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Algebraic Chrono-Dynamics: Stratified Covariant Phase Space and Boundary Algebra

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Abstract

We provide a coordinate-free characterisation of phase boundaries in field theory on globally hyperbolic spacetimes with boundary. For a complex scalar field, we prove that a diffeomorphism-invariant local scalar functional $P[\Phi, g]$, partitioning spacetime into strata across a level set $\mathcal{H} = \{P = P_\star\}$, induces a stratified covariant phase space in the sense of Sjamaar-Lerman, in which the admissible variation class jumps discontinuously across $\mathcal{H}$. Concretely, on the dense stratum $\mathcal{M}_{\text{dense}} = \{P \geq P_\star\}$ a diverging phase-stiffness functional $\kappa(P)$ enforces, by a finite-action selection rule, the vanishing of phase variations $\delta\theta = 0$, restricting the tangent space to amplitude fluctuations alone. The principal result, which we call the Phase Boundary Characterisation Theorem, states that this single energetic condition produces two algebraically equivalent effects on the augmented covariant phase space: it enlarges the presymplectic kernel of the augmented form $\Omega_\Sigma^{\text{aug}}$ in the phase sector, and it suppresses the explicitly represented boundary 2-cocycle of the boundary charge algebra, $K_{\text{dense}} = 0$. The phase boundary $\mathcal{H}$ is identified intrinsically as the unique locus of this stiffness-induced phase-sector degeneracy, with no reference to the trigger functional once the construction is complete. Along the way, we exhibit a concrete mechanism by which the Iyer-Wald-Zoupas freedom in a phase-dependent boundary density is removed on the constrained stratum by explicit mixed boundary conditions; this is a model calculation within the IWZ setting, not a general resolution of the ambiguity. We show that the algebraic structure on each stratum is compatible with reduced phase-space quantization carried out independently on each stratum, and verify that the unstratified limit $P_\star \to \infty$ recovers the standard Lee-Wald/Iyer-Wald formalism identically. Throughout the paper the metric is treated as fixed Lorentzian background data, and the trigger functional $P[\Phi, g]$ is prescribed for purposes of the variational problem; dynamical metric variation and fully dynamical trigger functionals are identified as natural extensions rather than assumptions of the present theorem.

Keywords

Covariant Phase Space, Boundary Symplectic Algebra, Central Extension, Stratified Symplectic Spaces, Presymplectic Reduction, Edge Modes

1. Introduction

1.1. Setting and Motivation

Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth, oriented, globally hyperbolic Lorentzian four-manifold, possibly with smooth boundary $\partial\mathcal{M}$. The covariant phase-space formalism, originating in the work of Crnković-Witten [1] and Zuckerman [2], developed by Lee-Wald [3], and Iyer-Wald [4], and extended to spacetimes with boundary in the systematic treatment of Harlow-Wu [5] and further refined for fluctuating boundaries by Adami *et al.* [6], associates with any local Lagrangian field theory a presymplectic current ω^{μ} on the space of solutions, a presymplectic form $\Omega_{\Sigma} = \int_{\Sigma} \omega^{\mu} n_{\mu}$ on each Cauchy hypersurface Σ, and, after quotient by the kernel of Ω_{Σ}, a symplectic phase space $\mathcal{P}$. When $\partial\mathcal{M} = \varnothing$ this construction is hypersurface independent: $\Omega_{\Sigma_1} = \Omega_{\Sigma_2}$ for any two Cauchy slices, by virtue of the on-shell conservation $\nabla_{\mu}\omega^{\mu} = 0$. When $\partial\mathcal{M} \neq \varnothing$, hypersurface independence generically fails because of symplectic flux through the timelike portion of the boundary. A by-now standard remedy is to augment Ω_{Σ} by a boundary 2-form $\tilde{\Omega}_{\partial\Sigma}$, recovering a hypersurface-independent total form $\Omega_{\Sigma}^{\text{aug}}$ at the cost of introducing edge-mode degrees of freedom and, generically, a nontrivial central extension in the algebra of boundary charges [5] [7]-[11].

The present paper addresses a natural and physically widespread generalisation of this setup that, to the author's knowledge, has not been treated in a coordinate-free mathematical form for the scalar case. Suppose that the admissible variation class—the subspace of the tangent bundle to the solution space that is relevant to the canonical structure—changes discontinuously across a codimension-one hypersurface $\mathcal{H} \subset \mathcal{M}$ defined by a diffeomorphism-invariant scalar condition $P(x) = P_{\star}$. Such a discontinuity arises whenever an effective stiffness modulus governing one sector of tangent directions diverges across $\mathcal{H}$, so that any finite-energy variation in the region $\mathcal{M}_{\text{dense}} = \{P \geq P_{\star}\}$ is forced into the complementary sector. The result is what one ought to call a stratified covariant phase space: a single global solution space whose tangent structure—and hence whose presymplectic structure—has different ranks on the two sides of $\mathcal{H}$. The aim of this paper is to develop this framework rigorously and to characterise the phase boundary $\mathcal{H}$ algebraically.

1.2. Main Result

The principal theorem is most cleanly stated in advance of the technical setup; the rest of the paper is devoted to proving it.

Main Theorem (Stratified Covariant Phase Structure; Phase Boundary Characterisation). Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth, oriented, globally hyperbolic Lorentzian four-manifold with boundary, carrying a complex scalar field Φ satisfying Assumptions 1-3 of Section 2. Then:

(i) **Hypersurface independence on each stratum.** The augmented presymplectic form $\Omega_\Sigma^{\text{aug}}$ is hypersurface independent and descends, under presymplectic reduction, to a non-degenerate symplectic form on the reduced phase space $\mathcal{P}$ on each stratum.

(ii) **Centrally extended algebra on the regular stratum.** On $\mathcal{P}_{\text{reg}} = \{P < P_\star \text{ everywhere}\}$, the boundary charge algebra is centrally extended,

$$\{Q_\xi, Q_\eta\} = Q_{[\xi,\eta]} + K(\xi,\eta),$$

with K an explicitly represented boundary 2-cocycle on $\mathfrak{g}_\partial$.

(iii) **Cocycle suppression on the dense stratum.** On $\mathcal{P}_{\text{dense}}$ the cocycle vanishes identically, $K_{\text{dense}} \equiv 0$, and the reduced phase-space dimension strictly decreases relative to $\mathcal{P}_{\text{reg}}$.

(iv) **Algebraic characterisation of $\mathcal{H}$.** The phase boundary $\mathcal{H}$ is the unique locus at which the stiffness-induced phase-sector kernel of $\Omega_\Sigma^{\text{aug}}$ enlarges and the cocycle simultaneously vanishes; conditions (ii) and (iii) are activated together at $\mathcal{H}$ and absent away from it. (Other, unrelated degeneracies of $\Omega_\Sigma^{\text{aug}}$—zeros of Φ, topological sectors, boundary-condition-induced degeneracies—are not excluded by this statement; the uniqueness is uniqueness of the stiffness-induced phase-sector mechanism.)

(v) **Bulk dynamics unaffected.** Each of (i)-(iv) holds without any modification of the Euler-Lagrange equations: the transition is purely algebraic.

The novelty here is not the augmentation $\Omega \mapsto \Omega^{\text{aug}}$, which is by now classical, nor the appearance of a 2-cocycle K in the boundary algebra, which is generic. The novelty is the assertion (iv): that the geometric locus $\mathcal{H} = \{P = P_\star\}$ equals the locus on which two ostensibly unrelated algebraic features—a stiffness-induced phase-sector kernel jump and a phase-sector cocycle collapse—simultaneously occur. Once the construction is complete, $\mathcal{H}$ is recoverable from $\Omega_\Sigma^{\text{aug}}$ alone, with no reference to the trigger functional P that originally produced it. This is what allows us to call the result a characterisation: the phase boundary is an intrinsic feature of the augmented covariant phase space, not of any particular dynamical scalar that happens to be used to define it. (The qualification of (iv) is important: the augmented phase space generically supports several distinct sources of degeneracy—zeros of Φ, topological sectors, boundary-condition-induced degeneracies—and we do not claim $\mathcal{H}$ is the only locus where any kernel direction appears. The claim is that the specific combined transition described by (ii)-(iii) is supported at $\mathcal{H}$ and nowhere else.)

1.3. Strategy of Proof and Organisation

The proof proceeds in three stages, organised into the body of the paper as follows.

Stage 1: covariant phase space with boundary (Section 2-3). We collect the geometric setup, state the three minimal assumptions on which the entire paper rests, and develop the standard Lee-Wald / Iyer-Wald construction with explicit attention to the boundary. We exhibit the variational origin of the diverging phase-stiffness through a stiffness-coupled Lagrangian (Proposition 3.1), so that the finite-action selection rule of Assumption 3 arises from a concrete action principle rather than being postulated as an external kinematic rule. We prove the boundary flux obstruction to hypersurface independence, introduce the augmentation $\tilde{\Omega}_{\partial\Sigma}$, give an explicit formula for the boundary symplectic density α in polar variables, and exhibit a concrete pair of mixed boundary conditions under which the Iyer-Wald-Zoupas freedom in this α is removed (a model calculation, not a general resolution of the IWZ ambiguity). The presymplectic reduction $\mathcal{P} = \mathcal{S} / \ker \Omega_{\Sigma}^{\text{aug}}$ and the associated Poisson structure are constructed in this stage.

Stage 2: charges and the cocycle (Section 4). We construct integrable Hamiltonian generators of boundary symmetries, derive the boundary charge algebra under the Poisson bracket, and identify the central extension explicitly. The polar decomposition $\Phi = \sqrt{\hat{t}}\mathrm{e}^{i\theta}$ exposes the cocycle as a bilinear form on phase- and amplitude-sector variations; this representation is the key technical input to Stage 3.

Stage 3: stratification and algebraic transition (Sections 5-6). We make Assumption 3 concrete by introducing the diffeomorphism-invariant trigger functional P and the diverging phase-stiffness coefficient $\kappa(P)$. We show that the resulting finite-action condition forces $\delta\theta = 0$ on $\mathcal{M}_{\text{dense}}$, prove the kernel-enlargement theorem (Theorem 6.1) and the cocycle-suppression theorem (Theorem 6.2) using the explicit cocycle formula from Stage 2, and assemble these into the Phase Boundary Characterisation Theorem (Theorem 6.3). The paper concludes (Section 7) with the global structural theorem and a discussion of compatibility with reduced phase-space quantization.

The construction is entirely self-contained. Assumptions 1-3 are the complete logical input; no appeal to any specific physical programme is required, although in Section 5 we discuss canonical choices of P (e.g., curvature scalars) that make contact with strong-gravity and dense-matter regimes.

1.4. Relation to the Literature

The covariant phase-space methodology we use is that originating in Crnković-Witten [1] and Zuckerman [2], developed by Lee-Wald [3] and Iyer-Wald [4], and given a systematic treatment of boundary contributions by Harlow-Wu [5] and Adami et al. [6]. The boundary flux issue and its resolution by an edge-mode form $\tilde{\Omega}_{\partial\Sigma}$ have been developed extensively by Donnelly-Freidel [8], Speranza [9], Wald-Zoupas [11], and Barnich-Brandt [10], among many others; a pedagogical exposition of the closely related asymptotic-symmetry framework for soft charges in gravity and gauge theory is given by Strominger [12]. The role of surface integrals as charge generators in the Hamiltonian formulation goes back to Regge-Teitelboim [13], and the appearance of nontrivial central charges in canonical realisations of asymptotic symmetries was first established in the seminal three-dimensional analysis of Brown-Henneaux [7]; the present paper exhibits an analogous central extension in a different geometric setting (timelike-boundary scalar field theory), together with its suppression at a phase boundary. Stratified symplectic spaces in the sense relevant here were introduced by Sjamaar-Lerman [14], generalising the classical (smooth) Marsden-Weinstein reduction [15]; the broader framework of stratified spaces with smooth structures and singular symplectic reductions is developed systematically by Pflaum [16] and by Ortega-Ratiu [17]. We use exactly the Sjamaar-Lerman notion of a stratification by smooth manifold pieces, each carrying a compatible presymplectic form, and ordered by closure. What is new in the present work is the combination: a stratification of the covariant phase space induced by an energetic (rather than topological or symmetry-reductive) selection rule on tangent vectors, together with the simultaneous algebraic effects (kernel enlargement and cocycle suppression) that this selection rule produces.

Remark 1.1 (Scope of applicability). The coordinate-free characterisation of $\mathcal{H}$ proved in Theorem 6.3—as the unique locus at which the symplectic rank of $\Omega_{\Sigma}^{\text{aug}}$ drops and the central extension K simultaneously vanishes—is intrinsic to the covariant phase space and makes no reference to the trigger functional P once the construction is complete. The framework therefore applies wherever a diffeomorphism-invariant scalar partitions a field theory into strata with different finite-energy variation classes. Strong-gravity interiors (with P a curvature scalar), dense-matter phases (with P an energy-density scalar), and condensed-matter systems with sharp phase boundaries (with P an order-parameter functional) are concrete settings; the structural mechanism is independent of the particular illustrative choice of trigger functional, within the class of models satisfying the stratification and admissibility assumptions of Section 2.

2. Geometric Setup and Minimal Assumptions

This section assembles the geometric background and states the three minimal assumptions on which every subsequent result depends. The reader will note that no further structural input is invoked anywhere in the paper: every theorem in Sections 3 - 6 is a consequence of Assumptions 1-3 and the formal apparatus of the covariant phase-space formalism.

2.1. Spacetime, Fields, and Configuration Space

Status of the metric. Throughout this paper, the Lorentzian metric $g_{\mu\nu}$ is treated as fixed background data, not as a dynamical field. The action principle of Section 3 contains only the complex scalar Φ as a dynamical degree of freedom, and all variations hold $g_{\mu\nu}$ fixed ($\delta g_{\mu\nu} \equiv 0$). The phrase "diffeomorphism-invariant scalar functional $P[\Phi, g]$" (Definition 2.2) is read in the standard equivariance sense: $P[\varphi^*\Phi,\varphi^*g](x)=P[\Phi,g](\varphi(x))$ under simultaneous pull-back of Φ and g by background diffeomorphisms. The extension to a dynamical metric—including gravitational backreaction and the disformal-coupling regime—is a natural direction for future work; here we work in the fixed-background regime throughout.

Geometric assumptions. Let $(\mathcal{M},g_{\mu\nu})$ be a smooth, oriented, four-dimensional Lorentzian manifold with smooth timelike boundary $\partial\mathcal{M}$, globally hyperbolic in the sense appropriate to manifolds with timelike boundary [18]-[20]. The boundaryless case follows the classical formulation of Geroch [21], with smooth-Cauchy-hypersurface and metric-splitting refinements due to Bernal-Sánchez [22], both adapted in [19] [20] to the timelike-boundary setting used here. Specifically:

(G1) $\partial\mathcal{M}$ is a smooth embedded timelike submanifold of codimension one, with Lorentzian induced metric.

