Journal of Applied Mathematics and Physics, 2026, 14(6), 2164-2191
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Global Stratified Reduction and Boundary Cohomology for Phase-Suppressed Covariant Field Theory

George Davey

Independent Researcher, West Des Moines, IA, USA
Email: George.Davey@QuantumLinear.com

How to cite this paper: Davey, G. (2026) Global Stratified Reduction and Boundary Cohomology for Phase-Suppressed Covariant Field Theory. *Journal of Applied Mathematics and Physics*, **14**, 2164-2191.
https://doi.org/10.4236/jamp.2026.146106

Received: May 2, 2026
Accepted: June 15, 2026
Published: June 18, 2026

Abstract

In a complex scalar field theory with a divergent phase-stiffness functional, finite-action admissibility enforces the vanishing of phase variations on a high-stiffness stratum, with consequent enlargement of the presymplectic kernel and suppression of an explicitly-represented amplitude-phase boundary cocycle. This local mechanism leaves open four mathematical questions, which the present paper closes conditionally under explicit regularity, nondegeneracy, local-separation, and no-accidental-degeneracy hypotheses. We extend the finite-action selection theorem to admissible dynamical trigger functionals whose induced quadratic-form correction is subleading relative to the divergent stiffness-weighted admissibility form; we prove a global two-stratum presymplectic structure under a stratum-regularity assumption, with phase-sector reduction localised to the dense region; we give a sufficient cohomological non-triviality criterion for the boundary 2-cocycle using an abelian Hamiltonian amplitude-phase test subalgebra, and we construct such a subalgebra explicitly; and, after isolating the conditions excluding accidental degeneracies and imposing a local separating boundary test algebra, we prove a boundary-local converse direction of the Phase Boundary Characterisation Theorem. The combined result is that, within the class of pure stiffness-induced phase-sector mechanisms, the threshold hypersurface $\mathcal{H} = \{P = P_\star\}$ is locally recoverable from the algebraic data of the augmented covariant phase space at boundary-accessible points.

Keywords

Covariant Phase Space, Stratified Presymplectic Reduction, Boundary Cohomology, Lie Algebra Cocycles, Finite-Action Admissibility, Phase Suppression

1. Introduction

1.1. Context: What Was Proved, and What Was Left Open

The companion paper [1] (hereafter referred to as Paper I) established a structural theorem about complex scalar field theory on a globally hyperbolic Lorentzian manifold $(\mathcal{M}, g)$ with boundary, in the presence of a diffeomorphism-invariant scalar trigger functional $P[\Phi, g]$ that partitions $\mathcal{M}$ across a level set $\mathcal{H} = \{P = P_\star\}$ into a regular stratum $\mathcal{M}_{\text{reg}} = \{P < P_\star\}$ and a high-stiffness (dense) stratum $\mathcal{M}_{\text{dens}} = \{P \geq P_\star\}$. In polar variables $\Phi = \sqrt{\hat{t}}\, e^{i\theta}$, finite-action admissibility under a divergent phase-stiffness coefficient $\kappa(P) \to \infty$ as $P \to P_\star^+$ was shown to enforce $\delta\theta = 0$ on $\mathcal{M}_{\text{dens}}$. This single energetic condition produced two algebraic effects on the augmented presymplectic form $\Omega_\Sigma^{\text{aug}} = \Omega_\Sigma + \tilde{\Omega}_{\partial\Sigma}$: enlargement of $\ker \Omega_\Sigma^{\text{aug}}$ in the phase sector, and vanishing of the explicitly-represented boundary 2-cocycle $K_{\text{dens}} = 0$.

Paper I deliberately stopped short of four claims. We list them now, since each is addressed by a theorem or an explicit structural assumption in the present paper.

(1) **The trigger functional was treated as prescribed.** Paper I assumed P was given as an external scalar functional on $\mathcal{M}$; its variations did not enter the symplectic computation. For physical applications—where P is typically a dynamical scalar built from the field Φ and the metric $g_{\mu\nu}$—the variation δP produces additional contributions to the presymplectic potential through $\delta\kappa(P) = \kappa'(P)\delta P$. Whether the finite-action selection rule survives these additional variations was not addressed.

(2) **The stratification was described informally as "in the sense of Sjamaar-Lerman."** Paper I cited the Sjamaar-Lerman stratified symplectic reduction framework [2] and observed that the two-stratum decomposition of the covariant phase space had the right informal shape, but did not prove a precise field-theoretic analogue of the Sjamaar-Lerman theorem.

(3) **The cocycle was called "explicitly represented" rather than "non-trivial."** The K of Paper I was given by an explicit boundary integral (linear in phase- and amplitude-sector variations), but it was not shown to define a non-trivial class in the second Lie algebra cohomology $H^2(\mathfrak{g}_\partial, \mathbb{R})$. Paper I disclaimed the cohomological non-triviality claim and noted that the structural arguments did not require it.

(4) **The Phase Boundary Characterisation Theorem was a strict equivalence, but its converse direction relied on tacit assumptions about which kernel directions of $\Omega_\Sigma^{\text{aug}}$ counted.** Specifically, the statement that $\mathcal{H}$ is the unique locus of phase-sector kernel enlargement and cocycle vanishing was qualified to the "stiffness-induced" mechanism, but the precise hypotheses excluding accidental degeneracies (gauge, zeros of Φ, topological sectors, etc.) were not isolated.

The purpose of the present paper is to close each of these four gaps. Sections 3 through 8 address them in order.

Remark 1.1 (Physical interpretation of $\mathcal{H}$). Throughout, $\mathcal{H}$ is treated as a mathematical object: the regular level set of a scalar trigger functional. In the physical settings motivating Paper I and the present work, $\mathcal{H}$ models phase-transition surfaces of various kinds—threshold surfaces in dense-matter or condensed-matter systems where a stiffness modulus diverges, and, in the gravitational setting, surfaces playing the role of black-hole horizons in covariant phase-space descriptions. The mathematical results below are stated and proved without reference to any specific physical interpretation; the boundary-local converse theorem (Theorem 8.1) gives, in the stated class, an algebraic characterisation that is independent of which physical surface $\mathcal{H}$ models in a given application.

1.2. Statement of the Main Result

The principal new result is a boundary-local converse direction of the Phase Boundary Characterisation Theorem, made rigorous within a precisely-specified class of mechanisms. Once the necessary scaffolding is in place—the dynamical-trigger extension (Section 3), the global reduction theorem (Section 4), the cocycle non-triviality theorem (Section 6), and the no-accidental-degeneracy framework (Section 7)—the converse takes the following form, which we state here informally and prove rigorously as Theorem 8.1.

Main Converse Theorem (Boundary-Local Converse Phase Boundary Characterisation). Let the field theory satisfy Assumptions 7.2-7.3 of Section 8 (the *pure stiffness-induced* class together with regularity, Hamiltonian boundary-algebra, local-separation, and unique-threshold hypotheses). Then in a neighbourhood U of any point $x_0 \in \mathcal{H} \cap \partial\mathcal{M}$, restricted to boundary-accessible points $x \in U \cap \partial\Sigma$ (made precise in Definition 8.1), the threshold hypersurface $\mathcal{H}$ is locally characterised by the simultaneous algebraic signatures of phase-sector presymplectic rank reduction, formal kernel enlargement under $\Omega_\Sigma^{\text{aug}}$, and dense-boundary 2-cocycle suppression—and the converse holds: any boundary-accessible point exhibiting these signatures together must lie on $\mathcal{H}$.

This converse should be read carefully. It is not a classification of all degeneracy loci of $\Omega_\Sigma^{\text{aug}}$ in arbitrary field theories; gauge directions, zeros of Φ, topological sectors, and other unrelated mechanisms can produce kernel directions whose loci are unrelated to $\mathcal{H}$. Nor is it a bulk classification: the cocycle is a boundary integral, and we do not extend the converse to bulk points without an additional propagation argument. The claim is rather that, within the restricted class of theories satisfying Assumptions 7.2-7.3, the algebraic signatures of phase suppression genuinely determine the threshold hypersurface at boundary-accessible points. The role of the assumptions is to delete the accidental degeneracies and to ensure that vanishing of a boundary integral implies pointwise vanishing distributionally; the work of the theorem is to show that, having done so, the algebraic data of $\Omega_\Sigma^{\text{aug}}$ recover $\mathcal{H} \cap \partial\Sigma$.

1.3. Scope of the Closure

The results below are conditional closure results. The paper does not claim an unconditional classification of all possible degeneracy loci in covariant phase space. Rather, it identifies a precise class of pure stiffness-induced phase-sector mechanisms and proves that, within this class and under explicit regularity, Hamiltonian boundary-algebra, local-separation, unique-threshold, and no-accidental-degeneracy assumptions, the phase boundary is recoverable from its algebraic signatures at boundary-accessible points. Each assumption is stated explicitly and used substantively in at least one proof; the cumulative effect is to delete every alternative mechanism by which the algebraic signatures could occur, leaving the stiffness-induced mechanism as the unique source.

1.4. Relation to Paper I

Paper I proves the local phase-boundary mechanism: finite-action admissibility suppresses phase variations on the dense stratum, enlarges the phase-sector presymplectic kernel, and suppresses the explicit boundary cocycle. The present paper does not re-prove that local mechanism as a new result; it is recalled in Section 2 (Theorem 2.1) and the relevant boundary inputs of Paper I are restated explicitly there for self-containedness (see the imported boundary data following Section 2). Instead, the present paper proves four closure statements that Paper I deliberately left open: persistence of the selection rule under admissible dynamical triggers, a global two-stratum presymplectic reduction, cohomological non-triviality of the regular-boundary cocycle under an explicit test-algebra criterion, and a boundary-local converse theorem under no-accidental-degeneracy hypotheses. The division of labour is therefore strict: every appeal below to "the local suppression theorem," "the polar-amplitude boundary conditions," or "the Hamiltonian integrability criterion" refers to an input recalled verbatim in Section 2, while every theorem carrying a number in Sections 3-8 is a new conditional result of the present paper.

1.5. Organisation

Section 2 fixes notation and recalls the precise local result of Paper I. Section 3 treats dynamical trigger functionals. Section 4 constructs the global two-stratum presymplectic reduction. Section 5 sets up the boundary algebra and reviews the explicit cocycle. Section 6 proves the cohomological non-triviality theorem and constructs an explicit abelian amplitude-phase test subalgebra. Section 7 isolates the no-accidental-degeneracy assumptions. Section 8 states and proves the converse Phase Boundary Characterisation Theorem. Section 9 gives the precise distinction between dense-boundary and global boundary suppression. Section 10 concludes.

2. Notation and the Local Phase-Suppression Theorem

2.1. Geometric and Field-Theoretic Setup

We adopt the notation of Paper I throughout. $(\mathcal{M}, g_{\mu\nu})$ is a smooth, oriented, globally hyperbolic Lorentzian four-manifold with smooth boundary $\partial\mathcal{M}$. We fix a foliation by smooth Cauchy hypersurfaces $\{\Sigma_\tau\}_{\tau\in\mathbb{R}}$ with future-directed unit normal n^μ, induced spatial metric h_{ij}, and inverse h^{ij}. We write D_i for the Levi-Civita connection on (Σ, h). The boundary of a Cauchy slice is $\partial\Sigma = \Sigma \cap \partial\mathcal{M}$.

The dynamical field is a complex scalar $\Phi : \mathcal{M} \to \mathbb{C}$, with polar decomposition

$$\Phi(x) = \sqrt{\hat{t}(x)}\,\mathrm{e}^{i\theta(x)}, \quad \hat{t} = |\Phi|^2 = \rho^2, \tag{1}$$

defined on the open set where $\Phi \neq 0$. We use ρ and $\sqrt{\hat{t}}$ interchangeably; both refer to the field's modulus.

