mathbf{G}}}$, $\hat{\mathbf{G}}(0) = 0$, and $\rho \in \mathcal{C}^1(\mathbb{R}^+; \mathbb{R}^+)$, $\rho^* = \sup_{t \ge 0} \rho'(t) < 1$.

Remark 4. From Remark 3, $\mathbf{G}(t, \xi)$ satisfying Condition 1 also satisfies Assumption 1.

Theorem 3. Let $\mathbf{F} \in \hat{H}$, Condition 1 hold, $\delta_1 > \gamma^{-1} L_{\hat{\mathbf{G}}}$, and

$$\delta_1 > \frac{c_3 \gamma^{-1} \|\mathbf{F}\|}{\delta_1 - \gamma^{-1} L_{\hat{\mathbf{G}}}} + \frac{(2 - \rho^*) \gamma^{-1} L_{\hat{\mathbf{G}}}}{2(1 - \rho^*)}, \tag{18}$$

then the initial value problem (14)-(15) has a stationary solution $\mathbf{v}^* \in D(A)$. When $\mathbf{F}(t) \equiv \mathbf{F}$, $\mathbf{\Phi} \in \mathcal{C}_0(\hat{H})$, there exists a unique weak solution $\mathbf{v}(t)$ of (14)-(15) satisfying

$$\left\|\mathbf{v}(t)-\mathbf{v}^*\right\|^2 \le C\left(\left\|\mathbf{\Phi}(0)-\mathbf{v}^*\right\|^2+\left\|\mathbf{\Phi}-\mathbf{v}^*\right\|^2_{L^2\left((-\rho(0),0);\hat{V}\right)}\right),\ \forall t\ge 0.$$

Proof. By Condition 1, $\mathbf{F}\in\hat{H}$, $\delta_1>\gamma^{-1}L_{\hat{\mathbf{G}}}$ and Theorem 1, when $\mathbf{F}(t)\equiv\mathbf{F}$, $\mathbf{\Phi}\in\mathcal{C}_0\left(\hat{H}\right)$, there exists a unique weak solution $\mathbf{v}=(\mathbf{u},\omega)$ of (14)-(15). From Theorem 2, there exists a unique stationary solution $\mathbf{v}^*\in D(A)$. Let $\tilde{\mathbf{v}}(t)=\mathbf{v}(t)-\mathbf{v}^*$. Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{v}}(t)+A\tilde{\mathbf{v}}(t)+B\left(\mathbf{u}(t),\mathbf{v}(t)\right)-B\left(\mathbf{u}^*,\mathbf{v}^*\right)+N\left(\mathbf{v}(t)\right)-N\left(\mathbf{v}^*\right)+J\left(\tilde{\mathbf{v}}(t)\right)\\&=\mathbf{G}\left(\mathbf{v}\left(t-\rho(t)\right)\right)-\mathbf{G}\left(\mathbf{v}^*\right).\end{aligned}\tag{19}$$

Taking the duality product of (19) with $\tilde{\mathbf{v}}(t)$ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\mathbf{v}}(t)\right\|^2=&-2\left\langle A\tilde{\mathbf{v}}(t),\tilde{\mathbf{v}}(t)\right\rangle-2\left\langle B\left(\mathbf{u}(t),\mathbf{v}(t)\right)-B\left(\mathbf{u}^*,\mathbf{v}^*\right),\tilde{\mathbf{v}}(t)\right\rangle\\&-2\left\langle N\left(\mathbf{v}(t)\right)-N\left(\mathbf{v}^*\right),\tilde{\mathbf{v}}(t)\right\rangle-2\left\langle J\left(\tilde{\mathbf{v}}(t)\right),\tilde{\mathbf{v}}(t)\right\rangle\\&+2\left(\mathbf{G}\left(\mathbf{v}\left(t-\rho(t)\right)\right)-\mathbf{G}\left(\mathbf{v}^*\right),\tilde{\mathbf{v}}(t)\right).\end{aligned}\tag{20}$$

Considering Condition 1, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\mathbf{v}}(t)\right\|^2\le&-2\delta_1\left\|\tilde{\mathbf{v}}(t)\right\|^2_{\hat{V}}+\frac{2c_3\gamma^{-1}\|\mathbf{F}\|}{\delta_1-\gamma^{-1}L_{\hat{\mathbf{G}}}}\left\|\tilde{\mathbf{v}}(t)\right\|^2_{\hat{V}}+\gamma^{-1}L_{\hat{\mathbf{G}}}\left\|\tilde{\mathbf{v}}(t)\right\|^2_{\hat{V}}\\&+L_{\hat{\mathbf{G}}}\left\|\tilde{\mathbf{v}}\left(t-\rho(t)\right)\right\|^2\\=&\left(\frac{2c_3\gamma^{-1}\|\mathbf{F}\|}{\delta_1-\gamma^{-1}L_{\hat{\mathbf{G}}}}+\gamma^{-1}L_{\hat{\mathbf{G}}}-2\delta_1\right)\left\|\tilde{\mathbf{v}}(t)\right\|^2_{\hat{V}}+L_{\hat{\mathbf{G}}}\left\|\tilde{\mathbf{v}}\left(t-\rho(t)\right)\right\|^2,\end{aligned}\tag{21}$$