(G2) There exists a smooth Cauchy temporal function $\tau:\mathcal{M}\to\mathbb{R}$ whose level sets Σ_τ are smooth spacelike Cauchy hypersurfaces with boundary, meeting $\partial\mathcal{M}$ transversally in smooth $(n-2)$-submanifolds $\partial\Sigma_\tau=\Sigma_\tau\cap\partial\mathcal{M}$ [19].

(G3) The foliation $\{\Sigma_\tau\}$ is smooth with induced Riemannian metric h_{ij}, uniformly bounded on compact τ-intervals.

(G4) Cobordism regions $\mathcal{R}_{\tau_1,\tau_2}\subset\mathcal{M}$ bounded by two slices and the corresponding boundary portion $\mathcal{B}_{\tau_1,\tau_2}$ are piecewise smooth manifolds with corners; Stokes' theorem applies in the form $\partial\mathcal{R}=\Sigma_{\tau_2}-\Sigma_{\tau_1}+\mathcal{B}$, with corner contributions controlled by 2.

We fix one such foliation throughout, denote a generic Cauchy slice by Σ, with future-directed unit normal n^μ, induced Riemannian metric h_{ij}, Levi-Civita connection D_i, and corner $\partial\Sigma=\Sigma\cap\partial\mathcal{M}$. (G1)-(G4) ensure that all bulk and boundary integrals below are well defined and that the Stokes-theorem manipulations in Corollary 3.1 are valid. Traces of Φ and its first derivatives onto $\partial\mathcal{M}$ are well-defined elements of $H^k(\partial\mathcal{M})$ for $k\geq 2$ (Definition 2.1) by the trace theorem on Lipschitz domains.

The dynamical field is a complex scalar $\Phi:\mathcal{M}\to\mathbb{C}$. We work with the configuration space and admissible variations

Definition 2.1 (Configuration space and admissible variations). The configuration space is

$$\mathcal{C}=\left\{\Phi:\mathcal{M}\to\mathbb{C}\mid\Phi\in C^\infty(\mathcal{M}),\Phi|_{\partial\mathcal{M}}\in H^k(\partial\mathcal{M}),k\geq 2\right\}.$$

A tangent vector at $\Phi\in\mathcal{C}$—an admissible variation—is a smooth complex function

$$\delta\Phi\in C^\infty(\mathcal{M}),\ \delta\Phi|_{\partial\mathcal{M}}\in H^k(\partial\mathcal{M}),\ k\geq 2.$$

Throughout the paper we identify the tangent space $T_\Phi\mathcal{C}$ with the space of such variations.

The Sobolev regularity at the boundary ensures that every boundary integral encountered below is well defined: bulk integrals require only C^∞ regularity, while boundary integrals involving derivatives of the field—including the normal derivatives $\nabla_n \hat{t}$ and $\nabla_n \theta$ that appear in the boundary symplectic density—require traces in H^k for $k \ge 2$. The choice $k = 2$ suffices for the variational analysis below; the construction is unchanged for higher k.

It will be convenient throughout to use the polar decomposition of the field on the open set where it is non-vanishing,

$$\Phi(x) = \sqrt{\hat{t}(x)}\,\mathrm{e}^{i\theta(x)}, \tag{1}$$

with $\hat{t} = |\Phi|^2 \ge 0$ the squared modulus and θ the phase, defined modulo 2π. The polar decomposition is regular wherever $\Phi \ne 0$, and we restrict the analysis to that open set; zeros of Φ are excluded from the polar chart and are treated by the regularization procedure in **Appendix A**; on each positive-amplitude stratum the polar decomposition is used without ambiguity. Variations in this representation decompose as $\delta\Phi = \frac{1}{2}\hat{t}^{-1/2}\mathrm{e}^{i\theta}\delta\hat{t} + i\sqrt{\hat{t}}\,\mathrm{e}^{i\theta}\delta\theta$, separating cleanly into amplitude and phase sectors. This separation is essential for the analysis of the boundary cocycle in Section 4 and the suppression mechanism in Section 6.

2.2. The Three Minimal Assumptions

The entire construction rests on three assumptions, stated here precisely. The first identifies the geometric stratification, the second guarantees that the phase boundary is regular, and the third encodes the energetic selection rule that drives the algebraic transition.

Definition 2.2 (Trigger functional). A trigger functional is a scalar function $P[\Phi, g]: \mathcal{M} \to \mathbb{R}$ constructed locally from the metric $g_{\mu\nu}$, the field Φ, and a finite number of their covariant derivatives at each point. That is, there exists an integer $r \ge 0$ such that

$$P[\Phi, g](x) = F\Big(g_{\mu\nu}(x), \nabla_{\mu_1}\cdots\nabla_{\mu_j}\Phi(x), j \le r; \\ R_{\mu\nu\rho\sigma}(x) \text{ and its covariant derivatives up to order } r\Big)$$

for some smooth function F. In particular, $P(x)$ depends only on values at the single point x; it is not an integral or otherwise nonlocal functional. We further require P to be smooth wherever $(g_{\mu\nu}, \Phi)$ are smooth, and strictly diffeomorphism invariant: $P[\varphi^*\Phi, \varphi^* g](x) = P[\Phi, g](\varphi(x))$ for all diffeomorphisms $\varphi: \mathcal{M} \to \mathcal{M}$.

Assumption 1 (Phase Stratification). There exists a trigger functional P as in Definition 2.2 and a fixed threshold $P_\star > 0$ such that P partitions $\mathcal{M}$ into two open regions and a separating hypersurface,

$$\mathcal{M} = \mathcal{M}_{\text{reg}} \cup \mathcal{H} \cup \mathcal{M}_{\text{dense}}, \tag{2}$$

where $\mathcal{M}_{\text{reg}} = \{P < P_\star\}$ is the oscillatory (or regular) stratum, $\mathcal{M}_{\text{dense}} = \{P \ge P_\star\}$ is the suppressed (or dense) stratum, and $\mathcal{H} = \{P = P_\star\}$ is the phase boundary.

Assumption 2 (Smooth Phase Boundary). The gradient $\nabla_\mu P$ is non-vanishing on $\mathcal{H}$, so that $\mathcal{H}$ is a smooth codimension-one embedded submanifold of $\mathcal{M}$ by the regular level-set theorem.

Assumption 3 (Energetically Stratified Variation Class). There exists a smooth, positive function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$, called the phase-stiffness coefficient, satisfying $\kappa(P) \to \infty$ as $P \to P_\star^+$. A tangent vector $\delta\Phi \in T_\Phi \mathcal{C}$ is admissible on the dense stratum if and only if it satisfies the finite-action condition

$$\int_{\Sigma \cap \mathcal{M}_{\text{dense}}} \kappa(P(x)) \hat{t}(x) \left[h^{ij}(x) (D_i \delta\theta)(x) (D_j \delta\theta)(x) + |\delta\theta(x)|^2 \right] \sqrt{h(x)} \mathrm{d}^3 x < \infty, \quad (3)$$

for every Cauchy slice Σ meeting $\mathcal{M}_{\text{dense}}$, where h_{ij} is the induced spatial metric on Σ, h^{ij} its inverse, and D_i the associated Levi-Civita connection. The integrand uses the positive-definite spatial metric, not the indefinite spacetime metric $g^{\mu\nu}$, since the condition functions as an energy bound on a Cauchy slice. The zero-order term $|\delta\theta|^2$ is retained so that constant phase variations are also controlled by the dense-stratum finite-action condition. The constant global $U(1)$ phase direction is then handled by the phase-redundancy convention adopted in this paper (see the discussion following Definition 3.6). On $\mathcal{M}_{\text{reg}}$ no such restriction on $\delta\theta$ is imposed.

Lemma 2.1 (Finite-action phase suppression). Let $\Omega \subset \Sigma \cap \mathcal{M}_{\text{dense}}$ be open, and suppose $\hat{t} \geq t_0 > 0$ on Ω. Let $\kappa_n(P) > 0$ be a sequence of phase-stiffness coefficients satisfying $\kappa_n(P) \to \infty$ uniformly on compact subsets of Ω (representing the limiting regime $\kappa(P) \to \infty$ as $P \to P_\star^+$). If $\delta\theta_n \in H^1_{\text{loc}}(\Omega)$ satisfies the uniform finite-action bound

$$\sup_n \int_\Omega \kappa_n(P) \hat{t} \left[h^{ij} D_i \delta\theta_n D_j \delta\theta_n + |\delta\theta_n|^2 \right] \sqrt{h} \mathrm{d}^3 x < \infty,$$

then $\delta\theta_n \to 0$ in $L^2_{\text{loc}}(\Omega)$. Consequently, the limiting finite-action admissible tangent space satisfies

$$\delta\theta(x) = 0 \text{ for all } x \in \mathcal{M}_{\text{dense}}. \quad (4)$$

Proof. Let $K \Subset \Omega$ be compact. Since $\hat{t} \geq t_0 > 0$ on Ω and $\kappa_n(P) \to \infty$ uniformly on K, the quantity

$$m_n(K) = \inf_K \kappa_n(P) \hat{t}$$

satisfies $m_n(K) \to \infty$. By the assumed uniform bound, there is a constant $C > 0$ such that

$$\int_\Omega \kappa_n(P) \hat{t} \left[h^{ij} D_i \delta\theta_n D_j \delta\theta_n + |\delta\theta_n|^2 \right] \sqrt{h} \mathrm{d}^3 x \leq C$$

for all n. Restricting to K and dropping the non-negative gradient term gives

$$\int_K |\delta\theta_n|^2 \sqrt{h} \mathrm{d}^3 x \leq \frac{1}{m_n(K)} \int_K \kappa_n(P) \hat{t} |\delta\theta_n|^2 \sqrt{h} \mathrm{d}^3 x \leq \frac{C}{m_n(K)}.$$

Since $m_n(K) \to \infty$, the right-hand side tends to zero, so $\delta\theta_n \to 0$ in $L^2(K)$. Since $K \Subset \Omega$ was arbitrary, the convergence is local. The limiting admissible tangent space therefore satisfies $\delta\theta = 0$ on $\mathcal{M}_{\text{dense}}$.

Remark 2.1 (The selection rule is derived, not postulated). We emphasise that (4) is a theorem, not an assumption: it is the unique consequence of the finite-action condition (3) together with the divergence of κ. We have stated Assumption 3 in this form deliberately, to avoid the appearance that $\delta\theta = 0$ is imposed by fiat. The mathematical content is the diverging stiffness; the kinematic restriction follows. This distinction matters in Section 6, where the kernel enlargement and cocycle suppression are derived from (4) and would carry no force if (4) had merely been declared.

Remark 2.2 (Two compatible descriptions: finite-energy admissibility and presymplectic quotient). The condition $\delta\theta = 0$ characterises the finite-energy sector of the tangent space at $\Phi \in \mathcal{S}_{\text{dense}}$. There are two equivalent ways to use this fact in the symplectic analysis, and we will need both. (a) In the finite-energy admissibility description, the dense-stratum tangent space is restricted to those $\delta\Phi$ with $\delta\theta = 0$ on $\mathcal{M}_{\text{dense}}$; phase-only variations supported in $\mathcal{M}_{\text{dense}}$ are not finite-energy admissible and are excluded. (b) In the presymplectic quotient description, one starts with the unconstrained tangent bundle to $\mathcal{S}$, allows phase-only variations as formal tangent vectors, identifies them as null directions of $\Omega_\Sigma^{\text{aug}}$ on $\mathcal{S}_{\text{dense}}$, and quotients them out. The two descriptions agree on the reduced phase space $\mathcal{P}_{\text{dense}}$. Description (a) is the one we use when computing finite-energy actions; description (b) is the one used in Theorem 6.1 and the kernel-enlargement argument, since the statement that a variation lies in $\ker\Omega_\Sigma^{\text{aug}}$ is, by definition, a statement about its pairing under the unconstrained presymplectic form. We will switch between the two as convenient and indicate which is in use when the distinction matters.

2.3. The Stratified Phase Space, in the Sense of Sjamaar-Lerman

We use the term stratified phase space in the precise sense of Sjamaar-Lerman [14]: a decomposition of a presymplectic space into a finite collection of smooth manifold strata, each carrying a compatible presymplectic form, with the strata ordered by closure inclusion. The broader theory of stratified spaces with smooth structures, including their de Rham theory and the analytic machinery used in singular symplectic reductions, is developed by Pflaum [16]; the systematic treatment of momentum maps and Hamiltonian reduction on such spaces, including singular reductions in the finite-dimensional setting, is given by Ortega-Ratiu [17]. In the present work the stratification is two-piece,

$$\mathcal{P} = \mathcal{P}_{\text{reg}} \cup \mathcal{P}_{\text{dense}}, \tag{5}$$

where each stratum is a smooth (infinite-dimensional) manifold equipped with the restriction of the augmented presymplectic form $\Omega_\Sigma^{\text{aug}}$ constructed in Section 3. The corresponding admissible tangent spaces satisfy the strict inclusion

$$T\mathcal{P}_{\text{dense}} \subsetneq T\mathcal{P}_{\text{reg}}, \tag{6}$$

proved as a consequence of Lemma 2.1 together with the kernel-enlargement theorem (Theorem 6.1). No Whitney regularity beyond smoothness of each stratum is required; in particular, the stratification does not depend on any embedding into a larger smooth manifold. This is the precise sense in which the covariant phase space admits a two-stratum decomposition as a piecewise-smooth presymplectic space whose presymplectic structure degenerates along the constrained stratum.

3. Covariant Phase Space with Boundary

This section develops the standard Lee-Wald/Iyer-Wald covariant phase-space construction with explicit attention to the timelike boundary, leading to the augmented presymplectic form $\Omega_{\Sigma}^{\text{aug}}$ and its presymplectic reduction. Three results in this section deserve emphasis: the boundary flux obstruction (Proposition 3.2), the explicit construction of the boundary symplectic density (Proposition 3.3), and the resolution of the Iyer-Wald-Zoupas ambiguity by mixed boundary conditions (Proposition 3.4). The first two are well known in the literature; the third is the form in which we use them here. With these in hand, hypersurface independence is restored and the reduced phase space is a well-defined symplectic manifold on each stratum.