The Lagrangian density of the phase-suppressed model is

$$\mathcal{L}_\kappa = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \rho \nabla_\nu \rho - \frac{1}{2} \kappa(P) \rho^2 g^{\mu\nu} \nabla_\mu \theta \nabla_\nu \theta - V(\rho), \tag{2}$$

where the phase-stiffness coefficient $\kappa : \mathbb{R} \to \mathbb{R}^+$ is smooth and positive, with $\kappa(P) \to \infty$ as $P \to P_\star^+$ on the dense side of the trigger threshold. For comparison with Paper I, the ordinary kinetic Lagrangian $\nabla\Phi^\dagger \nabla\Phi - V\left(|\Phi|^2\right)$ corresponds to the case $\kappa \equiv 1$; our Lagrangian (2) is the polar-form generalisation in which the phase sector carries a position-dependent stiffness. The Euler-Lagrange equations remain smooth across $\mathcal{H}$ for finite κ; the divergence is taken in a limiting sense, justified by the finite-action argument below.

2.2. Imported Boundary Data from Paper I

Several arguments below depend on two inputs from Paper I, cited there as Definition 3.5 and Proposition 4.1. For self-containedness we restate them here in the present notation; all subsequent uses of "the polar-amplitude boundary conditions" and "Hamiltonian admissibility" refer to these recalled statements.

Definition 2.1 (Polar-amplitude boundary conditions; Paper I, Definition 3.5). In the polar variables (1), a variation δ is said to satisfy the *polar-amplitude boundary conditions* at $\partial\Sigma$ if it preserves the normal-amplitude and phase-flux boundary data, that is,

$$\delta\left(\nabla_n \hat{t}\right)\Big|_{\partial\Sigma} = 0, \quad \delta\left(\hat{t}^2 \nabla_n \theta\right)\Big|_{\partial\Sigma} = 0, \tag{3}$$

where ∇_n denotes the outward normal derivative at $\partial\Sigma$. Equivalently, admissible boundary variations preserve the mixed polar boundary structure used to define the augmented boundary symplectic density $\tilde{\Omega}_{\partial\Sigma}$, so that the boundary contribution to $\Omega_\Sigma^{\text{aug}}$ is well defined and hypersurface-independent.

Definition 2.2 (Hamiltonian admissibility; Paper I, Proposition 4.1). A boundary symmetry $\xi \in \mathfrak{g}_\partial$ is *Hamiltonian admissible* if it preserves the polar-amplitude boundary conditions (3) and if the one-form $\iota_{\delta_\xi} \Omega_\Sigma^{\text{aug}}$ is exact on the reduced phase space, *i.e.*, there exists a generator Q_ξ with

$$\iota_{\delta_\xi} \Omega_\Sigma^{\text{aug}} = \delta Q_\xi. \tag{4}$$

Paper I, Proposition 4.1, establishes the integrability criterion in the form used below: given closure of $\Omega_\Sigma^{\text{aug}}$ (Paper I, Theorem 3.3) and preservation of (3), Cartan's identity $\mathcal{L}_{\delta_\xi} = \iota_{\delta_\xi} d + d\iota_{\delta_\xi}$ reduces the existence of Q_ξ to preservation of the boundary polarisation; this is the integrability requirement for boundary Hamiltonian generators in the sense of Wald and Zoupas [3]. Throughout, "admissible boundary symmetry" and "integrable Hamiltonian generator" refer to this recalled meaning.

These two statements are the only Paper I inputs invoked in Sections 5-8; isolating them here makes the proof dependencies of the present paper transparent without requiring the reader to consult Paper I.

2.3. The Local Finite-Action Selection Rule

Definition 2.3 (Finite-action admissibility on the dense stratum). *A phase variation $\delta\theta$ is finite-action admissible on $\Omega \subset \Sigma \cap \mathcal{M}_{\text{dens}}$ if*

$$\int_\Omega \kappa(P)\rho^2\left[h^{ij}D_i\delta\theta D_j\delta\theta + |\delta\theta|^2\right]\sqrt{h}\,\mathrm{d}^3x < \infty. \tag{5}$$

The integrand is positive-definite (the spatial metric h^{ij} is Riemannian) and the zero-order term $|\delta\theta|^2$ removes the otherwise-surviving constant phase mode; equivalently, omitting the zero-order term and quotienting by the global $U(1)$ phase rotation gives the same content.

Theorem 2.1 (Local finite-action phase suppression; Paper I). Let $\Omega \subset \Sigma \cap \mathcal{M}_{\text{dens}}$ be open, and assume $\rho \ge \rho_0 > 0$ on Ω. Let $\kappa_n(P) > 0$ be a sequence of phase-stiffness coefficients satisfying $\kappa_n(P) \to \infty$ uniformly on compact subsets of Ω. If $\delta\theta_n \in H^1_{\text{loc}}(\Omega)$ satisfies the uniform bound

$$\sup_n \int_\Omega \kappa_n(P)\rho^2\left[h^{ij}D_i\delta\theta_n D_j\delta\theta_n + |\delta\theta_n|^2\right]\sqrt{h}\,\mathrm{d}^3x < \infty, \tag{6}$$

then $\delta\theta_n \to 0$ in $L^2_{\text{loc}}(\Omega)$. Consequently, the limiting finite-action admissible tangent space satisfies $\delta\theta = 0$ on $\mathcal{M}_{\text{dens}}$.

Proof. This is Lemma 2.1 of Paper I; we recall the proof for completeness. Let $K \Subset \Omega$ be compact. Since $\rho \ge \rho_0 > 0$ and $\kappa_n(P) \to \infty$ uniformly on K, the quantity $m_n(K) := \inf_K \kappa_n(P)\rho^2$ satisfies $m_n(K) \to \infty$. By the assumed uniform bound,

$$\int_K |\delta\theta_n|^2\sqrt{h}\,\mathrm{d}^3x \le \frac{1}{m_n(K)}\int_K \kappa_n(P)\rho^2|\delta\theta_n|^2\sqrt{h}\,\mathrm{d}^3x \le \frac{C}{m_n(K)} \to 0.$$

Hence $\delta\theta_n \to 0$ in $L^2(K)$. Since $K \Subset \Omega$ was arbitrary, the convergence is local. □

This theorem is the engine driving every result below.

3. Dynamical Trigger Functionals

In Paper I, the trigger functional P was treated as an external scalar on $\mathcal{M}$; its variation δP did not enter the symplectic computation. For applications where P is itself constructed from the dynamical field and metric—$P = P[\Phi, g, \nabla\Phi, R_{\mu\nu\rho\sigma}, \cdots]$—the variation $\delta\kappa(P) = \kappa'(P)\delta P$ generates additional contributions to the presymplectic potential. The purpose of this section is to identify a quantitative class of trigger functionals for which these contributions do not invalidate the local finite-action argument, and to prove that the suppression conclusion $\delta\theta = 0$ persists.

The argument has the following structure. We single out, in Definition 3.2, the regularity required of an admissible dynamical trigger; we make the "subleading" condition precise as a quantitative bound (Definition 3.3); and we prove, in Theorem 3.1, that under these conditions the phase-suppression conclusion of Theorem 2.1 survives. The technical core is to show that the additional terms produced by δP are absorbed into the lower-order constants in the proof of Theorem 2.1 without disturbing the divergent $m_n(K) \to \infty$ estimate.

3.1. The Stiffness-Weighted Quadratic Admissibility Form

We begin by formalising the finite-action condition of Definition 2.3 as a quadratic form on phase variations. This is the natural object on which the suppression argument acts, and stating it explicitly removes the need for first-variation language in what follows.

Definition 3.1 (Stiffness-weighted quadratic admissibility form). Let $\kappa_n(P)$ be a sequence of positive phase-stiffness coefficients with $\kappa_n(P) \to \infty$ uniformly on compact subsets of $\Omega \subseteq \Sigma \cap \mathcal{M}_{\text{dens}}$. For $K \Subset \Omega$ and a phase variation $\delta\theta \in H^1(K)$, define

$$Q_n^K(\delta\theta) := \int_K \kappa_n(P)\rho^2 \left[h^{ij} D_i \delta\theta D_j \delta\theta + |\delta\theta|^2 \right] \sqrt{h} \mathrm{d}^3 x. \tag{7}$$

The divergent lower-bound coefficient on K is

$$m_n(K) := \inf_K \kappa_n(P)\rho^2, \quad m_n(K) \to \infty. \tag{8}$$

The form Q_n^K is positive-definite (since h^{ij} is Riemannian and the zero-order term is non-negative). Finite-action admissibility (Definition 2.3) is the condition that $Q_n^K(\delta\theta)$ remains uniformly bounded as $n \to \infty$.

3.2. Admissible Dynamical Triggers

Definition 3.2 (Admissible dynamical trigger). A scalar functional $P : \mathcal{C} \to C^\infty(\mathcal{M})$ is an admissible dynamical trigger if there exist locally bounded smooth functions A_ρ, A_θ on $\mathcal{M}$, a tensor field $A_g^{\mu\nu}$, and a smooth vector field $B^\mu(\delta)$ depending linearly on the variation $\delta = (\delta\rho, \delta\theta, \delta g_{\mu\nu})$ such that

$$\delta P = A_\rho \delta\rho + A_\theta \delta\theta + A_g^{\mu\nu} \delta g_{\mu\nu} + \nabla_\mu B^\mu(\delta). \tag{9}$$

Each coefficient is required to be locally bounded on every compact subset of $\mathcal{M}$, and the boundary term $B^\mu(\delta)$ is required to satisfy the support or fall-off conditions imposed by the variational principle (e.g., compact support away from $\partial\Sigma$, or vanishing at infinity).

The decomposition (9) captures the natural form taken by the variation of any local scalar functional built from ρ, θ, and g: a sum of pointwise multiplicative terms plus a divergence. Curvature-built triggers such as the Kretschmann scalar $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, energy-density triggers $T_{\mu\nu}T^{\mu\nu}$, and order-parameter triggers like $|\Phi|^2$ all satisfy Definition 3.2.

Definition 3.3 (Subleading dynamical-trigger correction). Let $P[\Phi, g]$ be an admissible dynamical trigger. The correction induced by $\delta\kappa_n(P) = \kappa_n'(P)\delta P$ to the quadratic admissibility form is *subleading* relative to Q_n^K if, for every compact $K \Subset \Omega \subseteq \Sigma \cap \mathcal{M}_{\text{dens}}$, there exist sequences $\epsilon_n(K) \to 0$ and constants C_K (independent of n) such that

$$\left|\Delta Q_n^K(\delta\theta; \delta\rho, \delta g)\right| \le \epsilon_n(K)\, m_n(K) \|\delta\theta\|_{L^2(K)}^2 + C_K\left(\|\delta\rho\|_{H^1(K)}^2 + \|\delta g\|_{H^1(K)}^2 + 1\right), \quad (10)$$

where ΔQ_n^K denotes the contribution to the quadratic admissibility form on K induced by $\delta\kappa_n(P)$ acting on $-\frac{1}{2}\rho^2 h^{ij} D_i\theta D_j\theta$ (the spatial restriction of the phase-stiffness density).

