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Estimating the Lower Bounds on the Eigenvalue of the Smallest Modulus in the Case of Mixed Boundary Conditions

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Abstract

We derive a lower bound on the eigenvalue of the smallest modulus associated with a mixed boundary condition problem in the general case of a regular Sturm-Liouville problem. Using the Fredholm integral operator and estimates on its norm, we derive bounds for the eigenvalue of the smallest modulus under the assumption that the coefficient function $q(x)$ and the weight function $r(x)$ exhibit no sign restrictions. The results contribute to a broader understanding of eigenvalue problems with mixed boundary conditions in the general case of the regular Sturm-Liouville problems. We also recommend similar studies in higher order Sturm-Liouville problems.

Keywords

Eigenvalue, Eigenfunction, Fredholm Integral Operator, Greens Function, Integral Equations, Non-Definite, Right-Definite, Left-Definite, Mixed Boundary Conditions

1. Introduction

The Sturm-Liouville Problem (SLP) is about finding all values of $\lambda \in \mathbb{C}$ and corresponding nonzero functions u which satisfy the differential equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right)+q(x)u=\lambda r(x)u \tag{1}$$

and the separated boundary conditions

$$\alpha_1 u(a)+\alpha_2 p(a)u'(a)=0,\ \ \beta_1 u(b)+\beta_2 p(b)u'(b)=0 \tag{2}$$

defined on an interval (a,b), $-\infty \le a < b \le \infty$. Here, $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{R}$ and are such that α_1 and α_2 do not vanish at the same time, similarly, for β_1 and β_2 (see ([1]) chapter 12). The parameter λ is called the eigenvalue and the corresponding function u is called an eigenfunction of problem (1)-(2). The set comprising all eigenvalues of problem (1)-(2) is called the spectrum, [2]-[4]. Each eigenvalue may have at least one eigenfunction. An eigenvalue with only one eigenfunction is called a simple eigenvalue, otherwise it is non-simple. The functions p', p, q, and r are continuous on (a,b) and p^{-1}, q, $r \in L_{loc}(a,b)$. The solution of the problem, is a generally complex-valued function $u(x)$ of the real variable x such that u and pu' are absolutely continuous on (a,b) that satisfies equation (1) and the boundary conditions (2). If the endpoints a and b are finite and $p(x)>0$ for all $x \in (a,b)$ the problem is called regular, otherwise it is singular if any of the two regularity conditions (or both) are not satisfied, [4] [5].

We consider the general weighted regular Sturm-Liouville problem (GWRSLP) in which the coefficient function $q(x)$ and the weight function $r(x)$ have no sign restrictions. The paper focuses on estimating the lower bound on the eigenvalue of the smallest modulus of the problem:

$$-y''(x)+q(x)y(x)=\lambda r(x)y(x), \quad a \le x \le b \tag{3}$$

$$y'(a)=y(b)=0. \tag{4}$$

It should be noted that problem (3)-(4) is in the form of problem (1)-(2), with $p(x)\equiv 1$, $\alpha_1 = 0 = \beta_2$. Our work extends the findings of Kikonko and Mingarelli [6] and Mingarelli [7] who examined similar bounds under Dirichlet boundary conditions. Mingarelli [7] obtained estimates with the assumption that $q \in L^\infty(a,b)$, while Kikonko and Mingarelli [6] extended these results to $q \in L^1(a,b)$. In our study, we replace Dirichlet boundary conditions with mixed boundary conditions while maintaining the assumptions on $q(x)$ and $r(x)$. In the next section, we classify the GWRSLP problems in order to show the nature of the spectrum and of the smallest eignvalue, in particular.

2. Preliminaries

2.1. Classification of the General Weighted Regular Sturm-Liouville Problems

The weight function $r(x)$ and the coefficient function $q(x)$ are critical in determining the nature of the spectrum of the GWRSLP. In the this section, we cover the classification of GWRSLPs.

When $r(x)>0$ a.e, and $q(x)$ takes on both positive and negative values on the interval (a, b) the problem is called right definite (RD). In this case , the eigenvalues are real and ordered such that $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ and so there is a smallest eigenvalue, λ_0 which is also the eigenvalue of the smallest modulus. The corresponding eigenfunction does not vanish on the interval (a,b). This is the classical case which is covered in many texts, see for example [8] [9].

When $q(x) \geq 0$ a.e. and $r(x)$ takes on both signs, then the problem is left definite (LD). In the LD case, the spectrum is made up of two sequences of real eigenvalues $\lambda_n^{\pm}$ where $\lambda_n^{\pm} \to \pm\infty$ as $n \to \infty$, and λ_0 is then the first positive (and or negative) eigenvalue (whose corresponding eigenfunction has no zero in (a, b)), and λ_0 may not be unique, because of symmetry. The case is covered in detail in [6], see also [10].

When $q(x) < 0$ a.e and $r(x)$ changes sign, problem 3-4 is nondefinite (or indefinite). In the non-definite case, the spectrum consists of a discrete, doubly infinite sequence of real eigenvalues, with at most a finite and even number of non-real eigenvalues (occurring in complex conjugate pairs). In this case, nonreal eigenvalues may exist and so the eigenvalue of the smallest modulus may be either real or nonreal. If real, then the corresponding eigenfunction can have any number of zeros on the interval (a, b), in contrast with the other two cases in which the corresponding eigenfunction has no interior zero on the given interval [2] [3] [11]-[17].

