

Two Dimensional Stable and Unstable Manifolds in Modified Lornez System

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Abstract

The chaos phenomenon is observed in the extended Lornez system in four dimensional phase space. The trivial solution is calculated as a saddle. With seasonal varying effects, the Lornez shaped stable manifold of the zero solution is simulated which is two dimensional. The alike Lornez shaped stable manifold is also observed in a developed autonomous Lornez system in four dimensional phase space. The modified Lornez system has a singular attractor. The manifold is drawn by adopting the BVPs problem combined with time homogenous method. The two dimensional unstable manifold is deemed as the neighborhood of the attractor and drawn as a colorful picture. The modified Lornez system has a mirror symmetry structure.

Keywords

Modified Lornez System, Chaos, Two Dimensional Stable Manifold, Lornez Shaped Manifold, Two Dimensional Unstable Manifold

1. Introduction

The dynamics of a nonlinear system evolve into a variety of kinds of bifurcation problems and complex phenomena. Chaos phenomena appear in every textbook, including [1]-[4]. The research on chaos and chaos control has inspired people's big enthusiasm in the dynamical research field [5]. The route to chaos is continuously discovered by Poincare section method by periodicity bifurcation of limit cycles [3] [4]. In 1960's, the Lornez attractor with chaos phenomena was explored [6], and people have been evoked with intuition in their novel thoughts of the skeleton of dynamics. The Lornez system was developed in a lab with atmospheric phenomena, however, it has been famous everywhere. We learn Lornez system from some related chaotic systems with different geometrical topologies. The

chaos with butterfly shapes is observed by symmetry property in specified parameter domain [6] [7].

The dynamics of Lornez system is studied by its equilibrium solutions and manifold structure. In recent years, a large number of papers have reported the stable manifold of zero saddle with the same parameters [8]-[10]. The zero saddle has two dimensional eigenvector subspace, and by choosing an initial manifold, people draw the two dimensional stable manifold of Lornez structure in different views under three dimensional space. In general, people develop the triangular methodology with strict test conditions to draw Lornez shapes manifold by supervision of their colors [8] [11]-[13]. With a peep of the boundary manifold, it seems that we understand what we do with the two dimensional unstable and stable manifolds. The computation of the two dimensional manifold is capable with the auxiliary modern computation tools, alike Auto tools and Xppaut [13]-[15]. By choosing a unique orbit from infinitely phase shifted orbits, the computation of periodical solutions is developed in Auto and Matcont [13] [16]. The computation of the phase orbit is possible by solving a difficult BVPs problem and we also adopt the method herein.

The usual Lornez system is described as

$$\begin{aligned}x' &= a(y-x) \\ y' &= \rho x - y - xz \\ z' &= -bz + xy\end{aligned}\tag{1}$$

with parameters chosen as $a = 10, b = 8/3, \rho = 28$. The zero equilibrium is a saddle with two positive characteristic roots.

With a sudden motion, we want to draw the two dimensional Lornez system with its modified version. Hence, in four dimensional phase space, the two dimensional stable manifold of Lornez manifold alike are gotten with the supposed Lornez system. Therefore, the modified Lornez version is written as

$$u' = f(u, t)\tag{2}$$

with $u \in R^4$. The examples are listed as the following

Example 1.

$$\begin{aligned}x' &= a(y-x) - w + 5x \\ y' &= \rho x - y - xz \\ z' &= -bz + xy + xw \\ w' &= -w \sin(t) + y\end{aligned}\tag{3}$$

with $a = 10, b = 8/3, \rho = 28$.

Example 2.

$$\begin{aligned}x' &= a(y-x) - 0.3w \\ y' &= \rho x - y - xz \\ z' &= -bz + xy \\ w' &= -w + a(y-x)\end{aligned}\tag{4}$$

with $a = 10, b = 8/3, \rho = 28$.

Example 3.

$$\begin{aligned}
 x' &= a(y-x) + w - 2x \\
 y' &= \rho x - y - xz \\
 z' &= -bz + xy \\
 w' &= -0.3xz - 3y
 \end{aligned} \tag{5}$$

with $a = 14, b = 8/3, \rho = 18$.