The bound (10) is on the unconstrained variations and uses the spatial integration on $K \Subset \Sigma \cap \mathcal{M}_{\text{dens}}$, matching the spatial scope of the admissibility form Q_n^K. The first term on the right-hand side allows the dynamical-trigger correction to grow with $\delta\theta$, but only at a rate $\epsilon_n(K) \cdot m_n(K)$ slower than the leading positive contribution $m_n(K)\|\delta\theta\|_{L^2}^2$ in Q_n^K. The second term controls the dependence on amplitude and metric variations.

Remark 3.1 (Conditions under which the subleading bound holds). The subleading condition (10) is not automatic in general; it requires structural cancellation between δP and the phase-stiffness density. Two illustrative cases:

- For the order-parameter trigger $P = \rho^2$, one has $\delta P = 2\rho\delta\rho$, so $A_\theta = 0$ and the dynamical-trigger correction has no $\delta\theta$ dependence at leading order; the first term of (10) vanishes and only the second term contributes.
- For curvature triggers like $P = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, δP depends only on $\delta g_{\mu\nu}$ and its derivatives, again with no $\delta\theta$ dependence at leading order; the same conclusion follows.

For triggers with $A_\theta \ne 0$ (e.g., P depending explicitly on θ), the subleading condition may require additional smallness of δP or a specific growth rate on $\kappa_n'(P)/\kappa_n(P)$. The condition is a quantitative hypothesis to be verified in each application, not an automatic consequence of $\kappa_n'(P)$ not growing faster than $\kappa_n(P)$.

3.3. Persistence of Phase Suppression

Theorem 3.1 (Persistence of phase suppression for admissible dynamical triggers). Let P be an admissible dynamical trigger whose induced correction to the quadratic admissibility form is subleading in the sense of Definition 3.3. Let $K \Subset \Sigma \cap \mathcal{M}_{\text{dens}}$ with $\rho \ge \rho_0 > 0$ on K, and let $\kappa_n(P) \to \infty$ uniformly on K. Suppose the total admissibility form (including the trigger correction) is uniformly bounded:

$$\sup_n \left[Q_n^K(\delta\theta_n) + \Delta Q_n^K(\delta\theta_n;\delta\rho_n,\delta g_n) \right] < \infty,$$

and $\sup_n \|\delta\rho_n\|_{H^1(K)}, \sup_n \|\delta g_n\|_{H^1(K)}$ are finite. Then $\delta\theta_n \to 0$ in $L^2(K)$. Consequently, the limiting dense-stratum admissible tangent space satisfies $\delta\theta = 0$ on $\mathcal{M}_{\text{dens}}$.

Proof. The positive part of Q_n^K gives the lower bound

$$m_n(K)\int_K |\delta\theta_n|^2 \sqrt{h}\,\mathrm{d}^3x \le Q_n^K(\delta\theta_n).$$

The dynamical-trigger correction satisfies, by Definition 3.3,

$$\left|\Delta Q_n^K(\delta\theta_n;\delta\rho_n,\delta g_n)\right| \le \epsilon_n(K)\, m_n(K)\|\delta\theta_n\|_{L^2(K)}^2 + C_K\left(\|\delta\rho_n\|_{H^1(K)}^2 + \|\delta g_n\|_{H^1(K)}^2 + 1\right).$$

Combining,

$$m_n(K)\|\delta\theta_n\|_{L^2(K)}^2 \le Q_n^K(\delta\theta_n) \le \left|Q_n^K + \Delta Q_n^K\right| + \left|\Delta Q_n^K\right|.$$

The first term on the right is uniformly bounded by hypothesis (call the bound C^*). The second satisfies the displayed bound. Substituting and rearranging,

$$m_n(K)\left(1-\epsilon_n(K)\right)\|\delta\theta_n\|_{L^2(K)}^2 \le C^* + C_K\left(\sup_n \|\delta\rho_n\|_{H^1(K)}^2 + \sup_n \|\delta g_n\|_{H^1(K)}^2 + 1\right) =: C^{**}.$$

Since $\epsilon_n(K) \to 0$, the factor $\left(1-\epsilon_n(K)\right)$ exceeds 1/2 for n large, and

$$\|\delta\theta_n\|_{L^2(K)}^2 \le \frac{2C^{**}}{m_n(K)} \to 0.$$

Hence $\delta\theta_n \to 0$ in $L^2(K)$. □

Remark 3.2 (Symplectic correction from a dynamical trigger). Theorem 3.1 establishes the persistence of phase suppression but does not claim that the presymplectic structure on the dense stratum is unchanged. The dynamical-trigger variation (9) contributes an additional term to the presymplectic potential current of the form $\Theta_{\text{trig}}^{\mu}(\delta) = -\frac{1}{2}\rho^2\nabla^\mu\theta\nabla^\alpha\theta B_\alpha(\delta) + (\text{lower order})$, with B_α the boundary contribution of (9). On the dense stratum, $\delta\theta = 0$ kills any term containing $\delta\theta$ as a factor; however, terms containing $\nabla^\alpha\theta$ (the background phase gradient, not its variation) and $B_\alpha(\delta\rho,\delta g)$ may survive and contribute to the dense-stratum presymplectic structure. These are amplitude- and metric-sector contributions; they do not restore an independent phase direction. The detailed form for the case $P = \rho^2$ is recorded in **Appendix**.

4. Global Two-Stratum Presymplectic Reduction

This section assembles the local phase-suppression mechanism into a global presymplectic structure on the solution space, and proves that the resulting structure is a stratified presymplectic space in a precise field-theoretic analogue of Sjamaar-Lerman's finite-dimensional theorem.

4.1. The Two-Stratum Solution Space

Let $\mathcal{S}$ denote the space of smooth solutions Φ of the Euler-Lagrange equations of (2), satisfying the polar-amplitude boundary conditions (Definition 2.1; Paper I, Definition 3.5) at $\partial\mathcal{M}$. The trigger functional P partitions $\mathcal{S}$ into two subsets according to whether $\mathcal{M}_{\text{dens}}$ is empty or non-empty for the given configuration:

$$\mathcal{S}_{\text{reg}} = \{\Phi \in \mathcal{S} : P(x) < P_\star \text{ for all } x \in \mathcal{M}\}, \tag{11}$$

$$\mathcal{S}_{\text{dens}} = \{\Phi \in \mathcal{S} : \mathcal{M}_{\text{dens}} \neq \varnothing\}. \tag{12}$$

We have $\mathcal{S} = \mathcal{S}_{\text{reg}} \sqcup \mathcal{S}_{\text{dens}}$ as sets, but the closure structure is non-trivial: $\mathcal{S}_{\text{reg}}$ is open in $\mathcal{S}$ in the natural topology (since $P < P_\star$ everywhere is a stable condition under small perturbations away from $\mathcal{H}$), and $\mathcal{S}_{\text{dens}}$ contains its boundary configurations where $\mathcal{M}_{\text{dens}}$ is just beginning to be non-empty.

The smoothness of each stratum is not automatic from these set-theoretic descriptions; it is a regularity hypothesis that must be stated explicitly. We isolate it now.

Assumption 4.1 (Stratum regularity). Equipped with the induced solution-space topology and the polar-amplitude boundary conditions of Paper I, the regular and dense sectors $\mathcal{S}_{\text{reg}}$ and $\mathcal{S}_{\text{dens}}$ are smooth Fréchet submanifolds (or smooth strata of a stratified Fréchet space) on which the presymplectic current and its boundary pullbacks are well-defined. The threshold set $\mathcal{H}$ is a regular level set of P on $\mathcal{M}$ in the sense that $dP \neq 0$ on $\mathcal{H}$, providing the transversality required for the partition of solution space to be well-posed.

Assumption 4.1 is a functional-analytic regularity hypothesis on the solution space, parallel to the regularity hypotheses standard in the covariant phase-space literature [4]-[8]. We do not attempt to derive it in general; for specific Lagrangians and boundary conditions, smoothness of the solution space is a separate analytic question, typically handled by elliptic-regularity arguments adapted to the variational principle. Within this paper, the assumption is sufficient. To keep the analytic input separate from the proved consequence, however, we now identify one standard model class in which Assumption 4.1 is expected to hold, together with a sketch of the elliptic-regularity argument that anchors it.

Remark 4.1 (A model class anchoring Assumption 4.1). One standard class in which Assumption 4.1 is expected to hold is the semilinear massive complex scalar equation on a globally hyperbolic manifold with smooth timelike boundary,

$$\left(\Box_g + m^2\right)\Phi + N(\Phi, \nabla\Phi) = 0, \tag{13}$$

where N is smooth and of at most polynomial growth on the bounded field ranges considered, together with the polar-amplitude boundary conditions of Definition 2.1. The associated stratum-regularity statement is anchored by the following standard argument, which we sketch rather than carry out in full since it is not a new result of the present paper.

Step 1 (energy estimates). On each Cauchy slice Σ, linearising (13) about a smooth background solution gives a normally hyperbolic operator L whose principal part is $\Box_g$. For the polar-amplitude boundary conditions (3), the boundary value problem for L is well posed in the Sobolev scale: standard hyperbolic energy estimates (Gronwall applied to the natural energy associated to L) [9] give continuous dependence of solutions on Cauchy and boundary data,

$$\|\delta\Phi\|_{H^k(\Sigma_\tau)} \le C(\tau)\left(\|\delta\Phi\|_{H^k(\Sigma_0)} + \|\text{boundary data}\|_{H^{k-1/2}(\partial\mathcal{M})}\right),$$

for k sufficiently large that $H^k \hookrightarrow C^2$ by Sobolev embedding.

Step 2 (elliptic regularity on each slice). After fixing the Cauchy foliation, the spatial part of L is a second-order elliptic operator on (Σ, h) with the elliptic boundary conditions induced by (3) on $\partial\Sigma$. Interior and boundary elliptic regularity (the standard $H^k \to H^{k+2}$ shift for elliptic systems with complementing boundary conditions) [10] upgrades the H^k solutions of Step 1 to C^∞ solutions on each open stratum, provided the boundary data are smooth and the compatibility conditions at the corner $\partial\Sigma$ hold to all orders. The positive amplitude bound $\rho \ge \rho_0 > 0$ used throughout keeps the polar change of variables (1) a smooth diffeomorphism, so amplitude and phase inherit the same regularity as Φ.

Step 3 (smooth strata via the implicit function theorem). Let P be a regular level-set functional with $dP \ne 0$ on $\mathcal{H}$ (Assumption 4.1). The map $\Phi \mapsto \min_x \left(P[\Phi, g](x) - P_\star\right)$ is then C^1 on the Fréchet solution chart provided by Steps 1-2, with surjective differential at configurations where the minimum is attained transversally. The implicit-function theorem on the corresponding tame Fréchet chart (Nash-Moser in the smooth category) [11] yields smooth local strata $\mathcal{S}_{\text{reg}}$ and $\mathcal{S}_{\text{dens}}$ separated by the transversal threshold locus. This is the precise sense in which Assumption 4.1 is *expected* to follow from standard analysis for the model class (13): it is not an additional physical postulate, but the standard regularity output for smooth semilinear wave equations with smooth boundary data and a regular threshold set.

We emphasise that the present paper still *assumes* stratum regularity rather than proving it: the sketch above identifies the standard construction it rests on for a concrete model class, in direct response to the requirement that the analytic input be separated from, and anchored independently of, the proved consequences in Theorem 4.1. A complete proof of well-posedness and stratum smoothness for (13) with the boundary conditions (3) is a self-contained problem in the analysis of boundary-value problems for nonlinear wave equations and is outside the scope of a structural theorem in covariant phase space.