3.1. Action Principle and Presymplectic Potential

The dynamics is specified, in polar variables $\Phi = \sqrt{\hat{t}}\mathrm{e}^{i\theta} = \rho\mathrm{e}^{i\theta}$, by the stiffness-coupled Lagrangian

$$\mathcal{L}_{\kappa} = -\frac{1}{2}\nabla_{\mu}\rho\nabla^{\mu}\rho - \frac{1}{2}\kappa(P)\rho^2\nabla_{\mu}\theta\nabla^{\mu}\theta - V(\rho),\ V \in C^{\infty}\left(\mathbb{R}^{+}\right), \tag{7}$$

with $\kappa(P) \to \infty$ as $P \to P_{\star}^{+}$ as in Assumption 3, and with $P[\Phi, g]$ prescribed (Definition 2.2); the variation of P does not contribute to the field variation in this paper. The standard complex-scalar Lagrangian

$$\mathcal{L} = \nabla_{\mu}\Phi^{\dagger}\nabla^{\mu}\Phi - V\left(|\Phi|^2\right) \tag{8}$$

is the case $\kappa \equiv 1$ (up to the conventional overall factor of $\frac{1}{2}$, which we absorb into V). For finite κ the Euler-Lagrange equations from (7) remain smooth across $\mathcal{H}$; the divergence $\kappa \to \infty$ on $\mathcal{M}_{\text{dense}}$ is approached through a sequence $\kappa_n \to \kappa$. Variation of (7) gives, after integration by parts,

$$\delta S = \int_{\mathcal{M}} E_{\kappa}(\rho, \theta)\delta\Phi\sqrt{-g}\mathrm{d}^4x + \int_{\partial\mathcal{M}}\theta_{\kappa}, \tag{9}$$

with bulk Euler-Lagrange operator (in polar variables) coming from (7). The presymplectic potential current produced by varying the trigger-coupled phase-sector Lagrangian is

$$\theta_{\kappa}^{\mu}(\delta) = -\nabla^{\mu}\rho\delta\rho - \kappa(P)\rho^2\nabla^{\mu}\theta\delta\theta, \tag{10}$$

and the corresponding phase-sector presymplectic current is

$$\omega^{\mu}_{\theta,\kappa}\left(\delta_1,\delta_2\right) = -\delta_1\left(\kappa(P)\rho^2\nabla^{\mu}\theta\right)\delta_2\theta + \delta_2\left(\kappa(P)\rho^2\nabla^{\mu}\theta\right)\delta_1\theta. \tag{11}$$

On the regular stratum, after the normalisation $\kappa = 1$, expressions (10)-(11) reduce to the standard complex-scalar presymplectic potential,

$$\theta^{\mu} = \nabla^{\mu}\Phi^{\dagger}\delta\Phi + \nabla^{\mu}\Phi\delta\Phi^{\dagger}, \tag{12}$$

in non-polar variables, with bulk Euler-Lagrange operator

$$E(\Phi) = -\nabla_{\mu}\nabla^{\mu}\Phi + V'\left(|\Phi|^2\right)\Phi. \tag{13}$$

On the dense stratum, finite-action admissibility forces $\delta\theta = 0$ in the stiffness limit, so the divergent phase-sector contribution in (11) has zero pairing against admissible dense-stratum variations. The displayed standard form (12) should therefore be understood as the effective admissible-sector presymplectic potential after imposing the fixed-background finite-action condition: it is correct on $\mathcal{M}_{\text{reg}}$ exactly, and on $\mathcal{M}_{\text{dense}}$ up to terms that vanish on admissible variations. We use (12) throughout Sections 3-4 for the boundary-flux and augmentation analysis on this understanding.

Proposition 3.1 (Action-level origin of the finite-action selection rule). The finite-action condition (3) of Assumption 3 arises as the natural energetic restriction associated with (7) in the prescribed-P setting. The phase-sector contribution of (7) to the second variation on $\Sigma\cap\mathcal{M}_{\text{dense}}$, on a pure phase variation $\delta\Phi$ with $\delta\rho = 0$, is

$$\begin{aligned} S^{(2)}_{\kappa}[\delta\theta]\Big|_{\Sigma\cap\mathcal{M}_{\text{dense}}} &= \frac{1}{2}\int_{\Sigma\cap\mathcal{M}_{\text{dense}}}\kappa(P)\rho^2 h^{ij}D_i\delta\theta D_j\delta\theta\sqrt{h}\,\mathrm{d}^3x \\ &\quad + (\text{positive time-derivative and boundary terms}), \end{aligned} \tag{14}$$

the phase-sector kinetic energy in $\delta\theta$. As $\kappa\to\infty$ on $\mathcal{M}_{\text{dense}}$, finiteness of $S^{(2)}_{\kappa}$ forces $\delta\theta\to 0$ in L^2_{loc} on $\mathcal{M}_{\text{dense}}$, recovering the conclusion of Lemma 2.1. Equivalently, in the $\kappa\to\infty$ limit, the term $\frac{1}{2}\kappa(P)\rho^2\nabla\theta\cdot\nabla\theta$ acts as a Lagrange multiplier enforcing $\nabla\theta = 0$ on $\mathcal{M}_{\text{dense}}$; configurations with $\nabla\theta\neq 0$ there are removed from the finite-action space.

Proof. The Lagrangian (7) is bilinear in $\nabla\theta$ in its phase-sector contribution. The second variation at fixed ρ and prescribed P is $S^{(2)}_{\kappa}[\delta\theta] = -\frac{1}{2}\int_{\mathcal{M}}\kappa(P)\rho^2 g^{\mu\nu}\nabla_{\mu}\delta\theta\nabla_{\nu}\delta\theta\sqrt{-g}\,\mathrm{d}^4x$. Decomposing ∇_{μ} on a Cauchy slice and reducing to the slice volume form gives the spatial-gradient integrand $\kappa(P)\rho^2 h^{ij}D_i\delta\theta D_j\delta\theta$ in (14); time-derivative and boundary contributions are non-negative on a spacelike slice and do not weaken the divergence-driven suppression. The zero-order $|\delta\theta|^2$ term in (3) ensures that constant phase variations are also controlled on the dense stratum (cf. Assumption 3); the constant global $U(1)$ phase direction is handled by the phase-redundancy convention adopted in this paper (see the discussion following Definition 3.6).

Remark 3.1 (Prescribed vs.dynamical trigger). When $P[\Phi,g]$ is allowed to vary dynamically—so that δP contributes through $\delta\kappa(P)=\kappa'(P)\delta P$—additional terms enter the presymplectic potential and must be controlled by a subleading-correction hypothesis. The dynamical case, including a persistence theorem for phase suppression under admissible dynamical triggers, is a natural extension of the framework and is left for future work. For the prescribed case treated here, no such terms arise.

The solution space is

Definition 3.1 (Solution space).

$$\mathcal{S}=\{\Phi\in\mathcal{C}\mid E(\Phi)=0\}.$$

We will view $\mathcal{S}$ as a (formal) submanifold of $\mathcal{C}$, with tangent space at $\Phi\in\mathcal{S}$ given by the linearised solutions $\delta\Phi$ of $\delta E(\Phi)=0$.

3.2. Presymplectic Current and Form

The presymplectic current is the antisymmetrisation of θ^μ over two independent variations:

Definition 3.2 (Presymplectic current). For two tangent vectors $\delta_1,\delta_2\in T_\Phi\mathcal{S}$,

$$\omega^\mu(\delta_1,\delta_2)=\delta_1\theta^\mu(\delta_2)-\delta_2\theta^\mu(\delta_1).$$

Substituting (12) and using $\delta_1\delta_2=\delta_2\delta_1$ on configuration space, one obtains the explicit expression

$$\omega^\mu(\delta_1,\delta_2)=\delta_1\left(\nabla^\mu\Phi^\dagger\right)\delta_2\Phi-\delta_2\left(\nabla^\mu\Phi^\dagger\right)\delta_1\Phi+\text{c.c.} \tag{15}$$

The presymplectic form on Σ is then

Definition 3.3 (Presymplectic form).

$$\Omega_\Sigma(\delta_1,\delta_2)=\int_\Sigma\omega^\mu(\delta_1,\delta_2)n_\mu\sqrt{h}\,\mathrm{d}^3x.$$

Theorem 3.1 (On-shell conservation). For $\Phi\in\mathcal{S}$ and $\delta_1,\delta_2\in T_\Phi\mathcal{S}$, one has $\nabla_\mu\omega^\mu(\delta_1,\delta_2)=0$.

Proof. Take the divergence of (15). Using the Leibniz rule,

$$\begin{aligned}\nabla_\mu\omega^\mu=&\ \delta_1\left(\nabla_\mu\nabla^\mu\Phi^\dagger\right)\delta_2\Phi-\delta_2\left(\nabla_\mu\nabla^\mu\Phi^\dagger\right)\delta_1\Phi+\delta_1\left(\nabla^\mu\Phi^\dagger\right)\nabla_\mu\delta_2\Phi\\&-\delta_2\left(\nabla^\mu\Phi^\dagger\right)\nabla_\mu\delta_1\Phi+\text{c.c.}\end{aligned}$$

The four cross terms cancel pairwise after exchanging $1\leftrightarrow2$ in the antisymmetrised expression. The remaining terms are $\left(\delta_1\Box\Phi^\dagger\right)\delta_2\Phi-\left(\delta_2\Box\Phi^\dagger\right)\delta_1\Phi+\text{c.c.}$ Linearising $E(\Phi)=0$ gives $\delta\Box\Phi^\dagger=\delta\left(V'\Phi^\dagger\right)=V'\delta\Phi^\dagger+V''\left(\Phi\delta\Phi^\dagger+\Phi^\dagger\delta\Phi\right)\Phi^\dagger$, and substitution shows that the remaining terms are symmetric in $1\leftrightarrow2$, hence vanish under antisymmetrisation. Hence $\nabla_\mu\omega^\mu=0$ on shell.

Corollary 3.1 (Hypersurface dependence in the presence of boundary). Let Σ_1,Σ_2 bound a spacetime region $\mathcal{R}\subset\mathcal{M}$ with $\partial\mathcal{R}=\Sigma_2-\Sigma_1+\mathcal{B}$, where $\mathcal{B}=\mathcal{R}\cap\partial\mathcal{M}$ is the timelike boundary segment. Then by Theorem 3.1 and Stokes' theorem,

$$\Omega_{\Sigma_2}(\delta_1,\delta_2)-\Omega_{\Sigma_1}(\delta_1,\delta_2)=-\int_{\mathcal{B}}\omega^{\mu}(\delta_1,\delta_2)\tilde{n}_{\mu}\sqrt{|\tilde{h}|}\,\mathrm{d}^3x, \tag{16}$$

where $\tilde{n}^{\mu}$ is the outward normal to $\mathcal{B}$.

Proposition 3.2 (Boundary flux obstruction). If $\omega^{\mu}\tilde{n}_{\mu}\big|_{\mathcal{B}}\neq 0$ for some pair of tangent vectors, then Ω_{Σ} is not independent of the choice of Cauchy hypersurface.

Proof. Direct from (16): a non-vanishing right-hand side forces the left-hand side to differ.

This is the obstruction we now resolve.

3.3. Boundary Symplectic Augmentation

To restore hypersurface independence, we augment Ω_{Σ} by a 2-form supported on the boundary $\partial\Sigma=\Sigma\cap\partial\mathcal{M}$.

Definition 3.4 (Edge symplectic form). The edge symplectic form is

$$\tilde{\Omega}_{\partial\Sigma}(\delta_1,\delta_2)=\int_{\partial\Sigma}\alpha(\Phi;\delta_1,\delta_2),$$

where α is a bilinear, antisymmetric form on $T_{\Phi}\mathcal{S}\times T_{\Phi}\mathcal{S}$ taking values in 2-forms on $\partial\Sigma$, satisfying the matching condition

$$\delta\tilde{\Omega}_{\partial\Sigma}(\delta_1,\delta_2)=\int_{\partial\Sigma}\omega^{\mu}(\delta_1,\delta_2)\tilde{n}_{\mu}. \tag{17}$$

The condition (17) is exactly what is needed for $\tilde{\Omega}_{\partial\Sigma}$ to absorb the flux on the right-hand side of (16). The existence of such an α is not automatic; we exhibit one explicitly below. The construction proceeds in two stages, the first specifying a boundary polarisation in the polar variables $(\theta,\hat{t})$, and the second exhibiting the matching density.

Definition 3.5 (Polar-amplitude boundary conditions). At $\partial\Sigma$, with n the outward unit normal, the polar-amplitude boundary conditions are

$$\delta\left(\nabla_n\hat{t}\right)\big|_{\partial\Sigma}=0, \tag{18}$$

$$\delta\left(\hat{t}^{2}\nabla_n\theta\right)\big|_{\partial\Sigma}=0. \tag{19}$$

Condition (18) is a Neumann condition on the amplitude: the normal gradient of $\hat{t}$ is fixed at $\partial\Sigma$. Condition (19) is a mixed Dirichlet-Neumann condition on the phase, weighted by $\hat{t}^{2}$ to capture the conserved phase current. Tangent vectors $\delta\Phi$ in $T_{\Phi}\mathcal{S}$ are required to satisfy these conditions throughout the boundary analysis below.

The choice (18)-(19) is one polarisation among several admissible choices in the IWZ framework; we discuss its scope and uniqueness in Proposition 3.4 below.

Proposition 3.3 (Explicit boundary density). In the polar decomposition (1), and under the boundary conditions (18)-(19) of Definition 3.5, the boundary symplectic density

$$\alpha(\Phi;\delta_1,\delta_2)=\delta_1\theta\,\delta_2\hat{t}-\delta_2\theta\,\delta_1\hat{t} \tag{20}$$

satisfies the matching condition (17).

Proof. We compute the polar form of the presymplectic current and identify which terms reduce to $\delta\alpha$ on the boundary.

Step 1: polar form of θ^μ. Substituting $\Phi = \sqrt{\hat{t}}e^{i\theta}$ into the field gradient gives

$$\nabla^\mu \Phi = \frac{1}{2}\hat{t}^{-1/2} e^{i\theta} \nabla^\mu \hat{t} + i\sqrt{\hat{t}} e^{i\theta} \nabla^\mu \theta,$$

with the conjugate analogous. Substituting into the presymplectic potential current $\theta^\mu = \nabla^\mu \Phi^\dagger \delta\Phi + \nabla^\mu \Phi \delta\Phi^\dagger$ and using $e^{-i\theta}e^{i\theta} = 1$, the imaginary cross-terms cancel between the two summands, leaving

$$\theta^\mu(\delta) = \frac{1}{2}\hat{t}^{-1} \nabla^\mu \hat{t}\delta\hat{t} + 2\hat{t}\nabla^\mu \theta\delta\theta. \tag{21}$$

This is the manifestly real polar expression for θ^μ, decomposed cleanly into amplitude and phase sectors.

Step 2: polar form of ω^μ. The presymplectic current is $\omega^\mu(\delta_1, \delta_2) = \delta_1 \theta^\mu(\delta_2) - \delta_2 \theta^\mu(\delta_1)$. Acting with δ_1 on (21) and antisymmetrising in $1 \leftrightarrow 2$, the symmetric (diagonal) terms cancel and the surviving structure is

$$\begin{aligned}\omega^\mu(\delta_1, \delta_2) = &\frac{1}{2}\hat{t}^{-1}\left[\nabla^\mu(\delta_1\hat{t})\delta_2\hat{t} - \nabla^\mu(\delta_2\hat{t})\delta_1\hat{t}\right] \\ &+ 2\hat{t}\left[\nabla^\mu(\delta_1\theta)\delta_2\theta - \nabla^\mu(\delta_2\theta)\delta_1\theta\right] \\ &- 2\nabla^\mu\theta\left[\delta_1\theta\delta_2\hat{t} - \delta_2\theta\delta_1\hat{t}\right].\end{aligned} \tag{22}$$

The first group is a pure-amplitude bracket, the second a pure-phase bracket, and the third is the mixed $(\theta, \hat{t})$ bracket whose structure matches the proposed α in (20).

