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Two Dimensional Unstable Manifolds in Chen System and Chen-Lv System

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Abstract

The complex dynamics, like chaos induced by period-doubling bifurcation of periodical oscillation phenomena are often examples of nonlinear dynamical science, nevertheless for the Chen system, Chen-Lv system and Lorenz system. The heteroclinic chaos phenomena and the scroll wave chaos phenomena and the Lorenz attractor, respectively of three chaos systems, are often proud complex elements. The two dimensional stable manifold of Lornez system is well known as a successful example of a manifold. The two dimensional unstable manifolds of the Chen system and the Chen-Lv system are computed by the manifold computation method which satisfies the tangency condition. As for the case of the heteroclinic chaos, the twin manifolds are pictured which originate from the two different saddles. Whilst for the case of the Chen-Lv system, the single scroll wave manifold, and the interaction scroll wave manifold and the double scroll wave manifold are computed, which are supposed to be the two dimensional unstable manifolds. The manifold's picture embodies system symmetry character, which means the manifold has mirror symmetry under the single parameter symmetry.

Keywords

Chaos, Unstable Manifold, Chen System, Chen-Lv System

1. Introduction

System dynamics is complex if the stability property is changed and periodical oscillation leads to quasi-periodical motion and even chaos. With minute parameter perturbation, the periodical solution loss its stability through period-doubling bifurcation, which paved the way alike by periodical doubling bifurcation to chaos [1]-[3]. With the control of the input constant variable, the Lv system exhibits its complex dynamics with strange attractors or chaos [4]-[6]. The similar chaos behavior always appears in the Chen system and Lornez system simultaneously. The quantitatively dynamical method is applied to study the complex behavior and chaos which list its examples as Lornez system, Chen system and Lv system.

We are taught by the chaos phenomena by the above three systems. For example, Chen chaos and Lv chaos and Lornez attractor. As for Lornez system, most papers have plotted the two dimensional stable manifold originating from zero saddle. We also draw the two dimensional stable manifold of the Chen system, cited by [7] [8]. Herein, we are interested in the Chen system with twins manifold, in which one is the two dimensional unstable manifold from one saddle (the first saddle) prolonged to the two dimensional stable eigen space of another saddle (the second saddle); whilst another two dimensional stable manifold from the second saddle expanded to the two dimensional unstable eigen space of the first saddle. The heteroclinic chaos is formed by the corresponding time evolution solution and we plot the phase portraits in phase space.

The manifold is the continuous chart drawn by solution portraits which are supposed to be time homogenous [9] [10]. In general, we write the autonomous ODEs system as

$$\dot{x} = f(x) \tag{1}$$

with $x \in R^n$. The stable and unstable manifold is defined as

$$\begin{aligned} W^s(x_0) &= \left\{\Phi(t;x_0) \mid \lim_{t\to+\infty} \Phi(t;x_0) = x_0\right\} \\ W^u(x_0) &= \left\{\Phi(t;x_0) \mid \lim_{t\to-\infty} \Phi(t;x_0) = x_0\right\} \end{aligned} \tag{2}$$

For computing the manifold, we often choose the initial value manifold as a circle

$$C(x_0) = x_0 + \text{circle}\left\{\text{spaned by } \{e_1, e_2\} \text{ eigenvector}\right\} \tag{3}$$

For example, with $\theta \in [0, 2\pi]$, the initial value manifold is written as

$$C(x_0) = x_0 + R\left(e_1 \sin(\theta) + e_2 \cos(\theta)\right) \tag{4}$$

with R being the initial radius of the circle.

Suppose ODEs (1) has mirror symmetry about z -axis and parameter u -symmetry, that is, assume

$$J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5}$$

we have

$$f\left(J\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}\right) = Jf\left(\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}\right) \tag{6}$$

Furthermore, suppose the chaos solution of Equation (1) is $\Gamma(x, y, z, u)$, the we have

$$\Gamma(x,y,z,u)=\Gamma(-x,-y,z,-u).$$

For example, the Chen system is listed as

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=a(y-x)\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=(c-a)x-xz+cy-ex^2\\ \frac{\mathrm{d}z}{\mathrm{d}t}&=xy-bz\end{aligned} \tag{7}$$

We discuss the stability property of the equilibrium solution of Equation (7), and calculate the two dimensional unstable manifolds of the saddles. The manifold calculation scheme is introduced in the papers [11]-[14]. System (7) has three saddles, therefore it has the two dimensional unstable manifold from two different saddles.

System (7) has chaos solution, which tends to the saddle equilibrium as time goes to infinity positively, hence we call it heteroclinic chaos. We emphasize the two dimensional unstable manifold originating from the saddle, then expanded to be the neighborhood of the heteroclinic chaos. The expanding method of the manifold is necessary to compute the tangency condition referred to the papers [10] [13] [14]. Suppose the manifold is expressed as

$$z=g(x,y) \tag{8}$$

which must satisfy the tangency condition which is described as the orthogonality condition of the tangent vector at $(x,y,g(x,y))$ on the manifold z-surface and the normal vector, that is

$$f(x,y,g(x,y))\left[g_x(x,y),g_y(x,y),-1\right]=0 \tag{9}$$

Furthermore, by calculating the derivatives, one gets that

$$\begin{aligned}g_x(x,y)&=\frac{yf_3(x,y,g(x,y))-g(x,y)\,f_2(x,y,g(x,y))}{yf_1(x,y,g(x,y))-xf_2(x,y,g(x,y))}\\ g_y(x,y)&=\frac{g(x,y)\,f_1(x,y,g(x,y))-xf_3(x,y,g(x,y))}{yf_1(x,y,g(x,y))-xf_2(x,y,g(x,y))}\end{aligned} \tag{10}$$