2.2. Some Key Results

We employ the Fredholm integral operator $(Tf)(x) = \int_a^b G(x,t) f(t) r(t) \mathrm{d}t$ associated with the problem 5-6 in the Hilbert space $L^2_{|r|} \equiv \mathcal{H}$ where $G(x,t)$ is the Green's function of problem 5-6. The norm of T is used in conjuction with estimates of the solutions to a Cauchy problem related to 5-6 to derive lower bounds.

The problem:

$$-y''(x) + q(x) y(x) = 0 \tag{5}$$

$$y'(a) = y(b) = 0. \tag{6}$$

admits a unique Green's function, $G(x,t)$ given by

$$G(x,t) = \begin{cases} \dfrac{y(x) z(t)}{y(b)}, & \text{if } a \leq x < t, \\ \dfrac{y(t) z(x)}{y(b)}, & \text{if } t < x \leq b, \end{cases} \tag{7}$$

where y, z are (real) linearly independent solutions of 5 satisfying the initial conditions

$$y'(a) = 0, \; y(a) = 1 \tag{8}$$

$$z'(b) = 1, \; z(b) = 0, \tag{9}$$

respectively. This is possible as long as $\lambda = 0$ is not an eigenvalue of the problem.

We define the inner product as:

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} |r(x)| \mathrm{d}x, \quad f, g \in H,$$

and the associated norm $\|\cdot\|$ by:

$$\|f\| = \left(\int_a^b |f(x)|^2 |r(x)| \mathrm{d}x \right)^{1/2}.$$

Theorem 1. [9] *Let* T *be the compact integral operator defined by* $(Tu)(x) = \int_a^b G(x,t) r(t) u(t) \mathrm{d}t$ *on the Hilbert space* $L_r^2[a,b]$, *with operator norm* $\|T\|$. *Let* λ *be a nonzero eigenvalue of the Sturm-Liouville problem*

$$-u''(x) + q(x)u(x) = \lambda r(x)u(x), \quad a \le x \le b$$

$$u'(a) = u(b) = 0.$$

Then

$$|\lambda| \ge \frac{1}{\|T\|}.$$

In particular, if λ_0 *is an eigenvalue of smallest modulus, then*

$$|\lambda_0| \ge \frac{1}{\|T\|}.$$

Lemma 2.1. [6] [7] *An eigenvalue of*

$$-y''(x) + q(x)y(x) = \lambda r(x)y(x), \quad a \le x \le b \tag{10}$$

$$y'(a) = y(b) = 0. \tag{11}$$

of the smallest modulus satisfies

$$|\lambda_0| \ge \frac{|y(b)|}{\sqrt{2}\|y\|\|z\|}, \tag{12}$$

where y *and* z *are solutions of* (10)-(18), *satisfying the conditions* $y'(a) = 0$, $y(a) = 1$ *and* $z'(b) = 1$, $z(b) = 0$, *respectively, and* $\|y\|$, $\|z\|$ *are their respective* $\mathcal{H}$ *-norms.*

Proof. We prove the lemma using the definition of the norm of the operator $(Tf)(x) = \int_a^b G(x,t) f(t) r(t) \mathrm{d}t$ given by

$$\|T\| = \left(\int_a^b \int_a^b |G(x,t)|^2 |r(x)||r(t)| \mathrm{d}x \mathrm{d}t \right)^{\frac{1}{2}},$$

where

$$G(x,t) = \begin{cases} \dfrac{y(x)z(t)}{y(b)}, & \text{if } a \le x < t, \\ \dfrac{y(t)z(x)}{y(b)}, & \text{if } t < x \le b, \end{cases}$$

and $r(t)$ is the weight function.

Thus,

$$\|T\| = \left(\int_a^b \left[\int_a^t \left| \frac{y(x)z(t)}{y(b)} \right|^2 |r(x)||r(t)| \mathrm{d}x + \int_t^b \left| \frac{y(t)z(x)}{y(b)} \right|^2 |r(x)||r(t)| \mathrm{d}x \right] \mathrm{d}t \right)^{\frac{1}{2}}$$

$$= \left(\int_a^b \left[\frac{|z(t)|^2 |r(t)|}{|y(b)|^2} \int_a^t |y(x)|^2 |r(x)| \mathrm{d}x + \frac{|y(t)|^2 |r(t)|}{|y(b)|^2} \int_t^b |z(x)|^2 |r(x)| \mathrm{d}x \right] \mathrm{d}t \right)^{\frac{1}{2}}$$

$$\le \left(\int_a^b \left[\frac{|z(t)|^2 |r(t)|}{|y(b)|^2} \|y(x)\|^2 + \frac{|y(t)|^2 |r(t)|}{|y(b)|^2} \|z(x)\|^2 \right] \mathrm{d}t \right)^{\frac{1}{2}}$$

$$= \left(\frac{\|y(x)\|^2}{|y(b)|^2} \int_a^b |z(t)|^2 |r(t)| \mathrm{d}t + \frac{\|z(x)\|^2}{|y(b)|^2} \int_a^b |y(t)|^2 |r(t)| \mathrm{d}t \right)^{\frac{1}{2}}$$

$$= \left(\frac{\|y\|^2 \|z\|^2}{|y(b)|^2} + \frac{\|z\|^2 \|y\|^2}{|y(b)|^2} \right)^{\frac{1}{2}}$$

$$= \left(2 \frac{\|y\|^2 \|z\|^2}{|y(b)|^2} \right)^{\frac{1}{2}}$$

$$\therefore \|T\| \le \frac{\sqrt{2}\|y\|\|z\|}{|y(b)|} \Rightarrow \frac{1}{\|T\|} \ge \frac{|y(b)|}{\sqrt{2}\|y\|\|z\|}.$$