Set $J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, then one gets $f(J(x)) = J(f(x))$, which shows

that system (3) and (4) have mirror symmetry. The chaos solutions are simulated, as shown in **Figure 1** in Section 2. As is well known, the equilibrium solution and periodical solution are an invariant set of the orbits of the system. For one dimensional curve manifold of equilibrium solution, which is either eventually positively or negatively tends to the equilibrium itself is called as $W^s(p_0)$ or $W^u(p_0)$. For system (3) and (4), the zero saddle has two stable eigenvectors, which form the initial manifold as a curve immersed in two dimensional eigen space. The global two dimensional manifold can be drawn by solving BVPs problem to compute an orbit solution starting from one point in the initial manifold. The global stable manifold is drawn as shown in **Figure 2** and **Figure 3** in Section 3. For system (5), the zero saddle has two unstable eigenvectors, and the global unstable manifold eventually tends to the invariant attractor, as shown in Section 4.

The whole paper is organized as the follows, the chaos solution and the singular attractors of the modified Lornez systems are drawn in Section 2. The global manifold is computed by the given initial manifold and the frontier manifold with BVPs problem. The two dimensional stable manifold of system (3) and (4) is drawn in Section 3 and the two unstable manifolds with singular attractor boundary of system (5) are drawn in Section 4.

2. Chaos and Singular Attractors

The solution orbits of system (3) and (4) are simulated by ODEs algorithm and the chaos solutions are observed. It is easily verified that system (3) has only one trivial solution as a saddle. With parameter excitation as seasonal varying effects, the chaos solution is observed and pictured under X - Y - Z view and X - Y - W view, respectively, as shown in **Figure 1(a)** and **Figure 1(b)**. System (4) still has the other two symmetrical equilibria. We simulate the chaos solution of system (4) with the ODEs algorithm, and plot the phase portraits under X - Y - Z view and X - Y - W view, as shown in **Figure 1(c)** and **Figure 1(d)**. For system (5), the singular attractor is observed by calculating the phase portrait, as shown in **Figure 1(e)** and **Figure 1(f)**.

3. Two Dimensional Stable Manifold of Modified System

The Lornez shape manifold of system (3) and (4) is simulated as the two dimensional stable manifolds of the saddle. It is calculated that the trivial solution of