Integrating (21) over $[0,t]$ gives

$$\begin{aligned}\left\|\tilde{\mathbf{v}}(t)\right\|^2\le&\left\|\tilde{\mathbf{v}}(0)\right\|^2+\left(\frac{2c_3\gamma^{-1}\|\mathbf{F}\|}{\delta_1-\gamma^{-1}L_{\hat{\mathbf{G}}}}+\gamma^{-1}L_{\hat{\mathbf{G}}}-2\delta_1\right)\int_0^t\left\|\tilde{\mathbf{v}}(s)\right\|^2_{\hat{V}}\,\mathrm{d}s\\&+L_{\hat{\mathbf{G}}}\int_0^t\left\|\tilde{\mathbf{v}}\left(s-\rho(s)\right)\right\|^2\,\mathrm{d}s.\end{aligned}\tag{22}$$

For the delay integral term, set $\eta=s-\rho(s)=\tau(s)$, then

$$L_{\hat{\mathbf{G}}}\int_0^t\left\|\tilde{\mathbf{v}}\left(s-\rho(s)\right)\right\|^2\,\mathrm{d}s\le\frac{L_{\hat{\mathbf{G}}}}{1-\rho^*}\int_{-\rho(0)}^t\left\|\tilde{\mathbf{v}}(\eta)\right\|^2\,\mathrm{d}\eta,\tag{23}$$

From (22)-(23) and (7), we obtain

$$\begin{aligned}\left\|\tilde{\mathbf{v}}(t)\right\|^2\le&\left\|\tilde{\mathbf{v}}(0)\right\|^2+\left(\frac{2c_3\gamma^{-1}\|\mathbf{F}\|}{\delta_1-\gamma^{-1}L_{\hat{\mathbf{G}}}}+\gamma^{-1}L_{\hat{\mathbf{G}}}-2\delta_1+\frac{\gamma^{-1}L_{\hat{\mathbf{G}}}}{1-\rho^*}\right)\int_0^t\left\|\tilde{\mathbf{v}}(s)\right\|^2_{\hat{V}}\,\mathrm{d}s\\&+\frac{\gamma^{-1}L_{\hat{\mathbf{G}}}}{1-\rho^*}\int_{-\rho(0)}^0\left\|\tilde{\mathbf{v}}(s)\right\|^2_{\hat{V}}\,\mathrm{d}s\\=&\left\|\tilde{\mathbf{v}}(0)\right\|^2+\left(\frac{2c_3\gamma^{-1}\|\mathbf{F}\|}{\delta_1-\gamma^{-1}L_{\hat{\mathbf{G}}}}-2\delta_1+\frac{\left(2-\rho^*\right)\gamma^{-1}L_{\hat{\mathbf{G}}}}{1-\rho^*}\right)\int_0^t\left\|\tilde{\mathbf{v}}(s)\right\|^2_{\hat{V}}\,\mathrm{d}s\\&+\frac{\gamma^{-1}L_{\hat{\mathbf{G}}}}{1-\rho^*}\int_{-\rho(0)}^0\left\|\tilde{\mathbf{v}}(s)\right\|^2_{\hat{V}}\,\mathrm{d}s.\end{aligned}$$

By (18), we have

$$\frac{2c_3\gamma^{-1}\|\mathbf{F}\|}{\delta_1-\gamma^{-1}L_{\hat{\mathbf{G}}}}-2\delta_1+\frac{\left(2-\rho^*\right)\gamma^{-1}L_{\hat{\mathbf{G}}}}{1-\rho^*}<0,$$

thus

$$\begin{aligned}\left\|\tilde{\mathbf{v}}(t)\right\|^2&\le\left\|\tilde{\mathbf{v}}(0)\right\|^2+\frac{\gamma^{-1}L_{\hat{\mathbf{G}}}}{1-\rho^*}\int_{-\rho(0)}^{0}\left\|\tilde{\mathbf{v}}(s)\right\|_{\hat{V}}^2\,\mathrm{d}s\\&\le C\left(\left\|\mathbf{\Phi}(0)-\mathbf{v}^*\right\|^2+\left\|\mathbf{\Phi}-\mathbf{v}^*\right\|_{L^2\left((-\rho(0),0);\hat{V}\right)}^2\right),\end{aligned}\tag{24}$$

where $C=\max\left\{1,\dfrac{\gamma^{-1}L_{\hat{\mathbf{G}}}}{1-\rho^*}\right\}$. This completes the proof of Theorem 3. □

Next, we establish the asymptotic stability of weak solutions with respect to the trivial stationary solution for the initial value problem (14)-(15) by constructing a Lyapunov functional.