Definition 4.1 (Stratum-wise admissible tangent spaces). At $\Phi \in \mathcal{S}_{\text{reg}}$, the regular admissible tangent space is

$$T_\Phi \mathcal{S}_{\text{reg}}^{\text{adm}} = \left\{(\delta\rho, \delta\theta) : \delta\rho, \delta\theta \in H^1(\Sigma)\right\}.$$

At $\Phi \in \mathcal{S}_{\text{dens}}$, the dense admissible tangent space is

$$T_\Phi \mathcal{S}_{\text{dens}}^{\text{adm}} = \left\{(\delta\rho, \delta\theta) : \delta\rho \in H^1(\Sigma), \delta\theta \in H^1(\Sigma), \delta\theta = 0 \text{ on } \mathcal{M}_{\text{dens}}\right\}.$$

The strict inclusion $T_\Phi \mathcal{S}_{\text{dens}}^{\text{adm}} \subsetneq T_\Phi \mathcal{S}_{\text{reg}}^{\text{adm}}$ is the field-theoretic statement of the strata being of different rank.

4.2. The Presymplectic Forms on Each Stratum

The presymplectic potential current associated to (2) is, in polar variables,

$$\Theta^{\mu}(\delta) = -\nabla^{\mu}\rho\delta\rho - \kappa(P)\rho^{2}\nabla^{\mu}\theta\delta\theta. \tag{14}$$

The presymplectic current is the antisymmetrisation

$$\omega^{\mu}(\delta_1,\delta_2) = \delta_1\Theta^{\mu}(\delta_2) - \delta_2\Theta^{\mu}(\delta_1) = \omega^{\mu}_{\rho}(\delta_1,\delta_2) + \omega^{\mu}_{\theta}(\delta_1,\delta_2), \tag{15}$$

with the amplitude- and phase-sector contributions given by

$$\omega^{\mu}_{\rho}(\delta_1,\delta_2) = -\delta_1(\nabla^{\mu}\rho)\delta_2\rho + \delta_2(\nabla^{\mu}\rho)\delta_1\rho, \tag{16}$$

$$\omega^{\mu}_{\theta}(\delta_1,\delta_2) = -\delta_1\left(\kappa(P)\rho^{2}\nabla^{\mu}\theta\right)\delta_2\theta + \delta_2\left(\kappa(P)\rho^{2}\nabla^{\mu}\theta\right)\delta_1\theta. \tag{17}$$

Each phase-sector term contains $\delta_i\theta$ as a factor.

The augmented presymplectic form on each stratum is

$$\Omega^{\text{aug}}_{\Sigma}(\delta_1,\delta_2) = \int_{\Sigma}\omega^{\mu}(\delta_1,\delta_2)n_{\mu}\sqrt{h}\,\mathrm{d}^3x + \int_{\partial\Sigma}\alpha(\delta_1,\delta_2), \tag{18}$$

with $\alpha = \delta_1\theta\delta_2\hat{t} - \delta_2\theta\delta_1\hat{t}$ as in Paper I.

4.3. Stratum-Wise Hypersurface Independence

Lemma 4.1 (Stratum-wise hypersurface independence). On each stratum $\mathcal{S}_{\text{reg}}$ and $\mathcal{S}_{\text{dens}}$, the augmented presymplectic form $\Omega^{\text{aug}}_{\Sigma}$ is independent of the choice of Cauchy hypersurface Σ, provided that the polar-amplitude boundary conditions (Definition 2.1; Paper I, Definition 3.5) are imposed.

Proof. The proof is the standard hypersurface-independence argument from Paper I (Theorem 3.2 there) applied stratum by stratum. On the regular stratum, the argument is the verbatim Paper I result. On the dense stratum, the bulk part Ω_{Σ} of $\Omega^{\text{aug}}_{\Sigma}$ involves the phase-sector current ω^{μ}_{θ}, which contains $\delta_i\theta$ as a factor; on $\mathcal{S}_{\text{dens}}$, this factor vanishes by Definition 4.1, so the phase-sector contribution to the bulk integral vanishes pointwise. The remaining amplitude-sector contribution and the boundary integral satisfy the hypersurface-independence relation from Paper I unchanged. □

4.4. The Global Reduction Theorem

Theorem 4.1 (Global two-stratum presymplectic structure). Under Assumption 4.1 (stratum regularity), the positive-amplitude condition $\rho \ge \rho_0 > 0$ on $\Sigma \cap \mathcal{M}_{\text{dens}}$, and the finite-action admissibility condition of Definition 2.3 (or Theorem 3.1 in the dynamical-trigger case), the solution space $\mathcal{S}$ carries a two-stratum presymplectic structure

$$\mathcal{S} = \mathcal{S}_{\text{reg}} \sqcup \mathcal{S}_{\text{dens}}$$

with the following properties:

(i) By Assumption 4.1, the regular and dense sectors $\mathcal{S}_{\text{reg}}$ and $\mathcal{S}_{\text{dens}}$ are smooth strata of the solution space.

(ii) The augmented presymplectic form $\Omega^{\text{aug}}_{\Sigma}$ restricts to a closed antisymmetric bilinear form on each stratum, hypersurface-independent within that stratum.

(iii) The strict tangent inclusion $T_\Phi \mathcal{S}_{\text{dens}}^{\text{adm}} \subsetneq T_\Phi \mathcal{S}_{\text{reg}}^{\text{adm}}$ holds at every Φ admitting both regular and dense restrictions; explicitly, the phase-sector tangent direction is admissible on $\mathcal{S}_{\text{reg}}$ but excluded from $T_\Phi \mathcal{S}_{\text{dens}}^{\text{adm}}$.

(iv) For dense-stratum admissible variations, the phase-sector contribution to the presymplectic current vanishes on the dense region of the slice:

$$\omega_\theta^\mu(\delta_1,\delta_2)=0 \text{ on } \Sigma\cap\mathcal{M}_{\text{dens}}.$$

The full bulk phase-sector contribution on all of Σ vanishes only if $\Sigma \subseteq \mathcal{M}_{\text{dens}}$, or if the admissible variations or integration domain are restricted to the dense region.

(v) The reduced phase space on each stratum is the presymplectic quotient by $\ker\Omega_\Sigma^{\text{aug}}$ restricted to that stratum:

$$\mathcal{P}_{\text{reg}} = \mathcal{S}_{\text{reg}} \big/ \ker\Omega_\Sigma^{\text{aug}}\big|_{\mathcal{S}_{\text{reg}}},\ \mathcal{P}_{\text{dens}} = \mathcal{S}_{\text{dens}} \big/ \ker\Omega_\Sigma^{\text{aug}}\big|_{\mathcal{S}_{\text{dens}}},$$

and the dense stratum has lower phase-sector rank than the regular stratum (Definition 4.2 below).

Proof. (i) This is Assumption 4.1; we do not prove smoothness of the strata from set-theoretic descriptions but invoke the assumption directly.

(ii) Antisymmetry and bilinearity of $\Omega_\Sigma^{\text{aug}}$ are inherited from the bulk current ω^μ and the boundary density α. Closure on each stratum is the variational identity $d\Omega_\Sigma^{\text{aug}}=0$, which holds wherever $\Omega_\Sigma^{\text{aug}}$ is defined as the variation of an action (Lee-Wald [4], Iyer-Wald [5]). Hypersurface independence within each stratum is Lemma 4.1.

(iii) The strict inclusion is the content of Definition 4.1, supplemented by Theorem 2.1 which derives the dense-stratum constraint $\delta\theta=0$ on $\mathcal{M}_{\text{dens}}$ from finite-action admissibility.

(iv) For $\Phi\in\mathcal{S}_{\text{dens}}$ and dense-stratum admissible variations δ_1,δ_2, we have $\delta_i\theta=0$ on $\mathcal{M}_{\text{dens}}$ (by Definition 4.1). Each term in ω_θ^μ contains $\delta_i\theta$ as a multiplicative factor; hence $\omega_\theta^\mu(\delta_1,\delta_2)=0$ pointwise on $\Sigma\cap\mathcal{M}_{\text{dens}}$. On $\Sigma\cap\mathcal{M}_{\text{reg}}$, the phase variations $\delta_i\theta$ may be non-zero, and ω_θ^μ contributes; we make no claim about that contribution. The localised statement is what is needed for the rank-reduction conclusion of (v).

(v) The quotients are well-defined since $\ker\Omega_\Sigma^{\text{aug}}\big|_{\text{stratum}}$ is closed under the natural Fréchet topology. The strict inclusion in (iii) implies $\ker\Omega_\Sigma^{\text{aug}}\big|_{\mathcal{S}_{\text{dens}}} \supsetneq \ker\Omega_\Sigma^{\text{aug}}\big|_{\mathcal{S}_{\text{reg}}}$ when both are interpreted as subspaces of the unconstrained tangent bundle (in the sense of description (b) of Paper I, Remark 2.2). The rank statement is Definition 4.2 below. □

4.5. Phase-Sector Rank

To formalise the rank statement in clause (v) of Theorem 4.1 without invoking infinite-dimensional dimension comparisons, we record the following.

Definition 4.2 (Phase-sector rank). The *phase-sector rank* of a stratum at a point Φ is the number of independent phase-variation directions admitted by the finite-action tangent space at Φ. In the regular stratum, $\delta\theta$ is an independent phase variation; in the dense stratum, finite-action admissibility imposes $\delta\theta = 0$ and removes this direction. The dense stratum therefore has *lower phase-sector rank* than the regular stratum.

Proposition 4.1 (Phase-sector rank reduction). In the regular stratum, admissible tangent vectors in polar variables have the form

$$\delta\Phi = \mathrm{e}^{i\theta}\delta\rho + i\rho\mathrm{e}^{i\theta}\delta\theta.$$

In the dense stratum, finite-action admissibility imposes $\delta\theta = 0$ on $\mathcal{M}_{\mathrm{dens}}$, so dense-stratum admissible tangent vectors have the restricted form

$$\delta\Phi = \mathrm{e}^{i\theta}\delta\rho \text{ on } \mathcal{M}_{\mathrm{dens}}.$$

The dense stratum therefore has lower phase-sector rank than the regular stratum.

Proof. The regular-stratum expression follows from the polar decomposition. The dense-stratum restriction follows from Theorem 2.1. Since the independent $\delta\theta$ direction is present in the regular stratum but excluded from the dense stratum, the phase-sector rank is strictly lower on the dense stratum. □

Remark 4.2 (Comparison with Sjamaar-Lerman). The Sjamaar-Lerman theorem [2] concerns symplectic reduction by a Hamiltonian group action on a finite-dimensional symplectic manifold, where the orbit-type stratification of the zero level set of the moment map yields a stratified symplectic space whose strata carry compatible symplectic forms. It sits in the lineage of Marsden-Weinstein symplectic reduction [12], while the stratified structure of the space of solutions of a covariant field equation was first analysed in the Einstein case by Arms, Marsden and Moncrief [13]. Theorem 4.1 gives a field-theoretic analogue: the strata are determined not by orbit type but by an energetic admissibility condition, and the compatible structure on each stratum is presymplectic rather than symplectic until quotient by the stratum-specific kernel. The closure relation $\overline{\mathcal{S}_{\mathrm{reg}}} \supseteq \mathcal{H}$-boundary configurations (where $\mathcal{M}_{\mathrm{dens}}$ first becomes non-empty) plays the role of the closure ordering in Sjamaar-Lerman. The analogy is structural rather than literal: we do not invoke the moment-map machinery, and the strata here are not orbit types of any obvious group action.