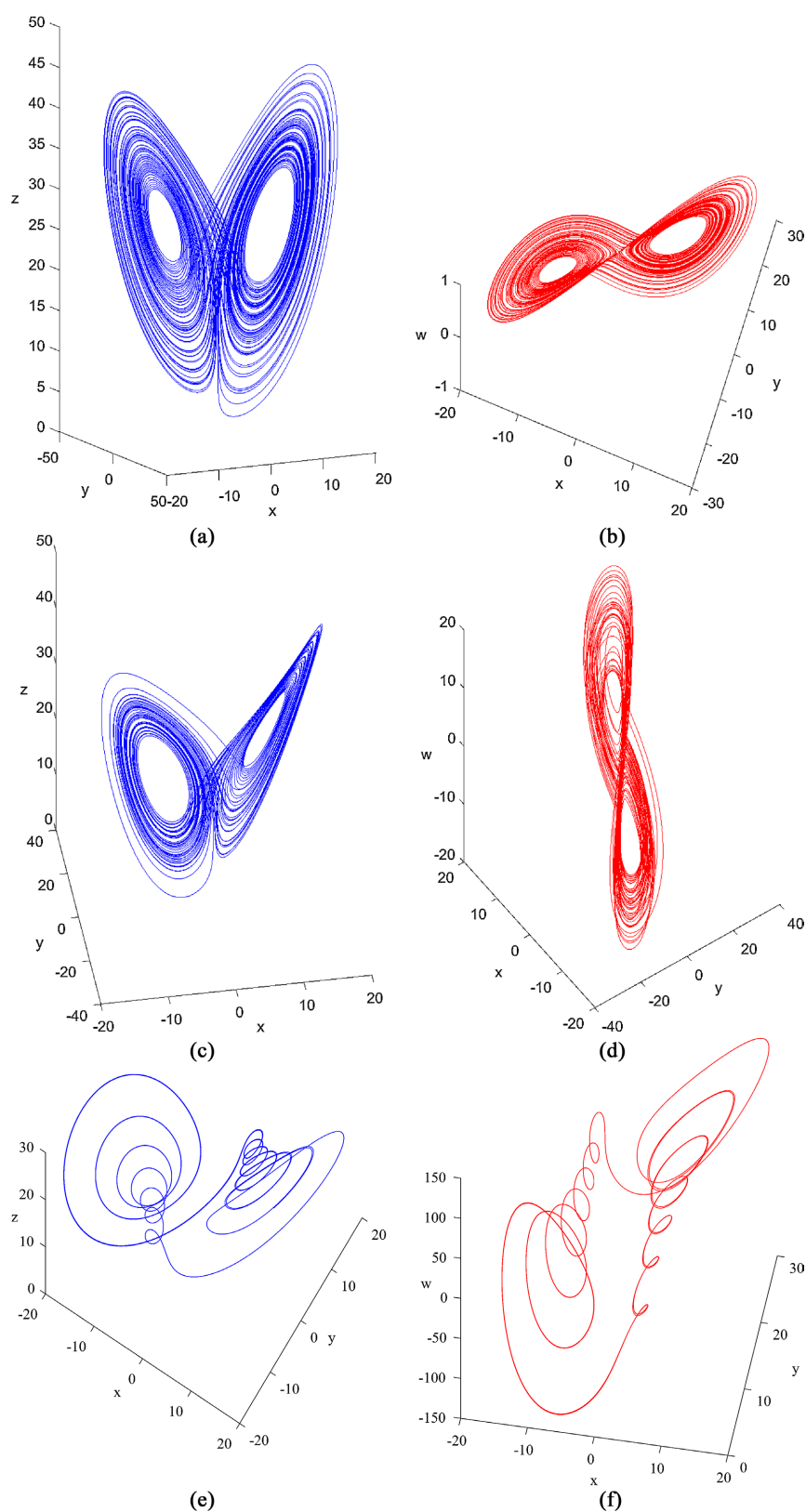


Figure 1. The chaos for modified Lornez system. (a) Chaos in $X-Y-Z$ view for system (3); (b) Chaos in $X-Y-W$ view for system (3); (c) Chaos in $X-Y-W$ view for system (4); (d) Chaos in $X-Y-W$ view for system (4); (e) Singular attractor in $X-Y-Z$ view for system (5); (f) Singular attractor in $X-Y-W$ view for system (5).

system (3) is unstable, which is a saddle with two eigenvectors corresponding to the positive characteristic roots. We choose the initial manifold with a circle lying in the stable eigen subspace. The stable manifold is calculated by choosing the initial manifold point and solving BVPs problem with the fixed arclength Δ_0 , which is called as the frontier point of the manifold. As computing the manifold with time homogenous, the μ -factor of the increasing orbits is supposed as refer to papers ([15] [16]). By choosing two frontier points P_1, P_2 , we list that μ -factor is calculated by the formula

$$\begin{aligned} L &= P_2 - P_1 \\ v &= (L'Lf - f'LL) \\ \mu &= \frac{C_0 \|L\|}{\|v\|} \end{aligned} \quad (6)$$

herein C_0 is the regulation constant for μ -increasing factor.

The BVPs problem is put forward as the following

$$\begin{aligned} u(-t)' &= -\mu f(-t, u(-t)) \\ \|u(-1) - u(0)\| &= \Delta_0 \\ G(u, \mu) &= 0 \end{aligned} \quad (7)$$

with the test condition $G(\mu, u) = 0$.

