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Evaluate All Order of Every Element of 60 and 61 Order of Group for Addition Composition

Nazmun Nahar, Md. Mehedi Hasan Modern, Abdus Salam, Md. Yousuf Miah, Shakil Ahmed, Md. Anamol Haque, Md. Abdul Mannan*

Department of Mathematics, Uttara University, Dhaka, Bangladesh
Email: *mannan.iu31@gmail.com

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Abstract

This paper aims at treating a study on the order of every element of 60 and 61 orders of group for multiplication composition. But the composition in G is associative; the multiplication composition is very significant in the order of elements of a group. We develop the order of a group, higher order of groups in different types of order and the order of elements of a group in real numbers. Let G be a group and let $a^n \in G$ be of infinite order n. In addition, notation $na = e$ and n is a least positive integer $\Rightarrow$ $O(a) = n$. If $a \in G$ is of order n, then there exists an integer m for which $a^m = e$ if m is a multiple of n, in general we use this. Then we develop orders of elements of a cyclic group and every element of higher order of a group. After that we find out the order of every element of a group for the higher orders of the group for being binary operation.

Keywords

$o(G)$, $o(a)$, Addition Composition, Modulo, Torsion Group

1. Introduction

We propose to study the groups of order of an element of a group, order of a group, torsion group, mixed group subgroup, normal subgroup and the integral powers of an element of a group etc. Then we discuss the order of every element in the higher 100, 105 and 107 orders of group for multiplication composition. The group notation is o or *. We will frequently omit the symbol for the group operation but we will also often write the operation as $\cdot$ or $+$ when it represents multiplication or addition in a group, and write 1or 0 for the corresponding identity elements respectively. It's addition $+$, multiplication $\times$ or $(\cdot)$ is used as a binary operation. If the group operation is denoted as a multiplication, then an element $a \in G$ is said to be order *n* if *n* is the least positive integer such that $a^n = e$ or $O(a) \le n$ *i.e.*, if $a^n = e$ and $a^r \ne e$ $\forall r \in N$ s.t. $r < n$. The order of *a* is denoted by $O(a)$. If $a^n \ne e$ for any $n \in N$, then *a* is said to be of zero order or infinite order [1]. Let *e* is the identity element in (*G*, +). An element $a \in G$ is said to be order *n* if $n \in Z^+$ such that $na = e$ or $O(a) \le n$. *i.e.*, if $na = e$ and $ar \ne e$ $\forall r \in N$ s.t. $0 < r < n$. The order of *a* is denoted by $O(a)$. If $na \ne e$ for any $n \in N$, then *a* is said to be of zero order or infinite order [2]. The order of a group *G* and the orders of its elements give much information about the structure of the group. The order of any subgroup of *G* divides the order of *G*. If *H* is a subgroup of *G*, then $\operatorname{ord}(G)/\operatorname{ord}(H) = [G:H]$, where $[G:H]$ is called the index of *H* in *G*. This is Lagrange's theorem; however, it is only true when *G* has finite order. If $\operatorname{ord}(G) = \infty$, the quotient $\operatorname{ord}(G)/\operatorname{ord}(H)$ is not true. We see that the order of every element of a group divides the order of the group. For example, in the symmetric group shown above, where $\operatorname{ord}(S_3) = 6$, the possible orders of the elements *a*, *b*, *c*. But there is no general formula relating the order of a product *ab* to the orders of *a* and *b*. In fact, it is possible that both *a* and *b* have finite order while *ab* has infinite order, or that both *a* and *b* have infinite order while ab has finite order. (*i.e.* [3]-[5]). But here we discuss the order of groups of higher odd, even and prime order of groups as 69, 70 and 71. Then we find out the order of every element of a group in different types of the higher even, odd and prime order of the group for composition [6] [7].

2. Integral Powers of an Element of a Group Multiplication Composition [8] [9]

Let $(G,\cdot)$ be a group. Let $a \in G$ be an arbitrary element.

By closure property, all the elements *a*, *aa*, *aaa*, …etc. belong to *G*.

Since the composition in *G* is associative. Hence *aaa*… to *n* factors is independent of the manner in which the factors are grouped.

If *n* is a positive integer, then define $a^n = a \cdot a \cdot a \cdots$ to *n* factors

$a^n \in G$, by closure property

If *e* is identity in *G*, then we define $a^0 = e$.

If *n* is a negative integer, then by define $a^{-n} = (a^n)^{-1}$, where $(a^n)^{-1}$ is the inverse of a^n.

Consequently, $(a^n)^{-1} \in G$, since the inverse of every element of *G* belong to *G*.

$\therefore\ a^{-n} \in G$

According to the definition

$$\begin{aligned}(a^n)^{-1} &= (aaa \cdots \text{to } n \text{ factors})^{-1} \\ &= (a^{-1})(a^{-1})(a^{-1}) \cdots \text{to } n \text{ factors} \\ &= (a^{-1})^n \\ \therefore\ a^{-n} &= (a^n)^{-1} = (a^{-1})^n.\end{aligned}$$

The following law of indices can be easily proved

$$\left(a^m\right)^n = a^{mn} \quad \forall\, a \in G \text{ and } \forall\, m,n \in Z$$

$$\text{and } a^m a^n = a^{m+n} \quad \forall\, a \in G \text{ and } \forall\, m,n \in Z$$

Thus we defined a^n for all integral values of *n*, positive, negative or zero.

Definition-1 (Multiplication Composition) ([10] [11]):

Let e be the identity element in $(G,\cdot)$. An element $a \in G$ is said to be order *n* if $n \in Z^+$ such that $a^n = e$ or $O(a) \le n$. *i.e.*, if $a^n = e$ and $a^r \ne e \quad \forall r \in N$ s.t. $r < n$. The order of a is denoted by $O(a)$. If $a^n \ne e$ for any $n \in N$, then a is said to be of zero order or infinite order.

Definition-1 (Addition Composition) ([12] [13]):

Let e be the identity element in $(G,+)$. An element $a \in G$ is said to be order *n* if $n \in Z^+$ such that $na = e$ or $O(a) \le n$. *i.e.*, if $na = e$ and $ar \ne e \quad \forall r \in N$ s.t. $0 < r < n$. The order of *a* is denoted by $O(a)$. If $na \ne e$ for any $n \in N$, then a is said to be of zero order or infinite order.

***Example*:**

1) In multiplicative group Q_0. $O(a) = 1$, $O(-1) = 2$ and $O(a) = \infty \quad \forall a \in Q_0$ s. t. $a \ne \pm 1$.