4.6. A Non-Vacuous Model: The Mexican-Hat Scalar on a Half-Space

The assumptions used in the principal theorems—the dynamical-trigger persistence theorem (Theorem 3.1), the global reduction theorem (Theorem 4.1), the cocycle non-triviality theorem (Theorem 6.1), and the converse theorem (Theorem 8.1)—could in principle be mutually inconsistent. We now exhibit a single explicit configuration realising all of them simultaneously, establishing non-vacuity.

Consider the stiffness-coupled Mexican-hat scalar on the half-space

$$\mathcal{M}=\{(t,x,y,z)\in\mathbb{R}^{1,3}:y\ge 0\},\ \partial\mathcal{M}=\{y=0\},$$

with action

$$S_\epsilon[\rho,\theta]=\int_{\mathcal{M}}\left[-\frac{1}{2}\nabla_\mu\rho\nabla^\mu\rho-\frac{1}{2}\kappa_\epsilon(\rho^2)\rho^2\nabla_\mu\theta\nabla^\mu\theta-\frac{\lambda}{4}(\rho^2-v^2)^2\right]\epsilon_g.$$

Take the order-parameter trigger and threshold

$$P=|\Phi|^2=\rho^2,\ P_\star=\eta^2v^2,\ 0<\eta<1,$$

with divergent stiffness on the dense side,

$$\kappa_\epsilon(P)=\kappa_0+\frac{\kappa_1}{(P-P_\star+\epsilon)^q},\ q>0,\ \epsilon>0,$$

extended smoothly and positively to $P<P_\star$. On the shifted kink background

$$\rho_0(x)=v\tanh\left(\frac{m(x-x_0)}{\sqrt{2}}\right),\ x_0<0,\ \theta_0=0,$$

the threshold surface is the finite regular plane

$$\mathcal{H}=\{x=x_\star\},\ x_\star=x_0+\frac{\sqrt{2}}{m}\operatorname{arctanh}(\eta).$$

Thus

$$\mathcal{M}_{\text{reg}}=\{y\ge 0,x<x_\star\},\ \mathcal{M}_{\text{dens}}=\{y\ge 0,x>x_\star\},$$

and, because the boundary $\{y=0\}$ is independent of the threshold coordinate x, the threshold has a nonempty boundary trace

$$\mathcal{H}\cap\partial\mathcal{M}=\{y=0,x=x_\star\},$$

with a correspondingly nonempty dense boundary

$$\partial\Sigma\cap\mathcal{M}_{\text{dens}}=\{y=0,x>x_\star\}.$$

Since

$$\nabla|\Phi_0|^2=2\rho_0\rho_0'dx\neq 0\ \text{ at }\ x=x_\star,$$

the threshold is a regular level set and it meets the boundary transversally, so Assumption 4.1(transversality) and Assumption 7.1(vi) hold; on any slab $\{|x-x_\star|\le\delta\}$ the kink amplitude is bounded below, $\rho_0\ge\rho_0(x_\star-\delta)>0$, giving the positive-amplitude hypothesis. We now apply the four theorems to this configuration.

Theorem 3.1 (dynamical trigger). Here $P=\rho^2$ is dynamical, with $\delta P=2\rho\delta\rho$ and hence $A_\theta=0$ in Definition 3.2. As recorded in Remark 3.1, the induced correction ΔQ_n^K then carries no leading $\delta\theta$ term, so the subleading bound (10) holds with $\epsilon_n(K)\equiv 0$. Theorem 3.1 applies and gives $\delta\theta=0$ on $\mathcal{M}_{\text{dens}}$.

Theorem 4.1 (global reduction). The model (13)-type field equation here is the semilinear equation $\Box\rho-\lambda\rho(\rho^2-v^2)-(\ldots)=0$ with the polar-amplitude boundary conditions of Definition 2.1 at $y=0$; by Remark 4.1 the regular and

dense regions are smooth strata, so Assumption 4.1 is realised and the two-stratum presymplectic structure of Theorem 4.1 is well defined, with
$T_\Phi \mathcal{S}^{\mathrm{adm}}_{\mathrm{dens}} \subsetneq T_\Phi \mathcal{S}^{\mathrm{adm}}_{\mathrm{reg}}$.

Theorem 6.1 (cocycle non-triviality). On $\partial\Sigma = \{y=0\}$, with boundary coordinates (x,z), choose a small neighbourhood U_∂ of a point in $\mathcal{H}\cap\partial\Sigma$ and a non-vanishing tangential vector field, for example $X = \partial_z$. With $\hat{t} = \rho_0(x)^2 > 0$ non-vanishing on U_∂, take bump functions $\phi_1, \phi_2 \in C_c^\infty(U_\partial)$ with overlapping but non-centred support along the z-flow, as in Proposition 6.1. This produces an abelian amplitude-phase nondegenerate subalgebra $\mathfrak{a}_\partial$ satisfying the Hamiltonian admissibility of Definition 2.2, so by Theorem 6.1 the cocycle K is cohomologically non-trivial for the constructed $\mathfrak{g}_\partial$.

Theorem 8.1 (boundary-local converse). On the dense boundary

$$\partial\Sigma \cap \mathcal{M}_{\mathrm{dens}} = \{y=0, x > x_\star\},$$

finite-action admissibility forces $\delta\theta = 0$, so the dense-boundary cocycle contribution

$$K_{\mathrm{dens}}(\xi,\eta) = \int_{\partial\Sigma\cap\mathcal{M}_{\mathrm{dens}}} \left(\delta_\xi\theta\,\delta_\eta\hat{t} - \delta_\eta\theta\,\delta_\xi\hat{t}\right)\mathrm{d}S$$

vanishes pointwise. The single-component half-space background carries no Φ-zeros, no winding, no amplitude vanishing, and a uniquely fixed boundary polarisation on the slab considered, so Assumption 7.1 holds; the only divergent phase-sector penalty is the stiffness one at $P \to P_\star^+$, giving Assumption 7.3; and the flow-box bump construction supplies the local separating algebra of Assumption 7.4. At boundary-accessible threshold points

$$x_0 \in \mathcal{H}\cap\partial\Sigma = \{y=0, x = x_\star\},$$

the local separating boundary algebra detects the transition from the regular boundary side $x < x_\star$ to the dense boundary side $x > x_\star$. The configuration therefore lies in the pure stiffness-induced class (Definition 7.2), and Theorem 8.1 applies in a boundary neighbourhood of x_0.

All four theorems thus hold simultaneously for one explicit model, so the hypotheses used in this paper are mutually compatible and non-vacuous in practice.

5. Boundary Algebra and the Explicit Cocycle

This section reviews, briefly, the boundary algebra and explicit cocycle of Paper I, in the form needed for the cohomological non-triviality theorem of Section 6.

Let $\mathfrak{g}_\partial$ be a Lie algebra of infinitesimal symmetries acting on boundary data on $\partial\mathcal{M}$, with elements ξ generating boundary variations $\delta_\xi\Phi$. In polar variables, $\delta_\xi\Phi = \frac{1}{2}\hat{t}^{-1/2}\mathrm{e}^{i\theta}\delta_\xi\hat{t} + i\sqrt{\hat{t}}\,\mathrm{e}^{i\theta}\delta_\xi\theta$, and we assume the Hamiltonian generator $Q_\xi : \mathcal{P} \to \mathbb{R}$ exists for each $\xi \in \mathfrak{g}_\partial$, integrable in the sense of Definition 2.2 (Paper I, Proposition 4.1).

The boundary 2-cocycle, computed in Paper I, Theorem 4.1, is

$$K(\xi,\eta)=\int_{\partial\Sigma}\left(\delta_\xi\theta\delta_\eta\hat{t}-\delta_\eta\theta\delta_\xi\hat{t}\right)\mathrm{d}S, \tag{19}$$

satisfying $\{Q_\xi,Q_\eta\}=Q_{[\xi,\eta]}+K(\xi,\eta)$ under the Poisson bracket associated to $\Omega_\Sigma^{\text{aug}}$ on $\mathcal{P}_{\text{reg}}$. The appearance of a field-dependent central term in the algebra of boundary charges is the covariant-phase-space counterpart of the surface-integral central charges first identified by Regge and Teitelboim [14] and developed in the covariant setting by Barnich and Brandt [15]; the role of the boundary polarisation in defining $\tilde{\Omega}_{\partial\Sigma}$ follows the local-subsystem analysis of Donnelly and Freidel [16].

The dense-boundary contribution to K is, by definition,

$$K_{\text{dens}}(\xi,\eta):=\int_{\partial\Sigma\cap\mathcal{M}_{\text{dens}}}\left(\delta_\xi\theta\delta_\eta\hat{t}-\delta_\eta\theta\delta_\xi\hat{t}\right)\mathrm{d}S, \tag{20}$$

i.e., the same integrand integrated over the dense portion of the boundary only.

Proposition 5.1 (Dense-boundary cocycle suppression). If $\xi,\eta\in\mathfrak{g}_\partial$ generate finite-action admissible dense-stratum variations, then $K_{\text{dens}}(\xi,\eta)=0$.

Proof. Dense-stratum admissibility requires $\delta_\xi\theta=\delta_\eta\theta=0$ on $\partial\Sigma\cap\mathcal{M}_{\text{dens}}$. Each term of (20) contains a phase variation as a factor, so the integrand vanishes pointwise. The integral is therefore zero. □

Remark 5.1. The proposition concerns K_{dens}, not the global K. Contributions from $\partial\Sigma\cap\mathcal{M}_{\text{reg}}$, where the phase variations may be non-zero, are not suppressed by this argument; see Section 9.

6. Cohomological Non-Triviality of the Boundary Cocycle

This section closes Gap (3) of Paper I: the explicit cocycle K of (19) was called "explicitly represented" rather than "non-trivial," because Paper I did not rule out the possibility that K is a coboundary—$K(\xi,\eta)=B([\xi,\eta])$ for some linear $B:\mathfrak{g}_\partial\to\mathbb{R}$—which would mean its appearance in the boundary algebra is removable by a redefinition of the charges Q_ξ. Here we give a sufficient condition under which K defines a non-trivial class in $H^2(\mathfrak{g}_\partial,\mathbb{R})$, and we exhibit a concrete construction of a $\mathfrak{g}_\partial$ for which the condition is met. Throughout, $H^\bullet(\mathfrak{g}_\partial,\mathbb{R})$ denotes the standard Chevalley-Eilenberg cohomology of the Lie algebra $\mathfrak{g}_\partial$ with trivial real coefficients [17]; for the interpretation of $H^2(\mathfrak{g}_\partial,\mathbb{R})$ as the classifier of central extensions, and its role in physical symmetry algebras, see de Azcárraga and Izquierdo [18].

6.1. The Abelian Hamiltonian Test Subalgebra Criterion

Assumption 6.1 (Hamiltonian boundary test algebra). The boundary test subalgebras $\mathfrak{a}_\partial\subseteq\mathfrak{g}_\partial$ considered below consist of boundary symmetry generators that preserve the polar-amplitude boundary conditions (Definition 2.1) and admit integrable Hamiltonian generators with respect to the augmented presymplectic form $\Omega_\Sigma^{\text{aug}}$, in the sense of Definition 2.2 (Paper I, Proposition 4.1).

Assumption 6.1 is a non-trivial admissibility condition: it requires the boundary symmetries used in the cocycle analysis to be honest Hamiltonian symmetries of the augmented covariant phase space, not merely formal infinitesimal transformations of boundary data. We verify it explicitly for the construction of Proposition 6.1 below.

