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A Liouville Invariance Theorem for Blackbody Spectrum Preservation

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Abstract

The preservation of the cosmic microwave background (CMB) Planck spectrum across cosmological time is commonly interpreted as requiring metric expansion. This note demonstrates that such preservation follows from Liouville transport under a homogeneous, isotropic, frequency-independent redshift mapping, without assuming any specific cosmological model or spacetime dynamics. For collisionless photon propagation, a generic mapping of the form $\nu_o = \nu_e / R(t_e, t_o)$, with R independent of frequency, suffices to conserve the spectral shape of an initial blackbody distribution. The derivation yields the standard temperature scaling $T(z) = T_0(1+z)$, preserves the zero chemical potential condition ($\mu = 0$), and reproduces the phase space and energy-momentum conservation scalings $\dot{n} = -3Hn$, $\dot{\rho} = -4H\rho$, under an effective rate defined by $H \equiv \mathrm{d}(\ln R)/\mathrm{d}t$. The result establishes a conditional invariance statement. Metric expansion is sufficient for blackbody preservation, but the spectrum shape alone does not uniquely identify expansion as the underlying cause.

Keywords

Cosmological Redshift, Cosmic Microwave Background, Liouville Theorem, Kinetic Redshift, Scalar Field Dynamics, Observational Degeneracy, Blackbody Spectrum, Static Cosmology, Alternative Cosmology, CMB Thermodynamics

1. Introduction

The observation that the cosmic microwave background (CMB) retains a near-perfect blackbody spectrum across cosmological redshifts is commonly presented as strong evidence for an expanding universe. In standard cosmology, this preservation is understood to follow from the redshifting of photon wavelengths due to metric expansion, as governed by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric [1] [2]. The temperature-redshift relation $T(z) = T_0(1+z)$, along with the absence of detectable spectral distortions [3] [4], is naturally explained by this expansion; however, the underlying transport logic permits a narrower inference. The preservation of a Planck distribution under redshift evolution does not, in itself, require metric expansion or any particular spacetime geometry. What is required is the conservation of the phase space density of collisionless photons and a redshift mapping that acts uniformly across frequencies. In such cases, the distribution function remains invariant along geodesics, and an initially Planckian spectrum retains its form. This result follows directly from the Liouville theorem in curved spacetime, which states that the distribution function $f(x^\mu, p^\nu)$ is conserved along photon trajectories in the absence of collisions [5]. The present note formalizes this point. It proves a theorem establishing that, under a homogeneous, isotropic, frequency-independent redshift mapping, the Liouville equation implies full preservation of the CMB blackbody spectrum. The derivation recovers the standard temperature scaling and energy-momentum conservation laws without invoking metric expansion, specific dynamics, or underlying mechanisms. No cosmological model is assumed. No appeal is made to gravitational field equations, spacetime curvature, or global geometry. The result is conditional. It states that if redshift acts uniformly across photon energies, then blackbody form is preserved. This does not imply that expansion is incorrect, but it does establish that expansion is not logically required to explain the observed spectrum.

This analysis serves two purposes. First, it isolates the mathematical structure responsible for spectral invariance in photon transport. Second, it closes a gap in the literature by formally separating the role of redshift kinematics from assumptions about metric expansion. Prior work has explored non-expanding explanations for the CMB, but these have generally lacked a clean mathematical demonstration of spectrum preservation from first principles. The result presented here is not a model and does not propose a physical mechanism for redshift. It is a conditional theorem. A statement about what follows from Liouville transport given a frequency-independent redshift mapping. A full physical realization of such a mapping is presented in a separate analysis [6], which derives frequency-independent redshift from cumulative kinematic effects without metric expansion. That derivation is not repeated here. The present note is limited in scope and framed deliberately in conventional terms to clarify the minimal requirements for CMB spectral preservation.

2. Setup

The analysis begins with the transport of collisionless photons in the post-decoupling regime. After last scattering, photon trajectories evolve along null geodesics in the absence of scattering or absorption. In this context, the photon phase space distribution function $f(x^\mu, p^\nu)$ satisfies the collisionless Boltzmann equation, also known as the Liouville equation:

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = 0,$$

where λ is an affine parameter along the geodesic and f is defined over the invariant phase space volume. This equation expresses the conservation of phase space density for freely streaming photons [5] [7]. To describe the kinematic evolution of the photon distribution, a general redshift mapping is introduced of the form

$$\nu_o = \frac{\nu_e}{R\left(t_e, t_o\right)},$$

where ν_e and ν_o denote the emission and observation frequencies respectively, and $R\left(t_e, t_o\right)$ is a dimensionless function of the emission and observation times. No specific form of R is assumed, and no geometric interpretation is imposed. The only restriction applied in this analysis is that R be independent of frequency. For the theorem below, the mapping is also assumed to be homogeneous and isotropic with respect to the observer congruence. Thus $R\left(t_e, t_o\right)$ depends only on the emission and observation times along the photon path, and not on photon frequency, spatial position, or propagation direction. This assumption is not a metric-expansion assumption. It is the minimal condition needed for the redshift map to act as an isotropic rescaling of the locally measured photon momentum. This condition guarantees that all photons, regardless of energy, undergo the same fractional shift in momentum. Such mappings preserve the form of any distribution function that depends solely on the modulus of the photon 3-momentum. The time evolution of the redshift mapping is characterized by an effective rate $H(t)$, defined as the logarithmic derivative of the redshift function,

$$H(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t} \ln R\left(t, t_0\right).$$

