

# Interpreting the Electric Field with Complexified Quaternions

Daniel Kovach

Kovach Technologies, Tampa, USA

Email: [daniel@kovachtech.com](mailto:daniel@kovachtech.com)

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## Abstract

This paper revisits classical electromagnetism using complexified quaternions (biquaternions), an associative algebra isomorphic to  $M_2(\mathbb{C})$ . Building on Maxwell's original quaternionic insights, we define a generalized four-gradient  $\partial_r$  and four-potential  $A$  in biquaternionic form. The electric field  $E$  and magnetic field  $B$  emerge from the anticommutator and commutator of  $\partial_r$  and  $A$ , respectively. Explicit computation yields the standard expressions for  $B = \nabla \times A$  and  $E = -\nabla\phi - (1/c)\partial_t A$  plus a gauge-dependent scalar term  $(1/c)\partial_t\phi + \nabla \cdot A$ . In a specific gauge where  $A = -\nabla S$  and  $\phi = (1/c)\partial_t S$ , this reduces to the d'Alembertian wave equation  $\square S = 0$  under the Lorentz condition. The biquaternionic framework unifies scalar and vector components more symmetrically than pure real-quaternionic or vector formulations, offering compact notation and potential insights into wave propagation, invariants, and extensions to relativistic or chiral contexts.

## Keywords

Biquaternions, Complexified Quaternions, Electromagnetic Potentials, Lorentz Gauge

## 1. Introduction

The theory of electromagnetism, as formulated by James Clerk Maxwell in the latter half of the nineteenth century, stands as one of the most profound achievements in classical physics. Notably, Maxwell's original presentation of his equations in [1] employed the quaternionic algebra developed by William Rowan Hamilton. This approach allowed for a compact notation that integrated scalar and vector components in a unified four-dimensional framework.

Subsequently, the equations underwent significant reformulation by Oliver Heaviside (and independently by Josiah Willard Gibbs) [2], who recast them in the language of modern vector calculus. This Heaviside-Gibbs version, consisting of the familiar four differential equations, was purported to offer greater computational simplicity and facilitated the rapid advancement of electromagnetic theory and its applications. Nevertheless, arguments have persisted in the literature regarding the relative merits of the two formulations. Some researchers have suggested that the quaternionic approach may retain geometric and algebraic symmetries or additional physical insights that are not immediately apparent in the vectorial representation.

In this paper, we explore some of the ramifications of returning to a quaternionic approach to electromagnetism, with a particular emphasis on a complexified quaternionic (biquaternionic) formulation. By incorporating complex coefficients into the quaternion structure, this method promises to yield more natural interpretations of electromagnetic phenomena while maintaining the elegance of the original quaternion algebra.

Our work builds upon previous studies employing the pure quaternionic form, such as that detailed in “Deductions from the Quaternion Form of Maxwell’s Electromagnetic Equations” by Dunning-Davies and Norman [3], wherein the authors identify an additional scalar field component. In contrast, the complexified quaternionic approach adopted herein leads to what we believe are slightly more natural conclusions, potentially offering deeper unification or simplified derivations of key results. The following sections detail the mathematical framework and its implications for electromagnetic theory.

## 2. The Cayley-Dickson Construction and the Biquaternionic Structure

The Cayley-Dickson construction provides a recursive algebraic procedure for generating a tower of hypercomplex number systems by systematically doubling the dimension of a given  $\ast$ -algebra. Formally, given an algebra  $A$  over  $R$  equipped with an involution (conjugation)  $\bar{\phantom{x}}$ , the doubled algebra consists of ordered pairs  $(a, b) \in A \times A$  with componentwise addition and the multiplication rule