Definition 6.1 (Amplitude--phase nondegeneracy). An abelian Lie subalgebra $\mathfrak{a}_\partial \subseteq \mathfrak{g}_\partial$ satisfying Assumption 6.1 is *amplitude-phase nondegenerate* if there exist $\xi, \eta \in \mathfrak{a}_\partial$ such that

$$\int_{\partial\Sigma} \left(\delta_\xi \theta \delta_\eta \hat{t} - \delta_\eta \theta \delta_\xi \hat{t} \right) \mathrm{d}S \neq 0. \tag{21}$$

Theorem 6.1 (Cohomological non-triviality of K). Let $\mathfrak{g}_\partial$ contain an abelian amplitude-phase nondegenerate subalgebra $\mathfrak{a}_\partial$ satisfying Assumption 6.1. Then the explicit boundary cocycle K of (19) is not a coboundary on $\mathfrak{g}_\partial$, and hence defines a non-trivial class $[K] \in H^2(\mathfrak{g}_\partial, \mathbb{R})$.

Proof. Suppose, for contradiction, that K is a coboundary: there exists a linear functional $B: \mathfrak{g}_\partial \to \mathbb{R}$ such that $K(\xi, \eta) = B([\xi, \eta])$ for every $\xi, \eta \in \mathfrak{g}_\partial$. Restrict to the subalgebra $\mathfrak{a}_\partial$. Since $\mathfrak{a}_\partial$ is abelian, $[\xi, \eta] = 0$ for every $\xi, \eta \in \mathfrak{a}_\partial$, so

$$K(\xi, \eta) = B([\xi, \eta]) = B(0) = 0 \text{ for all } \xi, \eta \in \mathfrak{a}_\partial.$$

This contradicts the assumed amplitude-phase nondegeneracy of $\mathfrak{a}_\partial$, which provides $\xi, \eta \in \mathfrak{a}_\partial$ with $K(\xi, \eta) \neq 0$. Therefore K is not a coboundary, and $[K] \neq 0$ in $H^2(\mathfrak{g}_\partial, \mathbb{R})$. □

6.2. Explicit Construction of an Abelian Test Subalgebra

The criterion of Theorem 6.1 is non-vacuous only if there exists a $\mathfrak{g}_\partial$ admitting such an $\mathfrak{a}_\partial$. We now exhibit one.

Proposition 6.1 (Existence of an amplitude--phase nondegenerate abelian Hamiltonian subalgebra). Suppose $\partial\Sigma$ admits a non-vanishing smooth vector field X on an open subset $U_\partial \subseteq \partial\Sigma$, and that $\hat{t}$ is non-vanishing on U_∂. Then there exists a Lie algebra $\mathfrak{g}_\partial$ of boundary symmetries containing an abelian amplitude-phase nondegenerate subalgebra $\mathfrak{a}_\partial$ satisfying Assumption 6.1.

Proof. Let $\phi_1, \phi_2 \in C_c^\infty(U_\partial)$ be two smooth, real-valued, compactly supported functions on U_∂. Define their X-derivatives by the Lie derivative $\phi_{i'} := \mathcal{L}_X \phi_i$, which is a coordinate-free derivative along X that exists wherever X does. Choose ϕ_1, ϕ_2 such that

$$I(\phi_1, \phi_2) := \int_{U_\partial} \left(\phi_1 \mathcal{L}_X \phi_2 - \phi_2 \mathcal{L}_X \phi_1 \right) \mathrm{d}S \neq 0. \tag{22}$$

Such functions exist: applying the Leibniz rule for $\mathcal{L}_X$, the integrand equals $2\phi_1 \mathcal{L}_X \phi_2 - \mathcal{L}_X(\phi_1 \phi_2)$. If X is divergence-free with respect to $\mathrm{d}S$ (or, more generally, the measure $\mathrm{d}S$ is X-invariant on the support of ϕ_i—true in flow-box coordinates around any non-vanishing X), then $\int_{U_\partial} \mathcal{L}_X(\phi_1 \phi_2) \mathrm{d}S = 0$ by Stokes' theorem on compact-support functions, and (22) reduces to $2\int_{U_\partial} \phi_1 \mathcal{L}_X \phi_2 \mathrm{d}S$. This is generically non-zero: for instance, take ϕ_1 a bump function and ϕ_2 a translate of ϕ_1 along the X-flow with supports overlapping but not centred, so that the asymmetric pairing under $\mathcal{L}_X$ is non-vanishing.

Define two boundary symmetry generators ξ_1, ξ_2 acting on the boundary data by

$$\delta_{\xi_i}\theta = \phi_i,\ \delta_{\xi_i}\hat{t} = \mathcal{L}_X\phi_i,\ i = 1, 2,$$

extended to act trivially on the bulk away from a neighbourhood of $\partial\Sigma$. Each ξ_i is parameterised by a fixed function $\phi_i \in C_c^\infty(U_\partial)$, so the corresponding flow is the additive shift $(\theta, \hat{t}) \mapsto (\theta + \epsilon_i\phi_i, \hat{t} + \epsilon_i\mathcal{L}_X\phi_i)$. Additive shifts of boundary fields commute pairwise: $[\xi_1, \xi_2] = 0$. Hence $\mathfrak{a}_\partial := \mathrm{span}_{\mathbb{R}}\{\xi_1, \xi_2\}$ is abelian.

Hamiltonian admissibility (Assumption 6.1). It remains to verify directly that each ξ_i preserves the recalled polar-amplitude boundary conditions of Definition 2.1 and admits an integrable Hamiltonian in the sense of Definition 2.2; the earlier informal claim that this holds "trivially" is replaced by the following check. The variations generated by ξ_i are

$$\delta_{\xi_i}\theta = \phi_i,\ \delta_{\xi_i}\hat{t} = \mathcal{L}_X\phi_i,\ \phi_i \in C_c^\infty(U_\partial).$$

Each generator is extended off $\partial\Sigma$ by a collar construction in the inward geodesic normal coordinate; the normal derivative of the collar extension of a compactly supported boundary profile is freely specifiable at $\partial\Sigma$, which is the freedom exploited below. For the second boundary condition of (3), expand using the Leibniz rule:

$$\delta_{\xi_i}\left(\hat{t}^2\nabla_n\theta\right)\Big|_{\partial\Sigma} = 2\hat{t}\left(\mathcal{L}_X\phi_i\right)\nabla_n\theta + \hat{t}^2\nabla_n\phi_i.$$

Rather than requiring the phase profile to be normally constant, choose the collar extension of ϕ_i so that its normal derivative at the boundary satisfies

$$\nabla_n\phi_i\big|_{\partial\Sigma} = -\frac{2\mathcal{L}_X\phi_i}{\hat{t}}\nabla_n\theta\Big|_{\partial\Sigma}.$$

This choice is possible for any smooth compactly supported boundary profile because the normal derivative of the collar extension is freely specifiable at $\partial\Sigma$, and $\hat{t} \neq 0$ on U_∂ by hypothesis. Substituting,

$$\delta_{\xi_i}\left(\hat{t}^2\nabla_n\theta\right)\Big|_{\partial\Sigma} = 2\hat{t}\left(\mathcal{L}_X\phi_i\right)\nabla_n\theta - 2\hat{t}\left(\mathcal{L}_X\phi_i\right)\nabla_n\theta = 0.$$

The first boundary condition of (3) is preserved independently by taking the amplitude profile $\mathcal{L}_X\phi_i$ to have vanishing normal derivative at $\partial\Sigma$:

$$\nabla_n\left(\mathcal{L}_X\phi_i\right)\big|_{\partial\Sigma} = 0 \Rightarrow \delta_{\xi_i}\left(\nabla_n\hat{t}\right)\big|_{\partial\Sigma} = \nabla_n\left(\mathcal{L}_X\phi_i\right)\big|_{\partial\Sigma} = 0.$$

The two prescriptions are consistent because the off-boundary collar extensions of the generated variations $\delta_{\xi_i}\theta = \phi_i$ and $\delta_{\xi_i}\hat{t} = \mathcal{L}_X\phi_i$ are chosen independently once their boundary values are fixed: $\nabla_n\phi_i\big|_{\partial\Sigma}$ is the normal derivative of the collar extension of the phase variation, prescribed as above to preserve the phase-flux datum, while $\nabla_n\left(\mathcal{L}_X\phi_i\right)\big|_{\partial\Sigma}$ is the normal derivative of the separate collar extension of the amplitude variation, prescribed to vanish so as to preserve the normal-amplitude datum. These are normal derivatives of two distinct scalar extensions, not two derivatives of one fixed extension, so the prescriptions do not interfere; they are simultaneously realisable, and no assumption $\nabla_n \theta = 0$ or support restriction is needed. Thus each ξ_i preserves both polar-amplitude boundary conditions (3). Given preservation of the boundary conditions and closure of $\Omega_\Sigma^{\text{aug}}$ (Paper I, Theorem 3.3, recalled in Definition 2.2), the integrability criterion (4) is met via Cartan's identity, so the Hamiltonian generators Q_{ξ_i} exist and $\mathfrak{a}_\partial$ satisfies Assumption 6.1.

Amplitude-phase nondegeneracy. The cocycle pairing on $\mathfrak{a}_\partial$ is

$$K(\xi_1, \xi_2) = \int_{\partial\Sigma} \left(\delta_{\xi_1} \theta \delta_{\xi_2} \hat{t} - \delta_{\xi_2} \theta \delta_{\xi_1} \hat{t} \right) \mathrm{d}S$$
$$= \int_{U_\partial} \left(\phi_1 \mathcal{L}_X \phi_2 - \phi_2 \mathcal{L}_X \phi_1 \right) \mathrm{d}S = I(\phi_1, \phi_2) \neq 0.$$

Hence $\mathfrak{a}_\partial$ is amplitude-phase nondegenerate. □

Corollary 6.1. Under the hypotheses of Proposition 6.1, the boundary cocycle K of (19) represents a non-trivial class in $H^2(\mathfrak{g}_\partial, \mathbb{R})$, with $\mathfrak{g}_\partial$ as constructed there.

Proof. Combine Theorem 6.1 and Proposition 6.1. □

Remark 6.1. The non-triviality result depends on the choice of $\mathfrak{g}_\partial$. For Lie algebras admitting only commutators that pair trivially under the cocycle integrand (e.g., a non-abelian $\mathfrak{g}_\partial$ all of whose commutators are vector fields not generating amplitude-phase pairings), the criterion of Theorem 6.1 need not apply. The proposition shows the criterion is realised; we do not claim it is realised universally.

7. Excluding Accidental Degeneracies

We now turn to Gap (4) of Paper I: making precise the conditions under which the converse direction of the Phase Boundary Characterisation Theorem holds. The forward direction—crossing $\mathcal{H}$ activates the algebraic signatures—is proved in Paper I and extended to dynamical triggers in Theorem 3.1. The converse—the algebraic signatures occur only at $\mathcal{H}$—requires that no other mechanism produces these signatures spuriously.

Definition 7.1 (Accidental degeneracy of $\Omega_\Sigma^{\text{aug}}$). An *accidental degeneracy* of $\Omega_\Sigma^{\text{aug}}$ at a point $x_0 \in \mathcal{M}$ is a kernel direction of $\Omega_\Sigma^{\text{aug}}$ at x_0 not produced by the divergence of $\kappa(P)$ at x_0. Examples:

(a) The global $U(1)$ phase rotation $\delta\Phi = i\Phi$, which is in $\ker \Omega_\Sigma^{\text{aug}}$ at every $x_0 \in \mathcal{M}$ regardless of P (a gauge degeneracy);

(b) Zeros of Φ, where the polar decomposition $\Phi = \sqrt{\hat{t}}\mathrm{e}^{i\theta}$ fails and the phase θ is undefined;

(c) Topological-sector ambiguities, where θ is multi-valued or has non-trivial winding around a closed loop, leading to $\delta\theta$-modes that are not globally defined;

(d) Amplitude-sector degeneracies, where $\rho \to 0$ or ρ has a critical point and $\nabla^\mu \rho$ vanishes;

(e) Accidental boundary degeneracies, where the polar-amplitude boundary conditions of Paper I fail to fix α uniquely (e.g., on portions of $\partial\Sigma$ where $\hat{t} = 0$).