Then substituting Equation (10) into Equation (9) to get

$$\begin{aligned}&f(x,y,g(x,y))\\ &\cdot\left[\frac{yf_3(x,y,g(x,y))-g(x,y)\,f_2(x,y,g(x,y))}{yf_1(x,y,g(x,y))-xf_2(x,y,g(x,y))},\frac{g(x,y)\,f_1(x,y,g(x,y))-xf_3(x,y,g(x,y))}{yf_1(x,y,g(x,y))-xf_2(x,y,g(x,y))},-1\right]\\ &=0\end{aligned} \tag{11}$$

With the tangency condition, the twins manifolds are calculated by the difficult manifold algorithm, which are originated from the different saddle E_0 and E_1 respectively. The manifold is usual the neighborhood of the heteroclinic chaos.

Without loss of generality, Chen-Lv system has mirror symmetry about z-axis and parameter symmetry, which is written as

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= a(y-x) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x - xz + cy + u \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= xy - bz \end{aligned} \tag{12}$$

In the Chen-Lv system, the usual findings of chaos are manifested at the different parameter values of input constant u. As is seen, the single scroll wave chaos are found with $|u| > 15$, whilst the interaction scroll wave chaos are found with $|u| = [11,15]$, and the double scroll wave chaos are found with $|u| \in [10,11]$. Applying the manifold algorithm with Chen-Lv system, the two dimensional unstable manifolds are expanded which are originated from two different saddles E_0 and E_1 respectively. As we have done, the two dimensional manifolds manifest their wave character just like the single scroll wave, interaction scroll wave and double scroll wave manifold.

The whole paper is organized as the following. In section 2, the heteroclinic chaos are exhibited and the twins manifolds of Chen system are also computed in section 3, which are further expanded to be the neighbourhood of the heteroclinic chaos. As for Chen-Lv system, the scroll wave manifolds are computed and drawn. As is verified, the manifolds have mirror symmetry and parameter symmetry.

2. Heteroclinic Chaos of Chen System

For Chen system (7), the fixed parameters are chosen as $a = 40$, $b = 3$, $c = 28$, and the free parameter is chosen as $e = \pm 0.5$. System (7) has the equilibrium solutions $E_0 = [6.2187; 6.2187; 12.8907]$ and $E_1 = [-7.7187; -7.7187; 19.8593]$. Suppose the initial value as $x_0 = E_0 + [0.1; 0.1; 0.1]$ and $x_1 = E_1 + [0.1; 0.1; 0.1]$. Starting from E_0, the heteroclinic chaos is drawn with long time evolution behavior. The phase portraits are drawn in **Figure 1(a)** and **Figure 1(b)**, respectively with the initial value x_0 and x_1. As shown in **Figure 1(a)** and **Figure 1(b)**, the two heteroclinic chaos are mirror symmetry about z-axis, and parameter symmetry with $e = 0.5$ and $e = -0.5$. We describe the solution orbit in **Figure 1(a)** as $\Phi(t; x_0) = \Gamma(x, y, z, e)$ with $e = 0.5$, then solution orbit in **Figure 1(b)** is described as $\Phi(t; x_0) = \Gamma(-x, -y, z, -e)$. And we have $\lim_{t \to +\infty} \Phi(t; x_0) = x_1$ for $e = \pm 0.5$. The twins heteroclinic chaos are also found in **Figure 1(c)** and **Figure 1(d)** with $e = 0.5$ and $e = -0.5$. The solution orbits in **Figure 1(c)** and **Figure 1(d)** are respectively represented as $\Phi(t; x_1) = \Theta(x, y, z, e)$ and $\Phi(t; x_1) = \Theta(-x, -y, z, -e)$, and we have $\lim_{t \to +\infty} \Phi(t; x_1) = x_0$ for $e = \pm 0.5$. The produced phase portraits in **Figure 1(a)** and **Figure 1(b)** are calculated by time evolution positively; whilst the phase portraits in **Figure 1(c)** and **Figure 1(d)** are produced by computing the solution orbits time reversely. Therefore, we have