By theorem (1), we get:

$$|\lambda_0| \ge \frac{1}{\|T\|} \ge \frac{|y(b)|}{\sqrt{2}\|y\|\|z\|}$$

$$\therefore |\lambda_0| \ge \frac{|y(b)|}{\sqrt{2}\|y\|\|z\|}.$$

□

3. Integral Equations

In this section, we find the integral representations of the two Cauchy problems:

$$-y''(x) + q(x)y(x) = 0 \quad y'(a) = 0,\ y(a) = 1, \tag{13}$$

$$-z''(x) + q(x)z(x) = 0 \quad z'(b) = 1,\ z(b) = 0. \tag{14}$$

We convert problem (13) to an integral equation:

Integrating we get:

$$\int_a^x y''(s)\mathrm{d}s = \int_a^x q(s)y(s)\mathrm{d}s$$

$$y'(x) = \int_a^x q(s)y(s)\mathrm{d}s.$$

Repeating the process yields

$$y(x) - y(a) = \int_a^x \int_a^t q(s)y(s)\mathrm{d}s\mathrm{d}t$$

$$\Rightarrow y(x)=1+\int_a^x\int_a^t q(s)y(s)\mathrm{d}s\mathrm{d}t.$$

Changing the order of integration yields:

$$\Rightarrow y(x)=1+\int_a^x\int_s^x q(s)y(s)\mathrm{d}t\mathrm{d}s,$$

$$\therefore y(x)=1+\int_a^x (x-s)q(s)y(s)\mathrm{d}s. \tag{15}$$

Similarly, the integral representation of (14) is

$$z(x)=x-b+\int_x^b\int_t^b q(s)z(s)\mathrm{d}s\mathrm{d}t$$

which simplifies to

$$z(x)=x-b+\int_x^b (s-x)q(s)z(s)\mathrm{d}s. \tag{16}$$

We now consider the two spaces for $q(x)$ to obtain the bounds, in the next section.

4. Obtaining the Bounds When $q\in L^1[a,b]$

Consider the integral Equation (15) given by

$$y(x)=1+\int_a^x (x-s)q(s)y(s)\mathrm{d}s,$$

where $q\in L^1(a,b)$. The solution can be expressed as a Neumann series:

$$y(x)=\sum_{n=0}^{\infty}A_n(x),$$

where the terms are defined iteratively by

$$A_0(x)=1,$$

$$A_n(x)=\int_a^x (x-s)q(s)A_{n-1}(s)\mathrm{d}s, n\geq 1. \tag{17}$$

For all $n\geq 1$, we have;

$$|A_n(x)|\leq \frac{(x-a)^n}{n^n}\|q\|_1^n, \tag{18}$$

where

$$\|q\|_1=\int_a^b |q(s)|\mathrm{d}s.$$

Proof. We prove this by mathematical induction. For $n=1$:

$$A_1(x)=\int_a^x (x-s)q(s)A_0\mathrm{d}s.$$

$$\Rightarrow |A_1(x)|\leq \int_a^x |x-s||q(s)|\mathrm{d}s\leq \max_{a\leq s\leq x}(x-s)\|q\|_1=(x-a)\|q\|_1.$$

$$\Rightarrow |A_1(x)|\leq \frac{(x-a)^1}{1^1}\|q\|_1$$

Hence it is true for $n=1$.

Assume (18) holds for $n=k$, that is

$$\left|A_k(x)\right| \le \frac{(x-a)^k}{k^k}\|q\|_1^k, \tag{19}$$

we show that (18) holds for $n=k+1$. From (17) we have

$$A_{k+1}(x)=\int_a^x (x-s)q(s)A_k(s)\,\mathrm{d}s.$$

$$\left|A_{k+1}(x)\right| \le \int_a^x |x-s|\|q(s)\|A_k(s)\,\mathrm{d}s.$$

Using the inductive hypothesis (19), we have

$$\begin{aligned}\left|A_{k+1}(x)\right| &\le \frac{\|q\|_1^k}{k^k}\int_a^x (x-s)(s-a)^k\left|q(s)\right|\mathrm{d}s \\ &\le \frac{\|q\|_1^k}{k^k}\max_{a\le s\le x}(x-s)(s-a)^k\int_a^x\left|q(s)\right|\mathrm{d}s \\ &\le \frac{\|q\|_1^{k+1}}{k^k}\max_{a\le s\le x}(x-s)(s-a)^k = \frac{(x-a)^{k+1}}{(k+1)^{k+1}}\|q\|_1^{k+1}.\end{aligned} \tag{20}$$

Thus, the result holds for $n=k+1$, and by induction it holds for all $n\ge 1$, $n\in\mathbb{Z}$. □

Let $c=\|q\|_1$,

$$\text{then } \left|A_n\right| \le c^n\frac{(x-a)^n}{n^n}, n=1,2,3,\cdots. \tag{21}$$