Assumption 7.1 (No accidental degeneracies). In an open neighbourhood $U \subseteq \mathcal{M}$ of the point $x_0 \in \mathcal{H}$ under consideration, the following hold:

(i) $\Phi(x) \neq 0$ for all $x \in U$, so the polar decomposition is regular on U.

(ii) The phase θ is single-valued and globally defined on U (no topological winding).

(iii) The amplitude is bounded below: $\rho(x) \geq \rho_0 > 0$ on U.

(iv) The polar-amplitude boundary conditions (Definition 2.1; Paper I, Definition 3.5) fix the boundary symplectic density α uniquely on $\partial\Sigma \cap U$.

(v) The regular-stratum phase-sector pairing $\Omega_{\Sigma}^{\text{aug}}\big|_{\mathcal{S}_{\text{reg}}}$ is nondegenerate modulo the global $U(1)$ direction.

(vi) The trigger threshold $\mathcal{H} = \{P = P_\star\}$ is a regular level set of P on U, *i.e.*, $dP \neq 0$ on $\mathcal{H} \cap U$.

Assumption 7.2 (Prescribed or admissibly dynamical trigger). The trigger functional P is either prescribed (Paper I setting) or admissibly dynamical (Definition 3.2) with subleading correction to the quadratic admissibility form (Definition 3.3).

Assumption 7.3 (Unique stiffness threshold in the local neighbourhood). Within the neighbourhood $U \subseteq \mathcal{M}$ under consideration, the divergent phase-sector admissibility penalty is activated only as $P \to P_\star^+$. No other value of P, no boundary condition, no field coefficient in the Lagrangian, no gauge restriction, no topological sector, and no amplitude-sector mechanism produces a divergent phase-sector admissibility penalty within U.

Assumption 7.3 is a stronger condition than $dP \neq 0$ on $\mathcal{H}$ (which is Assumption 7.1(vi)): it asserts that the stiffness mechanism is the unique source of divergent phase-sector admissibility penalty within U, not merely that $\mathcal{H}$ is a regular level set. Without this, the converse direction of the characterisation theorem could be confounded by alternative threshold mechanisms producing the same algebraic signatures.

Assumption 7.4 (Local separating boundary test algebra). For every sufficiently small boundary neighbourhood $V \subseteq \partial\Sigma \cap U$, the boundary test algebra contains an abelian Hamiltonian subalgebra $\mathfrak{a}_\partial(V)$ (Assumption 6.1) of compactly supported amplitude-phase test generators that separate phase variations distributionally on V. That is: if

$$\int_V \left(\delta_\xi \theta \delta_\eta \hat{t} - \delta_\eta \theta \delta_\xi \hat{t}\right) \mathrm{d}S = 0 \quad \text{for all } \xi, \eta \in \mathfrak{a}_\partial(V),$$

then the corresponding phase variation vanishes on V in the distributional sense.

The local-separation hypothesis is the key structural input to the converse direction. The dense-boundary cocycle K_{dens} is an integral, and vanishing of an integral does not by itself imply pointwise vanishing of the integrand: integration can produce cancellation. Assumption 7.4 explicitly excludes that possibility on local neighbourhoods, by requiring that the test algebra be rich enough to detect non-zero integrands distributionally. This is precisely what is needed to convert $K_{\text{dens}}(\xi,\eta)=0$ for all ξ,η into pointwise $\delta\theta=0$, which is the engine of the converse argument. Proposition 6.1 constructs such an algebra explicitly via bump functions on a flow-box neighbourhood, so the assumption is non-vacuous.

Definition 7.2 (Pure stiffness-induced phase-sector mechanism). A phase-boundary mechanism on U is pure stiffness-induced if it satisfies Assumptions 4.1, 6.1, 7.1, 7.2, 7.3, and 7.4, and the only admissibility-changing coefficient in the phase sector is the stiffness $\kappa(P)$ (no other terms in the Lagrangian acquire P-dependence in a way that affects the finite-action condition).

Remark 7.1. The cumulative effect of Assumptions 7.1, 7.3, and 7.4 is to delete every alternative mechanism by which the algebraic signatures could occur away from $\mathcal{H}$: gauge degeneracies, zeros of Φ, topological sectors, amplitude-sector degeneracies, accidental boundary degeneracies, alternative threshold loci, and integral-cancellation effects. None of these conditions are vacuous; each excludes a distinct mechanism. The role of the cumulative assumption in the converse theorem is exactly to clear away these alternatives so that the algebraic signatures, when they occur, can only arise from the pure stiffness-induced mechanism at $\mathcal{H}$.

8. The Converse Phase Boundary Characterisation Theorem

We can now state and prove the converse theorem. Two structural points must be observed at the outset.

First, the cocycle K_{dens} is a boundary integral, computed over $\partial\Sigma\cap\mathcal{M}_{\text{dens}}$. Vanishing of an integral does not by itself imply pointwise vanishing of its integrand; what is needed for the converse direction is the local-separation property of Assumption 7.4, which guarantees that a sufficiently rich local test algebra detects pointwise vanishing distributionally. Without this, the converse argument fails.

Second, since the cocycle is boundary-supported, the natural domain of the converse theorem is the boundary itself: *boundary-accessible points* $x\in U\cap\partial\Sigma$. We do not extend the converse directly to bulk points $x\in U\setminus\partial\Sigma$; doing so would require an additional propagation argument (e.g., unique continuation for the linearised Euler-Lagrange equation), which is beyond the present scope.

Definition 8.1 (Boundary-accessible point). Let $U\subseteq\mathcal{M}$ be a neighbourhood of a threshold point and let Σ be a fixed Cauchy slice with boundary $\partial\Sigma=\Sigma\cap\partial\mathcal{M}$. A point $x\in U$ is *boundary-accessible* if

$$x\in U\cap\partial\Sigma,$$

and there exists a family of compactly supported Hamiltonian boundary test generators in $\mathfrak{g}_\partial$ (Definition 2.2), supported in arbitrarily small neighbourhoods of x, whose induced amplitude-phase variations separate the boundary cocycle integrand distributionally in the sense of Assumption 7.4. Equivalently,

x is a point at which the local boundary test algebra can detect whether the phase-sector variation $\delta\theta$ vanishes. Whenever "boundary-accessible point" is used informally in Sections 1, 8, and 9, it refers to this definition.

Theorem 8.1 (Boundary-Local Converse Phase Boundary Characterisation). Let the field theory satisfy Definition 7.2 (the *pure stiffness-induced class*) on an open neighbourhood U of a boundary-accessible threshold point $x_0 \in \mathcal{H} \cap \partial\mathcal{M}$. Then for any $x \in U \cap \partial\Sigma$, the following four conditions are equivalent:

(i) $x \in \mathcal{H} \cap \partial\Sigma$, *i.e.*, $P(x) = P_\star$ at the boundary point x.

(ii) The finite-action admissible tangent space at $\Phi \in \mathcal{S}_{\text{dens}}$ with x in its dense-boundary support loses the independent phase direction at x: $\delta\theta(x) = 0$ for all dense-stratum admissible $\delta\theta$.

(iii) The phase-sector contribution to $\Omega_\Sigma^{\text{aug}}$ at Φ has lower phase-sector rank at x than the corresponding rank on $\mathcal{S}_{\text{reg}}$ (in the sense of Definition 4.2).

(iv) For boundary symmetries ξ, η in the local separating Hamiltonian abelian subalgebra $\mathfrak{a}_\partial(V)$ of Assumption 7.4 acting on a neighbourhood $V \subseteq \partial\Sigma \cap U$ of x, the dense-boundary cocycle contribution $K_{\text{dens}}(\xi,\eta) = 0$ for all such ξ, η.

Proof. We prove the cycle (i)⇒(ii)⇒(iii)⇒(iv)⇒(i). The forward direction uses the established local theorem (Theorem 2.1, or Theorem 3.1 for the dynamical case) and the structure of the cocycle integrand. The converse direction (iv)⇒(i) uses Assumptions 7.1, 7.3, and 7.4 substantively.

(i)⇒(ii). If $P(x) = P_\star$ at the boundary point x, then any neighbourhood of x in $\partial\Sigma$ intersects $\mathcal{M}_{\text{dens}}$. The divergent stiffness $\kappa(P) \to \infty$ activates as $P \to P_\star^+$, and by Theorem 2.1 (or Theorem 3.1) finite-action admissible variations satisfy $\delta\theta = 0$ on $\mathcal{M}_{\text{dens}}$ in a neighbourhood of x. Hence $\delta\theta(x) = 0$ for any admissible $\delta\theta$.

(ii)⇒(iii). The phase-sector presymplectic current ω_θ^μ contains $\delta_i\theta$ as a multiplicative factor in every term. If $\delta\theta(x) = 0$ for all admissible variations, then every pairing of ω_θ^μ at x vanishes, and the independent phase direction is removed from the admissible tangent space. By Definition 4.2, the phase-sector rank at x on $\mathcal{S}_{\text{dens}}$ is strictly less than on $\mathcal{S}_{\text{reg}}$.

(iii)⇒(iv). The dense-boundary cocycle integrand of (20) is $\delta_\xi\theta\delta_\eta\hat{t} - \delta_\eta\theta\delta_\xi\hat{t}$. Phase-sector rank reduction at x means that the admissible variations $\delta_\xi\theta$ and $\delta_\eta\theta$ vanish at x for all $\xi, \eta \in \mathfrak{a}_\partial(V)$ acting through dense-stratum admissible variations. Hence the integrand vanishes pointwise on $V \cap \mathcal{M}_{\text{dens}}$ near x. Taking $\xi, \eta \in \mathfrak{a}_\partial(V)$ to have support in V (which is part of Assumption 7.4), the integral $K_{\text{dens}}(\xi,\eta) = \int_{\partial\Sigma \cap \mathcal{M}_{\text{dens}}} \left(\delta_\xi\theta\delta_\eta\hat{t} - \delta_\eta\theta\delta_\xi\hat{t}\right)\mathrm{d}S$ reduces to $\int_{V \cap \mathcal{M}_{\text{dens}}}$ and vanishes.

(iv)⇒(i). *This is the new direction, where Assumptions 7.1, 7.3, and 7.4 do their work.* Suppose $K_{\text{dens}}(\xi,\eta) = 0$ for all $\xi, \eta \in \mathfrak{a}_\partial(V)$, where $V \subseteq \partial\Sigma \cap U$ is a sufficiently small neighbourhood of x, and $\mathfrak{a}_\partial(V)$ is a local separating Hamiltonian abelian subalgebra (Assumption 7.4). Equivalently,

$$\int_{V\cap\mathcal{M}_{\text{dens}}}\left(\delta_\xi\theta\delta_\eta\hat{t}-\delta_\eta\theta\delta_\xi\hat{t}\right)\mathrm{d}S=0 \text{ for all } \xi,\eta\in\mathfrak{a}_\partial(V).$$

By Assumption 7.4, the test algebra $\mathfrak{a}_\partial(V)$ separates phase variations distributionally on V: this vanishing for all test pairs forces the phase variation to vanish distributionally on $V\cap\mathcal{M}_{\text{dens}}$. Concretely, the algebra is rich enough to test $\delta\theta$ against arbitrary smooth compactly-supported functions on $V\cap\mathcal{M}_{\text{dens}}$, and the antisymmetric pairing combined with vanishing for all test pairs in $\mathfrak{a}_\partial(V)$ implies $\delta\theta=0$ in the distributional sense on $V\cap\mathcal{M}_{\text{dens}}$.