Theorem 4. Let $\mathbf{F}\equiv0$, Condition 1 hold, $\mathbf{\Phi}\in\mathcal{C}_0\left(\hat{H}\right)$, $\delta_1>\gamma^{-1}L_{\hat{\mathbf{G}}}$. Then the initial value problem (14)-(15) has a unique trivial stationary solution $\mathbf{v}^*\equiv0$ and a unique weak solution $\mathbf{v}(t)$. Moreover, if

$$\delta_1>\frac{\gamma^{-1}L_{\hat{\mathbf{G}}}}{\sqrt{1-\rho^*}},\tag{25}$$

then $\mathbf{v}(t)$ is asymptotically stable with respect to $\mathbf{v}^*\equiv0$.

Proof. By the above conditions and Theorem 2(a)-(b), the initial value problem (14)-(15) has a unique trivial stationary solution $\mathbf{v}^*\equiv0$ and a unique weak solution $\mathbf{v}(t)$. We prove the asymptotic stability in three steps.

Step 1 (Construction of a Lyapunov functional). Define $\mathbf{W}\left(t,\mathbf{v}(t)\right):\mathbb{R}^+\times C_0\left(\hat{H}\right)\to\mathbb{R}^+$ by

$$\mathbf{W}\left(t,\mathbf{v}(t)\right)=\left\|\mathbf{v}(t)\right\|^2+\frac{C'}{1-\rho^*}\int_{t-\rho(t)}^{t}\left\|\mathbf{v}(s)\right\|^2\,\mathrm{d}s,\tag{26}$$

where $C'>0$ will be determined later. From (26) and the nonnegativity of the integral term,

$$\mathbf{W}\left(t,\mathbf{v}(t)\right)\le\left\|\mathbf{v}(t)\right\|^2.\tag{27}$$

On the other hand, from (26),

$$\begin{aligned}\mathbf{W}\left(0,\mathbf{v}(0)\right)&=\left\|\mathbf{v}(0)\right\|^2+\frac{C'}{1-\rho^*}\int_{-\rho(0)}^{0}\left\|\mathbf{v}(s)\right\|^2\,\mathrm{d}s\\&\le\left\|\mathbf{\Phi}\right\|_{C_0(\hat{H})}^2+\frac{C'}{1-\rho^*}\int_{-\rho(0)}^{0}\left\|\mathbf{\Phi}(s)\right\|^2\,\mathrm{d}s\\&\le\left\|\mathbf{\Phi}\right\|_{C_0(\hat{H})}^2+\frac{C'\rho(0)}{1-\rho^*}\left\|\mathbf{\Phi}\right\|_{C_0(\hat{H})}^2.\end{aligned}$$

Thus there exists $\beta=1+\dfrac{C'\rho(0)}{1-\rho^*}>0$ such that

$$\mathbf{W}\left(0,\mathbf{v}\left(0\right)\right)\le\beta\left\|\mathbf{\Phi}\right\|_{C_0\left(\hat{H}\right)}^2. \tag{28}$$

Step 2 (Proof of stability). From (26), and Condition 1,

$$\begin{aligned}\frac{\mathrm{d}\mathbf{W}\left(t,\mathbf{v}\left(t\right)\right)}{\mathrm{d}t}&=\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{v}\left(t\right)\right\|^2+\frac{C'}{1-\rho^*}\left\|\mathbf{v}\left(t\right)\right\|^2-\frac{C'\left(1-\rho'\left(t\right)\right)}{1-\rho^*}\left\|\mathbf{v}\left(t-\rho\left(t\right)\right)\right\|^2\\&\le\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{v}\left(t\right)\right\|^2+\frac{C'}{1-\rho^*}\left\|\mathbf{v}\left(t\right)\right\|^2-C'\left\|\mathbf{v}\left(t-\rho\left(t\right)\right)\right\|^2.\end{aligned} \tag{29}$$

Since $\mathbf{F}\equiv 0$ and Condition 1 holds, Equation (14), becomes

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}=-A\mathbf{v}-B\left(u,\mathbf{v}\right)-N\left(\mathbf{v}\right)-J\left(\mathbf{v}\right)+\hat{\mathbf{G}}\left(\mathbf{v}\left(t-\rho\left(t\right)\right)\right),\ \forall t\ge 0,$$

Taking the duality product with $\mathbf{v}\left(t\right)$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{v}\left(t\right)\right\|^2=-2\left\langle -A\mathbf{v}-B\left(u,\mathbf{v}\right)-N\left(\mathbf{v}\right)-J\left(\mathbf{v}\right),\mathbf{v}\left(t\right)\right\rangle+2\left(\hat{\mathbf{G}}\left(\mathbf{v}\left(t-\rho\left(t\right)\right)\right),\mathbf{v}\left(t\right)\right).$$

Using (8), (10), Condition 1, the nonnegativity of N, and Hölder's inequality, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{v}\left(t\right)\right\|^2&\le-2\delta_1\left\|\mathbf{v}\left(t\right)\right\|_{\hat{V}}^2+2L_{\hat{\mathbf{G}}}\left\|\mathbf{v}\left(t-\rho\left(t\right)\right)\right\|\left\|\mathbf{v}\left(t\right)\right\|\\&\le-2\delta_1\left\|\mathbf{v}\left(t\right)\right\|_{\hat{V}}^2+\frac{L_{\hat{\mathbf{G}}}^2}{C'}\left\|\mathbf{v}\left(t\right)\right\|^2+C'\left\|\mathbf{v}\left(t-\rho\left(t\right)\right)\right\|^2.\end{aligned} \tag{30}$$