By using a similar approach as above and using the integral Equation (16) given by

$$z(x)=x-b+\int_x^b (s-x)q(s)z(s)\,\mathrm{d}s$$

we have:

$$z(x)=\sum_{n=0}^{\infty}B_n(x),$$

where

$$B_0(x)=x-b,$$

and

$$B_n(x)=\int_x^b (x-s)q(s)B_{n-1}(s)\,\mathrm{d}s,\ n\ge 1.$$

which gives a general expression as:

$$\left|B_n\right| \le \frac{(b-x)^{n+1}}{(n+1)^{n+1}}\|q\|_1^n, n=1,2,3,\cdots.$$

Let $c=\|q\|_1$, we have:

$$\left|B_n\right| \le c^n\frac{(b-x)^{n+1}}{(n+1)^{n+1}},\ n=1,2,3,\cdots. \tag{22}$$

5. Obtaining the Bounds When $q \in L^{\infty}[a,b]$

Using a similar approach in the previous section and using the integral equation in (15), *i.e.*

$$y(x) = 1 + \int_a^x (x-s) q(s) y(s) \mathrm{d}s,$$

we have:

$$y(x) = \sum_{n=0}^{\infty} A_n(x), \text{ where } A_0(x) = 1.$$

Then

$$|A_n(x)| \le \|q\|_\infty^n \frac{(x-a)^{2n}}{(2n)!}, \text{ for all } n \ge 1. \tag{23}$$

Similarly, using the integral equation

$$z(x) = x - b + \int_x^b (s-x) q(s) z(s) \mathrm{d}s$$

we have

$$z(x) = \sum_{n=0}^{\infty} B_n(x) \text{ where } B_0 = x - b,$$

then

$$|B_n| \le \frac{(x-b)^{2n+1}}{(2n+1)!} \|q\|_\infty^n, \ n \ge 1. \tag{24}$$

We prove (23) by induction.

Proof. For $n = 1$ and from the definition,

$$A_1(x) = \int_a^x (x-s) q(s) A_0 \mathrm{d}s.$$

Taking absolute values,

$$|A_1(x)| \le \int_a^x |x-s| |q(s)| \mathrm{d}s \le \int_a^x |x-s| \max_{a \le s \le x} |q(s)| \mathrm{d}s = \int_a^x \|q\|_\infty (x-s) \mathrm{d}s.$$

Thus,

$$|A_1(x)| \le \|q\|_\infty \frac{(x-a)^2}{2!}.$$

Hence, the result holds for $n = 1$.

Assume that (24) holds for $n = k$, *i.e.*,

$$|A_k(x)| \le \|q\|_\infty^k \frac{(x-a)^{2k}}{(2k)!}.$$

Then we prove that it is true for $n = k + 1$, as well. From the recursive definition,

$$A_{k+1}(x) = \int_a^x (x-s) q(s) A_k(s) \mathrm{d}s.$$

Thus,

$$
\begin{aligned}
\left|A_{k+1}(x)\right| &\le \int_a^x |x-s|\left\|q(s)\right\| A_k(s)\,\mathrm{d}s \\
&\le \int_a^x (x-s)\|q\|_\infty \left(\|q\|_\infty^k \frac{(s-a)^{2k}}{(2k)!}\right)\mathrm{d}s \\
&= \frac{\|q\|_\infty^{k+1}}{(2k)!}\int_a^x (x-s)(s-a)^{2k}\,\mathrm{d}s \\
&= \frac{\|q\|_\infty^{k+1}}{(2k)!}\cdot\frac{(x-a)^{2k+2}}{(2k+2)(2k+1)}.
\end{aligned}
$$

$$
\Rightarrow \left|A_{k+1}(x)\right| \le \|q\|_\infty^{k+1}\frac{(x-a)^{2k+2}}{(2k+2)!} = \|q\|_\infty^{k+1}\frac{(x-a)^{2(k+1)}}{\left[2(k+1)\right]!}.
$$

Thus, the result holds for $k+1$.

Using a similar approach for $B_n(x)$ will give us

$$
\left|B_n(x)\right| \le \|q\|_\infty^n \frac{(x-b)^{2n+1}}{(2n+1)!},\ n\ge 1.
$$

□

6. Estimates in the Case $q\in L^1$

Lemma 6.1. *Let* $y(x)$ *denote the solution of the Cauchy problem*

$$
-y''(x)+q(x)y(x)=0,\ a\le x\le b, \tag{25}
$$

satisfying

$$
y'(a)=0,\ y(a)=1. \tag{26}
$$

$$
B(c)=\left\{q:[a,b]\to\mathbb{R}, q\in L^1(a,b), \|q\|_1=c\right\}. \tag{27}
$$

Then for fixed $x\in[a,b]$ *we have,*

$$
\sup_{q\in B(c)}\left|y(x)\right| \le \mathrm{e}^{c(x-a)}. \tag{28}
$$

Proof. To prove (28), we use the inequality (21), to get:

$$
\begin{aligned}
\left|y(x)\right| &= \left|1+\sum_{n=1}^{\infty}A_n\right| \le 1+\sum_{n=1}^{\infty}\left|A_n\right| \\
&\le 1+\sum_{n=1}^{\infty}c^n\frac{(x-a)^n}{n^n} \le 1+\sum_{n=1}^{\infty}c^n\frac{(x-a)^n}{n!} \\
&= \sum_{n=0}^{\infty}c^n\frac{(x-a)^n}{n!}.
\end{aligned} \tag{29}
$$

□

Since the series

$$
\sum_{n=0}^{\infty}c^n\frac{(x-a)^n}{n!} \tag{30}
$$

converges, the Neumann series converges, too. This yields the following.

$$\left|y(x)\right| \le \mathrm{e}^{c(x-a)}, \tag{31}$$

which establishes the bound in (28).