Step 3: contraction with the boundary normal. On $\partial\mathcal{M}$, contracting (22) with the outward normal $\tilde{n}_\mu$ gives, with $\nabla_n := \tilde{n}_\mu \nabla^\mu$,

$$\begin{aligned}\omega^\mu \tilde{n}_\mu\big|_{\partial\Sigma} = &\frac{1}{2}\hat{t}^{-1}\left[(\nabla_n \delta_1\hat{t})\delta_2\hat{t} - (\nabla_n \delta_2\hat{t})\delta_1\hat{t}\right] \\ &+ 2\hat{t}\left[(\nabla_n \delta_1\theta)\delta_2\theta - (\nabla_n \delta_2\theta)\delta_1\theta\right] \\ &- 2(\nabla_n\theta)\left[\delta_1\theta\delta_2\hat{t} - \delta_2\theta\delta_1\hat{t}\right].\end{aligned} \tag{23}$$

Step 4: identification with $\delta\alpha$. With α as in (20), the variation is

$$\delta\alpha(\delta_1, \delta_2) = \delta\left(\delta_1\theta\delta_2\hat{t} - \delta_2\theta\delta_1\hat{t}\right).$$

Within the polar polarisation—in which θ and $\hat{t}$ are treated as the independent boundary variables and the field-space variation δ commutes with the variations δ_1, δ_2 in the standard way—this evaluates on the configuration to a sum involving $\nabla_n\theta$ and $\nabla_n\hat{t}$ (the normal derivatives that appear because $\delta\alpha$ is computed against the bulk symplectic flux). Specifically, $\delta\alpha$ produces precisely the third bracket of (23), $-2(\nabla_n\theta)\left[\delta_1\theta\delta_2\hat{t} - \delta_2\theta\delta_1\hat{t}\right]$, after the boundary conditions (18)-(19) are imposed. This is the term that exhibits the bilinear $(\theta, \hat{t})$ structure essential to the cocycle analysis.

Step 5: cancellation of the remaining brackets under (18)-(19). The first bracket of (23) is the amplitude-sector contribution. Under the Neumann condition (18), $\delta\left(\nabla_n\hat{t}\right)\big|_{\partial\Sigma}=0$, so $\left(\nabla_n\delta_i\hat{t}\right)=\delta_i\left(\nabla_n\hat{t}\right)=0$ on $\partial\Sigma$, and the first bracket vanishes pointwise. The second bracket is the phase-sector contribution. Under the mixed condition (19), $\delta\left(\hat{t}^2\nabla_n\theta\right)\big|_{\partial\Sigma}=0$ implies $\hat{t}^2\delta_i\left(\nabla_n\theta\right)+2\hat{t}\nabla_n\theta\delta_i\hat{t}=0$ on $\partial\Sigma$. Substituting $\delta_i\left(\nabla_n\theta\right)=-2\hat{t}^{-1}\left(\nabla_n\theta\right)\delta_i\hat{t}$ into the second bracket yields

$$2\hat{t}\left[\left(-2\hat{t}^{-1}\nabla_n\theta\delta_1\hat{t}\right)\delta_2\theta-\left(-2\hat{t}^{-1}\nabla_n\theta\delta_2\hat{t}\right)\delta_1\theta\right]=-4\left(\nabla_n\theta\right)\left[\delta_1\hat{t}\delta_2\theta-\delta_2\hat{t}\delta_1\theta\right],$$

which is $+4\left(\nabla_n\theta\right)\left[\delta_1\theta\delta_2\hat{t}-\delta_2\theta\delta_1\hat{t}\right]$. Combining with the third bracket of (23) (which is $-2\left(\nabla_n\theta\right)\left[\cdots\right]$), the total mixed-sector contribution under the boundary conditions is $+2\left(\nabla_n\theta\right)\left[\delta_1\theta\delta_2\hat{t}-\delta_2\theta\delta_1\hat{t}\right]$. This is the surviving term, and up to an overall normalisation constant absorbed into the conventions of $\tilde{\Omega}_{\partial\Sigma}$, it is exactly $\delta\alpha$ as computed in Step 4. Hence

$$\int_{\partial\Sigma}\omega^{\mu}\tilde{n}_{\mu}=\delta\int_{\partial\Sigma}\alpha,$$

which is the matching condition (17).

The decomposition into θ - and $\hat{t}$ -sectors visible in (20) is the structural feature that drives the cocycle suppression argument in Section 6: the cocycle K inherits the same bilinear $\left(\theta,\hat{t}\right)$ -structure, and the dense-stratum condition $\delta\theta=0$ thereby kills it directly.

3.4. A Model Calculation in the Iyer-Wald-Zoupas Setting

The boundary 2-form $\tilde{\Omega}_{\partial\Sigma}$ is defined by the matching condition (17), which determines its variation, but does not by itself fix α uniquely: there is the well-known Iyer-Wald-Zoupas freedom [4] [11] to shift the presymplectic potential by an exact piece, $\theta\mapsto\theta+d\beta$ for any local $(n-2)$-form β on $\mathcal{M}$. We do not attempt a general resolution of this ambiguity. The polar-amplitude boundary conditions of Definition 3.5 remove the freedom within the polar polarisation, in the sense made precise in the following proposition.

Proposition 3.4 (Removal of IWZ freedom within the polar polarisation). Within the polar-amplitude polarisation in which $\left(\theta,\hat{t}\right)$ are the independent boundary variables, and under the boundary conditions (18)-(19) of Definition 3.5, the boundary symplectic density α defined by Proposition 3.3 is unique up to terms that contribute trivially to the matching (17). In particular, the IWZ freedom within this polarisation is removed.

Proof. The proof of Proposition 3.3 (Steps 3-5) showed that under (18)-(19), the boundary contraction $\omega^{\mu}\tilde{n}_{\mu}\big|_{\partial\Sigma}$ reduces, modulo terms that vanish by the boundary conditions, to a multiple of the mixed bracket $\delta_1\theta\delta_2\hat{t}-\delta_2\theta\delta_1\hat{t}$. Any IWZ shift of α that is linear in amplitude- and phase-normal-derivative variations is thereby projected onto a vanishing contribution. Boundary modifications outside this linear class—in particular, polarisations that take different combinations of $\left(\theta,\hat{t},\nabla_n\theta,\nabla_n\hat{t}\right)$ as the independent boundary variables—are admissible in the IWZ framework but lead to different boundary symplectic densities; we do not address them here.

Remark 3.2 (Scope of Proposition 3.4). We are not claiming a general resolution of the Iyer-Wald-Zoupas ambiguity. The IWZ freedom permits a wide class of boundary modifications, and the proposition addresses one class—shifts of α within the polar-amplitude polarisation that are linear in amplitude- and phase-normal-derivative variations. Within this class, the boundary conditions (18)-(19) fix α to the form (20). Other choices of boundary polarisation are consistent with the formalism but lead to different boundary symplectic densities, with corresponding modifications of the cocycle K. The choice (20) is the one that admits the cleanest cocycle representation (30) and is suppressed cleanly on the dense stratum by the finite-energy selection rule.

The augmented presymplectic form is

$$\Omega_\Sigma^{\text{aug}} = \Omega_\Sigma + \tilde{\Omega}_{\partial\Sigma}. \tag{24}$$

Theorem 3.2 (Hypersurface independence). Under (17), the augmented form $\Omega_\Sigma^{\text{aug}}$ is hypersurface independent: $\Omega_{\Sigma_2}^{\text{aug}} = \Omega_{\Sigma_1}^{\text{aug}}$ for any two Cauchy hypersurfaces Σ_1,Σ_2.

Proof. Combine (16) and (17). The deformation of Ω_Σ between Σ_1 and Σ_2 is exactly cancelled by the variation of $\tilde{\Omega}_{\partial\Sigma}$:

$$\Omega_{\Sigma_2}^{\text{aug}} - \Omega_{\Sigma_1}^{\text{aug}} = \left(\Omega_{\Sigma_2} - \Omega_{\Sigma_1}\right) + \left(\tilde{\Omega}_{\partial\Sigma_2} - \tilde{\Omega}_{\partial\Sigma_1}\right) = -\int_{\mathcal{B}} \omega^\mu \tilde{n}_\mu + \int_{\mathcal{B}} \delta\alpha = 0,$$

where the final equality uses the matching condition integrated over $\mathcal{B}$ (Stokes between $\partial\Sigma_1$ and $\partial\Sigma_2$).

The augmentation is therefore not optional but mathematically necessary whenever the boundary flux is non-vanishing.

3.5. Presymplectic Reduction and Poisson Structure

The presymplectic form $\Omega_\Sigma^{\text{aug}}$ is antisymmetric and closed but generally degenerate; passing to the symplectic phase space requires quotienting by its kernel.

Definition 3.6 (Kernel of $\Omega_\Sigma^{\text{aug}}$)

$$\ker \Omega_\Sigma^{\text{aug}} = \left\{\delta_\lambda \in T_\Phi\mathcal{S} \mid \Omega_\Sigma^{\text{aug}}\left(\delta_\lambda,\delta\right) = 0 \text{ for all admissible } \delta\right\}.$$

Phase-redundancy convention for the global $U(1)$ direction. In this paper the global $U(1)$ phase direction $\Phi \mapsto e^{i\lambda}\Phi$, with $\lambda \in \mathbb{R}$ constant, is treated under a phase-redundancy convention for purposes of constructing the reduced phase space associated with the stiffness-induced boundary transition. This convention means that constant phase directions are not retained as independent dense-stratum degrees of freedom when the finite-action regulator suppresses them. The convention is a choice of reduced phase-space description for the present structural theorem, not a claim that every global $U(1)$ symmetry in scalar field theory is physically gauge in all contexts. If global $U(1)$ is instead treated as a physical charge symmetry, the same finite-action regulator suppresses constant dense-stratum phase variations, while the charge interpretation must be adjusted accordingly; the structural conclusion of the theorem is unchanged.

Definition 3.7 (Reduced phase space). The reduced phase space is defined by quotienting the solution space by the genuine degeneracy distribution of the augmented presymplectic form,

$$\mathcal{P} = \mathcal{S}/\ker \Omega_\Sigma^{\text{aug}}.$$

Under the phase-redundancy convention adopted in this paper, the constant global phase direction is included in the quotient whenever it lies in the presymplectic kernel or is removed by the finite-action dense-stratum admissibility condition. Under an alternative reading in which global $U(1)$ is treated as a physical charge symmetry, the quotient should be understood as removing only genuine kernel directions, with the associated charge retained on the regular stratum. Either reading gives the same structural conclusion below: dense-stratum admissibility eliminates phase variations supported in $\mathcal{M}_{\text{dense}}$ and enlarges the presymplectic degeneracy sector accordingly.

Theorem 3.3 (Symplectic reduction). $\Omega_\Sigma^{\text{aug}}$ descends to a non-degenerate symplectic form on $\mathcal{P}$.

Proof. Antisymmetry and closure of $\Omega_\Sigma^{\text{aug}}$ are inherited from Ω_Σ and $\tilde{\Omega}_{\partial\Sigma}$. Non-degeneracy on the quotient is by construction: any direction along which $\Omega_\Sigma^{\text{aug}}$ pairs to zero is, by Definition 3.6, in the kernel and is identified to zero in the quotient. The result is a closed, non-degenerate, antisymmetric bilinear form, i.e., a symplectic form on $\mathcal{P}$.

Remark 3.3 (Recovery of standard covariant phase space). When $\mathcal{M}_{\text{dense}} = \varnothing$—e.g. when $P(x) < P_\star$ everywhere, so Assumption 3 imposes no restriction on $\delta\theta$—the construction of this section reduces identically to the standard Lee-Wald [3] and Iyer-Wald [4] covariant phase-space formalism on a spacetime with boundary. The augmented form $\Omega_\Sigma^{\text{aug}}$ coincides with the standard presymplectic form, and $\mathcal{P}_{\text{reg}}$ carries the full complement of phase-wave degrees of freedom together with a centrally extended boundary algebra. The unstratified theory is therefore the $P_\star \to \infty$ limit of the construction; every formula of this paper specialises correctly to the Lee-Wald / Iyer-Wald result in that limit. The stratified theory is a strict extension, not a replacement.

The Poisson structure on $\mathcal{P}$ is determined by $\Omega_\Sigma^{\text{aug}}$ through the standard prescription. For a smooth functional $F : \mathcal{P} \to \mathbb{R}$, the Hamiltonian vector field X_F is defined by

$$\iota_{X_F} \Omega_\Sigma^{\text{aug}} = \delta F, \tag{25}$$

existing and being unique by Theorem 3.3. The Poisson bracket of two functionals is

$$\{F, G\} = \Omega_\Sigma^{\text{aug}}\left(X_F, X_G\right). \tag{26}$$

Proposition 3.5 (Poisson algebra). The bracket (26) is bilinear, antisymmetric, satisfies the Leibniz rule, and obeys the Jacobi identity.

Proof. Bilinearity and antisymmetry are immediate from the corresponding properties of $\Omega_\Sigma^{\text{aug}}$. The Leibniz rule follows from the chain rule applied to $\delta(FG) = G\delta F + F\delta G$ in (25), giving $X_{FG} = GX_F + FX_G$ and hence $\{FG,H\} = G\{F,H\} + F\{G,H\}$. The Jacobi identity is equivalent to closure of $\Omega_\Sigma^{\text{aug}}$ via the standard identity

$$d\Omega_\Sigma^{\text{aug}}\left(X_F, X_G, X_H\right) = \{F,\{G,H\}\} + (\text{cyclic}),$$

and $d\Omega_\Sigma^{\text{aug}} = 0$ since $\Omega_\Sigma^{\text{aug}}$ is the variation of an action functional augmented by a closed boundary form.

In a coordinate chart on Σ adapted to a time foliation, with conjugate momentum $\pi = \partial_t \Phi^\dagger$, the canonical equal-time bracket takes the standard form

$$\{\Phi(\vec{x}), \pi(\vec{y})\} = \delta^{(3)}(\vec{x} - \vec{y}), \tag{27}$$

with all other brackets vanishing modulo boundary terms. These relations establish the full canonical structure on each stratum.

4. Boundary Symmetries, Charges, and the Central Extension

We now construct integrable Hamiltonian generators of boundary symmetries and derive the algebra they satisfy under the Poisson bracket (26). The central question is whether the map $\xi \mapsto Q_\xi$ from the Lie algebra $\mathfrak{g}_\partial$ of boundary symmetries to functionals on $\mathcal{P}$ is a Lie algebra homomorphism. The answer, in general, is no: the boundary augmentation introduces a 2-cocycle K that we will identify explicitly. The explicit form of K obtained here, together with its bilinear $(\theta, \hat{t})$-structure inherited from the boundary symplectic density (20), is the key technical input to the cocycle suppression argument of Section 6.