Substituting (30) into (29), and using (7),

$$\begin{aligned}\frac{\mathrm{d}\mathbf{W}\left(t,\mathbf{v}\left(t\right)\right)}{\mathrm{d}t}&\le-2\delta_1\left\|\mathbf{v}\left(t\right)\right\|_{\hat{V}}^2+\frac{L_{\hat{\mathbf{G}}}^2}{C'}\left\|\mathbf{v}\left(t\right)\right\|^2+\frac{C'}{1-\rho^*}\left\|\mathbf{v}\left(t\right)\right\|^2\\&\le\left(-2\delta_1+\gamma^{-1}\left(\frac{L_{\hat{\mathbf{G}}}^2}{C'}+\frac{C'}{1-\rho^*}\right)\right)\left\|\mathbf{v}\left(t\right)\right\|_{\hat{V}}^2,\end{aligned} \tag{31}$$

Choosing $C'=L_{\hat{\mathbf{G}}}\sqrt{1-\rho^*}$, by (25), and (31), we have

$$\frac{\mathrm{d}\mathbf{W}\left(t,\mathbf{v}\left(t\right)\right)}{\mathrm{d}t}\le-2\left(\delta_1-\frac{\gamma^{-1}L_{\hat{\mathbf{G}}}}{\sqrt{1-\rho^*}}\right)\left\|\mathbf{v}\left(t\right)\right\|_{\hat{V}}^2<0, \tag{32}$$

Therefore, $\mathbf{W}\left(t,\mathbf{v}\left(t\right)\right)$ is strictly decreasing. From (27)-(28),

$$\left\|\mathbf{v}\left(t\right)\right\|^2\le\mathbf{W}\left(t,\mathbf{v}\left(t\right)\right)<\mathbf{W}\left(0,\mathbf{v}\left(0\right)\right)\le\beta\left\|\mathbf{\Phi}\right\|_{C_0\left(\hat{H}\right)}^2, \tag{33}$$

so $\mathbf{v}^*\equiv 0$ is stable.

Step 3 (Proof of attractivity). Integrate (32) over $\left[0,t\right]$:

$$\mathbf{W}\left(t,\mathbf{v}\left(t\right)\right)-\mathbf{W}\left(0,\mathbf{v}\left(0\right)\right)\le-2\left(\delta_1-\frac{\gamma^{-1}L_{\hat{\mathbf{G}}}}{\sqrt{1-\rho^*}}\right)\int_0^t\left\|\mathbf{v}\left(s\right)\right\|_{\hat{V}}^2\,\mathrm{d}s,$$

By (33),

$$
\begin{aligned}
\int_0^t \left\|\mathbf{v}(s)\right\|_{\hat{V}}^2 \mathrm{d}s &\le \frac{1}{2\left(\delta_1 - \dfrac{\gamma^{-1}L_{\hat{\mathbf{G}}}}{\sqrt{1-\rho^*}}\right)}\left(\mathbf{W}\left(t,\mathbf{v}(0)\right) - \mathbf{W}\left(\mathbf{v}(t)\right)\right) \\
&\le \frac{1}{2\left(\delta_1 - \dfrac{\gamma^{-1}L_{\hat{\mathbf{G}}}}{\sqrt{1-\rho^*}}\right)}\left(\beta\left\|\mathbf{\Phi}\right\|_{\mathcal{C}_0(\hat{H})}^2 - \left\|\mathbf{v}(t)\right\|^2\right) \\
&\le \frac{\beta}{2\left(\delta_1 - \dfrac{\gamma^{-1}L_{\hat{\mathbf{G}}}}{\sqrt{1-\rho^*}}\right)}\left\|\mathbf{\Phi}\right\|_{\mathcal{C}_0(\hat{H})}^2,
\end{aligned}
$$

which implies $\lim_{t\to\infty}\left\|\mathbf{v}(t)\right\|_V = 0$, *i.e.*, $\mathbf{v}^* \equiv 0$ is attractive. Thus $\mathbf{v}(t)$ is asymptotically stable with respect to $\mathbf{v}^* \equiv 0$. □

4.2. Stability: The Unbounded Variable Delay Driven by a Continuous Function

In this subsection, we consider delay functions satisfying the following condition.

Condition 2. Let

$$
\mathbf{G}(t,\xi) = \hat{\mathbf{G}}\left(\xi\left(-\rho(t)\right)\right),\ \forall (t,\xi) \in \mathbb{R}^+ \times \mathcal{C}_0(H),
$$

where $\hat{\mathbf{G}}: \hat{H} \to \hat{H}$ is a Lipschitz map with Lipschitz constant $L_{\hat{\mathbf{G}}}$, $\hat{\mathbf{G}}(0) = 0$, $\rho \in \mathcal{C}\left(\mathbb{R}^+;\mathbb{R}^+\right)$.

To establish the stability of strong solutions with respect to non-trivial stationary solutions, we need to modify Assumption 1 (i) as follows:

(i”) For any $\xi \in \mathcal{C}_0\left(\hat{H}\right)$, the map $[0,T] \ni t \mapsto \mathbf{G}(t,\xi) \in \hat{H}$ is continuous.