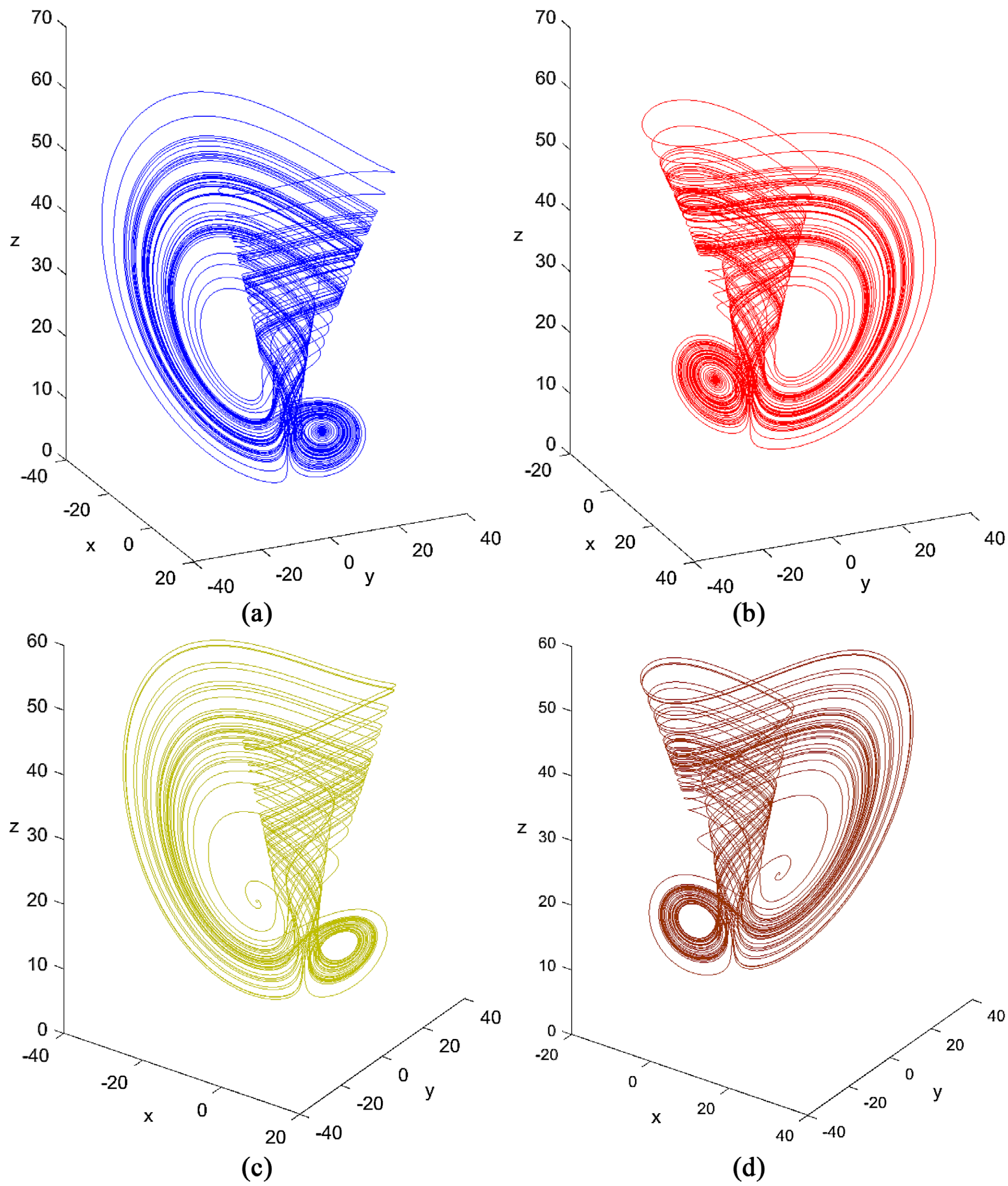

Figure 1. The heteroclinic chaos observed with (a) $e = 0.5$, (b) $e = -0.5$, which starts from x_0, then goes to the equilibrium solution E_1 as time tends to the infinity positively; The heteroclinic chaos observed with (c) $e = 0.5$, (d) $e = -0.5$, which starts from E_1, then goes to the chaos solution with time evolution reversely.

$$\begin{aligned} W^s(E_1) &= \Phi(t; x_0) = \Gamma(x, y, z, e) = \Gamma(-x, -y, z, -e) \\ W^s(E_0) &= \Phi(t; x_1) = \Theta(x, y, z, e) = \Theta(-x, -y, z, -e) \end{aligned} \tag{13}$$

3. Twins Manifold for Chen System

As described in section 2, the four heteroclinic chaos solutions are observed with Z_2 symmetry and parameter symmetry. The two heteroclinic chaos in **Figure 1(a)** and **Figure 1(c)** are also called as twins chaos, which are the stable manifolds of the saddle E_1 and E_0 respectively; whilst the two heteroclinic chaos in **Fig-**

ure 1(b) and Figure 1(d) are twins chaos too. It is easily calculated that the saddle E_0 has the eigenvalues $real(\lambda_{1,2})>0$ and $real(\lambda_3)<0$, and the saddle E_1 has the eigenvalues $real(\lambda_{1,2})>0$ and $real(\lambda_3)<0$. As referred to the manifold algorithm [15], the initial manifold is denoted as

$$C_0 = E_0 + R\left(\Re(e_1)\cos(\theta)+\Im(e_1)\sin(\theta)\right) \quad (14)$$

wherein $e_{1,2}$ are the eigenvectors of the complex roots $\lambda_{1,2}$. Noticed as calculating the two dimensional unstable manifold, the time homogenous of orbit solution must keep pace with arc length increasing. The tangent condition is deduced as

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t},\frac{\mathrm{d}y}{\mathrm{d}t},\frac{\mathrm{d}z}{\mathrm{d}t}\right)\cdot\left[\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},-1\right]=0 \quad (15)$$

which is the same condition as listed in Equations (9) and (11).