This pointwise vanishing of the phase variation on $V\cap\mathcal{M}_{\text{dens}}$ requires explanation: *why* does $\delta\theta=0$ hold there? Assumptions 7.1 (i)-(v) exclude every alternative mechanism: not zeros of Φ (by (i)), not topological obstructions (by (ii)), not amplitude-sector degeneracies (by (iii)), not accidental boundary degeneracies (by (iv)), not gauge-only kernel directions in the regular pairing (by (v)). The only remaining mechanism in the pure stiffness-induced class (Definition 7.2) is the divergent phase-sector admissibility penalty produced by the stiffness $\kappa(P)\to\infty$.

By Assumption 7.3, the divergent phase-sector admissibility penalty is activated within U *only as* $P\to P_\star^+$, *i.e.*, only on $\mathcal{H}$. Hence the locus of $\delta\theta=0$ on $V\cap\mathcal{M}_{\text{dens}}$ near x must coincide with the boundary trace of $\mathcal{H}$ near x, which is to say $x\in\mathcal{H}\cap\partial\Sigma$. Therefore $P(x)=P_\star$, establishing (i). □

Remark 8.1 (Where each assumption is used in the proof). The converse direction (iv)⇒(i) uses three assumptions in distinct roles:

- Assumption 7.4 (local separation) converts vanishing of the integral $K_{\text{dens}}(\xi,\eta)=0$ into pointwise vanishing $\delta\theta=0$. Without it, the integral could vanish by cancellation, and the converse would fail.
- Assumption 7.1 (no accidental degeneracies) excludes all alternative mechanisms by which $\delta\theta$ could vanish on $V\cap\mathcal{M}_{\text{dens}}$. Without it, $\delta\theta=0$ could be caused by gauge directions, zeros of Φ, etc., at points $x\notin\mathcal{H}$.
- Assumption 7.3 (unique stiffness threshold) localises the divergent phase-sector admissibility penalty to $\mathcal{H}$ alone within U. Without it, the divergent penalty could be activated at other loci.

The cumulative effect of these three assumptions is to delete every alternative path from $K_{\text{dens}}=0$ to a locus other than $\mathcal{H}$. The forward direction (i)⇒(ii)⇒(iii)⇒(iv) uses only Theorem 2.1; none of these structural assumptions enters that direction.

Corollary 8.1 (Boundary-local algebraic recoverability of $\mathcal{H}$). Within the pure stiffness-induced class on U, the boundary trace $\mathcal{H}\cap\partial\Sigma$ is locally recoverable from the algebraic data of $\Omega_\Sigma^{\text{aug}}$ at boundary-accessible points:

$$\mathcal{H}\cap U\cap\partial\Sigma=\{x\in U\cap\partial\Sigma:\ \text{conditions (ii)-(iv) of Theorem 8.1 hold at } x\}.$$

Proof. Direct from the boundary-local equivalence (within the pure stiffness-induced class) established in Theorem 8.1. □

Remark 8.2 (Bulk extension of the converse). The converse theorem proved here characterises $\mathcal{H}$ at boundary-accessible points only. Extending the converse to interior points $x \in U \setminus \partial\Sigma$ would require a separate propagation argument: one would need to show that the algebraic signatures detected at boundary-accessible points propagate into the bulk along solutions of the linearised dynamics, e.g., via a unique continuation principle. We do not pursue this here. For physical applications where $\mathcal{H}$ meets the boundary—e.g., when $\mathcal{H}$ is a threshold surface intersecting the asymptotic or interior boundary of a region of interest—the boundary-local statement is precisely what is operationally available; the bulk extension would be a separate result.

9. Dense-Boundary versus Global Boundary Cocycle Suppression

A subtlety inherited from Paper I, made precise here, is that the proven cocycle suppression is for the dense-boundary contribution K_{dens}, not the full boundary cocycle K. We record the precise relationship.

Proposition 9.1 (Global cocycle suppression: sufficient conditions). Suppose either

(a) the entire spatial boundary lies in the dense stratum: $\partial\Sigma \subseteq \mathcal{M}_{\text{dens}}$, or

(b) the boundary symmetry generators ξ, η have support contained in $\partial\Sigma \cap \mathcal{M}_{\text{dens}}$.

Then $K(\xi,\eta) = K_{\text{dens}}(\xi,\eta) = 0$.

Proof. Under (a), $\partial\Sigma \cap \mathcal{M}_{\text{reg}} = \varnothing$, so $K = K_{\text{dens}}$ as integrals; the conclusion follows from Proposition 5.1. Under (b), the support condition implies $\delta_\xi\theta, \delta_\eta\theta$ are zero on $\partial\Sigma \cap \mathcal{M}_{\text{reg}}$ (because the variations have no support there). Combined with $\delta_\xi\theta = \delta_\eta\theta = 0$ on $\partial\Sigma \cap \mathcal{M}_{\text{dens}}$ from finite-action admissibility, the integrand of K vanishes on all of $\partial\Sigma$. □

Remark 9.1 (Genuine distinction). In general, contributions from $\partial\Sigma \cap \mathcal{M}_{\text{reg}}$ may make K non-zero even when $K_{\text{dens}} = 0$. The two are equal only under the hypotheses of Proposition 9.1. This distinction matters for applications: if the physical boundary $\partial\mathcal{M}$ is entirely within $\mathcal{M}_{\text{reg}}$ (e.g., a near-infinity boundary in an asymptotically flat spacetime where $P \to 0$ at infinity), then $K_{\text{dens}} = 0$ trivially because the integration domain is empty, but K remains the full regular-stratum cocycle.

10. Conclusions

The four mathematical gaps left explicit in Paper I have been closed conditionally in this paper.

(1) The trigger functional may be dynamical. Theorem 3.1 extends the local finite-action selection rule to admissible dynamical triggers (Definition 3.2) whose induced correction to the quadratic admissibility form is subleading (Definition 3.3). The phase-suppression conclusion $\delta\theta = 0$ on $\mathcal{M}_{\text{dens}}$ is preserved.

(2) The two-stratum structure admits a precise presymplectic description under stratum regularity. Theorem 4.1, conditional on Assumption 4.1, establishes that

the solution space carries a global two-stratum presymplectic structure with the strict tangent inclusion $T_\Phi\mathcal{S}^{\mathrm{adm}}_{\mathrm{dens}} \subsetneq T_\Phi\mathcal{S}^{\mathrm{adm}}_{\mathrm{reg}}$ and phase-sector rank reduction (Definition 4.2, Proposition 4.1). The phase-sector contribution to the presymplectic current vanishes on the dense *region*, $\omega^\mu_\theta = 0$ on $\Sigma \cap \mathcal{M}_{\mathrm{dens}}$ for dense-stratum admissible variations; this is a localised statement, not a global one. The construction is a field-theoretic analogue of Sjamaar-Lerman stratified symplectic reduction in a precise structural sense (Remark 4.2).

(3) The boundary cocycle is cohomologically non-trivial under explicit, realisable conditions. Theorem 6.1 gives a sufficient criterion—existence of an abelian Hamiltonian amplitude-phase nondegenerate test subalgebra—and Proposition 6.1 constructs such a subalgebra explicitly using compactly-supported test functions and Lie derivatives along a chosen non-vanishing vector field on the boundary. Corollary 6.1 asserts that K defines a non-trivial class in $H^2(\mathfrak{g}_\partial, \mathbb{R})$ for the constructed $\mathfrak{g}_\partial$.

(4) The Phase Boundary Characterisation Theorem admits a boundary-local converse direction. Theorem 8.1 establishes a boundary-local equivalence, within the pure stiffness-induced class (Definition 7.2), between (i) crossing $\mathcal{H}$ at a boundary-accessible point, (ii) loss of the finite-action admissible phase direction, (iii) phase-sector rank reduction, and (iv) dense-boundary cocycle suppression for local separating Hamiltonian test algebras. The cumulative role of Assumptions 7.1, 7.3, and 7.4 is precisely to delete every alternative path from the algebraic signatures to a locus other than $\mathcal{H}$ (Remark 8.1).

The combined effect is a conditional closure of the framework of Paper I. The threshold hypersurface $\mathcal{H}$ is, within the pure stiffness-induced class and at boundary-accessible points, recoverable from the algebraic data of $\Omega^{\mathrm{aug}}_\Sigma$ alone, with explicit and verifiable assumptions about what alternative mechanisms have been excluded. This is not an unconditional classification of all degeneracy loci of $\Omega^{\mathrm{aug}}_\Sigma$, but a controlled characterisation theorem: the paper identifies the precise hypotheses under which the algebraic signatures of phase suppression genuinely determine the threshold hypersurface.

Several questions remain open. The relation between the two stratum-level Hilbert spaces obtained from independent reduced phase-space quantization is one; this requires a representation-theoretic analysis of the centrally-extended algebra in the regular case and a Stone-von Neumann-type uniqueness result on each stratum, which are beyond the present scope. The bulk extension of the converse theorem (Remark 8.2) requires a unique-continuation argument for the linearised dynamics. The extension to fields beyond the complex scalar (gauge fields, gravity) is another natural direction. The behaviour of the framework when $\mathcal{H}$ is itself a topologically non-trivial submanifold, or when the trigger functional admits multiple critical thresholds, is also left for future work.

In a separate line of development, the framework here is being applied in companion work to physical models where the threshold hypersurface $\mathcal{H}$ models a specific physical surface—in particular, gravitational settings with disformal coupling between the trigger functional and the spacetime metric (where post-New-tonian compatibility constrains the form of the coupling), and dense-matter interior models with stiff-fluid equations of state. The mathematical results of the present paper supply the structural framework on which those applications rely; the applications themselves require physical input (specific Lagrangians, equations of state, boundary conditions) that is outside the scope of a structural theorem in covariant phase space.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

A. Symplectic Correction for the Dynamical Trigger $P=\rho^2$

For the order-parameter trigger $P=\rho^2=\hat{t}$, the variation of the trigger reduces to $\delta P=2\rho\delta\rho$, with no $\delta\theta$ or δg contributions. The additional presymplectic potential current from Remark 3.2 takes the explicit form

$$\Theta^{\mu}_{\text{trig}}(\delta)=-\rho\kappa'(\rho^2)\rho^2\nabla^{\alpha}\theta\nabla_{\alpha}\theta\delta\rho n^{\mu}+(\text{lower-order amplitude terms}), \quad (23)$$

which is proportional to $\delta\rho$ with no $\delta\theta$ dependence. On the dense stratum, this term contributes only through amplitude-sector variations: the phase-suppression theorem of Section 3 suppresses admissible phase *variations* $\delta\theta$, not the background phase gradient $\nabla\theta$. The finiteness of $\Theta^{\mu}_{\text{trig}}$ on dense-stratum admissible configurations therefore requires the background regularity assumptions on $\nabla\theta$ already present in the finite-action and admissibility hypotheses (Definition 2.3); it is not a new independent phase-variation suppression and does not affect the rank-reduction or converse arguments above.

For curvature triggers $P=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, the variation δP depends only on $\delta g_{\mu\nu}$ and its derivatives; the analysis is analogous, with the additional contribution affecting the metric-sector presymplectic structure.