4.1. Boundary Symmetries and Hamiltonian Generators

Let $\mathfrak{g}_\partial$ denote a Lie algebra of infinitesimal transformations acting on boundary data on $\partial\mathcal{M}$. For $\xi \in \mathfrak{g}_\partial$, the induced variation of the field is denoted $\delta_\xi \Phi$; in the polar decomposition this decomposes as $\delta_\xi \Phi = \frac{1}{2}\hat{t}^{-1/2} e^{i\theta} \delta_\xi \hat{t} + i\sqrt{\hat{t}} e^{i\theta} \delta_\xi \theta$.

Definition 4.1 (Hamiltonian generator). A functional $Q_\xi : \mathcal{P} \to \mathbb{R}$ is a Hamiltonian generator of the boundary symmetry $\xi \in \mathfrak{g}_\partial$ if

$$\iota_{\delta_\xi} \Omega_\Sigma^{\text{aug}} = \delta Q_\xi. \tag{28}$$

The existence of such a Q_ξ is not automatic: the right-hand side of (28) must be an exact 1-form on $\mathcal{P}$, equivalently the linearisation δQ_ξ must satisfy the integrability condition $\delta(\delta Q_\xi) = 0$.

Definition 4.2 (Integrability). The expression δQ_ξ is integrable on $\mathcal{P}$ if $\delta(\delta Q_\xi) = 0$ as a 2-form on $\mathcal{P}$.

Proposition 4.1 (Integrability of boundary charges). If $\Omega_\Sigma^{\text{aug}}$ is closed and δ_ξ

preserves the boundary conditions (18)-(19), then δQ_ξ is integrable, and hence Q_ξ exists.

Proof. Computing $\delta\left(\delta Q_\xi\right)$ from (28) via Cartan's magic formula,

$$\delta\left(\delta Q_\xi\right)=\delta\left(\iota_{\delta_\xi}\Omega_\Sigma^{\text{aug}}\right)=\mathcal{L}_{\delta_\xi}\Omega_\Sigma^{\text{aug}}-\iota_{\delta_\xi}d\Omega_\Sigma^{\text{aug}}.$$

The second term vanishes because $\Omega_\Sigma^{\text{aug}}$ is closed (Theorem 3.3). The first term is the Lie derivative of the augmented form along the symmetry direction; if δ_ξ preserves the boundary polarisation (18)-(19), then $\mathcal{L}_{\delta_\xi}\Omega_\Sigma^{\text{aug}}=0$. Hence $\delta\left(\delta Q_\xi\right)=0$ and Q_ξ exists by the Poincaré lemma applied to the (formally infinite-dimensional) phase space.

4.2. The Boundary Algebra and the Central Extension

The boundary symmetries close under the bracket of vector fields on $\mathcal{C}$: $\left[\delta_\xi,\delta_\eta\right]=\delta_{[\xi,\eta]}$. The corresponding algebra of charges, however, generally fails to close exactly—it closes only up to a 2-cocycle.

Proposition 4.2 (Boundary algebra with central extension). Suppose $\left[\delta_\xi,\delta_\eta\right]=\delta_{[\xi,\eta]}$. Then under the Poisson bracket (26),

$$\left\{Q_\xi,Q_\eta\right\}=Q_{[\xi,\eta]}+K\left(\xi,\eta\right), \tag{29}$$

where $K\left(\xi,\eta\right)$ is a bilinear, antisymmetric expression depending only on the boundary data and not on the bulk values of the field.

Proof. By (26) and (28),

$$\left\{Q_\xi,Q_\eta\right\}=\Omega_\Sigma^{\text{aug}}\left(\delta_\xi,\delta_\eta\right)=\delta_\xi Q_\eta=-\delta_\eta Q_\xi,$$

where the second equality uses the definition of Q_η as Hamiltonian for δ_η. On the other hand, $Q_{[\xi,\eta]}$ is the Hamiltonian for $\delta_{[\xi,\eta]}=\left[\delta_\xi,\delta_\eta\right]$, so

$$\delta Q_{[\xi,\eta]}=\iota_{\left[\delta_\xi,\delta_\eta\right]}\Omega_\Sigma^{\text{aug}}=\mathcal{L}_{\delta_\xi}\iota_{\delta_\eta}\Omega_\Sigma^{\text{aug}}-\iota_{\delta_\eta}\mathcal{L}_{\delta_\xi}\Omega_\Sigma^{\text{aug}}=\delta\left(\delta_\xi Q_\eta\right)-0,$$

using closure of $\Omega_\Sigma^{\text{aug}}$ and preservation of boundary conditions. Hence $\delta_\xi Q_\eta-Q_{[\xi,\eta]}$ is a δ-closed quantity on $\mathcal{P}$, *i.e.* a constant on each connected component, depending only on ξ and η and not on the field. We denote this constant by $K\left(\xi,\eta\right)$, giving (29). Bilinearity follows from linearity of the construction in ξ and η; antisymmetry follows from antisymmetry of $\Omega_\Sigma^{\text{aug}}$. The constant depends only on boundary data because the bulk part of $\Omega_\Sigma^{\text{aug}}$ generates the closed bracket exactly; the offset arises entirely from $\tilde{\Omega}_{\partial\Sigma}$.

The term $K\left(\xi,\eta\right)$ is the central extension of the boundary charge algebra: it is the failure of the map $\xi\mapsto Q_\xi$ to be a Lie algebra homomorphism, and it arises because the boundary symplectic density $\tilde{\Omega}_{\partial\Sigma}$ contributes a piece that the bulk equations cannot absorb.

Theorem 4.1 (Explicit form of the cocycle). With the boundary symplectic density (20) of Proposition 3.3,

$$K(\xi,\eta)=\int_{\partial\Sigma}\left(\delta_{\xi}\theta\delta_{\eta}\hat{t}-\delta_{\eta}\theta\delta_{\xi}\hat{t}\right). \tag{30}$$

Proof. By Proposition 4.2, $K(\xi,\eta)$ is the boundary contribution to $\{Q_{\xi},Q_{\eta}\}-Q_{[\xi,\eta]}$. Substituting the explicit boundary symplectic density (20) into $\tilde{\Omega}_{\partial\Sigma}(\delta_{\xi},\delta_{\eta})$ gives

$$\tilde{\Omega}_{\partial\Sigma}(\delta_{\xi},\delta_{\eta})=\int_{\partial\Sigma}\left(\delta_{\xi}\theta\delta_{\eta}\hat{t}-\delta_{\eta}\theta\delta_{\xi}\hat{t}\right),$$

and tracking the construction of Proposition 4.2 shows that this is exactly the central-extension contribution. The bulk part of $\Omega_{\Sigma}^{\text{aug}}$ produces $Q_{[\xi,\eta]}$ exactly via the standard Hamiltonian-action computation, leaving (30) as the residual.

The representation (30) exposes two structural features that we will use heavily in Section 6: K is bilinear in the phase-sector variation $\delta\theta$ and the amplitude-sector variation $\delta\hat{t}$, and it is supported entirely on $\partial\Sigma$. Both features will be essential for the suppression argument.

Definition 4.3 (2-cocycle). A bilinear antisymmetric form $K:\mathfrak{g}_{\partial}\times\mathfrak{g}_{\partial}\to\mathbb{R}$ is a 2-cocycle on $\mathfrak{g}_{\partial}$ if

$$K([\xi,\eta],\zeta)+K([\eta,\zeta],\xi)+K([\zeta,\xi],\eta)=0$$

for all $\xi,\eta,\zeta\in\mathfrak{g}_{\partial}$.

Theorem 4.2 (Cocycle condition). K as in (30) is a 2-cocycle on $\mathfrak{g}_{\partial}$.

Proof. Apply the Jacobi identity for the Poisson bracket of three charges $Q_{\xi},Q_{\eta},Q_{\zeta}$. Cyclic permutation gives, after using (29),

$$\{Q_{\xi},\{Q_{\eta},Q_{\zeta}\}\}+\text{cyclic}=\{Q_{\xi},Q_{[\eta,\zeta]}+K(\eta,\zeta)\}+\text{cyclic}.$$

The Jacobi identity for the Poisson bracket (Proposition 3.5) makes the left-hand side vanish. The right-hand side, using (29) once more and the Jacobi identity in $\mathfrak{g}_{\partial}$ (which gives $[[\xi,\eta],\zeta]+\text{cyclic}=0$), reduces to $K([\xi,\eta],\zeta)+\text{cyclic}$. Equating to zero gives the cocycle condition.

The boundary algebra is therefore represented up to a central extension defined by a 2-cocycle, in the standard sense of Lie algebra cohomology.

Remark 4.1 (On the cohomological status of K). We do not claim here that K represents a non-trivial cohomology class in $H^2(\mathfrak{g}_{\partial},\mathbb{R})$. Establishing non-triviality would require fixing the symmetry algebra $\mathfrak{g}_{\partial}$ explicitly and showing that K cannot be written as a coboundary $K(\xi,\eta)=f([\xi,\eta])$ for any linear functional f on $\mathfrak{g}_{\partial}$. The structural results of this paper require only the weaker and explicit fact that the displayed boundary cocycle (30) is bilinear in phase- and amplitude-sector variations, and therefore vanishes on the dense stratum once admissible phase variations are suppressed. Whether K also represents a non-trivial cohomology class—a question of independent interest, depending on the choice of $\mathfrak{g}_{\partial}$—is not needed for any conclusion below.

4.3. Regulated Boundary Structure

For technical control of ultraviolet or singular behaviour at $\partial\Sigma$—e.g. if the boundary admits asymptotic limits or sharp corners—one may regulate the boundary symplectic density by a smooth, bounded multiplier $R\left(\chi;\epsilon,\Xi_1,\Xi_2\right)$, with χ a coordinate along $\partial\Sigma$ and ϵ,Ξ_1,Ξ_2 regulator parameters. The regulated edge form is

$$\tilde{\Omega}_{\partial\Sigma}^{(R)}=\int_{\partial\Sigma}R\left(\chi;\epsilon,\Xi_1,\Xi_2\right)\alpha\left(\Phi;\delta_1,\delta_2\right), \tag{31}$$

and the regulated augmented form is $\Omega_\Sigma^{\text{aug},(R)}=\Omega_\Sigma+\tilde{\Omega}_{\partial\Sigma}^{(R)}$. Multiplication of the integrand by a smooth bounded function preserves bilinearity, antisymmetry, and closure, so $\Omega_\Sigma^{\text{aug},(R)}$ remains closed and hypersurface independent for any fixed regulator. The regulated cocycle K_R depends continuously on the regulator parameters and reduces to K in the limit $R\to 1$. We use the regulated form only as a technical device: all algebraic statements in Section 6 are stated for the unregulated form and are stable under the regulator limit.

5. The Stratification Trigger and the Dense-Time Variation Class

We now make the abstract trigger functional P of Assumption 1 concrete by exhibiting natural physical realisations, and we develop the consequences of the dense-stratum admissibility condition derived in Lemma 2.1. The aim of this section is twofold: to show that the assumptions of Section 2 are not vacuous (they admit explicit examples), and to translate the abstract finite-action selection rule into a concrete restriction on the presymplectic structure that can be used in Section 6.

5.1. Canonical Choices of the Trigger Functional

The trigger functional $P[\Phi,g]$ of Definition 2.2 can take any of a wide range of forms; the structural mechanism is independent of the particular illustrative choice, within the class of models satisfying Definition 2.2 and Assumptions 2-3. Three natural choices arise in physical contexts.

Curvature scalar (strong-gravity setting). The Kretschmann scalar

$$P=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

is a natural choice in strong-gravity environments. It is a scalar curvature invariant—not a complete invariant of the Riemann tensor (the full set of independent algebraic curvature invariants in four dimensions is larger; see e.g. the Carminati-McLenaghan invariants), but it does provide a coordinate-free measure of curvature magnitude that is non-vanishing wherever spacetime curvature is dynamically significant and is well-defined even on non-stationary backgrounds. The threshold $P_\star$ then selects regions whose curvature scale exceeds a critical value, giving the dense stratum the precise geometric meaning of a strong-curvature interior. The use of the Kretschmann scalar here is illustrative; the formal construction requires only a scalar trigger P whose threshold set is a regular level surface, and other curvature invariants may be substituted.

Energy-momentum scalar (dense-matter setting). The square of the stress-energy tensor,

$$P = T_{\mu\nu} T^{\mu\nu},$$

is a natural choice when the stratification is driven by matter density rather than spacetime geometry. It picks out regions of high energy density and, with a suitable threshold, models phase transitions in dense matter (e.g. between hadronic and quark phases in compact-star interiors).

Order-parameter scalar (condensed-matter setting). When Φ has the interpretation of an order parameter of a phase transition, P may be taken as a local function of Φ and its derivatives that distinguishes the phases—for instance $P = |\Phi|^2$ or a more elaborate Landau-type scalar. The threshold $P_\star$ then realises the phase boundary as a level set of the order parameter.

All subsequent results hold for any P satisfying Definition 2.2 together with Assumptions 2 and 3. The Kretschmann scalar is the canonical illustration in the gravitational setting; in what follows, the reader may safely substitute any choice from the above list (or any other admissible one) without altering a single proof.

5.2. Stratification of Solution Space and Admissibility

Recall from (2) the partition $\mathcal{M} = \mathcal{M}_{\text{reg}} \cup \mathcal{H} \cup \mathcal{M}_{\text{dense}}$ induced by P. The solution space inherits a corresponding stratification.

Definition 5.1 (Stratified solution space).

$$\mathcal{S}_{\text{reg}} = \{\Phi \in \mathcal{S} \mid P(x) < P_\star \text{ for all } x \in \mathcal{M}\}, \ \mathcal{S}_{\text{dense}} = \{\Phi \in \mathcal{S} \mid \mathcal{M}_{\text{dense}} \neq \varnothing\},$$

giving $\mathcal{S} = \mathcal{S}_{\text{reg}} \cup \mathcal{S}_{\text{dense}}$.

The Euler-Lagrange equations (13) are unmodified across this partition: $\mathcal{S}$ is the joint zero locus of $E(\Phi) = 0$ and $E(\Phi^\dagger) = 0$ globally on $\mathcal{M}$, irrespective of which stratum the field configuration falls into. The stratification affects only the tangent structure of $\mathcal{S}$—which directions $\delta\Phi$ are admissible at which configurations—not the configurations themselves.

The relevant restriction on tangent directions is exactly that of Assumption 3, made concrete in Lemma 2.1. We re-state it here in the form most useful for the symplectic analysis.

Definition 5.2 (Dense-time admissibility). A tangent vector $\delta\Phi \in T_\Phi \mathcal{S}$ at a configuration $\Phi \in \mathcal{S}_{\text{dense}}$ is dense-time admissible if, in the polar decomposition (1),

$$\delta\theta(x) = 0 \text{ for all } x \in \mathcal{M}_{\text{dense}}. \tag{32}$$

On the regular stratum, no such restriction is imposed.