As is well known, the two dimensional unstable manifold of the saddle is prolonged and formed into a neighborhood of the heteroclinic chaos. As is explained in section 2, the heteroclinic chaos is the stable manifold of the saddle E_1 simultaneously. Therefore, we draw the two dimensional unstable manifold of the saddle $E_0=[6.2187;6.2187;12.8907]$ as shown in Figure 2(a) and Figure 2(c) for $e=0.5$, with a colorful view in *X*-*Y*-*Z* dimensional space. Both the manifold in Figure 2(a) and Figure 2(c) are the same manifold contrast to the view of colors, and with arc length $\delta_0=0.16$ and $\delta_0=0.8$. However, the mirror symmetry about z-axis aids to have a peek of the unstable manifold at $e=-0.5$, henceforth, we draw the symmetrical manifold of the saddle
$E_0=[-6.2187;-6.2187;12.8907]$, with a colorful view shown in Figure 2(b) and Figure 2d). The arc length is fixed as $\delta_0=0.16$ in Figure 2(b) and $\delta_0=0.8$ in Figure 2(d). By the mirror symmetry and parameter symmetry of a pair of opposite value of e, the two dimensional unstable manifold of the saddle
$E_0=[6.2187;6.2187;12.8907]$ for $e=0.5$ observed in Figure 2(a) and Figure 2c), and the two dimensional unstable manifold of the saddle
$E_0=[-6.2187;-6.2187;12.8907]$ for $e=-0.5$ shown in Figure 2(b) and Figure 2(d) give us a strong impression. The unstable manifolds are terminally bounded and become the stable manifold of the saddle E_1. Hence, can we deduce that the two dimensional unstable manifold of E_0 are in coincidence with the stable manifold of the saddle E_1?

Impressed by the heteroclinic chaos shown in Figure 1(c) and Figure 1(d), that is, the time reversely integration solution orbits starting from the saddle E_0, are mirror symmetry with parameter $e=\pm0.5$. Hence we draw the two dimensional unstable manifold of the saddle $E_1=(-7.7187,-7.7187,19.8593)$ for $e=0.5$, and the two dimensional unstable manifold of the saddle
$E_1=(7.7187,7.7187,19.8593)$ for $e=-0.5$. Due to the unstable manifold of E_1 is the neighborhood of the heteroclinic chaos in Figure 1(c) for $e=0.5$, the manifold is terminally bounded and becomes the stable manifold of the saddle E_0. We call a pair of twins manifold as shown in Figure 2(a) and Figure 3(a), and the