The corresponding linear matrix is listed as

$$\begin{pmatrix} D & f \\ \delta(-1) - \delta(0) & 0 \\ G_u & G_\mu \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta \mu \end{pmatrix} = 0 \quad (8)$$

The next frontier point P on the manifold is calculated by solving BVPs problem with solution $u(1) = P$, the given initial point $u(0)$ is lying on the segment P_0P_1 . The Newton-Raphson algorithm is applied to solve the convergence problem of the solutions. The global manifold of the saddle asymptotically tends to the saddle at time tends to infinity. As shown in **Figure 2**, the two dimensional stable manifold is viewed as shown in X - Y - Z space, X - Z plane, Y - Z plane and X - Y - W space, respectively. For system (4), the Lorenz shape manifold is also gotten by the computation method. Starting from the initial manifold, increasing frontier manifold with fixed arc length Δ_0 by keeping orbits time homogenous further, hence the stable manifold is prolonged. As shown in **Figure 3**, the Lorenz shape manifold is observed under X - Y - Z view, by projecting onto X - Z plane and Y - Z plane, and viewed in X - Y - W space.

4. Two Dimensional Unstable Manifold with Modified System

The trivial solution of system (5) is a saddle which has two dimensional unstable eigen subspace. The initial manifold is chosen as the circle lying inside the eigen subspace, and μ -factor is chosen for keeping the manifold time homogenous