Remark 5. From Remark 3, the delay term satisfying Condition 2 satisfies Assumption 1 and (I”).

Then, we can establish an important lemma.

Lemma 2. Let $\mathbf{G}$ satisfies Assumption 1 and (i”), $\mathbf{F} \in \hat{H}$, $\mathbf{v}^* \in D(A)$ be a stationary solution of (14)-(15). For any $\mathbf{\Phi} = (\boldsymbol{\phi},\phi_3) \in \mathcal{C}_0\left(\hat{H}\right)$ with $\mathbf{\Phi}(0) = \left(\boldsymbol{\phi}(0),\phi_3(0)\right) \in \hat{V}$, where $\boldsymbol{\phi} = (\phi_1,\phi_2) \in \mathcal{C}_0(H)$, $\boldsymbol{\phi}(0) = \left(\phi_1(0),\phi_2(0)\right) \in V$, and $\mathbf{\Phi} \ne \mathbf{v}^*$ satisfying

$$
\left\|\mathbf{\Phi} - \mathbf{v}^*\right\|_{\mathcal{C}_0(\hat{H})} = \left\|\mathbf{\Phi}(0) - \mathbf{v}^*\right\| \tag{34}
$$

such that

$$
\begin{aligned}
&-\left(A\left(\mathbf{\Phi}(0)-\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right) - \left(B\left(\boldsymbol{\phi}(0),\mathbf{\Phi}(0)\right) - B\left(\mathbf{u}^*,\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right) \\
&-\left(N\left(\mathbf{\Phi}(0)\right) - N\left(\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right) - \left(J\left(\mathbf{\Phi}(0)-\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right) \\
&+\left(\mathbf{G}(t,\mathbf{\Phi}) - \mathbf{G}\left(t,\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right) < 0.
\end{aligned} \tag{35}
$$

then the strong solution $\mathbf{v}(t)$ of (14)-(15) satisfies

$$\left\|\mathbf{v}(t)-\mathbf{v}^*\right\| \le \left\|\mathbf{\Phi}-\mathbf{v}^*\right\|_{\mathcal{C}_0(\hat{H})}, \ \forall t \ge 0. \tag{36}$$

Proof. When $\mathbf{\Phi}=\mathbf{v}^*$, the conclusion is trivial. When $\mathbf{\Phi}\ne\mathbf{v}^*$, we prove by contradiction. Suppose there exists an initial value $\mathbf{\Phi}$ satisfying $\mathbf{\Phi}\in\mathcal{C}_0(\hat{H})$, $\mathbf{\Phi}(0)\in\hat{V}$, $\mathbf{\Phi}\ne\mathbf{v}^*$ such that (36) fails. Then there exists $t>0$ with

$$\left\|\mathbf{v}(t)-\mathbf{v}^*\right\| > \left\|\mathbf{\Phi}-\mathbf{v}^*\right\|_{\mathcal{C}_0(\hat{H})}.$$

Define

$$\sigma=\inf\left\{t>0:\left\|\mathbf{v}(t)-\mathbf{v}^*\right\| > \left\|\mathbf{\Phi}-\mathbf{v}^*\right\|_{\mathcal{C}_0(\hat{H})}\right\},$$

Then for any $0\le s\le\sigma$,

$$\left\|\mathbf{v}(s)-\mathbf{v}^*\right\| \le \left\|\mathbf{v}(\sigma)-\mathbf{v}^*\right\| = \left\|\mathbf{\Phi}-\mathbf{v}^*\right\|_{\mathcal{C}_0(\hat{H})}, \tag{37}$$

and there exists a subsequence $\{t_k\}_{k\ge1}\subset(\sigma,\infty)$, $t_k\downarrow\sigma$ as $k\to\infty$ such that

$$\left\|\mathbf{v}(t_k)-\mathbf{v}^*\right\| > \left\|\mathbf{v}(\sigma)-\mathbf{v}^*\right\|. \tag{38}$$

From (37),

$$\left\|\mathbf{v}_\sigma-\mathbf{v}^*\right\|_{\mathcal{C}_0(\hat{H})} = \sup_{\theta\le0}\left\|\mathbf{v}(\sigma+\theta)-\mathbf{v}^*\right\| = \left\|\mathbf{v}(\sigma)-\mathbf{v}^*\right\|. \tag{39}$$

By (34)-(35) and (39), taking $\mathbf{\Phi}=\mathbf{v}_\sigma$ we have

$$\begin{aligned}&-\left(A\left(\mathbf{v}(\sigma)-\mathbf{v}^*\right),\mathbf{v}(\sigma)-\mathbf{v}^*\right)-\left(B\left(\mathbf{u}(\sigma),\mathbf{v}(\sigma)\right)-B\left(\mathbf{u}^*,\mathbf{v}^*\right),\mathbf{v}(\sigma)-\mathbf{v}^*\right)\\&-\left(N\left(\mathbf{v}(\sigma)\right)-N\left(\mathbf{v}^*\right),\mathbf{v}(\sigma)-\mathbf{v}^*\right)-\left(J\left(\mathbf{v}(\sigma)-\mathbf{v}^*\right),\mathbf{v}(\sigma)-\mathbf{v}^*\right)\\&+\left(\mathbf{G}\left(\sigma,\mathbf{v}_\sigma(\cdot)\right)-\mathbf{G}\left(t,\mathbf{v}^*\right),\mathbf{v}(\sigma)-\mathbf{v}^*\right)<0.\end{aligned}$$