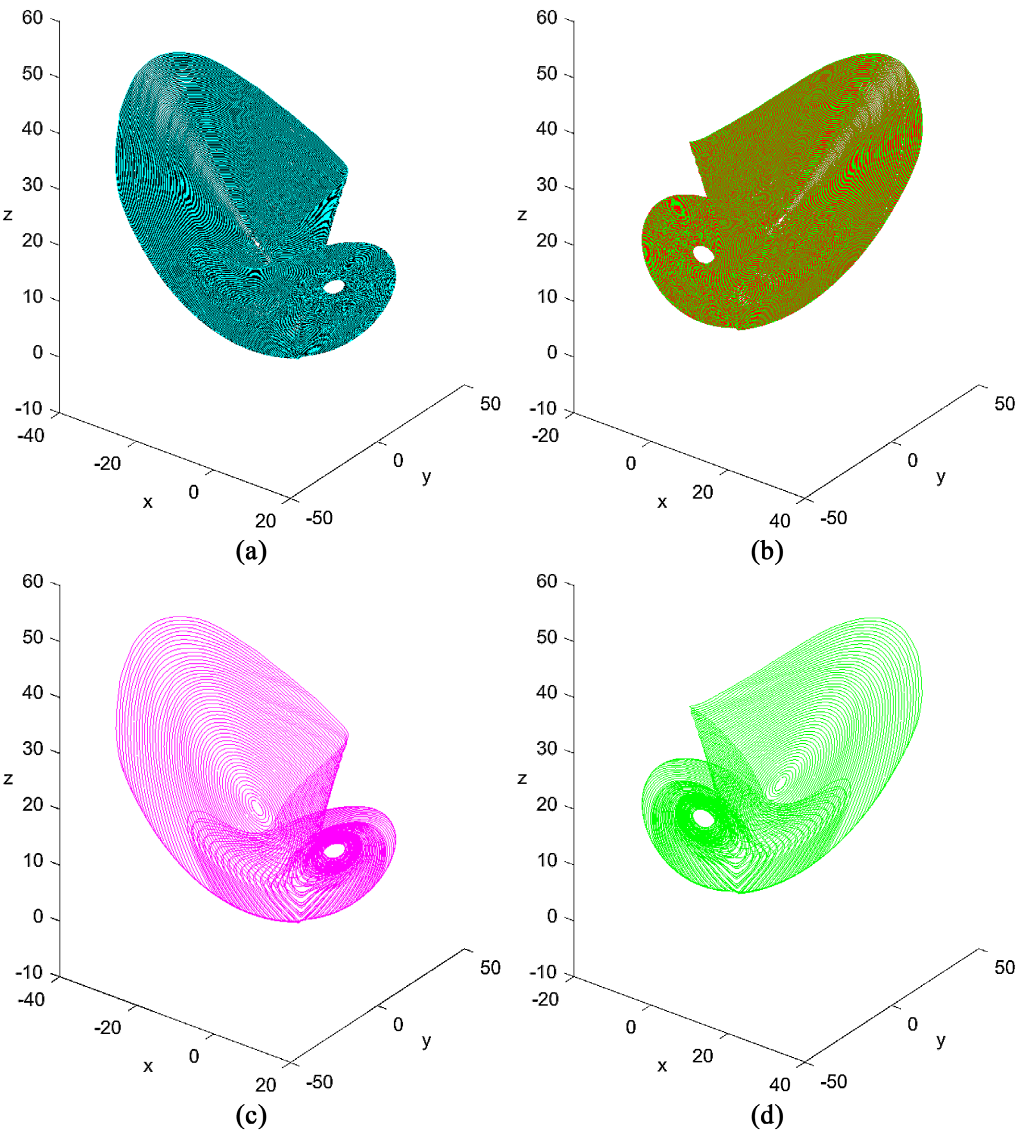

Figure 2. The two dimensional unstable manifold starts from the saddle E_0. (a) The manifold observed with $e = 0.5$, (b)The manifold observed with $e = -0.5$; (c) The manifold of (a); (d) The manifold of (b).

twins manifold of pink colors, as shown in Figure 2(c) and Figure 3(c). With the view of mirror symmetry, a pair of twins manifold as shown in Figure 2(b) and Figure 3(b) for $e = -0.5$ are drawn with manifold algorithm. The twins manifold of green colors are redrawn in Figure 2(d) and Figure 3(d).

4. Scroll Wave Manifolds for Chen-Lv System

Set $J = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, we have $f\left(J\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}\right) = J\left(f\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}\right)$. For Chen-Lv system, we choose the fixed parameters as $a = 36$, $c = 20$, $b = 3$, and u as the

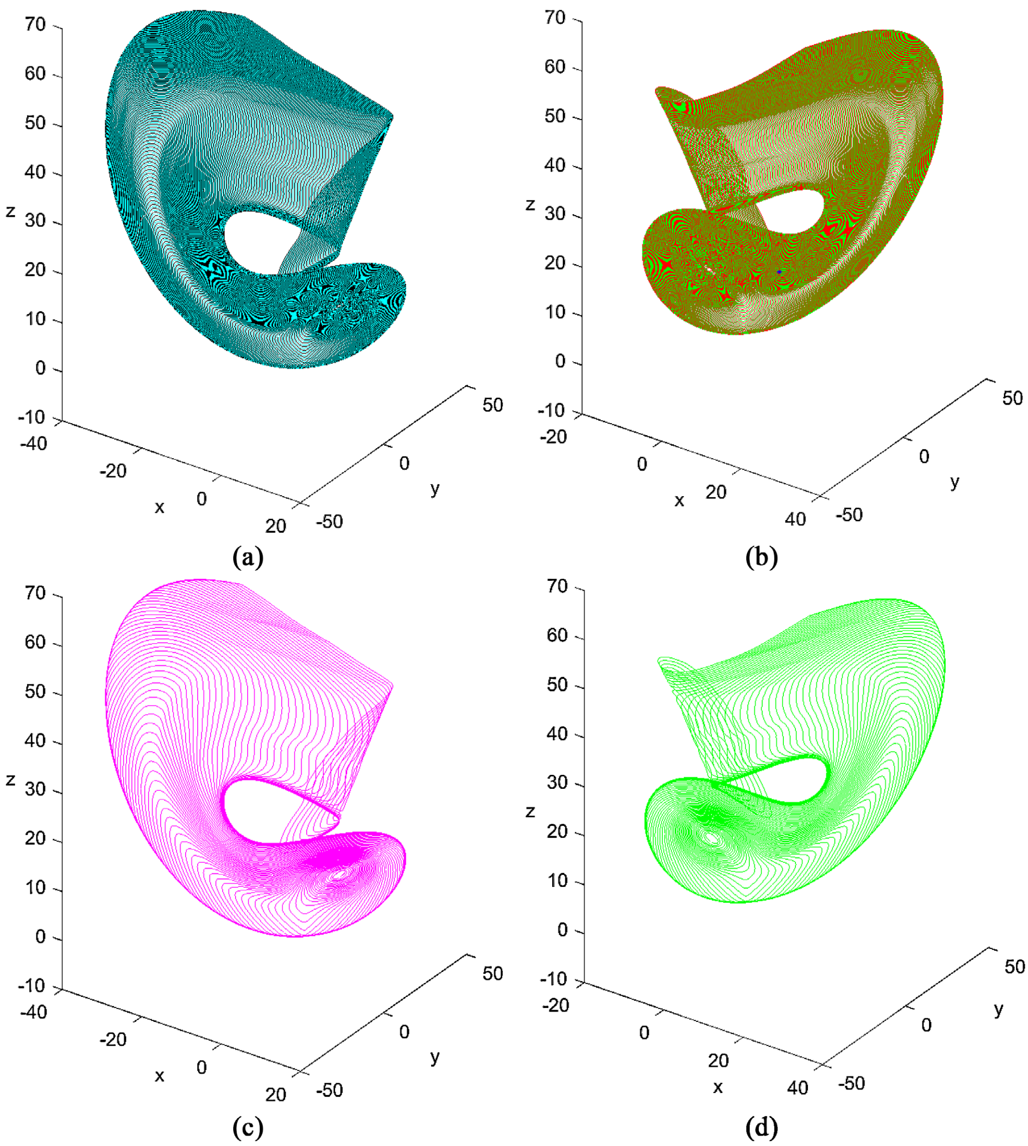

Figure 3. The two dimensional unstable manifold starts from the saddle E_1. (a) The manifold observed with $e = 0.5$, (b) The manifold observed with $e = -0.5$; (c) The manifold of (a); (d) The manifold of (b).