By Lemma 2.1, condition (32) is the unique consequence of the finite-action condition (3); we have not added any new content. Definition 5.2 is simply the form of the selection rule that makes its impact on the presymplectic structure most transparent.

Remark 5.1 (Interpretation: selection rule, not axiom). We re-emphasise the point of Remark 2.1: the dense-time admissibility condition (32) is a derived consequence of the diverging stiffness $\kappa(P) \to \infty$, not a freely-imposed axiom. The mathematical content lies in Assumption 3 and Lemma 2.1. The Euler-Lagrange equations are not modified; only the finite-energy sector of the tangent space at each $\Phi \in \mathcal{S}_{\text{dense}}$ is restricted, and that restriction is forced by the action functional itself. The phase boundary $\mathcal{H}$ is therefore not a singular dynamical surface but a locus where the finite-energy variation class changes rank.

5.3. Collapse of the Phase-Sector Contribution

Recall from (15), after substituting the polar decomposition, that the presymplectic current contains an amplitude-sector contribution proportional to $\delta\hat{t}$ and a phase-sector contribution proportional to $\delta\theta$. Inside $\mathcal{M}_{\text{dense}}$, the dense-time admissibility condition kills the phase-sector contribution.

Proposition 5.1 (Collapse of the phase sector). For dense-time admissible tangent vectors δ_1, δ_2 at a configuration $\Phi \in \mathcal{S}_{\text{dense}}$, the phase-sector contribution to the presymplectic current $\omega^\mu(\delta_1, \delta_2)$ vanishes pointwise on $\mathcal{M}_{\text{dense}}$, and hence

$$\left(\Omega_\Sigma^{\text{aug}}\right)_{\text{phase}}\Big|_{\Sigma \cap \mathcal{M}_{\text{dense}}} = 0.$$

Proof. The phase-sector contribution to ω^μ is bilinear in the phase variations $\delta_i\theta$. By Definition 5.2, both $\delta_1\theta$ and $\delta_2\theta$ vanish identically on $\mathcal{M}_{\text{dense}}$. Direct substitution gives zero pointwise; the integral over $\Sigma \cap \mathcal{M}_{\text{dense}}$ vanishes correspondingly.

This collapse is the geometric mechanism behind both the kernel enlargement and the cocycle suppression of Section 6.

6. Algebraic Transition at the Phase Boundary

This section assembles the principal results of the paper. The kernel of $\Omega_\Sigma^{\text{aug}}$ enlarges on the dense stratum (Theorem 6.1); the boundary cocycle K vanishes there (Theorem 6.2); and these two effects are different manifestations of the same geometric fact, encoded in the Phase Boundary Characterisation Theorem (Theorem 6.3). The proofs use only the explicit cocycle formula (30), the dense-time admissibility condition (32), and the boundary symplectic density (20); no further input is required.

6.1. Kernel Enlargement

Consider an infinitesimal phase-only variation supported in the interior of $\Sigma \cap \mathcal{M}_{\text{dense}}$:

$$\delta_{\text{phase}}\Phi = i\lambda(x)\Phi, \ \operatorname{supp}(\lambda) \Subset \operatorname{int}(\Sigma \cap \mathcal{M}_{\text{dense}}), \tag{33}$$

where $\lambda : \mathcal{M} \to \mathbb{R}$ is smooth and compactly supported in the open interior of $\Sigma \cap \mathcal{M}_{\text{dense}}$. In particular, λ vanishes on $\partial\Sigma \cap \mathcal{M}_{\text{dense}}$ and on a neighbourhood of $\mathcal{H}$. In polar variables, $\delta_{\text{phase}}\hat{t}=0$ and $\delta_{\text{phase}}\theta=\lambda$. On the regular stratum, such a variation generally pairs non-trivially with other admissible variations and is not in the kernel of $\Omega_\Sigma^{\text{aug}}$. On the dense stratum, the situation is different.

Theorem 6.1 (Kernel enlargement on the dense stratum). For any configuration $\Phi\in\mathcal{S}_{\text{dense}}$ and any δ_{phase} as in (33),

$$\delta_{\text{phase}}\in\ker\Omega_\Sigma^{\text{aug}}\big|_{\mathcal{S}_{\text{dense}}}.$$

Proof. The statement is about the kernel of $\Omega_\Sigma^{\text{aug}}$ acting on the unconstrained tangent bundle to $\mathcal{S}$, in description (b) of Remark 2.2: δ_{phase} is a formal pre-admissibility tangent direction, not a finite-energy admissible tangent vector at $\Phi\in\mathcal{S}_{\text{dense}}$. The claim is that the pairing of δ_{phase} against any finite-energy admissible variation δ_2 vanishes; this is the defining condition for $\delta_{\text{phase}}\in\ker\Omega_\Sigma^{\text{aug}}$. The two descriptions agree on the reduced phase space, with δ_{phase} identified to zero in either.

Let δ_2 be any finite-energy admissible variation. We must show $\Omega_\Sigma^{\text{aug}}\left(\delta_{\text{phase}},\delta_2\right)=0$. Decompose

$$\Omega_\Sigma^{\text{aug}}\left(\delta_{\text{phase}},\delta_2\right)=\Omega_\Sigma\left(\delta_{\text{phase}},\delta_2\right)+\tilde{\Omega}_{\partial\Sigma}\left(\delta_{\text{phase}},\delta_2\right).$$

Bulk term. From the polar form (22) of the presymplectic current with $\delta_1=\delta_{\text{phase}}$ (where $\delta_1\hat{t}=0$ and $\delta_1\theta=\lambda$), the surviving contributions on Σ are bilinear in $(\lambda,\delta_2\theta)$ in the phase-sector and mixed brackets of (22). By Definition 5.2 and Lemma 2.1, $\delta_2\theta=0$ on $\mathcal{M}_{\text{dense}}$, so $\delta_2\theta$ and $\nabla\delta_2\theta$ vanish there pointwise; the integrand vanishes on $\Sigma\cap\mathcal{M}_{\text{dense}}$. Outside $\mathcal{M}_{\text{dense}}$, the support condition $\operatorname{supp}(\lambda)\Subset\operatorname{int}(\Sigma\cap\mathcal{M}_{\text{dense}})$ kills λ and its gradient; the integrand vanishes there too. Hence $\Omega_\Sigma\left(\delta_{\text{phase}},\delta_2\right)=0$.

Boundary term. From the explicit boundary symplectic density (20),

$$\alpha\left(\delta_{\text{phase}},\delta_2\right)=\delta_{\text{phase}}\theta\,\delta_2\hat{t}-\delta_2\theta\,\delta_{\text{phase}}\hat{t}=\lambda\delta_2\hat{t}-0=\lambda\delta_2\hat{t}.$$

The compact-support condition on λ implies $\lambda|_{\partial\Sigma}=0$, so $\alpha\left(\delta_{\text{phase}},\delta_2\right)$ vanishes pointwise on $\partial\Sigma$. Hence $\tilde{\Omega}_{\partial\Sigma}\left(\delta_{\text{phase}},\delta_2\right)=0$.

Combining, $\Omega_\Sigma^{\text{aug}}\left(\delta_{\text{phase}},\delta_2\right)=0$ for every admissible δ_2, establishing the claim.

Corollary 6.1 (Strict enlargement of the kernel).

$$\ker\Omega_\Sigma^{\text{aug}}\big|_{\mathcal{S}_{\text{dense}}}\supsetneq\ker\Omega_\Sigma^{\text{aug}}\big|_{\mathcal{S}_{\text{reg}}}.$$

Proof. On $\mathcal{S}_{\text{reg}}$, the kernel of $\Omega_\Sigma^{\text{aug}}$ in the phase sector is generically narrow: the constant-phase direction is handled by the phase-redundancy convention (Section 3.1, discussion following Definition 3.6), and any spatially-varying phase mode pairs non-trivially against amplitude modes through the boundary symplectic density. On $\mathcal{S}_{\text{dense}}$, by contrast, dense-time admissibility forces $\delta\theta=0$ on

$\mathcal{M}_{\text{dense}}$ for every finite-action variation, including both spatially-varying modes (suppressed by the gradient term of (3)) and the constant-phase mode (suppressed by the zero-order term). The phase-only variations of (33) therefore lie in the kernel on $\mathcal{S}_{\text{dense}}$ by Theorem 6.1 but not on $\mathcal{S}_{\text{reg}}$. Hence the dense-stratum kernel strictly contains the regular-stratum kernel: specifically, it contains all $\lambda \in C_c^\infty\left(\text{int}\left(\Sigma \cap \mathcal{M}_{\text{dense}}\right)\right)$, including the locally-constant mode, none of which is in $\ker \Omega_\Sigma^{\text{aug}}\big|_{\mathcal{S}_{\text{reg}}}$.

The reduced phase space on the dense stratum is

$$\mathcal{P}_{\text{dense}} = \mathcal{S}_{\text{dense}} / \ker \Omega_\Sigma^{\text{aug}}\big|_{\mathcal{S}_{\text{dense}}}. \tag{34}$$

Proposition 6.1 (Dimension drop). $\dim \mathcal{P}_{\text{dense}} < \dim \mathcal{P}_{\text{reg}}$.

Proof. The quotient (34) divides $\mathcal{S}_{\text{dense}}$ by a strictly larger kernel than the quotient $\mathcal{P}_{\text{reg}} = \mathcal{S}_{\text{reg}} / \ker \Omega_\Sigma^{\text{aug}}\big|_{\mathcal{S}_{\text{reg}}}$ (Corollary 6.1). Strictly more null directions are quotiented out; the dimension of the quotient strictly decreases. Concretely, the phase-wave degrees of freedom $\lambda(x)$ supported in $\mathcal{M}_{\text{dense}}$ are present in $\mathcal{S}_{\text{reg}} / \ker$ but absent from $\mathcal{S}_{\text{dense}} / \ker$.

6.2. Cocycle Suppression

Recall the explicit cocycle from Theorem 4.1:

$$K(\xi, \eta) = \int_{\partial\Sigma} \left(\delta_\xi \theta \delta_\eta \hat{t} - \delta_\eta \theta \delta_\xi \hat{t}\right). \tag{35}$$

On the regular stratum, this is the explicitly represented boundary 2-cocycle of Theorem 4.2 (whose cohomological status, in the sense of $H^2(\mathfrak{g}_\partial, \mathbb{R})$, we do not address; see Remark 4.1). On the dense stratum, the structure of (35) together with dense-time admissibility forces it to vanish.

Theorem 6.2 (Dense-time cocycle suppression). For $\Phi \in \mathcal{S}_{\text{dense}}$ and any boundary symmetries $\xi, \eta \in \mathfrak{g}_\partial$ acting through dense-time admissible variations,

$$K_{\text{dense}}(\xi, \eta) = 0.$$

Proof. By Definition 5.2, both $\delta_\xi \theta$ and $\delta_\eta \theta$ vanish on $\mathcal{M}_{\text{dense}}$, and in particular on $\partial\Sigma \cap \mathcal{M}_{\text{dense}}$. The integrand of (35) thereby vanishes pointwise on the portion of $\partial\Sigma$ lying in $\mathcal{M}_{\text{dense}}$. On $\partial\Sigma \cap \mathcal{M}_{\text{reg}}$, the variations may be non-trivial, but boundary symmetries acting on configurations in $\mathcal{S}_{\text{dense}}$ are constrained to preserve the dense-time admissibility globally; the corresponding boundary action on $\partial\Sigma \cap \mathcal{M}_{\text{reg}}$ is a measure-zero contribution at the limiting locus $\partial\Sigma \cap \mathcal{H}$, where $\delta_\xi \theta$ vanishes by smoothness and matching to the dense stratum. Integrating over $\partial\Sigma$, both terms of (35) vanish, giving $K_{\text{dense}}(\xi, \eta) = 0$.

Corollary 6.2 (Boundary algebra on the dense stratum).

$$\left\{Q_\xi, Q_\eta\right\}_{\text{dense}} = Q_{[\xi, \eta]},$$

i.e. the boundary charge algebra on $\mathcal{S}_{\text{dense}}$ is centrally non-extended; the map $\xi \mapsto Q_\xi$ is a Lie algebra homomorphism. The Jacobi identity is preserved (Proposition 3.5, Theorem 4.2 with $K = 0$).

6.3. The Phase Boundary Characterisation Theorem

Theorems 6.1 and 6.2 each identify a distinct algebraic consequence of the dense-time admissibility condition. We now show that these are not independent results but two manifestations of a single geometric fact: the phase boundary $\mathcal{H}$ is precisely the locus of a simultaneous transition in symplectic rank and central extension.

Theorem 6.3 (Phase Boundary Characterisation). Under Assumptions 1-3, the following four conditions are equivalent:

(i) $x \in \mathcal{H}$, *i.e.* $P(x) = P_\star$.

(ii) The presymplectic kernel of $\Omega_\Sigma^{\text{aug}}$ enlarges across x:

$$\ker \Omega_\Sigma^{\text{aug}}\Big|_{\mathcal{S}_{\text{dense}} \ni \Phi} \supsetneq \ker \Omega_\Sigma^{\text{aug}}\Big|_{\mathcal{S}_{\text{reg}} \ni \Phi}.$$

(iii) The boundary charge cocycle vanishes at x: $K_{\text{dense}}(\xi, \eta) = 0$ for all $\xi, \eta \in \mathfrak{g}_\partial$.

(iv) The reduced phase-space dimension strictly drops at x: $\dim \mathcal{P}_{\text{dense}} < \dim \mathcal{P}_{\text{reg}}$.

In particular, conditions (ii)-(iv) are simultaneously activated at $\mathcal{H}$ and simultaneously absent away from it. The "uniqueness" implicit in this equivalence is uniqueness of the stiffness-induced phase-sector mechanism: the augmented presymplectic form $\Omega_\Sigma^{\text{aug}}$ may admit other, unrelated kernel directions (degeneracies at zeros of Φ where the polar decomposition fails, topological-sector ambiguities, boundary-condition-induced degeneracies, etc.); the theorem does not exclude these. What it asserts is that the specific combined transition described by (ii) (phase-sector kernel jump) and (iii) (phase-sector cocycle vanishing) is supported at $\mathcal{H}$ and nowhere else.

Proof. The implications are established by the preceding results. We prove the cycle (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i).

(i)⇒(ii). Crossing $\mathcal{H}$ from $\mathcal{M}_{\text{reg}}$ into $\mathcal{M}_{\text{dense}}$ activates Assumption 3: the diverging stiffness $\kappa(P) \to \infty$ on the dense side enforces, via Lemma 2.1, the dense-time admissibility condition $\delta\theta = 0$ on $\mathcal{M}_{\text{dense}}$. Theorem 6.1 then gives the kernel enlargement of (ii).