By the continuity of operators A,B,N,G, there exists $\epsilon>0$ such that for any $t\in[\sigma,\sigma+\epsilon]$,

$$\begin{aligned}&-\left(A\left(\mathbf{v}(t)-\mathbf{v}^*\right),\mathbf{v}(t)-\mathbf{v}^*\right)-\left(B\left(\mathbf{u}(t),\mathbf{v}(t)\right)-B\left(\mathbf{u}^*,\mathbf{v}^*\right),\mathbf{v}(t)-\mathbf{v}^*\right)\\&-\left(N\left(\mathbf{v}(t)\right)-N\left(\mathbf{v}^*\right),\mathbf{v}(t)-\mathbf{v}^*\right)-\left(J\left(\mathbf{v}(t)-\mathbf{v}^*\right),\mathbf{v}(t)-\mathbf{v}^*\right)\\&+\left(\mathbf{G}\left(t,\mathbf{v}_t\right)-\mathbf{G}\left(t,\mathbf{v}^*\right),\mathbf{v}(t)-\mathbf{v}^*\right)<0.\end{aligned}$$

Let $\tilde{\mathbf{v}}(t)=\mathbf{v}(t)-\mathbf{v}^*$. From (20), for any $t\in[\sigma,\sigma+\epsilon]$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\mathbf{v}}(t)\right\|^2 &= -\left(A\tilde{\mathbf{v}}(t),\tilde{\mathbf{v}}(t)\right)-\left(B\left(\mathbf{u}(t),\mathbf{v}(t)\right)-B\left(\mathbf{u}^*,\mathbf{v}^*\right),\tilde{\mathbf{v}}(t)\right)\\&\quad-\left(N\left(\mathbf{v}(t)\right)-N\left(\mathbf{v}^*\right),\tilde{\mathbf{v}}(t)\right)-\left(J\left(\tilde{\mathbf{v}}(t)\right),\tilde{\mathbf{v}}(t)\right)\\&\quad+\left(\mathbf{G}\left(t,\mathbf{v}_t\right)-\mathbf{G}\left(t,\mathbf{v}^*\right),\tilde{\mathbf{v}}(t)\right)\\&<0,\end{aligned} \tag{40}$$

Take $t_{k(\epsilon)}\in(\sigma,\sigma+\epsilon]$ and integrate (40) over $\left[\sigma,t_{k(\epsilon)}\right]$ to obtain

$$\begin{aligned}\left\|\tilde{\mathbf{v}}\left(t_{k(\epsilon)}\right)\right\|^2-\left\|\tilde{\mathbf{v}}(\sigma)\right\|^2 \le\ & 2\int_{\sigma}^{t_{k(\epsilon)}}-\left(A\tilde{\mathbf{v}}(t),\tilde{\mathbf{v}}(t)\right)-\left(B\left(\mathbf{u}(t),\mathbf{v}(t)\right)-B\left(\mathbf{u}^*,\mathbf{v}^*\right),\tilde{\mathbf{v}}(t)\right)\mathrm{d}t\\ &+2\int_{\sigma}^{t_{k(\epsilon)}}-\left(N\left(\mathbf{v}(t)\right)-N\left(\mathbf{v}^*\right),\tilde{\mathbf{v}}(t)\right)-\left(J\left(\tilde{\mathbf{v}}(t)\right),\tilde{\mathbf{v}}(t)\right)\mathrm{d}t\\ &+2\int_{\sigma}^{t_{k(\epsilon)}}\left(\mathbf{G}\left(t,\mathbf{v}_t\right)-\mathbf{G}\left(t,\mathbf{v}^*\right),\tilde{\mathbf{v}}(t)\right)\mathrm{d}t\\ <\ & 0,\end{aligned}$$

i.e., $\left\|\tilde{\mathbf{v}}\left(t_{k(\epsilon)}\right)\right\|<\left\|\tilde{\mathbf{v}}(\sigma)\right\|$, contradicting (38). Hence (36) holds. □

Theorem 5. Suppose Condition 2 holds, $\mathbf{F}\in\hat{H}$, $\delta_1>\gamma^{-1}L_{\hat{\mathbf{G}}}$, and

$$\delta_1>\gamma^{-1}L_{\hat{\mathbf{G}}}+\frac{c_3\cdot\gamma^{-1}\|\mathbf{F}\|}{\delta_1-\gamma^{-1}L_{\hat{\mathbf{G}}}}, \tag{41}$$

then for any $\mathbf{\Phi}\in\mathcal{C}_0\left(\hat{H}\right)$ with $\mathbf{\Phi}(0)\in\hat{V}$, the initial value problem (14)-(15) has a unique stationary solution $\mathbf{v}^*\in D(A)$ and a unique strong solution $\mathbf{v}(t)$ satisfying

$$\left\|\mathbf{v}(t)-\mathbf{v}^*\right\|\le\left\|\mathbf{\Phi}-\mathbf{v}^*\right\|_{\mathcal{C}_0(\hat{H})},\quad\forall t\ge0. \tag{42}$$