free parameter. As is referred in the papers, choosing $|u| > 11$, the single scroll wave chaos is observed; However, if choose $10 < |u| < 11$, the interaction scroll wave chaos is observed; and choosing $|u| < 10$, the double scroll wave chaos is simulated. What's fascination motivation to simulate a two dimensional unstable manifold is to exhibit a wave formation in chaos. Numerous unstable periodical solution orbits embeds in chaos solution. If the unstable manifold of the periodic orbit is a bounded attractor, which is part of the formation of the wave manifold, hence the scroll wave chaos is found. As referred, the two dimensional unstable manifold is the neighborhood of scroll wave chaos. Therefore, we try our best to simulate the beautiful scroll wave manifolds of Chen-Lv system, which encourages us to walk further. We choose $u = \pm 11.5$, then the single scroll wave chaos are observed. System (12) has saddles $E_0 = [-7.5247; -7.5247; 18.8737]$ and $E_1 = [8.2938; 8.2938; 22.9291]$. The corresponding eigenvalue for E_0 is calculated as $\Re(\lambda_{1,2}) > 0$, $\Re(\lambda_3) < 0$. The initial manifold starting from E_0 is described as [16]

$$C_0 = E_0 + R\left(e_1 \cos(\theta) + e_2 \sin(\theta)\right) \tag{16}$$

For $u = -11.5$, the two dimensional unstable manifold of E_0 is simulated as shown in **Figure 4(a)** and **Figure 4(c)**, with are length being $\delta_0 = 0.16$ and

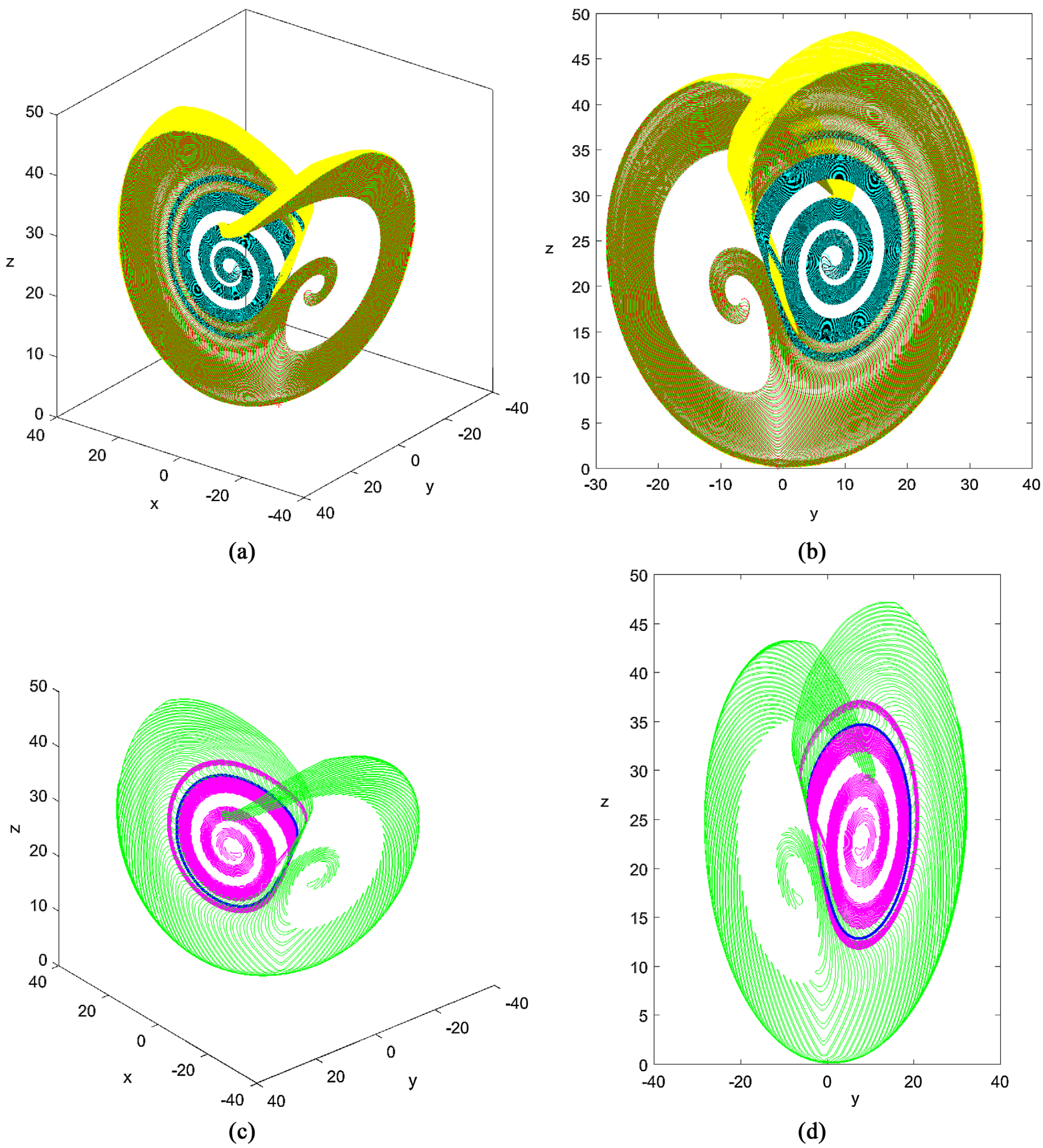

Figure 4. The single scroll wave manifold of Chen-Lv system under mirror symmetry and parameter symmetry. (a) The left scroll wave manifold for $u = -11.5$; (b) The right scroll wave manifold for $u = 11.5$; (c) The manifold of (a); (d) The manifold of (b).