This definition is purely kinematic and imposes no assumption of metric expansion. It allows redshift scaling to be analyzed in terms of local time derivatives of the momentum distribution. The sign convention here implies that when $R\left(t, t_0\right)$ increases with time, corresponding to photon redshift toward the observer, the effective rate H is positive. This matches the identification $\dot{p}/p = -H$ along photon trajectories, consistent with the standard derivation of redshift evolution in expanding universes [1] [2]. No assumption is made about the origin of the redshift mapping. It may arise from metric expansion, from a cumulative kinetic mechanism, or from another process entirely. The analysis is restricted to the logical implications of Liouville transport under a frequency-independent redshift mapping. The physical realization of such a mapping is presented elsewhere [6], and is not repeated in this note.

Under these assumptions, the conservation of the distribution function implies that f is constant along photon trajectories, and may be evaluated as a function of the local 3-momentum and redshift. The consequences for blackbody preservation follow directly and are derived in the next section.

3. Theorem and Proof

The following result formalizes the consequences of Liouville conservation under a frequency-independent redshift mapping.

Theorem. Let $f(t,\vec{p})$ denote the photon distribution function measured in the local orthonormal frame of a fundamental observer in the post-decoupling regime, where $\vec{p}$ is the observer-measured spatial photon momentum. Assume collisionless propagation governed by the Liouville equation and a homogeneous, isotropic, frequency-independent redshift mapping $\nu_o = \nu_e / R(t_e, t_o)$, with R independent of position, direction, and photon frequency. Then the distribution function evolves as

$$f(t,\vec{p}) = f(t_0, R(t,t_0)\vec{p}),$$

where t_0 is the initial reference epoch, with $t > t_0$ and $R(t,t_0)$ is the redshift mapping evaluated along the photon's path.

Proof. The collisionless Boltzmann equation implies that the distribution function is conserved along photon geodesics,

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \frac{\partial f}{\partial t} + \frac{\mathrm{d}p^i}{\mathrm{d}\lambda}\frac{\partial f}{\partial p^i} = 0,$$

where p^i denotes the spatial components of the photon's 3-momentum and λ is an affine parameter. For massless particles, the modulus $|\vec{p}|$ evolves as a function of time under redshift. By assumption, the redshift acts uniformly on all photons such that

$$|\vec{p}(t)| = \frac{|\vec{p}(t_0)|}{R(t,t_0)}.$$

Substituting into the distribution function yields

$$f(t,\vec{p}) = f(t_0, R(t,t_0)\vec{p}),$$

which satisfies the Liouville equation by construction. Since the mapping is isotropic and frequency-independent, angular dependence is preserved, and no spectral distortions are introduced.

Corollary. If the distribution function at time t_0 is Planckian,

$$f(t_0,\vec{p}) = \frac{1}{\mathrm{e}^{|\vec{p}|/T_0} - 1},$$

then at any later time t the distribution evolves as

$$f(t,\vec{p}) = \frac{1}{\mathrm{e}^{|\vec{p}|R(t,t_0)/T_0} - 1},$$

which is again Planckian, with temperature

$$T(t) = \frac{T_0}{R(t,t_0)}.$$

The blackbody form is preserved exactly. The occupation number is unchanged, and no chemical potential term is generated. This implies that $\mu = 0$ is conserved throughout photon transport, in agreement with observational constraints from COBE/FIRAS [3] [8] and Planck [4]. The result establishes that spectral preservation follows from Liouville conservation together with a homogeneous, isotropic, frequency-independent redshift mapping. No assumption of metric expansion or gravitational field equations is needed. This conclusion agrees with earlier general statements in Tolman, Lindquist, and Misner, Thorne, and Wheeler [5] [9] [10], but is here presented in constructive form. A physical derivation of such a mapping is provided separately [6].

4. Derived Relations

Given the frequency-independent redshift mapping $\nu_o = \nu_e / R(t_e, t_o)$ and the Planck spectrum preservation derived in Section 3, several standard thermodynamic scalings follow directly. These relations are typically associated with metric expansion in ΛCDM cosmology, but here arise purely from redshift kinematics and Liouville invariance.

Temperature Scaling. The blackbody temperature evolves as

$$T(t) = \frac{T_0}{R(t, t_0)}.$$

If the redshift mapping is related to observed redshift by $1 + z = R(t, t_0)$, then the temperature-redshift relation becomes

$$T(z) = T_0(1+z),$$

where T_0 now denotes the presently observed CMB temperature, consistent with observational constraints from the CMB spectrum [8] [11].