Lemma 6.2. *Let* y *denote a linearly independent solution of* (25) *satisfying* (26) *then*

$$\sup_{q\in B(c)} \left\|y(x)\right\| \le \left(\int_a^b \left(\mathrm{e}^{c(x-a)}\right)^2 \left|r(x)\right| \mathrm{d}x\right)^{\frac{1}{2}}, \tag{32}$$

Proof. We prove the above result by calculating the $\mathcal{H}$-norms of y and use the bounds in Lemma 6.1 as shown below.

$$\begin{aligned} \left\|y(x)\right\| &= \left(\int_a^b \left|y(x,q)\right|^2 \left|r(x)\right| \mathrm{d}x\right)^{\frac{1}{2}} \\ &< \left(\int_a^b \left|\mathrm{e}^{c(x-a)}\right|^2 \left|r(x)\right| \mathrm{d}x\right)^{\frac{1}{2}} = \left\|\mathrm{e}^{c(x-a)}\right\|, \end{aligned} \tag{33}$$

Taking the supremum on $\left\|y(x)\right\|$ in the inequality (33) yields the bounds in (32). □

Lemma 6.3. *Let* $z(x)$ *denotes the solution of the Cauchy problem*

$$-z''(x) + q(x)z(x) = 0 \tag{34}$$

$$z'(b) = 1,\ z(b) = 0 \tag{35}$$

then for fixed $x \in [a,b]$ *we have,*

$$\sup_{q\in B(c)} \left|z(x)\right| \le (b-x)\mathrm{e}^{c(b-x)}. \tag{36}$$

Proof. To prove (36), we use the inequality (22) as shown below;

$$\begin{aligned} \left|z(x)\right| &= \left|x-b+\sum_{n=1}^{\infty} B_n\right| \le \left|b-x\right| + \sum_{n=1}^{\infty}\left|B_n\right| \\ &\le b-x+\sum_{n=1}^{\infty} c^n \left(\frac{b-x}{n+1}\right)^{n+1} \le b-x+\sum_{n=1}^{\infty} c^n \frac{(b-x)^{n+1}}{(n+1)!} \\ &\le (b-x)\sum_{n=0}^{\infty} c^n \frac{(b-x)^n}{n!} \end{aligned} \tag{37}$$

which yields

$$\left|z(x)\right| \le (b-x)\mathrm{e}^{c(b-x)} \tag{38}$$

and the bounds in (36) is established.

Lemma 6.4. *Let* z *denote the linearly independent solutions of* (34) *satisfying* (35) *then*

$$\sup_{q\in B(c)} \left\|z(x)\right\| \le \left(\int_a^b \left((b-x)\mathrm{e}^{c(x-a)}\right)^2 \left|r(x)\right| \mathrm{d}x\right)^{\frac{1}{2}}, \tag{39}$$

Proof. We prove the above result by calculating the $\mathcal{H}$-norms of z and use the bounds in Lemma 6.3 as shown below.

$$\begin{aligned}\|z(x)\| &= \left(\int_a^b |z(x,q)|^2 |r(x)| \mathrm{d}x\right)^{\frac{1}{2}} \\ &< \left(\int_a^b \left|(b-x)\mathrm{e}^{c(x-a)}\right|^2 |r(x)| \mathrm{d}x\right)^{\frac{1}{2}} = \left\|(b-x)\mathrm{e}^{c(b-x)}\right\|,\end{aligned} \tag{40}$$

Taking the supremum on $\|z(x)\|$ in the inequality (40) yields the bounds in (39). □

7. Estimates in the Case $q \in L^\infty$

Lemma 7.1. *Let* $y(x)$ *denote the solution of the Cauchy problem*:

$$-y''(x) + q(x)y(x) = 0, \ a \le x \le b, \tag{41}$$

$$y'(a) = 0, \ y(a) = 1. \tag{42}$$

$$B(c) = \left\{q : [a,b] \to \mathbb{R}, q \in L^\infty(a,b), \|q\|_\infty = c\right\}. \tag{43}$$

Then for fixed x in [a, b]

$$\sup_{q \in B(c)} |y(x)| \le \cosh\left(\sqrt{c}(x-a)\right) \tag{44}$$

with equality in the case where $q(x) = c$ *a.e. on* (*a*, *b*)

Similarly if $z(x,q)$ *denotes the solution of* (41) *and* q *as defined in* (43) *satisfying*,

$$z(b) = 0, \ z'(b) = 1, \tag{45}$$

then for fixed x *in* $[a,b]$,

$$\sup_{q \in B(c)} |z(x,q)| \le \frac{\sinh\left(\sqrt{c}(b-x)\right)}{\sqrt{c}} \tag{46}$$

Proof. To prove (44), we use the Neumann series of (15) and inequality (23). Therefore we have that,

$$\begin{aligned}|y(x)| &= \left|1 + \sum_{n=1}^\infty A_n\right| \le 1 + \sum_{n=1}^\infty c^n \frac{(x-a)^{2n}}{(2n)!} \\ &= \sum_{n=0}^\infty c^n \frac{(x-a)^{2n}}{(2n)!} \\ &= \sum_{n=0}^\infty \frac{\left(\sqrt{c}(x-a)\right)^{2n}}{(2n)!} \\ &= \cosh\left(\sqrt{c}(x-a)\right)\end{aligned} \tag{47}$$

And the bounds in (44) are established.