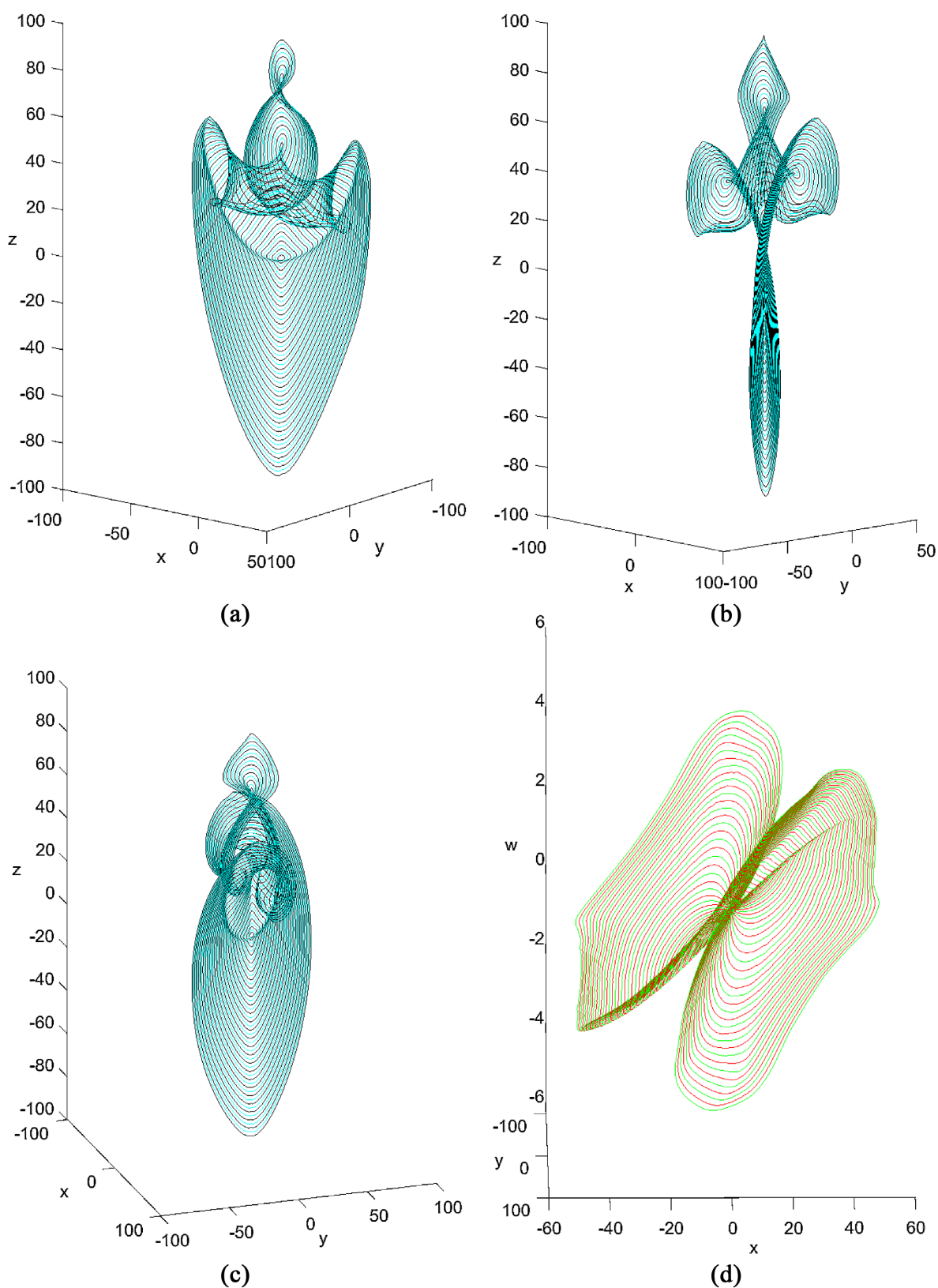


Figure 2. The stable manifold of the modified Lorenz system (3). (a) The two dimensional stable manifold shown in X - Y - Z view; (b) The manifold as observed in X - Z plane; (c) The manifold as observed in Y - Z plane; (d) The two dimensional stable manifold shown in X - Y - W view.