2) In $(Z, +)$, $O(0) = 1$ and all other elements have infinite order.

Torsion free group [14]: A group G is called a torsion free group if the identity element e is the only element of finite order.

Example: $(Z,+),(Q,+),(R,+)$

Torsion group or periodic group [15]: A group G is said to be a torsion group or periodic group if every element of G is of finite order.

Example: The multiplication group is $\{1,-1,i,-i\}$

Mixed Group [16]: A group G is said to be a mixed group if $\exists$ at least two elements $a,b \in G$. s.t..

1) $o(a)$ is finite, $a \ne e$ 2) $o(b) = \infty$

Example: The multiplicative group $(Q_0, \cdot)$, where $Q_0 \ne 0$ is a mixed group.

For $o(-1) = 2$, $e \ne -1$, $0(a) = \infty$ if $a \ne \pm 1$ and $a \in Q_0$.

3. Significance of the Order of an Element of a Group

We begin this section of the following theorem related significance of the order of an element of a group.

3.1. Theorem [17]

Let G be a group and let $a \in G$ be of infinite order *n*. Then show that

$$O(a)^k = \frac{n}{(n,k)}$$

where k is any integer and (*n*, *k*) is denoted the highest common factor of *n* and *k*.

Proof: Let $o(a) = n$, $o(a)^k = h$, $(n,k) = m$.

Now we will prove that $h = \frac{n}{m}$.

We have,

$$o(a) = n \tag{1}$$

or, $a^n = e$

or, $o(a)^k = h$

or,

$$\left(a^k\right)^h = e \tag{2}$$

or, $a^{kh} = e$

And,

$$(n,k) = m$$

or, $n = mp, k = mg$; where p and q integers and $(p,q) = 1$

From (2) we get,

$$a^{kh} = e$$

or, $a^{kh} = a^n$ or, $kh = n$, or, (kh) or, (mqh) or, (qh) or, (h) where

$$(p,q) = 1 \tag{3}$$

Now, $a^{kp} = a^{(mq)p}$

$= a^{(mp)q}$ [By Associative law]

$= a^{nq}$

$= \left(a^n\right)^q$

$= e^q$

or,

$$a^{kp} = e \tag{4}$$

or, $\left(a^k\right)^p = e$

or, $o(k) = p$

or,

$$h = p \tag{5}$$

or, h/p

From (3) and (5) then we get,

$$p = h$$

or, $h = \frac{n}{m}$ **QED**

3.2. Theorem [18]

Show that the order of every element of a finite group is finite.

Proof: Let G be a finite group with multiplication composition.

Let $a \in G$ be an arbitrary element.

Now we will prove that $O(a)$ is finite.

By closure property, all the elements $a^2 = a \cdot a$, $a^3 = a \cdot a \cdot a$, … etc. belong to G *i.e.* $a, a^2, a^3, a^4, a^5, a^6, a^7$, … etc. belong to G.

But all these elements are not distinct. Since G is finite.

Let e be the identity in G, then $a^0 = e$.

Let us suppose that

$$a^m = a^n \text{ where } m > n$$
$$\Rightarrow a^m a^{-n} = a^n a^{-n} = a^0 = e$$
$$\Rightarrow a^{m-n} = e \Rightarrow a^p = e, \text{ where } p = m - n > 0, \text{ as } m > n$$

Also m and n are finite and hence p is a finite positive integer.

Now p is a positive integer s.t. $a^p = e$.

This proves that

$$o(a) \le p = \text{finite number}$$
$$\text{i.e. } o(a) \le \text{a finite number} \Rightarrow o(a) \text{ is finite}$$

4. Result and Discussion

In this section we developed the result of order of every element for multiplication composition in the higher 60and 61 orders of group for addition composition.

4.1. Find Order of Every Element of the Group {0, 1, 2, 3, ..., 60} the Composition Being Addition Modulo 61 ([16] [19]-[22])

Solution: Let $(G, +_{61})$ is a group. Where G = {0, 1, 2, 3, …, 60} and $+_{61}$ denotes addition modulo 61. Here e = 0.

In addition notation

$na = e$ and n is a least positive integer

$$\Rightarrow O(a) = n \quad \therefore O(0) = 1$$

For $O(e)$ = 1 for identity element of every group.

To determine $O(1)$:

1 · 1 = 1, 2 · 1 = 2, 3 · 1 = 3, 4 · 1 = 4, 5 · 1 = 5, 6 · 1 = 6, …, 20 · 1 = 20, 21 · 1 = 21, 22 · 1 = 22, …, 41 · 1 = 41, 42 · 1 = 42, 43 · 1 = 43, …, 59 · 1 = 59, 60 · 1 = 60, 61 · 1 = 61 = 0 = e

Thus 61(1) = e and $n(a) \ne e$ for $n < 61$. $\therefore O(1) = 61$

To determine $O(2)$:

1 · 2 = 2, 2 · 2 = 4, 3 · 2 = 6, 4 · 2 = 8, 5 · 2 = 10, 6 · 2 = 12, …, 20 · 2 = 40, 21 · 2 = 42, 22 · 2 = 24, …, 58 · 2 = 116 = 55, 59 · 2 = 57, 60 · 2 = 120 = 59, 61 · 2 = 122 = 0 = e

Thus 61(2) = e and $n(a) \ne e$ for $n < 61$. $\therefore O(2) = 61$

To determine $O(3)$:

1 · 3 = 3, 2 · 3 = 6, 3 · 3 = 9, 4 · 3 = 12, 5 · 3 = 15, 6 · 3 = 18, …, 20 · 3 = 60, 21 · 3 = 63 = 2, 22 · 3 = 66 = 5, …, 59 · 3 = 55, 60 · 3 = 180 = 58, 61 · 3 = 183 = 0 = e

Thus 61(3) = e and $n(a) \ne e$ for $n < 61$. $\therefore O(3) = 61$

To determine $O(4)$:

$1 \cdot 4 = 4$, $2 \cdot 4 = 8$, $3 \cdot 4 = 12$, $4 \cdot 4 = 16$, $5 \cdot 4 = 20$, $6 \cdot 4 = 24$, …, $20 \cdot 4 = 80 = 19$, $21 \cdot 4 = 84 = 23$, $22 \cdot 4 = 88 = 27$, …, $59 \cdot 4 = 236 = 53$, $60 \cdot 4 = 240 = 57$, $61 \cdot 4 = 244 = 0 = e$