To prove (46) we use the Neumann series of (16) and inequality (24), thus;

$$\begin{aligned} |z(x)| &= \left| x-b+\sum_{n=1}^{\infty} B_n \right| \le |b-x| + \sum_{n=1}^{\infty} |B_n| \\ &\le b-x+\sum_{n=1}^{\infty} c^n \frac{(b-x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} c^n \frac{(x-b)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{\left(\sqrt{c}(x-b)\right)^{2n+1}}{\sqrt{c}(2n+1)!} \\ &= \frac{\sinh\left(\sqrt{c}(b-x)\right)}{\sqrt{c}} \end{aligned} \tag{48}$$

which yields that

$$\sup_{q\in B(c)} |z(x)| \le \frac{\sinh\left(\sqrt{c}(b-x)\right)}{\sqrt{c}} \tag{49}$$

and the bounds in (46) is established.

Combining the pointwise estimations on y, z in Lemma 7.1 we may obtain (sharp) estimates on the $\mathcal{H}$-norms of y, z which leads to the following results.

Lemma 7.2. *Let* y *and* z *denote the two linearly independent solutions of* (41) *satisfying* (42) *and* (45), *respectively, then,*

1.

$$\sup_{q\in B(c)} \|y(x)\| \le \left(\int_a^b \cosh^2\left(\sqrt{c}(x-a)\right)|r(x)|\,\mathrm{d}x\right)^{\frac{1}{2}} = \left\|\cosh\left(\sqrt{c}(x-a)\right)\right\|, \tag{50}$$

2.

$$\sup_{q\in B(c)} \|z(x)\| \le \left(c^{-1}\int_a^b \sinh^2\left(\sqrt{c}(b-x)\right)|r(x)|\,\mathrm{d}x\right)^{\frac{1}{2}} = \left\|\frac{\sinh\left(\sqrt{c}(b-x)\right)}{\sqrt{c}}\right\|, \tag{51}$$

Proof. We prove the above results by calculating the $\mathcal{H}$-norms of y and z by integrating the pointwise bounds given by (44) and (46) and use the bounds in (7.1) for a given $q \in B(c)$. □

8. Main Results

Applying the results in Lemma 6.2 and Lemma 6.4 to the lower bounds in (12), we get the main result when $q \in L^1$, which is a variant of Theorem 1 in [7].

Theorem 2. *Let* $\|q\|_1 = c$. *Then for problem* (3)-(4), *an eigenvalue* λ_0 *of the smallest modulus may be estimated by:*

$$|\lambda_0| \ge |y(b)| \left(\sqrt{2}\left\|\mathrm{e}^{c(x-a)}\right\|\left\|(b-x)\mathrm{e}^{c(b-x)}\right\|\right)^{-1}, \tag{52}$$

where $x \in (a,b)$ *and* $y(b)$ *is a solution of the Cauchy problem* (25) *and* (26) *evaluated at* $x = b$.

Applying the results in lemma (7.2) to the lower bounds in (12), gives our main result when $q \in L^\infty$.

Theorem 3. *Let* $\|q\|_\infty = c$. *Then for problem* (3)-(4), *an eigenvalue* λ_0 *of the* *smallest modulus may be estimated by;*

$$|\lambda_0| \geq |y(b)| \left(\sqrt{2} \left\| \cosh\left(\sqrt{c}(x-a)\right) \right\| \left\| \frac{\sinh\left(\sqrt{c}(b-x)\right)}{\sqrt{c}} \right\| \right)^{-1}, \tag{53}$$

$x \in (a,b)$ *where* $y(b)$ *is a solution of the Cauchy problem* (41) *and* (42) *evaluated at* $x = b$.

9. Examples

Here we give examples to verify if the theorems 2 and 3 hold. Without loss of generality, we consider the case where $q(x) = q \in \mathbb{R}$ on the interval [−1,1]. The eigenvalues are found using the Maple C package RootFinding [Analytic].