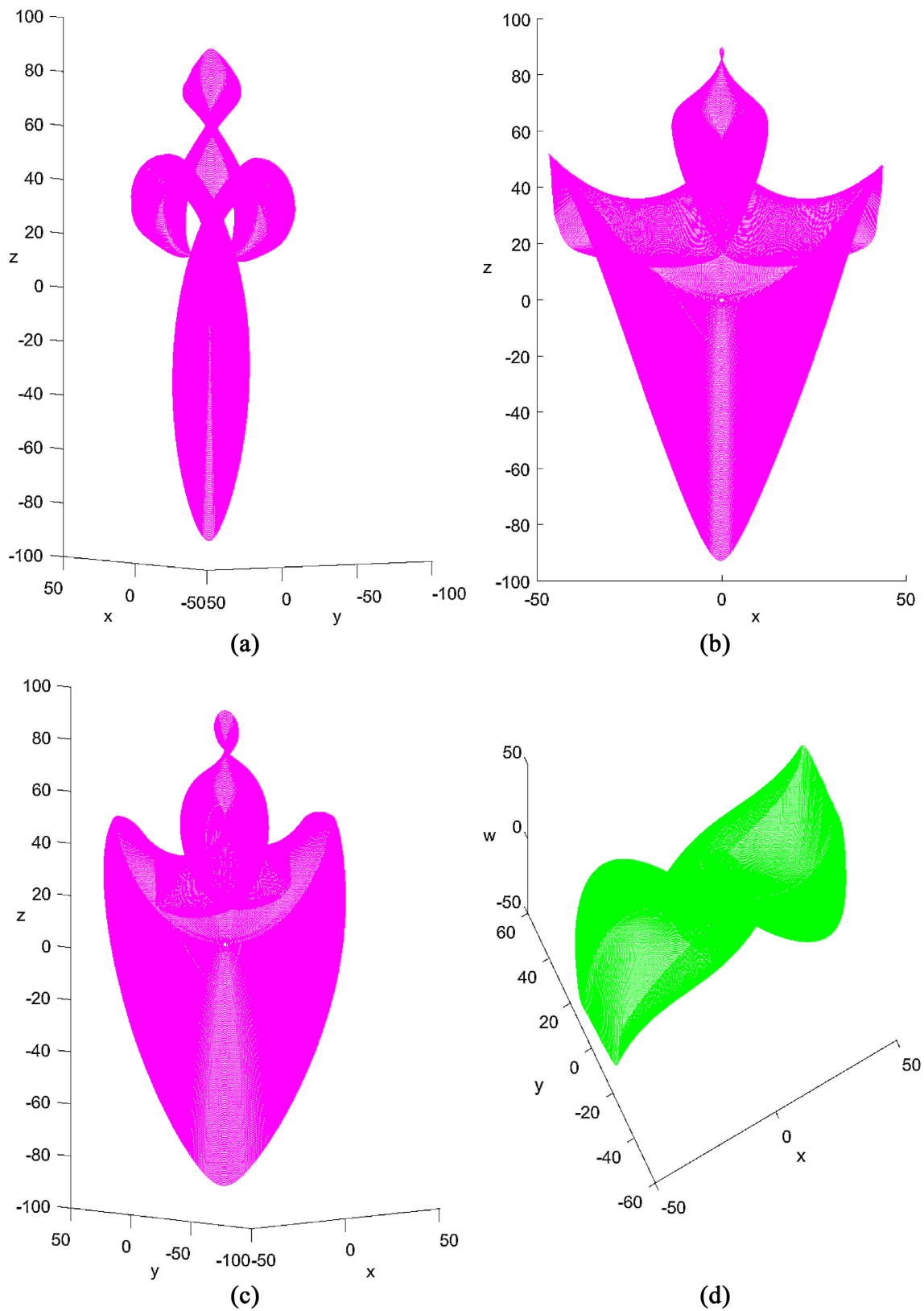


Figure 3. The stable manifold of the modified Lorenz system (4). (a) The two dimensional stable manifold shown in X - Y - Z view; (b) The manifold as observed in X - Z plane; (c) The manifold as observed in Y - Z plane; (d) The two dimensional stable manifold shown in X - Y - W view.

increased. The unstable manifold is prolonged which is deemed as the neighborhood of the singular attractor. As shown in **Figure 1(e)** and **Figure 1(f)**, system (5) has the singular attractor as observed in X - Y - Z space or in X - Y - W space. We often draw the two dimensional unstable manifold as a certain fact that the singular attractor is immersed within the manifold. By choosing one point from the prepared manifold, the manifold is increased by solving the BVPs problem to get the next frontier point again. By choosing two points P_0, P_1 on the initial manifold, to keep time homogenous, the μ -factor is given in formula (6). Hence the BVPs problem of the μ -parameter dynamical system is described as

$$\begin{aligned} u(t)' &= \mu f(t, u(t)) \\ \|u(1) - u(0)\| &= \Delta_0 \\ G(u, \mu) &= 0 \end{aligned} \quad (9)$$

with the test condition $G(\mu, u) = 0$.

The corresponding linear matrix is listed as

$$\begin{pmatrix} D & f \\ \delta(1) - \delta(0) & 0 \\ G_u & G_\mu \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta \mu \end{pmatrix} = 0 \quad (10)$$

By solving the BVPs problem, the next frontier point P on the manifold is obtained, and the two dimensional unstable manifold is prolonged by the new circle formed by the frontier points. The manifold surface is regarded as $z = g(x, y), w = h(x, y)$ which satisfies the tangency condition

$$\begin{aligned} f(x, y, g(x, y), h(x, y)) \begin{bmatrix} \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1, 0 \end{bmatrix} &= 0 \\ f(x, y, g(x, y), h(x, y)) \begin{bmatrix} \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, 0, -1 \end{bmatrix} &= 0 \end{aligned} \quad (11)$$

We observe the two dimensional unstable manifold by viewing in X - Y - Z space, and in X - Z plane and in Y - Z plane, and in X - Y - W space, as shown in **Figures 4(a)-(d)**. The unstable manifold is colorful pictured which is the neighborhood of the attractor shown in **Figure 1(e)** and **Figure 1(f)**.

5. Conclusion

Both the two dimensional stable and unstable manifolds were drawn by developing the modified Lorenz system in four dimensional phase space. The chaos solution was simulated. The Lorenz shaped stable manifolds were observed either with a non-autonomous system with parameter excitation, or with autonomous ODEs. The manifold was taken as the surface of (x, y) variables which satisfies the tangency condition. The two dimensional unstable manifold was drawn by extending the manifold by solving the BVPs problems. The manifolds were extended which was the phase orbits clusters to keep the time homogeneous solution