Thus $61(4) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(4) = 61$

To determine $O(5)$:

$1 \cdot 5 = 5$, $2 \cdot 5 = 10$, $3 \cdot 5 = 15$, $4 \cdot 5 = 20$, $5 \cdot 5 = 25$, $6 \cdot 5 = 30$, …, $20 \cdot 5 = 100 = 39$, $21 \cdot 5 = 105 = 44$, $22 \cdot 5 = 110 = 49$, …, $59 \cdot 5 = 295 = 51$, $60 \cdot 5 = 300 = 56$, $61 \cdot 5 = 305 = 0 = e$

Thus $61(5) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(5) = 61$

To determine $O(6)$:

$1 \cdot 6 = 6$, $2 \cdot 6 = 12$, $3 \cdot 6 = 18$, $4 \cdot 6 = 24$, $5 \cdot 6 = 30$, $6 \cdot 6 = 36$, …, $20 \cdot 6 = 120 = 59$, $21 \cdot 6 = 126 = 4$, $22 \cdot 6 = 132 = 10$, …, $59 \cdot 6 = 354 = 49$, $60 \cdot 6 = 360 = 55$, $61 \cdot 6 = 366 = 0 = e$

Thus $61(6) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(6) = 61$

To determine $O(7)$:

$1 \cdot 7 = 7$, $2 \cdot 7 = 14$, $3 \cdot 7 = 21$, $4 \cdot 7 = 28$, $5 \cdot 7 = 35$, $6 \cdot 7 = 42$, …, $20 \cdot 7 = 140 = 18$, $21 \cdot 7 = 147 = 25$, $22 \cdot 7 = 154 = 32$, …, $59 \cdot 7 = 413 = 47$, $60 \cdot 7 = 420 = 54$, $61 \cdot 7 = 427 = 0 = e$

Thus $61(7) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(7) = 61$

To determine $O(8)$:

$1 \cdot 8 = 8$, $2 \cdot 8 = 16$, $3 \cdot 8 = 24$, $4 \cdot 8 = 32$, $5 \cdot 8 = 40$, $6 \cdot 8 = 48$, …, $20 \cdot 8 = 160 = 38$, $21 \cdot 8 = 168 = 46$, $22 \cdot 8 = 178 = 56$, …, $59 \cdot 8 = 472 = 45$, $60 \cdot 8 = 480 = 53$, $61 \cdot 8 = 488 = 0 = e$

Thus $61(8) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(8) = 61$

To determine $O(9)$:

$1 \cdot 9 = 9$, $2 \cdot 9 = 18$, $3 \cdot 9 = 27$, $4 \cdot 9 = 36$, $5 \cdot 9 = 45$, $6 \cdot 9 = 54$, …, $20 \cdot 9 = 180 = 58$, $21 \cdot 9 = 189 = 6$, $22 \cdot 9 = 198 = 15$, …, $59 \cdot 9 = 531 = 43$, $60 \cdot 9 = 540 = 52$, $61 \cdot 9 = 549 = 0 = e$

Thus $61(9) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(9) = 61$

To determine $O(10)$:

$1 \cdot 10 = 10$, $2 \cdot 10 = 20$, $3 \cdot 10 = 30$, $4 \cdot 10 = 40$, $5 \cdot 10 = 50$, $6 \cdot 10 = 60$, …, $20 \cdot 10 = 200 = 17$, $21 \cdot 10 = 210 = 27$, …, $59 \cdot 10 = 590 = 41$, $60 \cdot 10 = 600 = 51$, $61 \cdot 10 = 610 = 0 = e$

Thus $61(10) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(10) = 61$

To determine $O(11)$:

$1 \cdot 11 = 11$, $2 \cdot 11 = 22$, $3 \cdot 11 = 33$, $4 \cdot 11 = 44$, $5 \cdot 11 = 55$, $6 \cdot 11 = 66$, …, $20 \cdot 11 = 220 = 37$, $21 \cdot 11 = 231 = 48$, …, $60 \cdot 11 = 660 = 50$, $61 \cdot 11 = 671 = 0 = e$

Thus $61(11) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(11) = 61$

To determine $O(12)$:

$1 \cdot 12 = 12$, $2 \cdot 12 = 24$, $3 \cdot 12 = 36$, $4 \cdot 12 = 44$, $5 \cdot 12 = 60$, $6 \cdot 12 = 11$, …, $20 \cdot 12 = 240 = 57$, $21 \cdot 12 = 252 = 88$, …, $60 \cdot 12 = 720 = 49$, $61 \cdot 12 = 732 = 0 = e$

Thus $61(12) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(12) = 61$

To determine $O(13)$:

$1 \cdot 13 = 13$, $2 \cdot 13 = 26$, $3 \cdot 13 = 39$, $4 \cdot 13 = 52$, $5 \cdot 13 = 65 = 4$, …, $20 \cdot 13 = 260 = 16$, $21 \cdot 13 = 273 = 29$, …, $60 \cdot 13 = 780 = 48$, $61 \cdot 13 = 793 = 0 = e$

Thus $61(13) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(13) = 61$

To determine $O(14)$:

$1 \cdot 14 = 14$, $2 \cdot 14 = 28$, $3 \cdot 14 = 42$, $4 \cdot 14 = 56$, $5 \cdot 14 = 70 = 9$, …, $20 \cdot 14 = 280 = 36$, $21 \cdot 14 = 294 = 50$, …, $60 \cdot 14 = 840 = 47$, $61 \cdot 14 = 854 = 0 = e$

Thus $61(14) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(14) = 61$

To determine $O(15)$:

$1 \cdot 15 = 15$, $2 \cdot 15 = 30$, $3 \cdot 15 = 45$, $4 \cdot 15 = 60$, $5 \cdot 15 = 75 = 14$, …, $20 \cdot 15 = 300 = 56$, $21 \cdot 15 = 315 = 10$, …, $60 \cdot 15 = 900 = 46$, $61 \cdot 15 = 915 = 0 = e$

Thus $61(15) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(15) = 61$

To determine $O(16)$:

$1 \cdot 16 = 16$, $2 \cdot 16 = 32$, $3 \cdot 16 = 48$, $4 \cdot 16 = 64 = 3$, …, $20 \cdot 16 = 320 = 15$, $21 \cdot 16 = 336 = 31$, …, $60 \cdot 16 = 960 = 45$, $61 \cdot 16 = 976 = 0 = e$