(ii)⇒(iii). The cocycle integrand (35) is bilinear in $\delta\theta$ and $\delta\hat{t}$. The kernel enlargement of (ii), via the construction of Theorem 6.1, is precisely the statement that phase-only variations δ_θ supported in $\mathcal{M}_{\text{dense}}$ become null directions of $\Omega_\Sigma^{\text{aug}}$. This is equivalent—by the matching of the bulk and boundary parts of $\Omega_\Sigma^{\text{aug}}$ proved in Proposition 3.3—to the vanishing of the integrand of (35) on $\partial\Sigma \cap \mathcal{M}_{\text{dense}}$. Hence $K_{\text{dense}} = 0$ as in (iii).

(iii)⇒(iv). If $K_{\text{dense}} = 0$, the cocycle integrand vanishes for all admissible $(\delta_\xi, \delta_\eta)$, which by the bilinear $(\theta, \hat{t})$-structure of (35) forces $\delta\theta = 0$ on the support of every admissible boundary variation. By the matching of bulk and boundary in $\Omega_{\Sigma}^{\text{aug}}$, this propagates to a vanishing phase-sector contribution to the bulk presymplectic form on $\mathcal{M}_{\text{dense}}$. The phase-only variations (33) are then null directions of $\Omega_{\Sigma}^{\text{aug}}\big|_{\mathcal{S}_{\text{dense}}}$, contributing to the kernel and being quotiented out. Thus the dense-stratum reduced phase space is strictly smaller than the regular one: $\dim\mathcal{P}_{\text{dense}} < \dim\mathcal{P}_{\text{reg}}$, as in (iv).

(iv)⇒(i) (contrapositive). Suppose $P(x) < P_{\star}$ at every $x \in \mathcal{M}$, so $\mathcal{M}_{\text{dense}} = \varnothing$ and $\mathcal{H} = \varnothing$. Then Assumption 3 imposes no restriction (vacuously); the unstratified covariant phase space of Remark 3.3 obtains. The kernel of $\Omega_{\Sigma}^{\text{aug}}$ contains only the generic non-stiffness-induced directions (degeneracies at zeros of Φ, boundary-condition-induced directions, etc.), with no stiffness-induced enlargement; and $\dim\mathcal{P}_{\text{dense}} = \dim\mathcal{P}_{\text{reg}}$ trivially since $\mathcal{S}_{\text{dense}}$ is empty (or, equivalently, $\mathcal{P}_{\text{dense}} = \mathcal{P}_{\text{reg}}$ as a degenerate case). The dimension drop in (iv) therefore requires $\mathcal{M}_{\text{dense}} \neq \varnothing$, *i.e.*, the existence of $x \in \mathcal{M}$ with $P(x) \geq P_{\star}$. By smoothness of P and connectedness arguments, such an x exists if and only if $\mathcal{H} = \{P = P_{\star}\}$ is non-empty, i.e. $\mathcal{H} \neq \varnothing$ and (i) holds at the threshold-attaining points.

The four conditions are therefore mutually equivalent.

Corollary 6.3 (Algebraic characterisation of $\mathcal{H}$). The phase boundary $\mathcal{H}$ admits a purely algebraic, coordinate-free characterisation as the locus where the stiffness-induced phase-sector kernel of $\Omega_{\Sigma}^{\text{aug}}$ first appears, equivalently the locus where the explicitly represented boundary cocycle (30) first vanishes. Among the algebraic features of the augmented covariant phase space, this combined transition is what selects $\mathcal{H}$; no reference to the trigger functional P or its threshold $P_{\star}$ is required for the identification once the covariant phase space has been constructed.

This is the principal structural content of the paper: the phase boundary is recoverable from the augmented covariant phase space alone, via the specific combination of phase-sector kernel jump and phase-sector cocycle vanishing. Different choices of trigger functional P (Section 5) produce the same algebraic characterisation, with $\mathcal{H}$ identified intrinsically. As emphasised in Theorem 6.3, this characterisation is uniqueness of the phase-sector mechanism, not a claim that $\mathcal{H}$ is the unique locus of any kernel direction in $\Omega_{\Sigma}^{\text{aug}}$.

Remark 6.1 (Geometric interpretation of $\mathcal{H}$). The phase boundary $\mathcal{H} = \{P = P_{\star}\}$ is therefore not a singular dynamical region. It is a degeneracy surface of the augmented presymplectic form: the locus in field space where the kernel of $\Omega_{\Sigma}^{\text{aug}}$ enlarges due to the onset of the dense-time admissibility condition. The Euler-Lagrange equations remain smooth across $\mathcal{H}$. The transition is purely canonical: phase-wave degrees of freedom become null directions of the presymplectic form, the central extension of the boundary charge algebra vanishes, and the boundary algebra reduces to a purely geometric Lie algebra free of cocycle.

7. Quantization Compatibility and Global Structure

We close by stating a global structural theorem that summarises the construction, and discussing compatibility with reduced phase-space quantization on each stratum.

7.1. The Global Structural Theorem

Theorem 7.1 (Stratified Covariant Phase Structure). The complex scalar field theory defined by Sections 2-3, under Assumptions 1-3, admits a covariant phase-space structure satisfying:

(1) Hypersurface independence via boundary symplectic augmentation (Theorem 3.2), with the Iyer-Wald-Zoupas freedom in the polar-amplitude α removed by the mixed boundary conditions of Proposition 3.4 (a model calculation; not a general resolution of the IWZ ambiguity).

(2) A well-defined reduced phase space $\mathcal{P} = \mathcal{P}_{\text{reg}} \cup \mathcal{P}_{\text{dense}}$ on each stratum, with non-degenerate symplectic form (Theorem 3.3).

(3) Integrable boundary charges Q_ξ generating boundary symmetries (Proposition 4.1), with charge algebra represented up to the explicitly represented boundary 2-cocycle K (Proposition 4.2, Theorem 4.1; cohomological status of K is not addressed, see Remark 4.1).

(4) Diffeomorphism-invariant local stratification by the trigger functional P (Section 5), with smooth phase boundary $\mathcal{H}$ (Assumption 2).

(5) Strict enlargement of the presymplectic kernel on the dense stratum (Theorem 6.1), as the inescapable consequence of the finite-energy selection rule of Assumption 3.

(6) Suppression of the boundary cocycle on the dense stratum (Theorem 6.2), without modification of bulk dynamics.

(7) Algebraic characterisation of the phase boundary as the unique degeneracy locus of $\Omega_\Sigma^{\text{aug}}$ (Theorem 6.3, Corollary 6.3).

(8) Compatibility with reduced phase-space quantization carried out independently on each stratum (Section 7.2 below).

The proof is the assembly of the cited results.

7.2. Quantization Compatibility

Although a full Hilbert-space construction lies beyond the scope of this paper, the algebraic structure on each stratum is compatible with reduced phase-space quantization in the standard sense [3] [4]. On the regular stratum, the canonical quantization correspondence $\{F, G\} \mapsto \frac{1}{i\hbar}\left[\hat{F}, \hat{G}\right]$ applied to the boundary charge algebra (29) gives

$$\left[\hat{Q}_\xi, \hat{Q}_\eta\right] = i\hbar \hat{Q}_{[\xi,\eta]} + i\hbar \hat{K}(\xi, \eta), \tag{36}$$

the standard centrally-extended commutator of an edge-mode boundary algebra. On the dense stratum, the cocycle suppression $K_{\text{dense}} = 0$ from Theorem 6.2 gives instead

$$\left[\hat{Q}_\xi, \hat{Q}_\eta\right]_{\text{dense}} = i\hbar \hat{Q}_{[\xi,\eta]}, \tag{37}$$

the uncentrally-extended commutator of a Lie algebra representation. Furthermore, for any $x \in \mathcal{M}_{\text{dense}}$,

$$\left[\hat{\theta}(x), \hat{A}\right] = 0 \text{ for any operator } \hat{A},$$

reflecting the collapse of phase-sector operator dynamics. The quantization is carried out independently on each stratum, with no obstruction from the framework.

Remark 7.1 (Two independent quantizations). The two strata $\mathcal{P}_{\text{reg}}$ and $\mathcal{P}_{\text{dense}}$ are quantized separately, each carrying its own representation of the appropriate algebra: (36) on the regular side, (37) on the dense side. The relationship between the two Hilbert spaces is itself an interesting question—in particular, whether they assemble into a single representation of a globally-defined algebra, or whether they remain distinct sectors—but it lies beyond the scope of the present structural analysis. What we have shown is that the algebraic transition is consistent with quantization on each stratum.

7.3. Scope and Limitations

The results are intentionally structural and fixed-background in scope. The metric $g_{\mu\nu}$ is not varied in the present covariant phase space; it supplies the Lorentzian geometry, curvature invariants, volume form, Cauchy foliation, and boundary structure. Likewise, the trigger scalar $P[\Phi, g]$ is prescribed for the variational problem developed here. This is sufficient for the algebraic theorem proved in the paper: a divergent phase-stiffness coefficient encoded in the scalar action forces the admissible phase variations to vanish on the dense stratum, which in turn enlarges the phase-sector kernel and suppresses the boundary cocycle.

If P is promoted to a fully dynamical functional of Φ or if $g_{\mu\nu}$ is varied as part of a gravitational theory, the presymplectic potential acquires additional terms involving δP and, in the gravitational case, $\delta g_{\mu\nu}$. Those terms are not contradictions of the present result; they define a larger dynamical-trigger problem. The present paper should therefore be read as the prescribed-trigger, fixed-background sector of the broader theory. The extension to dynamical triggers, gravitational coupling, and full metric variation is left to future work.

The boundary symplectic density α in (20) is a natural choice adapted to the polar-amplitude polarisation, and Proposition 3.4 fixes the IWZ freedom within that polarisation; other polarisations are admissible and yield other densities. We do not address the cohomological status of the cocycle K in the sense of $H^2(\mathfrak{g}_\partial, \mathbb{R})$ (Remark 4.1); the structural arguments require only its explicit representation (30) together with its phase-sector bilinearity. The treatment of zeros of Φ is developed in **Appendix A**; topological zero-loci (vortices, domain walls) lie outside the present framework, as noted in Remark A.1. The "uniqueness" of $\mathcal{H}$ asserted in Theorem 6.3(iv) is uniqueness of the stiffness-induced phase-sector mechanism, not exclusion of other unrelated kernel directions in $\Omega_\Sigma^{\text{aug}}$.

Natural directions for future work include:

- A global presymplectic reduction theorem for the two-stratum decomposition, in a precise field-theoretic analogue of the Sjamaar-Lerman stratified-reduction theorem.
- A persistence theorem for phase suppression under dynamical trigger functionals, where δP contributes to the presymplectic potential and a subleading-correction hypothesis is needed.
- A sufficient criterion for cohomological non-triviality of the boundary cocycle K in $H^2(\mathfrak{g}_\partial, \mathbb{R})$, complementing its explicit representation (30).
- A converse direction of the Phase Boundary Characterisation Theorem, identifying precise hypotheses under which the algebraic signatures of phase suppression occur only at $\mathcal{H}$.
- Treatment of topological zero-loci (vortices, domain walls) as a sector-by-sector extension of the present framework.
- Extension to a dynamical metric with gravitational backreaction, the relation between the two stratum-level Hilbert spaces obtained from independent quantization (Section 7.2), extension to gauge fields and to gravity, and the question of whether the present framework admits a natural smooth (rather than stratified) deformation as κ becomes finite.

8. Conclusions

We have constructed a self-contained covariant phase-space formulation for a complex scalar field theory stratified by a diffeomorphism-invariant local trigger functional. The construction rests on the three minimal assumptions of Section 2: phase stratification by a local invariant scalar (Assumption 1), regularity of the resulting phase boundary (Assumption 2), and an energetic selection rule on tangent variations driven by a diverging phase-stiffness functional (Assumption 3). No further physical input is required; in particular, the trigger functional P is left abstract throughout the structural development and only specialised to canonical examples (Kretschmann scalar, energy-momentum scalar, order-parameter scalar) at the level of illustration.

The principal results, assembled into the global structural theorem (Theorem 7.1), are:

1) Hypersurface independence is restored by boundary symplectic augmentation, with the Iyer-Wald-Zoupas freedom in the polar-amplitude α removed (within that polarisation) by the mixed boundary conditions $\delta(\nabla_n \hat{t})\big|_{\partial\Sigma} = 0$ and $\delta(\hat{t}^2 \nabla_n \theta)\big|_{\partial\Sigma} = 0$ (Proposition 3.4). This is a model calculation within the IWZ setting, not a general resolution of the ambiguity.

2) The boundary charge algebra is represented up to an explicitly given boundary 2-cocycle K, with explicit form $K(\xi,\eta) = \int_{\partial\Sigma} (\delta_\xi \theta \delta_\eta \hat{t} - \delta_\eta \theta \delta_\xi \hat{t})$ (Theorem 4.1) on the regular stratum. The cohomological non-triviality of K in $H^2(\mathfrak{g}_\partial, \mathbb{R})$ is not addressed; only its explicit phase-sector bilinearity is needed for the suppression argument.

3) On the dense stratum, the diverging phase-stiffness of Assumption 3 acts as a finite-action selection rule forcing $\delta\theta = 0$ (Lemma 2.1). This single condition simultaneously enlarges the presymplectic kernel (Theorem 6.1) and suppresses the cocycle (Theorem 6.2): $K_{\text{dense}} = 0$. In the unstratified limit $P_\star \to \infty$, the construction reduces identically to standard Lee-Wald/Iyer-Wald covariant phase space (Remark 3.3).

4) The Phase Boundary Characterisation Theorem (Theorem 6.3) unifies these results: the four conditions—(i) crossing the trigger threshold, (ii) kernel enlargement, (iii) cocycle suppression, and (iv) reduced phase-space dimension drop—are mutually equivalent. The phase boundary $\mathcal{H}$ therefore admits a purely algebraic, coordinate-free characterisation independent of the specific trigger functional (Corollary 6.3).

5) The algebraic structure on each stratum is compatible with reduced phase-space quantization carried out independently per stratum, giving a centrally extended commutator on the regular side and an uncentrally extended one on the dense side (Section 7.2).

The result is a general theorem about stratified field theories within the class of models satisfying Assumptions 1-3. The algebraic transition across a phase boundary—from centrally extended to non-extended boundary charge algebra—follows solely from the geometry of the augmented covariant phase space and the energetics of the admissible variation class. The framework applies, with the choice of trigger functional adapted to the setting, to strong-gravity interiors, dense-matter phases, and condensed-matter systems with sharp phase boundaries (Remark 1.1); the structural mechanism is independent of the particular illustrative choice of trigger functional within this class.

The fixed-background and prescribed-trigger assumptions isolate the mathematical layer treated in this paper. At this layer, the main result is algebraic: once the scalar action contains a divergent phase-stiffness coefficient and finite-action admissibility is imposed, dense-stratum phase variations disappear from the admissible tangent space, and the corresponding boundary algebra changes accordingly. Dynamical triggers, metric variation, gravitational backreaction, and model-specific defect sectors are natural extensions of this structure rather than hidden assumptions of the theorem proved here. The zero-locus and defect-sector discussion of **Appendix A** should be read as a regularization safeguard for the polar variables used in this paper, not as a classification of all possible vortex, wall, or topological-defect sectors.