Number Density Scaling. The photon number density $n(t)$ is computed from the phase space integral of the distribution function. Since $f(t, \vec{p})$ is preserved and momentum scales as $|\vec{p}| \propto 1/R$, the phase space volume element transforms as $\mathrm{d}^3 p \propto R^{-3}$. This yields

$$n(t) \propto \int f(t, \vec{p}) \mathrm{d}^3 p \propto \frac{1}{R^3}.$$

Differentiating with respect to time gives

$$\dot{n} = -3Hn,$$

where the effective rate is defined as $H(t) \equiv \mathrm{d}(\ln R)/\mathrm{d}t$ as in Section 2.

Energy Density Scaling. Similarly, the radiation energy density $\rho(t)$ is proportional to the fourth power of temperature, or equivalently

$$\rho(t) \propto \int |\vec{p}| f(t, \vec{p}) \mathrm{d}^3 p \propto \frac{1}{R^4}.$$

Taking the time derivative yields

$$\dot{\rho} = -4H\rho.$$

This relation is consistent with radiation dominance in the early universe, but is derived here without invoking expansion, Einstein field equations, or the FLRW metric. These results reproduce the standard scaling relations of an expanding radiation-dominated universe, but their logical origin is different. In standard cosmology, these relations are derived from the conservation of energy-momentum in a metric background with scale factor $a(t)$ [12] [13]. Here, they follow from Liouville conservation and an energy-independent redshift mapping, with $R(t,t_0)$ playing the role conventionally assigned to $a(t)$. The mathematical structure is preserved, but the causal interpretation is shifted.

The identification $R(t,t_0)=1+z$ is observational and requires no dynamical model. Provided the redshift is frequency-independent, the scaling relations for T, n, and ρ follow necessarily from kinematic principles. This reinforces the claim that metric expansion is sufficient, but not necessary, for blackbody preservation and standard thermodynamic scaling.

5. Discussion

It is emphasized that the result established in this note is a conditional invariance statement. If photons undergo redshift governed by a frequency-independent mapping $R(t_e,t_o)$, then the cosmic microwave background (CMB) Planck spectrum is preserved under Liouville transport. The blackbody form, temperature scaling, number density evolution, and energy density decay all follow directly from this condition, without invoking metric expansion or field equations. This is a logical implication, not a physical explanation. The analysis does not derive the redshift mapping $R(t_e,t_o)$ from first principles, nor does it specify the physical origin of its frequency independence. The mapping is introduced as a minimal kinematic input and treated as observationally fixed. As such, the theorem does not rule out expansion-based cosmologies, but it does demonstrate that expansion is not uniquely required to preserve the CMB spectrum.

The conclusion should therefore be separated from stronger claims about cosmological dynamics. The theorem concerns spectrum preservation only. It does not determine the gravitational field equations, the origin of the redshift mapping, structure formation, distance duality beyond the assumed mapping, or the physical stability of any non-expanding cosmology. Mechanisms that generate the same frequency-independent redshift would have to be distinguished by independent observables, including anisotropy transport, luminosity-distance relations, baryon acoustic scales, gravitational lensing, structure growth, and consistency with the full CMB angular power spectrum. The result here is only that the blackbody spectrum shape itself does not uniquely require metric expansion.

In standard cosmology, metric expansion provides the physical mechanism by which redshift acts equally on all wavelengths and therefore preserves the blackbody form. However, this equivalence can also emerge from other mechanisms. A separate derivation based on cumulative kinematic effects yields a frequency-independent redshift without metric expansion, and realizes $R(t)$ as the evolution of a cumulative motion kernel [6]. That work provides a dynamical realization of the condition assumed here and lies outside the present scope. The analysis here is deliberately restricted to the downstream consequence. Liouville transport plus frequency-independent redshift implies exact preservation of the photon spectrum. This result clarifies that the empirical properties of the CMB spectrum, including the absence of distortions and the scaling $T(z)=T_0(1+z)$, are not sufficient to determine the underlying cause of redshift. Expansion remains a valid explanation, but it is not a logically necessary one.

6. Conclusion

The analysis presented in this work establishes that preservation of the cosmic microwave background Planck spectrum does not require metric expansion. Under the stated conditions of collisionless Liouville transport and a homogeneous, isotropic, frequency-independent redshift mapping, the Liouville equation ensures the exact conservation of a blackbody distribution across cosmological time. All standard thermodynamic scalings follow from this assumption, including temperature evolution, number density dilution, and energy density decay. No physical mechanism is proposed or derived within this note. The redshift mapping is treated as an observational input, and its frequency independence is assumed rather than explained. A physical realization of this condition lies in a separate kinetic framework and is not addressed here. The result is testable. Any observed deviation from frequency-independent redshift would falsify the derivation. The preservation of the blackbody spectrum is therefore a necessary consequence of the stated assumptions, not of expansion itself. The distinction is logical, not interpretive.

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Ethical Approval

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Author Contributions

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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