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}), \quad (1)$$

where the precise sign convention may vary by normalization. Successive applications of this construction to the real numbers  $\mathbb{R}$  yield the complex numbers  $\mathbb{C}$ , followed by the (real) quaternions  $\mathbb{H}$ , as the Cayley-Dickson double of  $\mathbb{C}$ . In this representation, every quaternion admits the decomposition

$$q = a_1 + a_2 j, a_\mu \in \mathbb{C}, \quad (2)$$

with  $j^2 = -1$ ,  $j$  anticommuting appropriately with the complex unit. Using the standard quaternion basis  $\{1, i, j, k\}$ , we may rewrite (2) as

$$q = a_0 + a_1 i + a_2 j + a_3 k, a_\mu \in \mathbb{R}, \quad (3)$$

where we recall that  $\{1, i, j, k\}$  observe the multiplication rules in the following

table

$$\begin{array}{cccc} . & i & j & k \\ i & -1 & k & j \\ j & -k & -1 & i \\ k & -j & -i & -1 \end{array} \quad (4)$$

A subsequent application of this process yields a natural generalization is the complexification of the quaternion algebra itself. The resulting biquaternions, or *complexified quaternions*, form the algebra  $\mathbb{H}_C = \mathbb{C} \otimes_R \mathbb{H}$ , obtained simply by allowing the coefficients of the standard quaternion basis to take values in  $\mathbb{C}$ :

$$Q = a_0 + a_1 i + a_2 j + a_3 k, a_\mu \in \mathbb{C}, \quad (5)$$

This yields an associative algebra of dimension 4 over  $\mathbb{C}$ . Equivalently, we may write this as an algebra of dimension 8 over  $\mathbb{R}$ , in which case (5) may be written

$$Q = (a_0 + a_1 i_1 + a_2 j_1 + a_3 k_1) + i_2 (a_4 + a_5 i_1 + a_6 j_1 + a_7 k_1), a_\mu \in \mathbb{R}, \quad (6)$$

where the subscripts on the quaternionic units emphasize the difference between the new complexified complex unit  $i_2$ . This algebra is isomorphic to the matrix algebra  $M_2(\mathbb{C})$ . The complex imaginary unit commutes with the quaternion units, thereby introducing a central complex structure while preserving the non-commutative geometry of  $\mathbb{H}$ .

When reformulating Maxwell's equations, the biquaternionic framework retains the notational compactness and geometric insight of Maxwell's original quaternionic presentation while incorporating the flexibility of complex analysis. The resulting expressions are both more symmetric and more directly interpretable than either the classical vector-calculus form or the pure real-quaternionic version, often leading to natural derivations of wave solutions, energy densities, and field invariants without auxiliary scalar fields.

This complexified structure thus serves as the algebraic foundation for the present investigation, bridging the historical quaternionic approach with contemporary techniques in a manner that reveals deeper connections within electromagnetic theory

### 3. Quaternion Implementation

In the sections that follow, for simplicity, we use  $\{i_c, i, j, k\}$  to rewrite  $i_2(a_4 + a_5 i_1 + a_6 j_1 + a_7 k_1)$  from (6) as  $a_4 i_c + a_5 i + a_6 j + a_7 k$ , where  $i_c$  denotes the central complex unit (commuting with  $i, j, k$ ) to clearly distinguish it from the quaternion basis element  $i$ . Next define

$$\partial_r = \frac{1}{c} \partial_{i_c} + \partial_x i + \partial_y j + \partial_z k \quad (7)$$

and

$$\partial_r A = \phi i_c + a_x i + a_y j + a_z k. \quad (8)$$

Recall the following definitions from classical electrodynamics of the electric field  $E$ , and the magnetic field  $B$

$$E = -\{\partial_r, A\} = -\frac{1}{2}(\partial_r A + A\partial_r) \quad (9)$$

and

$$B = [\partial_r, A] = \frac{1}{2}(\partial_r A - A\partial_r). \quad (10)$$

To compute  $E$  and  $B$ , expand  $\partial_r A$  and  $A\partial_r$  using the quaternion multiplication rules defined in (4), in addition to  $i_c^2 = -1$ . This results in

$$E = \left( \frac{1}{c} \partial_t \phi + \nabla \cdot A \right) - i_c \left( \nabla \phi - \frac{1}{c} \partial_t A \right). \quad (11)$$