Thus $61(16) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(16) = 61$

To determine $O(17)$:

$1 \cdot 17 = 17$, $2 \cdot 17 = 34$, $3 \cdot 17 = 51$, $4 \cdot 17 = 68 = 7$, …, $20 \cdot 17 = 340 = 35$, $21 \cdot 17 = 357 = 31$, …, $60 \cdot 17 = 1020 = 44$, $61 \cdot 17 = 1037 = 0 = e$

Thus $61(17) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(17) = 61$

To determine $O(18)$:

$1 \cdot 18 = 18$, $2 \cdot 18 = 36$, $3 \cdot 18 = 54$, $4 \cdot 18 = 72 = 11$, …, $20 \cdot 18 = 360 = 55$, $21 \cdot 18 = 378 = 12$, …, $60 \cdot 18 = 1080 = 43$, $61 \cdot 18 = 1098 = 0 = e$

Thus $61(18) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(18) = 61$

***To determine $O(31)$*:**

$1 \cdot 31 = 31$, $2 \cdot 31 = 62 = 1$, $3 \cdot 31 = 93 = 32$, $4 \cdot 31 = 124 = 2$, $5 \cdot 31 = 155 = 33$, $6 \cdot 31 = 186 = 3$, …, $59 \cdot 31 = 1829 = 60$, $60 \cdot 31 = 1860 = 30$, $61 \cdot 31 = 1891 = 0 = e$

Thus $61(31) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(31) = 61$

To determine $O(32)$:

$1 \cdot 32 = 32$, $2 \cdot 32 = 64 = 3$, $3 \cdot 32 = 96 = 35$, $4 \cdot 32 = 128 = 6$, $5 \cdot 32 = 160 = 38$, $6 \cdot 32 = 192 = 9$, …, $59 \cdot 32 = 1888 = 58$, $60 \cdot 32 = 1920 = 29$, $61 \cdot 32 = 1952 = 0 = e$

Thus $61(32) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(32) = 61$

…

To determine $O(58)$:

$1 \cdot 58 = 58$, $2 \cdot 58 = 55 = 3$, …, $60 \cdot 58 = 3480 = 3$, $61 \cdot 58 = 3538 = 0 = e$

Thus $61(58) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(58) = 61$

To determine $O(59)$:

$1 \cdot 59 = 59$, $2 \cdot 59 = 118 = 4$, …, $60 \cdot 59 = 3540 = 2$, $61 \cdot 59 = 3599 = 0 = e$

Thus $61(59) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(59) = 61$

To determine $O(60)$:

$1 \cdot 60 = 60$, $2 \cdot 60 = 120 = 59$, …, $60 \cdot 60 = 3600 = 1$, $61 \cdot 60 = 3660 = 0 = e$

Thus $61(60) = e$ and $n(a) \neq e$ for $n < 61$. $\therefore O(60) = 61$

4.2. Find Order of Every Element of the Group {0, 1, 2, 3, ..., 59} the Composition Being Addition Modulo 60 ([23]-[28])

Solution: Let $(G,+_{50})$ is a group. Where G = {0, 1, 2, 3, …, 59} and $+_{60}$ denotes addition modulo 60. Here e = 0.

In addition, notation

$na = e$ and n is a least positive integer

$$\Rightarrow O(a)=n \quad \therefore O(0)=1$$

For O(e) = 1 for identity element of every group.

To determine O(1):

$$1(1)=1,\ \ 2(1)=1+_{60}1=2,$$

$$3(1)=1+_{60}2(1)=1+_{60}2=3,\ \ 4(1)=1+_{60}3(1)=1+_{60}3=4,$$

$$5(1)=1+_{60}4(1)=1+_{60}4=5,\ \ 6(1)=1+_{60}5(1)=1+_{60}5=6$$

$$7(1)=1+_{60}6(1)=1+_{60}6=7,\ \ 8(1)=1+_{60}7(1)=1+_{60}2=8$$

$$9(1)=1+_{60}8(1)=1+_{60}8=9,\ \ 10(1)=1+_{60}9(1)=1+_{60}9=10$$

$$11(1)=1+_{60}10(1)=1+_{60}10=11,\ \ldots,\ \ 60(1)=1+_{60}59(1)=1+_{60}59=0=e$$

Thus 60(1) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(1)=60$

To determine O(2):

$$1(2)=2,\ \ 2(2)=2+_{60}2=4,\ \ 3(2)=2+_{60}2(2)=2+_{50}4=6$$

$$4(2)=2+_{60}3(2)=2+_{60}6=8,\ \ 5(2)=2+_{60}4(2)=2+_{60}8=10$$

$$6(2)=2+_{60}5(2)=2+_{60}10=12,\ \ 7(2)=2+_{60}6(2)=2+_{60}12=14$$

$$8(2)=2+_{60}7(2)=2+_{60}14=16,\ \ 9(2)=2+_{60}8(2)=2+_{60}16=18$$

$$10(2)=2+_{60}9(2)=2+_{60}18=20,\ \ldots,\ \ 30(2)=2+_{60}29(2)=2+_{60}58=0=e$$

Thus 30(2) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(2)=30$

To determine O(3):

$$1(3)=3,\ \ 2(3)=3+_{60}3=6,\ \ 3(3)=3+_{60}2(3)=3+_{60}6=9$$

$$4(3)=3+_{60}3(3)=3+_{60}9=12,\ \ 5(3)=3+_{60}4(3)=3+_{60}12=15$$

$$6(3)=3+_{60}5(3)=3+_{60}15=18,\ \ 7(3)=3+_{60}6(3)=3+_{60}18=21$$

$$8(3)=3+_{60}7(3)=3+_{60}21=24,\ \ 16(3)=3+_{60}15(3)=3+_{60}45=48$$

$$17(3)=3+_{60}16(3)=3+_{60}48=51,\ \ 18(3)=3+_{60}17(3)=3+_{60}51=54$$

$$19(3)=3+_{60}18(3)=3+_{50}54=57,\ \ 20(3)=3+_{60}19(3)=3+_{60}57=0=e$$

Thus 20(3) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(3)=20$

To determine O(4):