$\delta_0 = 0.8$, respectively. We also draw the two dimensional unstable manifold E_0 under the mirror symmetry and parameter symmetry. For $u = 11.5$, the single scroll wave manifold is shown in **Figure 4(b)** and **Figure 4(d)** for different arc lengths. As shown in **Figure 4**, the left scroll wave manifold and the right scroll wave manifold are respectively simulated for $u = -11.5$ and $u = 11.5$. As shown in **Figure 5**, the interaction scroll wave manifold is drawn for $u = \pm 10.5$ and the

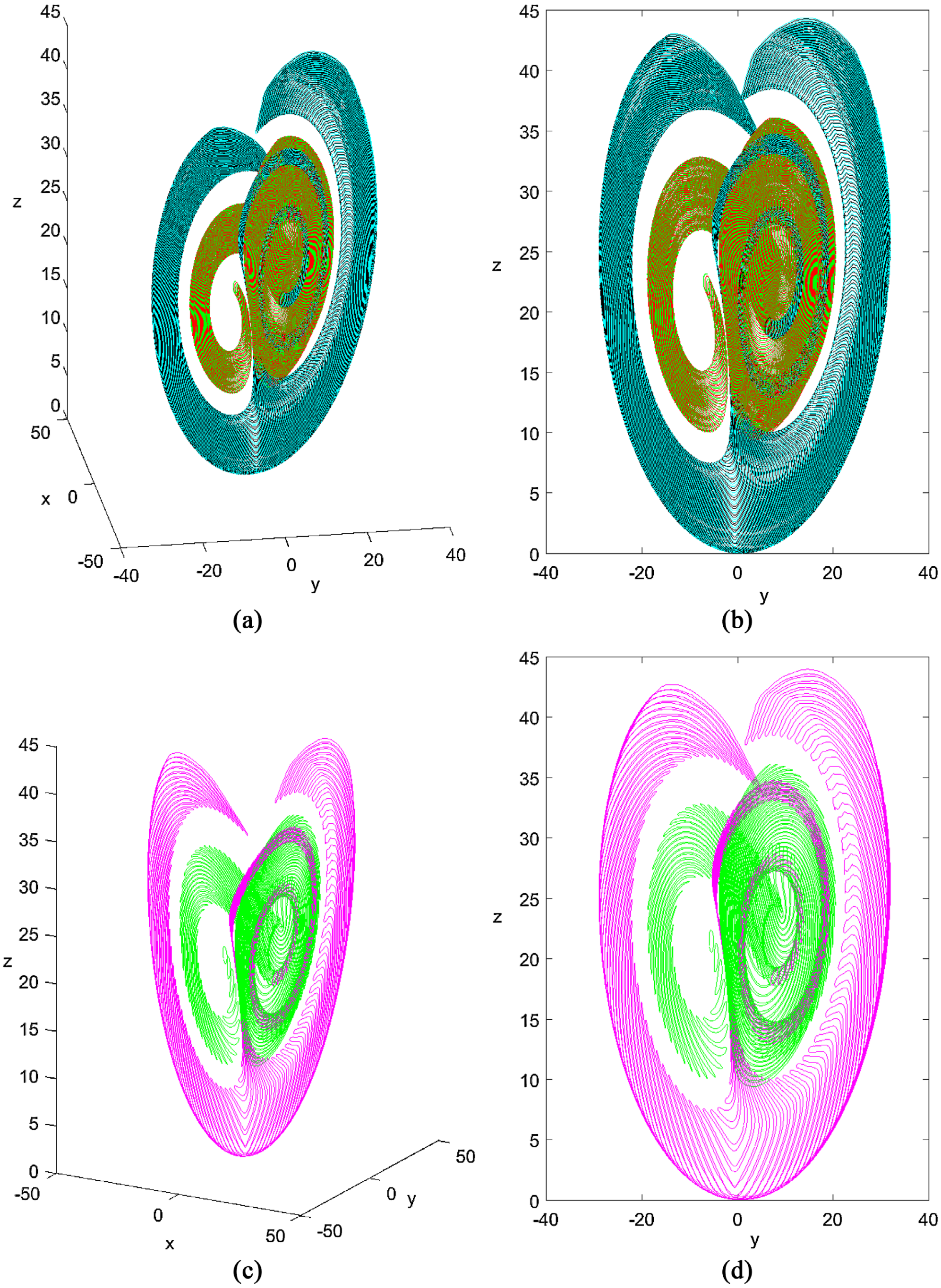

Figure 5. The interaction scroll wave manifold of Chen-Lv system under mirror symmetry and parameter symmetry. (a) The interaction scroll wave manifold for $u = -10.5$; (b) The interaction scroll wave manifold for $u = 10.5$; (c) The manifold of (a); (d) The manifold of (b).