Proof. By Condition 2, $\mathbf{F}\in\hat{H}$, $\delta_1>\gamma^{-1}L_{\hat{\mathbf{G}}}$, Theorem 1 and Theorem 2 imply the existence of a unique stationary solution $\mathbf{v}^*\in D(A)$ and a unique strong solution $\mathbf{v}(t)$ of (14)-(15). We now prove (42). Let $\mathbf{\Phi}=(\boldsymbol{\phi},\phi_3)\in\mathcal{C}_0\left(\hat{H}\right)$ with $\mathbf{\Phi}(0)=\left(\boldsymbol{\phi}(0),\phi_3(0)\right)\in\hat{V}$, where $\boldsymbol{\phi}=(\phi_1,\phi_2)\in\mathcal{C}_0(H)$, $\boldsymbol{\phi}(0)=\left(\phi_1(0),\phi_2(0)\right)\in V$, and $\mathbf{\Phi}\ne\mathbf{v}^*$ satisfies

$$\left\|\mathbf{\Phi}-\mathbf{v}^*\right\|_{\mathcal{C}_0(\hat{H})}=\left\|\mathbf{\Phi}(0)-\mathbf{v}^*\right\|. \tag{43}$$

Using (7), (10), (16), (43), we obtain

$$\begin{aligned}&-\left(A\left(\mathbf{\Phi}(0)-\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right)-\left(B\left(\boldsymbol{\phi}(0),\mathbf{\Phi}(0)\right)-B\left(\mathbf{u}^*,\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right)\\ &-\left(N\left(\mathbf{\Phi}(0)\right)-N\left(\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right)-\left(J\left(\mathbf{\Phi}(0)-\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right)\\ &+\left(\mathbf{G}(t,\mathbf{\Phi})-\mathbf{G}\left(t,\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right)\\ &\le-\delta_1\left\|\mathbf{\Phi}(0)-\mathbf{v}^*\right\|_{\hat{V}}^2+\left|\left(B\left(\mathbf{\Phi}(0)-u^*,\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right)\right|\\ &\quad+L_{\hat{\mathbf{G}}}\left\|\mathbf{\Phi}-\mathbf{v}^*\right\|_{\mathcal{C}_0(\hat{H})}\left\|\mathbf{\Phi}(0)-\mathbf{v}^*\right\|\\ &\le-\delta_1\left\|\mathbf{\Phi}(0)-\mathbf{v}^*\right\|_{\hat{V}}^2+\frac{c_3\gamma^{-1}\|\mathbf{F}\|}{\delta_1-\gamma^{-1}L_{\hat{\mathbf{G}}}}\left\|\mathbf{\Phi}(0)-\mathbf{v}^*\right\|_{\hat{V}}^2+\gamma^{-1}L_{\hat{\mathbf{G}}}\left\|\mathbf{\Phi}(0)-\mathbf{v}^*\right\|_{\hat{V}}^2.\end{aligned} \tag{44}$$

By (41) and (42),

$$\begin{aligned}&-\left(A\left(\mathbf{\Phi}(0)-\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right)-\left(B\left(\boldsymbol{\phi}(0),\mathbf{\Phi}(0)\right)-B\left(\mathbf{u}^*,\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right)\\ &-\left(N\left(\mathbf{\Phi}(0)\right)-N\left(\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right)-\left(J\left(\mathbf{\Phi}(0)-\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right)\\ &+\left(\mathbf{G}(t,\mathbf{\Phi})-\mathbf{G}\left(t,\mathbf{v}^*\right),\mathbf{\Phi}(0)-\mathbf{v}^*\right)\\ &\le\left(-\delta_1+\frac{c_3\gamma^{-1}\|\mathbf{F}\|}{\delta_1-\gamma^{-1}L_{\hat{\mathbf{G}}}}+\gamma^{-1}L_{\hat{\mathbf{G}}}\right)\left\|\mathbf{\Phi}(0)-\mathbf{v}^*\right\|_{\hat{V}}^2<0.\end{aligned}$$

Thus (35) holds. By Lemma 2, Theorem 5 follows. □

4.3. Polynomial Stability: Proportional Delay

In this subsection, we consider delay functions satisfying the following condition.

Condition 3. Let

$$\mathbf{G}(t,\xi)=\hat{\mathbf{G}}\left(\xi\left(-(1-\gamma_0)t\right)\right),\ (t,\xi)\in\mathbb{R}^+\times\mathcal{C}_0\left(\hat{H}\right),$$

where $\gamma_0\in(0,1)$, $\hat{\mathbf{G}}:\hat{H}\to\hat{H}$ is a Lipschitz map with Lipschitz constant $L_{\hat{G}}$ and $\hat{\mathbf{G}}(0)=0$.