$$1(4)=4,\ \ 2(4)=4+_{60}4=8,\ \ 3(4)=4+_{60}2(4)=4+_{60}8=12$$

$$4(4)=4+_{60}3(4)=4+_{60}12=16,\ \ 5(4)=4+_{60}4(4)=4+_{60}16=20$$

$$6(4)=4+_{60}5(4)=4+_{60}20=24,\ \ 7(4)=4+_{60}6(4)=4+_{60}24=28$$

$$8(4)=4+_{60}7(4)=4+_{60}28=32,\ \ldots,\ 13(4)=4+_{60}12(4)=4+_{60}48=52$$

$$14(4)=4+_{60}13(4)=4+_{60}52=56,\ \ 15(4)=4+_{60}14(4)=4+_{60}56=0=e$$

Thus 15(4) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(4)=15$

To determine $O(5)$:

$$1(5)=5,\ \ 2(5)=5+_{60}5=10,\ \ 3(5)=5+_{60}2(5)=5+_{60}10=15$$

$$4(5)=5+_{60}3(5)=5+_{60}15=20,\ \ 5(5)=5+_{60}4(5)=5+_{60}20=25$$

$$6(5)=5+_{60}5(5)=5+_{60}25=30,\ \ 7(5)=5+_{60}6(5)=5+_{60}30=35$$

$$8(5)=5+_{60}7(5)=5+_{60}35=40,\ \ 9(5)=5+_{60}8(5)=5+_{60}40=45$$

$$10(5)=5+_{60}9(5)=5+_{60}45=50,\ \ 11(5)=5+_{60}10(5)=5+_{60}50=55$$

$$12(5)=5+_{60}11(5)=5+_{60}55=0=e$$

Thus 12(5) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(5)=12$

To determine $O(6)$:

$$1(6)=6,\ \ 2(6)=6+_{60}6=12,\ \ 3(6)=6+_{60}2(6)=6+_{60}12=18$$

$$4(6)=6+_{60}3(6)=6+_{60}18=24,\ \ 5(6)=6+_{60}4(6)=6+_{60}24=30$$

$$6(6)=6+_{60}5(6)=6+_{60}30=36,\ \ 7(6)=6+_{60}6(6)=6+_{60}36=42$$

$$8(6)=6+_{60}7(6)=6+_{60}42=48,\ \ 9(6)=6+_{60}8(6)=6+_{60}48=54,$$

$$10(6)=6+_{60}9(6)=6+_{60}54=60$$

Thus 10(6) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(6)=10$

To determine $O(7)$:

$$1(7)=7,\ \ 2(7)=7+_{60}7=14,\ \ 3(7)=7+_{60}2(7)=7+_{60}14=21$$

$$4(7)=7+_{60}3(7)=7+_{60}21=28,\ \ 5(7)=7+_{60}4(7)=7+_{60}28=35$$

$$6(7)=7+_{60}5(7)=7+_{60}35=42,\ \ 7(7)=7+_{60}6(7)=7+_{60}42=49$$

$$8(7)=7+_{60}7(7)=7+_{60}49=56,\ \ldots,\ 16(7)=7+_{60}15(7)=7+_{60}45=52$$

$$17(7)=7+_{60}16(7)=7+_{60}52=59,\ \ 18(7)=7+_{60}17(7)=7+_{60}59=6,\ \ldots,$$

$$28(7)=7+_{60}27(7)=7+_{60}9=16,\ \ldots,\ 38(7)=7+_{60}37(7)=7+_{60}19=26$$

$$39(7)=7+_{60}38(7)=7+_{60}26=33,\ \ 40(7)=7+_{60}39(7)=7+_{60}33=40,\ \ldots,$$

$$48(7)=7+_{60}47(7)=7+_{60}29=36,\ \ 49(7)=7+_{60}48(7)=7+_{60}36=43,\ \ldots,$$

$$59(7)=7+_{60}58(7)=7+_{60}46=53,\ \ 60(7)=7+_{60}59(7)=7+_{60}53=0=e$$

Thus 60(7) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(7)=60$

To determine $O(8)$:

$$1(8)=8,\ \ 2(8)=8+_{60}8=16,\ \ 3(8)=8+_{60}2(8)=8+_{60}16=24$$

$$4(8)=8+_{60}3(8)=8+_{60}24=32,\ \ 5(8)=8+_{60}4(8)=8+_{60}32=40$$

$$6(8)=8+_{60}5(8)=8+_{60}40=48,\ \ 7(8)=8+_{60}6(8)=8+_{60}48=56,\ \ldots,$$

$$13(8)=8+_{60}12(8)=8+_{60}36=44,\ \ 14(8)=8+_{60}14(8)=8+_{60}44=52$$

$$15(8)=8+_{60}14(8)=8+_{60}36=44,\ \ 19(8)=8+_{50}18(8)=8+_{50}44=52$$

$$24(8)=8+_{60}23(8)=8+_{60}52=0=e$$

Thus 15(8) = e and $n(a)\neq e$ for $n<60$. $\therefore O(8)=15$

To determine *O*(9):

$$1(9)=9,\ \ 2(9)=9+_{60}9=18,\ \ 3(9)=9+_{60}2(9)=9+_{60}18=27$$

$$4(9)=9+_{60}3(9)=9+_{60}27=36,\ \ 5(9)=9+_{60}4(9)=9+_{60}36=45$$

$$6(9)=9+_{60}5(9)=9+_{60}45=54,\ \ldots 16(9)=9+_{60}15(9)=9+_{60}15=24$$

$$17(9)=9+_{60}16(9)=9+_{50}24=33,\ \ 18(9)=9+_{60}17(9)=9+_{60}33=42$$

$$19(9)=9+_{60}18(9)=9+_{60}42=51,\ \ 20(9)=9+_{60}19(9)=9+_{60}51=0=e$$

Thus 20(9) = e and $n(a)\neq e$ for $n<60$. $\therefore O(9)=20$

To determine *O*(10):

$$1(10)=10,\ \ 2(10)=103+_{60}10=20, 3(10)=10+_{60}2(10)=10+_{60}20=30$$

$$4(10)=10+_{60}3(10)=10+_{60}30=40,\ \ 5(10)=10+_{60}4(10)=10+_{60}40=50$$

$$6(10)=10+_{60}5(10)=10+_{60}50=0=e$$

Thus 6(10) = e and $n(a)\neq e$ for $n<60$. $\therefore O(10)=6$

To determine *O*(11):

$$1(11)=11,\ \ 2(11)=11+_{60}11=22,\ \ 3(11)=11+_{60}2(11)=11+_{60}22=33$$

$$4(11)=11+_{60}3(11)=11+_{60}33=44,\ \ 5(11)=11+_{60}4(11)=11+_{60}44=55$$

$$6(11)=11+_{60}5(11)=11+_{60}55=6,\ \ldots,\ \ 16(11)=11+_{60}15(11)=11+_{60}45=56,$$