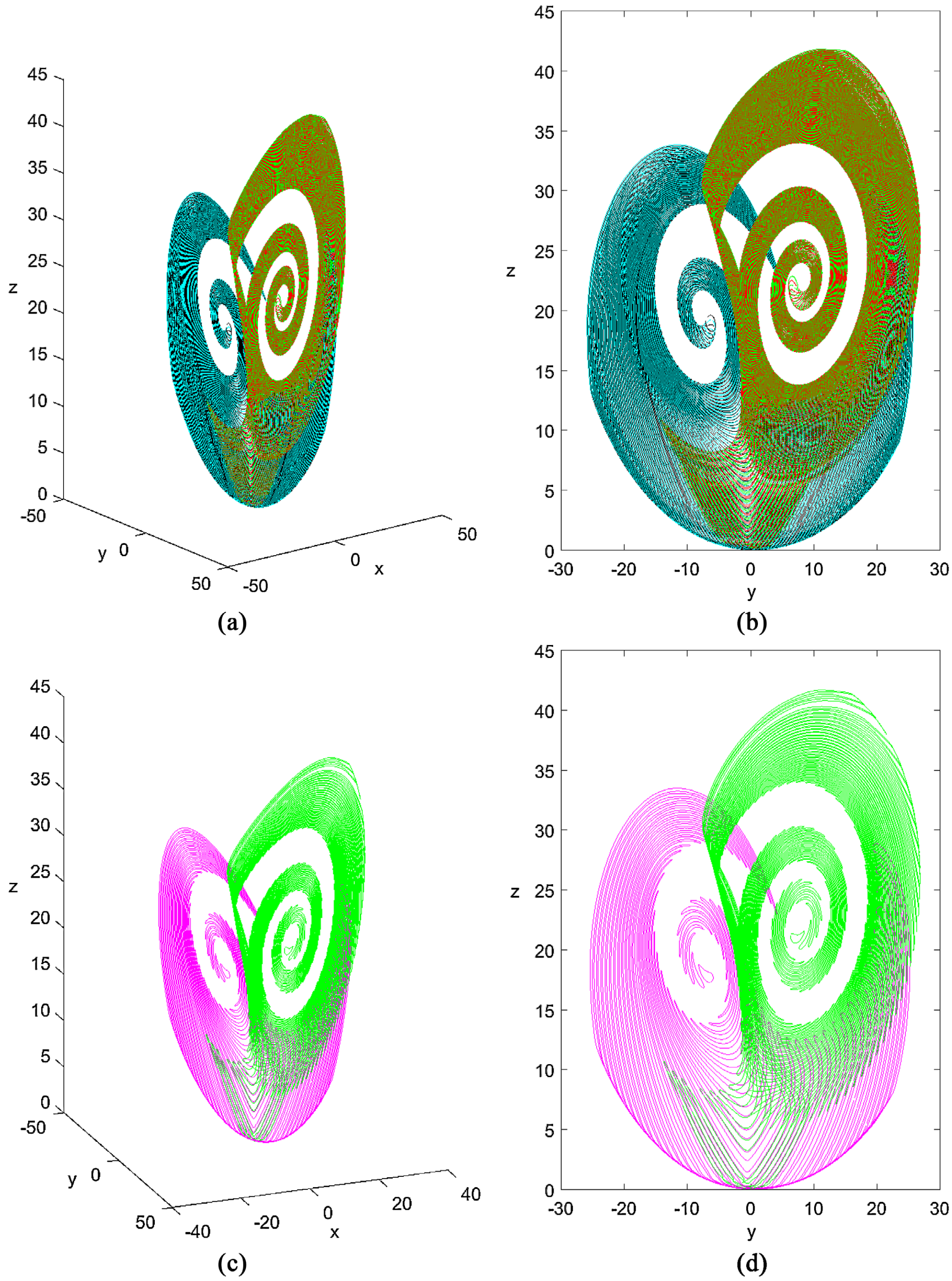

Figure 6. The double scroll wave manifold of Chen-Lv system under mirror symmetry and parameter symmetry. (a) The double scroll wave manifold for $u = -7.8$; (b) The double scroll wave manifold for $u = 7.8$; (c) The manifold of (a); (d) The manifold of (b).

double scroll wave manifold are plotted for $u = \pm 7.8$ (**Figure 6**). All the scroll wave manifold is colorful, which tell us to go further with colorful sight view.

5. Conclusion

Under mirror symmetry and parameter symmetry, the heteroclinic chaos was discussed for Chen system, and the manifold of the two saddles was drawn which looked like a pair of twin manifolds. On one respect, the twins manifold was bounded, and formed as the neighborhood of the heteroclinic chaos; and in addition, the twins manifold was the two unstable manifolds of the original saddle however, it eventually became the stable manifold of the second saddle. The enthusiasm for the manifolds calculation succeeded in simulating the scroll wave manifolds further. With the chosen free parameter, under mirror symmetry, the left and right scroll wave manifolds were simulated for Chen-Lv system; In addition, the interaction scroll wave manifolds were manifested; And furthermore, the double scroll wave manifolds were observed too. All the manifolds computation is helped by the manifold computation methodology, hence advised by the tangency condition on the manifold surface given in papers [13] [14].

Availability of Data and Materials

All data generated or analyzed during this study are included in this published article.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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