Remark 6. From Remark 3, the delay term satisfying Condition 3 satisfies Assumption 1.

We first recall the following important result.

Lemma 3. ([10] [29]) Consider the initial value problem for the Pantograph equation

$$\begin{cases}x'(t)=ax(t)+bx(\gamma_0 t),\ \forall t\ge 0,\ \gamma_0\in(0,1),\\ x(0)=x_0.\end{cases}\tag{45}$$

Let $a<0$, $b\in\mathbb{R}$, and x be the solution of (45). Then there exists $C=C(a,b,\gamma_0)>0$ such that

$$|x(t)|\le C|x(0)|(1+t)^{\mu'},\ t\ge 0,$$

where $\mu'\in\mathbb{R}$ satisfies $0=a+|b|\gamma_0^{\mu'}$.

Theorem 6. Suppose Condition 3 holds, $\mathbf{F}\equiv 0$, $\delta_1>\gamma^{-1}L_{\hat{\mathbf{G}}}$. Then for any $\boldsymbol{\Phi}\in\mathcal{C}_0\left(\hat{H}\right)$, the initial value problem (14)-(15) has a unique trivial stationary solution $\mathbf{v}^*=0$ and a unique weak solution $\mathbf{v}(t)$, and there exist $C=C\left(\delta_1,L_{\hat{\mathbf{G}}},\gamma_0\right)>0$, $\mu'<0$ such that

$$\|\mathbf{v}(t)\|^2<C\|\boldsymbol{\Phi}(0)\|^2(1+t)^{\mu'},\ \forall t\ge 0.$$

Proof. By $\mathbf{F}\equiv 0$, $\delta_1>\gamma^{-1}L_{\hat{\mathbf{G}}}$, Condition 3 and Theorem 2(a)-(b), the initial value problem (14)-(15) has a unique trivial stationary solution and a unique weak solution $\mathbf{v}(t)$. Taking the duality product of (14) with $\mathbf{v}(t)$ yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\|^2+\langle A\mathbf{v}(t),\mathbf{v}(t)\rangle+\langle B(u(t),\mathbf{v}(t)),\mathbf{v}(t)\rangle+\langle N(\mathbf{v}(t)),\mathbf{v}(t)\rangle\\&+\langle J(\mathbf{v}(t)),\mathbf{v}(t)\rangle=\left(\hat{\mathbf{G}}(\mathbf{v}(\gamma_0 t)),\mathbf{v}(t)\right),\end{aligned}\tag{46}$$

Using (8), (10), the nonnegativity of N, and Condition 3, together with the Cauchy inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\|^2+2\delta_1\|\mathbf{v}(t)\|_{\hat{V}}^2\le 2L_{\hat{\mathbf{G}}}\|\mathbf{v}(\gamma_0 t)\|\|\mathbf{v}(t)\|\le L_{\hat{\mathbf{G}}}\|\mathbf{v}(\gamma_0 t)\|^2+L_{\hat{\mathbf{G}}}\|\mathbf{v}(t)\|^2,$$

By (7),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(t)\|^2+2\gamma\delta_1\|\mathbf{v}(t)\|^2\le L_{\hat{\mathbf{G}}}\|\mathbf{v}(\gamma_0 t)\|^2+L_{\hat{\mathbf{G}}}\|\mathbf{v}(t)\|^2,$$

which simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{v}(t)\right\|^2 \le \left(-2\gamma\delta_1 + L_{\hat{G}}\right)\left\|\mathbf{v}(t)\right\|^2 + L_{\hat{G}}\left\|\mathbf{v}(\gamma_0 t)\right\|^2. \tag{47}$$

Since $\delta_1 > \gamma^{-1}L_{\hat{G}}$, we have $-2\gamma\delta_1 + L_{\hat{G}} < 0$. By Lemma 2 and (47), there exist $C = C\left(\delta_1, L_{\hat{G}}, \gamma_0\right) > 0$ and $\mu' \in \mathbb{R}$ such that

$$\left\|\mathbf{v}(t)\right\|^2 \le C\left\|\mathbf{\Phi}(0)\right\|^2 (1+t)^{\mu'},$$

where μ' satisfies

$$-2\delta_1\gamma + L_{\hat{G}} + L_{\hat{G}}\gamma_0^{\mu'} = 0,$$

Solving,

$$\mu' = \log_{\gamma_0}\left(\frac{2\delta_1\gamma}{L_{\hat{G}}} - 1\right). \tag{48}$$

Since $\delta_1\gamma > L_{\hat{G}}$, we have $\frac{2\delta_1\gamma}{L_{\hat{G}}} - 1 > 1$, and with $\gamma_0 \in (0,1)$, it follows that $\mu' < 0$. This completes the proof of Theorem 6. □

5. Conclusion and Outlook

At present, the proposed method is limited to two-dimensional bounded domains, and the admissible unbounded delay functions are required to possess certain specific structures. Future work will focus on extending the analysis to three-dimensional problems, considering more general boundary conditions (e.g., free boundaries), investigating stochastic perturbations, and establishing stability results for broader classes of delay functions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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