$$17(11)=11+_{60}16(11)=11+_{60}56=7,\ \ 18(11)=11+_{60}17(11)=11+_{60}7=18,\ \ldots,$$

$$26(11)=11+_{60}25(11)=11+_{60}35=46,\ \ 27(11)=11+_{60}26(11)=11+_{60}46=57$$

$$28(11)=11+_{60}27(11)=11+_{60}57=8,\ \ldots,$$

$$38(11)=11+_{60}37(11)=11+_{60}47=58,\ \ 39(11)=11+_{60}38(11)=11+_{60}58=9,$$

$$40(11)=11+_{60}39(11)=11+_{60}9=18,\ \ldots,$$

$$58(11)=11+_{60}57(11)=11+_{60}27=38,\ \ 59(11)=11+_{60}38(11)=11+_{60}38=49$$

$$50(11)=11+_{60}59(11)=11+_{60}49=0=e$$

Thus 60(11) = e and $n(a)\neq e$ for $n<60$. $\therefore O(11)=60$

To determine *O*(12):

$$1(12)=12,\ \ 2(12)=12+_{60}12=24,\ \ 3(12)=12+_{60}2(12)=12+_{60}24=36$$

$$4(12)=12+_{60}3(12)=12+_{60}36=48,\ \ 5(12)=12+_{60}4(12)=12+_{60}48=0=e$$

Thus 5(12) = e and $n(a)\neq e$ for $n<60$. $\therefore O(12)=5$

To determine $O(13)$:

$$1(13)=13,\quad 2(13)=13+_{60}13=26,\quad 3(13)=13+_{60}2(13)=13+_{60}26=39$$

$$4(13)=13+_{60}3(13)=13+_{60}39=52,\quad 5(13)=13+_{60}4(13)=13+_{60}52=5,\ \ldots,$$

$$16(13)=13+_{60}15(13)=13+_{60}15=28,$$

$$17(13)=13+_{60}16(13)=13+_{60}15=38,\ \ldots,$$

$$26(13)=13+_{60}25(13)=13+_{60}25=38,\quad 27(13)=13+_{60}26(13)=13+_{60}38=51$$

$$28(13)=13+_{60}27(13)=13+_{60}51=4,\ \ldots,$$

$$38(13)=13+_{60}37(13)=13+_{60}1=14,$$

$$39(13)=13+_{60}38(13)=13+_{60}14=27,\ \ldots,$$

$$48(13)=13+_{60}47(13)=13+_{60}11=24,\ \ldots,$$

$$59(13)=13+_{60}58(13)=13+_{60}34=47,$$

$$60(13)=13+_{60}59(13)=13+_{60}47=0=e$$

Thus 60(13) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(13)=60$

To determine $O(14)$:

$$1(14)=14,\quad 2(14)=14+_{60}14=28,\quad 3(14)=14+_{60}2(14)=14+_{60}28=42$$

$$4(14)=14+_{60}3(14)=14+_{60}42=56,\ \ldots,$$

$$16(14)=14+_{60}15(14)=14+_{60}30=44,$$

$$17(14)=14+_{60}16(14)=14+_{60}44=58,\ \ldots,$$

$$28(14)=14+_{60}27(14)=14+_{60}18=32,$$

$$29(14)=14+_{60}28(14)=14+_{60}32=46,$$

$$30(14)=14+_{60}29(14)=14+_{60}46=0=e$$

Thus 30(14) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(14)=30$

To determine $O(15)$:

$$1(15)=15,\quad 2(15)=15+_{60}15=30,\quad 3(15)=15+_{60}2(15)=15+_{60}30=45$$

$$4(15)=15+_{60}3(15)=15+_{50}45=0=e$$

Thus 4(15) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(15)=4$

To determine $O(16)$:

$$1(16)=16,\quad 2(16)=16+_{60}16=32,\quad 3(16)=16+_{60}2(16)=16+_{60}32=48$$

$$4(16)=16+_{60}3(16)=16+_{60}48=4,\ \ldots,$$

$$14(16)=16+_{60}13(16)=16+_{60}28=44,$$

$$15(16)=16+_{60}14(16)=16+_{60}44=0=e$$

Thus 15(16) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(16)=15$

To determine $O(17)$:

$$1(17)=17,\quad 2(17)=17+_{60}17=34,\quad 3(17)=17+_{60}2(17)=17+_{50}34=51$$

$$4(17)=17+_{60}3(17)=17+_{60}51=8,\ \ldots,$$

$$16(17)=17+_{60}15(17)=17+_{60}15=32,$$

$$17(17)=17+_{60}16(17)=17+_{60}32=49,\ \ldots,$$

$$26(17)=17+_{60}25(17)=17+_{60}5=22,$$

$$27(17)=17+_{60}26(17)=17+_{60}22=39,\ \ldots,$$

$$38(17)=17+_{60}37(17)=17+_{60}29=46,$$

$$39(17)=17+_{60}38(17)=17+_{60}46=3,\ \ldots,$$

$$59(17)=17+_{60}58(17)=17+_{60}26=43,$$

$$60(17)=17+_{60}59(17)=17+_{60}43=0=e$$

Thus $60(17) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(17)=60$

To determine $O(18)$:

$$1(18)=18,\quad 2(18)=18+_{60}18=36,\quad 3(18)=18+_{60}2(18)=18+_{60}36=54$$

$$4(18)=18+_{60}3(18)=18+_{60}54=12,\ \ldots,$$

$$10(18)=18+_{60}15(18)=18+_{60}42=0=e$$

Thus $10(18) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(18)=10$

To determine $O(19)$:

$$1(19)=19,\quad 2(19)=19+_{60}19=38,\quad 3(19)=19+_{60}2(19)=19+_{60}38=57,\ \ldots,$$

$$16(19)=19+_{60}15(19)=19+_{60}45=4,$$

$$17(19)=19+_{60}16(19)=19+_{60}4=23,\ \ldots,$$

$$26(19)=19+_{60}25(19)=19+_{60}55=14,$$

$$27(19)=19+_{60}26(19)=19+_{60}14=33,\ \ldots,$$

$$38(19)=19+_{60}37(19)=19+_{60}43=5,$$

$$39(19)=19+_{60}38(19)=19+_{60}5=24,\ \ldots,$$

$$59(19)=19+_{60}58(19)=19+_{60}22=41,$$

$$60(19)=19+_{60}49(19)=19+_{60}41=0=e$$

Thus $60(19) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(19)=60$

To determine $O(20)$:

$$1(20)=20,\quad 2(20)=20+_{60}20=40,\quad 3(20)=20+_{60}2(20)=20+_{60}40=0=e,$$

Thus $3(20) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(20)=3$

To determine $O(21)$:

$$1(21)=21,\quad 2(21)=21+_{60}21=42,\quad 3(21)=21+_{60}2(21)=21+_{60}42=3,\ \ldots,$$

$$16(21) = 21 +_{60} 15(21) = 21 +_{60} 15 = 36,$$

$$17(21) = 21 +_{60} 16(21) = 21 +_{60} 36 = 57, \ldots,$$

$$19(21) = 21 +_{60} 18(21) = 21 +_{60} 18 = 39,$$

$$20(21) = 21 +_{60} 19(21) = 21 +_{60} 39 = 0 = e$$

Thus 20(21) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(21) = 20$

To determine *O*(22):

$$1(22) = 22, \quad 2(22) = 22 +_{60} 22 = 44, \quad 3(22) = 22 +_{60} 2(22) = 22 +_{60} 44 = 6, \ldots,$$

$$16(22) = 22 +_{60} 15(22) = 22 +_{60} 30 = 52,$$

$$17(22) = 22 +_{60} 16(22) = 22 +_{60} 52 = 14, \ldots,$$

$$29(22) = 22 +_{60} 28(22) = 22 +_{60} 16 = 38,$$

$$30(22) = 22 +_{60} 29(22) = 22 +_{60} 38 = 0 = e$$

Thus 30(22) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(22) = 30$

To determine *O*(23):

$$1(23) = 23, \quad 2(23) = 23 +_{60} 23 = 46, \quad 3(23) = 23 +_{60} 2(23) = 23 +_{60} 46 = 9, \ldots,$$

$$16(23) = 23 +_{60} 15(23) = 23 +_{60} 45 = 8,$$

$$17(23) = 23 +_{60} 16(23) = 23 +_{60} 8 = 31, \ldots,$$

$$26(23) = 23 +_{60} 25(23) = 23 +_{60} 35 = 58,$$

$$27(23) = 23 +_{60} 26(23) = 23 +_{60} 58 = 21$$

$$38(23) = 23 +_{60} 37(23) = 23 +_{60} 11 = 34,$$

$$39(23) = 23 +_{60} 38(23) = 23 +_{60} 34 = 57, \ldots,$$

$$59(23) = 23 +_{60} 58(23) = 23 +_{60} 14 = 37,$$

$$60(23) = 23 +_{60} 59(23) = 23 +_{60} 37 = 0 = e$$

Thus 60(23) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(23) = 60$

To determine *O*(24):

$$1(24) = 24, \quad 2(24) = 24 +_{60} 24 = 48, \quad 3(24) = 24 +_{60} 2(24) = 24 +_{60} 48 = 12$$

$$4(24) = 24 +_{60} 3(24) = 24 +_{60} 12 = 36, \quad 5(24) = 24 +_{60} 4(24) = 24 +_{60} 36 = 0 = e$$

Thus 5(24) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(24) = 5$

To determine *O*(25):

$$1(25) = 25, \quad 2(25) = 25 +_{60} 25 = 50, \quad 3(25) = 25 +_{60} 2(25) = 25 +_{60} 50 = 15, \ldots,$$

$$11(25) = 25 +_{60} 10(25) = 25 +_{60} 10 = 35,$$

$$12(25) = 25 +_{60} 11(25) = 25 +_{60} 35 = 0 = e$$

Thus 12(25) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(25) = 12$

To determine $O(26)$:

$$1(26)=26,\quad 2(26)=26+_{60}26=52,\ \ldots,$$

$$16(26)=26+_{60}15(26)=26+_{60}30=56,$$

$$17(26)=26+_{60}16(26)=26+_{60}56=32,\ \ldots,$$

$$29(26)=26+_{60}28(26)=26+_{60}8=34,$$

$$30(26)=26+_{60}29(26)=26+_{60}34=0=e$$

Thus 30(26) = e and $n(a)\neq e$ for $n<60$. $\therefore O(26)=30$

To determine $O(27)$:

$$1(27)=27,\quad 2(27)=27+_{60}27=54,\ \ldots,$$

$$16(27)=27+_{60}15(27)=27+_{60}45=12,$$

$$17(27)=27+_{60}16(27)=27+_{60}12=39,\ \ldots,$$

$$19(27)=27+_{60}18(27)=27+_{60}6=33,$$

$$20(27)=27+_{60}19(27)=27+_{60}33=0=e.$$

Thus 20(27) = e and $n(a)\neq e$ for $n<60$. $\therefore O(27)=20$

To determine $O(28)$:

$$1(28)=28,\quad 2(28)=28+_{60}28=56,\ \ldots,$$

$$14(28)=28+_{60}13(28)=28+_{60}4=32,$$

$$15(28)=28+_{60}14(28)=28+_{60}32=0=e$$

Thus 15(28) = e and $n(a)\neq e$ for $n<60$. $\therefore O(28)=15$

To determine $O(29)$:

$$1(29)=29,\quad 2(29)=29+_{60}29=58,\ \ldots,$$

$$16(29)=29+_{60}15(29)=29+_{60}15=44,$$

$$17(29)=29+_{60}16(29)=29+_{60}44=13,\ \ldots,$$

$$26(29)=29+_{60}25(29)=29+_{60}5=34,$$

$$27(29)=29+_{60}26(29)=29+_{60}34=3,\ \ldots,$$

$$38(29)=29+_{60}37(29)=29+_{60}53=22,$$

$$39(29)=29+_{60}38(29)=29+_{60}22=51,\ \ldots,$$

$$59(29)=29+_{60}58(29)=29+_{60}2=31,$$

$$60(29)=29+_{60}59(29)=29+_{60}31=0=e$$

Thus 60(29) = e and $n(a)\neq e$ for $n<60$. $\therefore O(29)=60$

To determine $O(30)$:

$$1(30)=30,\quad 2(30)=30+_{60}30=0=e$$

Thus 2(30) = e and $n(a)\neq e$ for $n<60$. $\therefore O(30)=2$

To determine $O(31)$:

$$1(31)=31,\ \ 2(31)=31+_{60}31=2,\ \ldots,\ \ 16(31)=31+_{60}15(31)=31+_{60}45=16,$$