Example 1. (*the case* $q = -4\pi^2$ *and* $r(x)$ *changes sign*). *This is an example of the non-definite SLP.*

$$y'' + 4\pi^2 y = 0,\ y(-1) = 1,\ y'(-1) = 0, \tag{54}$$

$$y'' + 4\pi^2 y = -\lambda r(x) y,\ y'(-1) = 0,\ y(1) = 0. \tag{55}$$

we take,

$$r(x) = \begin{cases} -1, & \text{if } x \in [-1,0), \\ 1, & \text{if } x \in (0,1], \end{cases} \tag{56}$$

The solution to the problem (54) *is*

$$y = \cos 2\pi x \quad \text{from which we get} \quad y(1) = 1, \tag{57}$$

and

$$c = \|q\|_1 = \left(4\pi^2 \int_{-1}^{1} \mathrm{d}x\right)^{\frac{1}{2}} = \sqrt{8\pi^2} = 2\sqrt{2}\pi.$$

$$\left\| \mathrm{e}^{c(x+1)} \right\| = \left(\int_{-1}^{1} \left| \mathrm{e}^{2\pi\sqrt{2}(x+1)} \right|^2 |\pm 1| \mathrm{d}x \right)^{\frac{1}{2}} = \left(\left(\frac{\mathrm{e}^{4\pi\sqrt{2}(x+1)}}{4\pi\sqrt{2}} \right) \Bigg|_{-1}^{1} \right)^{\frac{1}{2}} \tag{58}$$
$$= 12394121.35650267$$

$$\left\| (1-x)\mathrm{e}^{c(1-x)} \right\| = \left(\int_{-1}^{1} \left((1-x)\mathrm{e}^{2\pi\sqrt{2}(1-x)} \right)^2 |\pm 1| \mathrm{d}x \right)^{\frac{1}{2}} \tag{59}$$
$$= 24100921.09759482$$

Substituting (57), (58) *and* (59) *in inequality* (52) *gives us*

$$|\lambda_0| \geq \left(\sqrt{2} (12394121.35650267)(24100921.09759482) \right)^{-1}$$
$$= 2.36720362412 \times 10^{-15}$$
$$\Rightarrow \lambda_0 \geq 2.36720362412 \times 10^{-15} \tag{60}$$

The spectrum of (55) *when* r *is given in* (56) *is*

−477.4, −343.8, −229.7, −134.8, −57.1, −39.5, −30.8, −7.315 ± 14.59*i*, 24.18 ± 9.767*i*, 36.45, 94.27, 179.9, 284.3, 408.2,… *and so* $|-7.315 \pm 14.59i| = 16.3$ *is the* *smallest modulus and we see that*

$$16.3 > 2.36720362412 \times 10^{-15}$$

Thus theorem 2 *is satisfied.*

Next we verify theorem 3

$$c = \|q\|_\infty = \|-4\pi^2\|_\infty = 4\pi^2.$$

$$\left\|\cosh\left(\sqrt{c}\left(x-a\right)\right)\right\|_\infty = \left\|\cosh\left(2\pi\left(x+1\right)\right)\right\|_\infty = \cosh 4\pi = 267.75 \tag{61}$$

and

$$\left\|\frac{\sinh\left(\sqrt{c}\left(b-x\right)\right)}{\sqrt{c}}\right\|_\infty = \frac{\sinh 4\pi}{2\pi} = 22818.94 \tag{62}$$

Substituting (57), (61) *and* (62) *in inequality* (53) *gives us*

$$|\lambda_0| \geq \left(\sqrt{2}\left(267.75\right)\left(22818.94\right)\right)^{-1} = 1.1573 \times 10^{-7} \tag{63}$$

using the spectra in (55), *we see that*

$$16.3 > 1.1573 \times 10^{-7}$$

Example 2. (*the case* $q = 4\pi^2$ *and* $r(x)$ *changes sign*). *This is an example of the left definite case.*

We consider the problem

$$-y'' + 4\pi^2 y = 0,\ \ y\left(-1\right) = 1,\ \ y'\left(-1\right) = 0, \tag{64}$$

$$-y'' + 4\pi^2 y = \lambda r\left(x\right) y,\ \ y'\left(-1\right) = 0,\ \ y\left(1\right) = 0. \tag{65}$$

where,

$$r\left(x\right) = \begin{cases} -1, & \text{if } x \in \left[-1, 0\right), \\ 1, & \text{if } x \in \left(0, 1\right]. \end{cases}$$

The solution to (64) *is*

$$y\left(x\right) = \cosh\left(2\pi\left(1+x\right)\right) \tag{66}$$

$$y\left(1\right) = \cosh\left(4\pi\right) = 267.746. \tag{67}$$

$$c = \|q\|_1 = \left(\int_{-1}^{1} \left|4\pi^2\right| \mathrm{d}x\right)^{\frac{1}{2}} = 2\sqrt{2}\pi.$$

And

$$\left\|\mathrm{e}^{c(x+1)}\right\| = 12394121.35650267 \tag{68}$$

$$\left\|\left(1-x\right)\mathrm{e}^{c(1-x)}\right\| = 24100921.09759482 \tag{69}$$

Substituting (67), (68) *and* (69) *in* (52) *gives*

$$\left|\lambda_0\right| > 6.33809301544 \times 10^{-13} \tag{70}$$

The spectrum for problem (65) *is*

−426.8, −313.6, −220.2, −146.5, −92.60, −57.98, −41.48, 39.48, 47.58, 72.93, 117.1, 180.89, 264.45, 367.7, 490.9, …

From the spectrum we get the eigenvalue of the smallest modulus, $\lambda_0 = 39.48$. *and satisfies* (70).