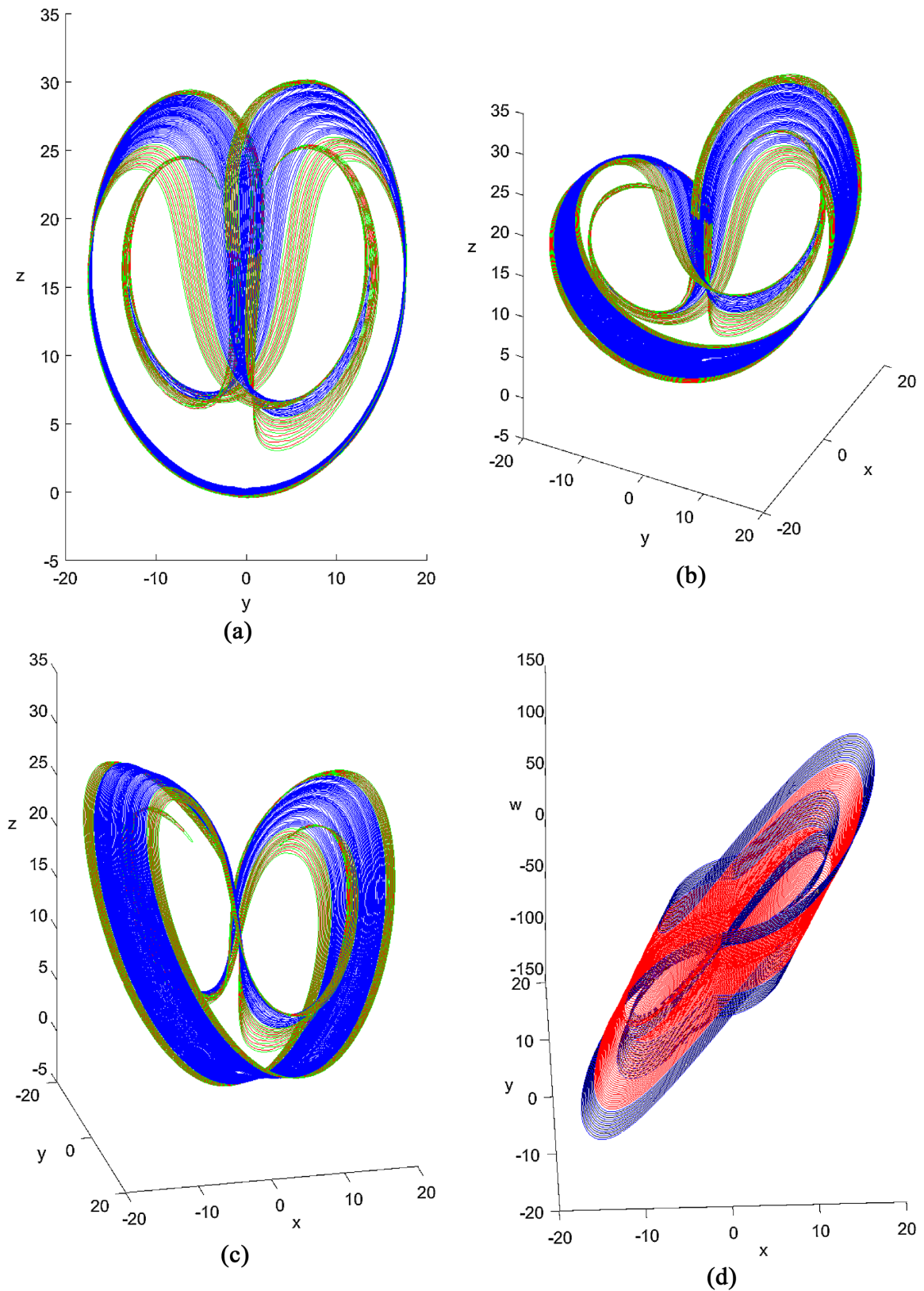


Figure 4. The unstable manifold of the modified Lorenz system (5). (a) The two dimensional unstable manifold shown in X - Y - Z view; (b) The manifold as observed in X - Z plane; (c) The manifold as observed in Y - Z plane; (d) The two dimensional unstable manifold shown in X - Y - W view.

increased. The modified Lorenz system has mirror symmetry.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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