Several natural directions for extension are enumerated in Section 7.3. The algebraic characterisation of $\mathcal{H}$ obtained here suggests that the framework should generalise without obstruction to any field theory in which an admissible variation class can be defined and an energetic selection rule formulated; we leave this for future work.

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Data Availability Statement

This manuscript contains no experimental data, observational data, or numerical simulations. All results are purely mathematical and are contained in full within the article itself. No data sets were generated or analysed during this study, and no data repository is associated with this work.

Conflicts of Interest

The author declares that he has no conflicts of interest relevant to the content of this article. This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors, and was conducted independently.

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Appendix

A. Behaviour of the Presymplectic Structure at Zero-Loci of Φ

The polar decomposition $\Phi = \sqrt{\hat{t}}\,\mathrm{e}^{i\theta}$ used in the body of the paper is regular only on $\mathcal{M}^* := \{\Phi \neq 0\}$. This appendix gives a tubular-neighbourhood regularization that handles isolated and codimension-≥ 1 measure-zero zeros directly, and records why topologically charged zero-loci (vortices, domain walls) fall outside the present framework.

Let $\mathcal{Z} := \{x \in \mathcal{M} : \Phi(x) = 0\}$ and $\mathcal{Z}_\Sigma := \mathcal{Z} \cap \Sigma$, and write $N_\epsilon(\mathcal{Z}_\Sigma)$ for the open ϵ-tubular neighbourhood of $\mathcal{Z}_\Sigma$ in Σ in the Riemannian metric h. The regularized form is

$$\Omega_\Sigma^{\mathrm{aug},\epsilon}(\delta_1,\delta_2) := \int_{\Sigma_\epsilon} \omega^\mu(\delta_1,\delta_2)\, n_\mu \sqrt{h}\,\mathrm{d}^3x + \int_{\partial\Sigma_\epsilon} \alpha(\delta_1,\delta_2), \tag{38}$$

with $\Sigma_\epsilon = \Sigma \setminus N_\epsilon(\mathcal{Z}_\Sigma)$ and $\partial\Sigma_\epsilon$ the corresponding piecewise-smooth boundary. On Σ_ϵ the polar decomposition is regular and Sections 3-6 apply verbatim.

Proposition A.1 (Tubular regularization). Suppose $\mathcal{Z}_\Sigma$ is a finite (or countable) union of submanifolds of Σ of codimension $k \geq 1$, with bounded variations $\delta_i\Phi$ and bounded first derivatives on a neighbourhood of $\mathcal{Z}_\Sigma$. Then, under these bounded-variation and bounded-trace assumptions, and using a fixed smooth tubular cutoff, the regularized forms $\Omega_\Sigma^{\mathrm{aug},\epsilon}$ admit a finite limit as $\epsilon \to 0^+$,

$$\Omega_\Sigma^{\mathrm{aug}}(\delta_1,\delta_2) = \int_\Sigma \omega^\mu(\delta_1,\delta_2)\, n_\mu \sqrt{h}\,\mathrm{d}^3x + \int_{\partial\Sigma} \alpha(\delta_1,\delta_2),$$

with ω^μ given by the non-polar form (15). This limit agrees with the non-polar expression for the presymplectic form on the complement of the zero set and is independent of the tubular regulator within this cutoff class.

Proof. The non-polar current (15) is bilinear in $\delta\Phi, \nabla\delta\Phi$ and is continuous wherever the variations are; the apparent $\hat{t}^{-1}$ singularity of the polar form (22) is cancelled by the corresponding $\hat{t}$-factor in $\delta\hat{t} = 2\,\mathrm{Re}\left(\Phi^\dagger\delta\Phi\right)$, so the polar and non-polar forms agree on $\mathcal{M}^*$ via continuous extension. The bulk volume estimate is $\mathrm{Vol}\left(N_\epsilon(\mathcal{Z}_\Sigma)\right) = O\left(\epsilon^k\right)$, giving $\left|\int_{N_\epsilon} \omega^\mu n_\mu \sqrt{h}\,\mathrm{d}^3x\right| = O\left(\epsilon^k\right) \to 0$. The corner contribution $\int_{\mathcal{C}_\epsilon} \alpha$ scales as $O\left(\epsilon^{k-1}\right)$. Hence it vanishes directly for codimension $k \geq 2$. For codimension-one zero sets, the limit requires the stated bounded-trace hypothesis together with the fixed smooth transverse cutoff; under those assumptions the regulator contribution is finite and belongs to the excluded defect-boundary sector rather than to the stiffness-induced phase-sector mechanism studied here. The same argument with one dimension fewer applies when $\mathcal{Z}_\Sigma \cap \partial\Sigma \neq \varnothing$ at isolated points: the boundary corner is a 1-sphere of length $2\pi\epsilon$, with $|\alpha|$ bounded by the variation regularity, giving an $O(\epsilon)$ correction that vanishes in the limit. Within the fixed tubular-cutoff class, the limit is independent of the regulator.

The purpose of this construction is not to classify all possible topological defect sectors, but to ensure that the stiffness-induced phase-sector mechanism is well-defined on each positive-amplitude stratum and has a controlled limiting interpretation near excluded zero-loci.

In summary: let $Z=\{x\in\mathcal{M}:\Phi(x)=0\}$ and $\mathcal{M}^{\times}=\mathcal{M}\setminus Z$ denote the positive-amplitude region on which the polar decomposition $\Phi=\rho e^{i\theta}$ is smooth. The phase-sector presymplectic forms are first evaluated on $\Sigma_{\epsilon}=\Sigma\setminus N_{\epsilon}(Z)$, where θ is well-defined. Under the stated bounded-variation and bounded-trace assumptions, and using a fixed smooth tubular cutoff, the regularized forms $\Omega_{\Sigma}^{\mathrm{aug},\epsilon}$ admit a finite limit as $\epsilon\to 0^{+}$. On the complement of Z, this limit agrees with the non-polar complex-field expression for the presymplectic form. Tubular boundary contributions vanish directly for zero sets of codimension $k\geq 2$. For codimension-one zero sets, any remaining finite regulator contribution is assigned to the excluded defect-boundary sector and is not part of the stiffness-induced phase-sector mechanism studied in this paper.

Remark A.1 (Topological sectors: vortices and domain walls). Codimension-2 zero-loci with non-trivial winding (vortices) and stable extended zero-loci of codimension 1 (domain walls) are qualitatively different. On a neighbourhood of such a locus, θ is globally multi-valued and cannot be lifted to a single-valued function even after excising $\mathcal{Z}$; the phase variation $\delta\theta$ supports a topological zero-mode (the discrete shift of winding number) which is not infinitesimal and is not removable by tubular regularization. The structural framework of this paper is stated within a single topological sector and does not address cross-sector statements. A sector-by-sector treatment of vortices and domain walls—including a positive theory of phase vortices as carriers of half-integer topological charge, with concrete realisations such as fractional-flux vortices in two-gap superconductors [23] and the related phase-stiffness phenomenology of superfluid systems [24]—is a natural direction for future work.

Remark A.2 (Positive-amplitude convention near the dense stratum). Unless otherwise stated, the dense-stratum suppression theorem (Lemma 2.1) is applied on compact subsets $K\Subset\Sigma\cap\mathcal{M}_{\mathrm{dense}}$ satisfying $\rho\geq\rho_0>0$ for some ρ_0. Points where $\rho=0$ are excluded from the polar phase analysis and treated by the excision/limiting procedure of Proposition A.1.

B. A Concrete Worked Example: Mexican-Hat Scalar on a Half-Space

This appendix specialises the construction to a fully explicit Lagrangian and computes the trigger threshold, the presymplectic form, and the boundary cocycle integrand. The model satisfies the assumptions of Section 2 and exhibits all the algebraic features established in Sections 3-6.

Specification. Take $\mathcal{M}=\{(t,x,y,z)\in\mathbb{R}^{1,3}:x\geq 0\}$ with the Minkowski metric, $\partial\mathcal{M}=\{x=0\}$ a timelike plane, and the stiffness-coupled scalar action

$$S_{\epsilon}[\rho,\theta]=\int_{\mathcal{M}}\left[-\frac{1}{2}\nabla_{\mu}\rho\nabla^{\mu}\rho-\frac{1}{2}\kappa_{\epsilon}(\rho^2)\rho^2\nabla_{\mu}\theta\nabla^{\mu}\theta-\frac{\lambda}{4}(\rho^2-v^2)^2\right]\epsilon_g,\quad(39)$$

with the Mexican-hat potential $V\left(|\Phi|^2\right)=\frac{\lambda}{4}\left(|\Phi|^2-v^2\right)^2$, of the form familiar from Ginzburg-Landau theory of phase transitions [25], vacuum amplitude $v>0$, order-parameter trigger $P=|\Phi|^2=\rho^2$, and threshold chosen below the vacuum amplitude,

$$P_\star=\eta^2 v^2=\eta^2\frac{m^2}{\lambda},\ 0<\eta<1, \tag{40}$$

where $m^2=\lambda v^2$ is the mass scale of the linearised theory near the broken vacuum. The regularized stiffness coefficient is

$$\kappa_\epsilon\left(P\right)=\kappa_0+\frac{\kappa_1}{\left(P-P_\star+\epsilon\right)^q},\ q>0,\ \epsilon>0, \tag{41}$$

on the region $P\geq P_\star$, with a smooth positive extension to $P<P_\star$; the stiffness limit is $\epsilon\to0^+$. This setup satisfies (G1)-(G4) and Assumptions 1-3.

Background and the geometry of $\mathcal{H}$. We work on the shifted ϕ^4-kink background $\Phi_0\left(x\right)=\rho_0\left(x\right)e^{i\theta_0}$ with

$$\rho_0\left(x\right)=v\tanh\left(m\left(x-x_0\right)/\sqrt{2}\right),\ x_0<0,\ \theta_0=0.$$

Then $\rho_0>0$ throughout $\mathcal{M}$ (no boundary-zero), and $\rho_0\left(x\right)\to v$ only as $x\to\infty$. The phase boundary is the finite regular level set

$$\mathcal{H}=\left\{x\in\mathcal{M}:\rho_0\left(x\right)=\eta v\right\}=\left\{x=x_\star\right\},\ x_\star=x_0+\frac{\sqrt{2}}{m}\operatorname{arctanh}\left(\eta\right),$$

which is well-defined and finite for any $0<\eta<1$. The dense stratum is $\mathcal{M}_{\text{dense}}=\left\{x>x_\star\right\}$, the regular stratum is the slab $\mathcal{M}_{\text{reg}}=\left\{0\leq x<x_\star\right\}$, and $\nabla|\Phi_0|^2=2\rho_0\rho_0'\neq0$ at $x_\star$, satisfying Assumption 2.

Presymplectic form on the kink background. Evaluated at Φ_0, on variations $\delta_i\Phi=e^{i\theta_0}\left(\delta_i\rho+i\rho_0\delta_i\theta\right)$, the polar current (22) simplifies (using $\nabla\theta_0=0$) to amplitude- and phase-sector contributions only:

$$\Omega_\Sigma\left(\delta_1,\delta_2\right)\Big|_{\Phi_0}=\int_\Sigma\left\{2\left[\dot{\delta}_1\rho\delta_2\rho-\dot{\delta}_2\rho\delta_1\rho\right]+2\rho_0^2\left[\dot{\delta}_1\theta\delta_2\theta-\dot{\delta}_2\theta\delta_1\theta\right]\right\}d^3x,$$

where $\dot{(\cdot)}=\partial_t$. The boundary symplectic density at $\partial\Sigma=\left\{x=0\right\}$ is

$$\alpha\left(\delta_1,\delta_2\right)\Big|_{\partial\Sigma,\Phi_0}=2\rho_0\left(0\right)\left[\delta_1\theta\delta_2\rho-\delta_2\theta\delta_1\rho\right],$$

non-degenerate in the $\left(\theta,\rho\right)$ pairing since $\rho_0\left(0\right)>0$. The dense-stratum finite-action condition takes the explicit form

$$\int_{\Sigma\cap\mathcal{M}_{\text{dense}}}\kappa_\epsilon\left(\rho^2\right)\rho^2\left[h^{ij}D_i\delta\theta D_j\delta\theta+|\delta\theta|^2\right]\sqrt{h}d^3x<\infty, \tag{42}$$

and as $\epsilon\to0^+$, $\kappa_\epsilon\left(\rho^2\right)\to\infty$ on $\rho^2\geq P_\star$, forcing $\delta\theta=0$ on $\Sigma\cap\mathcal{M}_{\text{dense}}$ under the positive-amplitude condition $\rho\geq\rho_0>0$. The phase-sector bulk term in the presymplectic current is consequently killed: $\omega_\theta^\mu=0$ on dense-stratum admissible variations; the amplitude sector survives.

Boundary algebra and the cocycle. Choose boundary test generators ξ_1, ξ_2 via bump functions $\phi_1, \phi_2 \in C_c^\infty(U_\partial)$ on a region of $\partial\Sigma$ where $\rho_0 > 0$, with $\delta_{\xi_i}\theta = \phi_i$ and $\delta_{\xi_i}\hat{t} = \mathcal{L}_X \phi_i$ for some boundary tangent vector X. This defines an abelian boundary symmetry algebra (the generators commute as additive shifts) compatible with the polar-amplitude boundary conditions of Definition 3.5. The cocycle pairing evaluates to

$$K(\xi_1, \xi_2) = \int_{\partial\Sigma} (\phi_1 \mathcal{L}_X \phi_2 - \phi_2 \mathcal{L}_X \phi_1)\, \mathrm{d}S,$$

which is non-zero for generic ϕ_1, ϕ_2, confirming that the cocycle (30) is explicitly represented on the regular stratum. If instead ϕ_1, ϕ_2 are supported on $\partial\Sigma \cap \mathcal{M}_{\text{dense}}$, dense-time admissibility forces $\delta_{\xi_i}\theta = 0$ on the support and the integrand vanishes pointwise: $K_{\text{dense}}(\xi_1, \xi_2) = 0$. This is Theorem 6.2, made concrete in this model.

Summary. For the Mexican-hat scalar with $P = |\Phi|^2$ on a half-space and threshold $P_\star = \eta^2 v^2 = \eta^2 m^2 / \lambda$ with $0 < \eta < 1$: the phase boundary $\mathcal{H} = \{|\Phi|^2 = P_\star\}$ is realised on the kink background as the finite plane $\{x = x_\star\}$ with $x_\star = x_0 + (\sqrt{2}/m)\operatorname{arctanh}(\eta)$; the presymplectic form and boundary density take the displayed explicit forms; and the boundary cocycle is non-zero on regular-boundary test pairs and identically zero on dense-boundary test pairs, realising the kernel-enlargement and cocycle-suppression theorems of Section 6 in a concrete physical model.