Thus,

$$\left|\lambda_0\right| = 39.48 > 6.33809301544 \times 10^{-13}$$

and theorem 2 *is verified.*

Similarly, from the terms of theorem 3 *we have*

$$c = \|q\|_\infty = \left\|4\pi^2\right\|_\infty = 4\pi^2.$$

$$\left\|\cosh\left(\sqrt{c}\left(x-a\right)\right)\right\|_\infty = \left\|\cosh\left(2\pi\left(x+1\right)\right)\right\|_\infty = \cosh 4\pi = 267.75 \tag{71}$$

and

$$\left\|\frac{\sinh\left(\sqrt{c}\left(b-x\right)\right)}{\sqrt{c}}\right\|_\infty = \frac{\sinh 4\pi}{2\pi} = 22818.94 \tag{72}$$

Substituting (71) *and* (72) *in to theorem* 3 *we get*

$$\left|\lambda_0\right| \geq 3.098725 \times 10^{-5} \tag{73}$$

And from the spectrum it follows that

$$\left|\lambda_0\right| = 39.48 > 3.098725 \times 10^{-5}$$

which verifies theorem 3.

Example 3. (*the case* $q = 4\pi^2$ *and* $r(x) \equiv 1$). *This is an example of the right definite case.*

We consider the problem

$$-y'' + 4\pi^2 y = 0,\ \ y(-1) = 1,\ \ y'(-1) = 0, \tag{74}$$

$$-y'' + 4\pi^2 y = \lambda r(x) y,\ \ y'(-1) = 0,\ \ y(1) = 0, \tag{75}$$

Solving (74) *gives the solution*

$$\begin{gathered} y(x) = \cosh\left(2\pi\left(x+1\right)\right) \\ \Rightarrow y(1) = \cosh 4\pi = 267.746 \end{gathered} \tag{76}$$

The spectrum of problem (75) *is,*

39.48, 40.09, 45.03, 54.90, 69.70, 89.44, 114.1, … *Here,* $\lambda_0 = 39.48$

$$c = \|q\|_1 = \left(\int_{-1}^{1}\left|4\pi^2\right|\mathrm{d}x\right)^{\frac{1}{2}} = 2\sqrt{2}\pi.$$

And

$$\left\|\mathrm{e}^{c(x+1)}\right\| = 12394121.35650267 \tag{77}$$

$$\left\|\left(1-x\right)\mathrm{e}^{c(1-x)}\right\| = 24100921.09759482 \tag{78}$$

Substituting (76), (77) *and* (78) *in* (52)

$$\left|\lambda_0\right| \geq 6.33809301544 \times 10^{-13} \tag{79}$$

Thus;

$$\left|\lambda_0\right| = 39.48 > 6.33809301544 \times 10^{-13}$$

which verifies theorem 2.

Similarly, from the terms of inequality (53) *we have*

$$c = \|q\|_\infty = \left\|4\pi^2\right\|_\infty = 4\pi^2.$$

$$\left\|\cosh\left(\sqrt{c}\left(x-a\right)\right)\right\|_\infty = \left\|\cosh\left(2\pi\left(x+1\right)\right)\right\|_\infty = \cosh 4\pi = 267.75 \tag{80}$$

and

$$\left\|\frac{\sinh\left(\sqrt{c}\left(b-x\right)\right)}{\sqrt{c}}\right\|_\infty = \frac{\sinh 4\pi}{2\pi} = 22818.94 \tag{81}$$

Substituting (80) *and* (81) *in* (53) *we get*

$$\left|\lambda_0\right| \geq 3.098725 \times 10^{-5} \tag{82}$$

And from the spectrum

$$\left|\lambda_0\right| = 39.48 > 3.098725 \times 10^{-5}$$

which verifies theorem 3.

10. Discussion on the Looseness of the Bounds

The numerical examples presented in this paper verify that the bounds obtained in Theorems 2 and 3 hold, but are not tight enough. It is observed that the estimated lower bounds are significantly smaller than the corresponding eigenvalues of smallest modulus. The loss is due to various factors, some of which we highlight below.

The looseness of the bounds is due to the estimation of the norm $\|T\|$ of the operator, T because at most of the steps inequalities are introduced, which lead to overestimation of $\|T\|$. Another reason is the use of the Neumann series expansions to estimate the solutions, $y(x)$ and $z(x)$. The bounding of the terms of the series which replace the exact structure lead to exponential and hyperbolic terms, thereby inflating the values of $\|y\|$ and $\|z\|$. In future studies we could use alternative methods to improve the tightness of the inequalities. For example, we could use methods that are based on the Rayleigh quotient, *i.e.*,

$$\lambda_0 = \inf \frac{\int\left(\left|y'\right|^2 + qy^2\right)}{\int ry^2},$$

when admissible test functions are carefully chosen and further refinement of the above outlined steps may lead to more accurate results.

11. Conclusions

In this paper, we extend the results obtained by Kikonko and Mingarelli [6] and Mingarelli [7], which established lower bounds for an eigenvalue of the smallest modulus corresponding to the problem (3) in the case of Dirichlet BCs and different conditions on the coefficient function, $q(x)$.

Our main work was to extend these results to mixed boundary conditions, while maintaining the assumptions on q in both $L^{\infty}(a,b)$ and $L^{1}(a,b)$. The lower bounds for the eigenvalue of the smallest modulus were obtained in both cases, thus consolidating the results previously obtained under Dirichlet boundary conditions. This work provides a significant step towards a more comprehensive understanding of general weighted regular Sturm-Liouville problems with mixed boundary conditions. Future work may involve further investigation into higher-order differential equations and the use of the Rayleigh quotient techniques.

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Conflicts of Interest

The authors declare that there are no competing interests regarding the publication of this article.

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