The first (scalar) term in parentheses is precisely the Lorentz gauge condition  $\partial_\mu A^\mu = 0$  (in units where  $c$  appears explicitly). In the Lorentz gauge this scalar part vanishes, and the  $i_c$ -term recovers the standard expression for  $E$

$$E = -\nabla \phi - \frac{1}{c} \partial_t A. \quad (12)$$

Note that there are no  $i_c$ -terms on the magnetic component, and it is easy to see that

$$B = \nabla \times A. \quad (13)$$

Returning to Equation (11), we return to the gauge term  $\frac{1}{c} \partial_t \phi + \nabla \cdot A$ . Now, assign the gauge

$$A = -\nabla S, \quad (14)$$

$$\phi = \frac{1}{c} \frac{\partial}{\partial t} S. \quad (15)$$

Then Equation (11) becomes

$$E = \square S - \left( \nabla \phi + \frac{1}{c} \partial_t A \right) i_c, \quad (16)$$

where  $\square S = \frac{1}{c^2} \partial_t^2 - \nabla^2$  is the d'Alembertian (wave) operator (signature convention  $(+, -, -, -)$ ). Imposing the Lorentz condition then requires  $\square S = 0$ . Thus, the scalar potential  $S$  itself satisfies the homogeneous wave equation. Setting the vector potential to the gradient of a scalar in this gauge zeroes out the magnetic field  $B = \nabla \times A = 0$ .

## 4. Conclusions

In this paper, we have revisited the foundations of classical electromagnetism through the lens of a complexified quaternionic (biquaternionic) formalism. By leveraging the algebraic structure afforded by the Cayley-Dickson construction and the resulting biquaternion algebra, we have demonstrated a compact representation of the electromagnetic potentials and fields that unifies scalar and vector components while naturally incorporating complex analysis.

Central to our approach is the definition of the generalized derivative operator  $\partial_r$  and the four-potential  $A$ , in terms of the complexified quaternionic frame-

work. This allows the electric field  $E$  and magnetic field  $B$  to emerge from commutator and anticommutator relations in a manner reminiscent of Maxwell's original quaternionic insights. The explicit computation reveals that the electric field includes the familiar gauge-dependent terms  $-\nabla\phi - \frac{1}{c}\partial_t A$ , augmented by the Lorentz gauge condition term  $\frac{1}{c}\partial_t\phi + \nabla \cdot A$ . By imposing the gauge choice  $A = -\nabla S$  and  $\phi = \frac{1}{c}\partial_t S$ , this extra term reduces to the d'Alembertian operator applied to the scalar function  $S$ , yielding the homogeneous wave equation  $\square S = 0$ , under the Lorentz condition. This result highlights how the biquaternionic framework can elegantly encode wave propagation in scalar potentials without introducing extraneous auxiliary fields, in contrast to certain pure real-quaternionic formulations that yield additional scalar components.

The biquaternionic approach preserves the geometric compactness and symmetry of Maxwell's historical quaternion-based presentation while benefiting from the analytic power of complex coefficients. This yields expressions for field invariants, energy-momentum relations, and wave solutions that appear more symmetric and directly interpretable than their vector-calculus counterparts or purely real-quaternionic versions. The isomorphism to matrix algebra further suggests potential extensions to relativistic contexts, chiral media, or generalizations involving monopoles and massive photons, areas where biquaternionic techniques have shown promise in related literature.

While the Heaviside-Gibbs vector formulation remains indispensable for computational efficiency in many applications, the complexified quaternionic perspective offers deeper algebraic unification and may reveal hidden symmetries or facilitate novel derivations in theoretical and applied electromagnetism. Future work could explore explicit solutions in inhomogeneous or nonlinear regimes, connections to quantum field theory via Clifford algebras, or numerical implementations leveraging the matrix representation.

Ultimately, this investigation reaffirms that alternative algebraic structures, when thoughtfully applied, continue to illuminate the profound unity underlying electromagnetic phenomena, a unity first glimpsed by Maxwell over a century ago.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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