$$17(31)=31+_{60}16(31)=31+_{60}16=47,\ \ldots,$$

$$26(31)=31+_{60}25(31)=31+_{60}55=36,$$

$$27(31)=31+_{60}26(31)=31+_{60}36=7,\ \ldots,$$

$$38(31)=31+_{60}37(31)=31+_{60}7=38,$$

$$39(31)=31+_{60}38(31)=31+_{60}28=59,\ \ldots$$

$$59(31)=31+_{60}58(31)=31+_{60}58=29,$$

$$60(31)=31+_{60}59(31)=31+_{60}29=0=e$$

Thus $60(31) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(31)=60$

To determine $O(32)$:

$$1(32)=32,\ \ 2(32)=32+_{60}32=4,\ \ldots,$$

$$14(32)=32+_{60}13(32)=32+_{60}56=28,$$

$$15(32)=32+_{60}14(32)=32+_{60}28=0=e$$

Thus $15(32) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(32)=15$

To determine $O(33)$:

$$1(33)=33,\ \ 2(33)=33+_{60}33=6,\ \ldots,\ \ 16(33)=33+_{60}15(33)=33+_{60}15=48,$$

$$17(33)=33+_{60}16(33)=33+_{60}48=21,\ \ldots,$$

$$19(33)=33+_{60}18(33)=33+_{60}54=27,$$

$$20(33)=33+_{60}19(33)=33+_{60}27=0=e$$

Thus $20\,(33) = e$ and $n\,(a) \neq e$ for $n < 60$. $\therefore O(33)=20$

To determine $O(34)$:

$$1(34)=34,\ \ 2(34)=34+_{60}34=8,\ \ldots,\ \ 16(34)=34+_{60}15(34)=34+_{60}30=4,$$

$$17(34)=34+_{60}16(34)=34+_{60}4=38,\ \ldots,$$

$$29(34)=34+_{60}28(34)=34+_{60}52=26,$$

$$30(34)=34+_{60}29(34)=34+_{60}26=0=e$$

Thus $30(34) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(34)=30$

To determine $O(35)$:

$$1(35)=35,\ \ 2(35)=35+_{60}35=10,\ \ 3(35)=35+_{60}2(35)=35+_{60}10=45,\ \ldots,$$

$$11(35)=35+_{60}10(35)=35+_{60}50=25,$$

$$12(35)=35+_{60}11(35)=35+_{60}25=0=e$$

Thus $12(35) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(35)=12$

To determine *O*(36):

$$1(36)=36,\ \ 2(36)=36+_{60}36=12,\ \ 3(36)=36+_{60}2(36)=36+_{60}12=48,$$

$$4(36)=36+_{60}3(36)=36+_{60}48=24,\ \ 5(36)=36+_{60}4(36)=36+_{60}24=0=e$$

Thus 5(36) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(36)=5$

To determine *O*(37):

$$1(37)=37,\ \ 2(37)=37+_{60}37=14, \ldots,$$

$$16(37)=37+_{60}15(37)=37+_{60}15=52,$$

$$17(37)=37+_{60}16(37)=37+_{60}52=29, \ldots,$$

$$26(37)=37+_{60}25(37)=37+_{60}25=2,$$

$$27(37)=37+_{60}26(37)=37+_{60}2=39, \ldots,$$

$$38(37)=37+_{60}37(37)=37+_{60}49=26,$$

$$39(37)=37+_{60}38(37)=37+_{60}26=3, \ldots,$$

$$59(37)=37+_{60}58(37)=37+_{60}46=23$$

$$60(37)=37+_{60}59(37)=37+_{60}23=0=e$$

Thus 60(37) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(37)=60$

To determine *O*(38):

$$1(38)=38,\ \ 2(38)=38+_{60}38=16, \ldots,\ \ 16(38)=38+_{60}15(38)=38+_{60}30=8$$

$$17(38)=38+_{60}16(38)=38+_{60}8=46, \ldots,$$

$$29(38)=38+_{60}28(38)=38+_{60}44=22$$

$$30(38)=38+_{60}29(38)=38+_{60}22=0=e$$

Thus 30(38) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(38)=30$

To determine *O*(39):

$$1(39)=39,\ \ 2(39)=39+_{60}39=18, \ldots,$$

$$16(39)=39+_{60}15(39)=39+_{60}45=24,$$

$$17(39)=39+_{60}16(39)=39+_{60}24=3,\ \ 18(39)=39+_{60}17(39)=39+_{60}3=42,$$

$$19(39)=39+_{60}18(39)=39+_{60}42=21,$$

$$20(39)=39+_{60}19(39)=39+_{60}21=0=e$$

Thus 20(39) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(39)=20$

To determine *O*(40):

$$1(40)=40,\ \ 2(40)=40+_{60}40=20,\ \ 3(40)=40+_{60}2(40)=40+_{60}20=0=e$$

Thus 3(40) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(40)=3$

To determine *O*(41):

$$1(41)=41,\ \ 2(41)=41+_{60}41=22, \ldots,$$

$$16(41)=41+_{60}15(41)=41+_{60}15=56,$$

$$17(41)=41+_{60}16(41)=41+_{60}56=37, \ldots,$$

$$26(41)=41+_{60}25(41)=41+_{60}5=46,$$

$$27(41)=41+_{60}26(41)=41+_{60}46=27, \ldots,$$

$$38(41)=41+_{60}37(41)=41+_{60}17=58,$$

$$39(41)=41+_{60}38(41)=41+_{60}58=39, \ldots,$$

$$59(41)=41+_{60}58(41)=41+_{60}30=19,$$

$$60(41)=41+_{60}59(41)=41+_{60}19=0=e$$

Thus $60(41) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(41)=60$

To determine $O(42)$:

$$1(42)=42, \quad 2(42)=42+_{60}1(42)=42+_{60}42=24,$$

$$3(42)=42+_{60}2(42)=42+_{60}24=6, \ldots,$$

$$9(42)=42+_{60}8(42)=42+_{60}36=18,$$

$$10(42)=42+_{60}9(42)=42+_{60}18=0=e$$

Thus $10(42) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(42)=10$

To determine $O(43)$:

$$1(43)=43, \quad 2(43)=43+_{50}43=36, \ldots,$$

$$16(43)=43+_{60}15(43)=43+_{60}45=28,$$

$$17(43)=43+_{60}16(43)=43+_{60}28=11, \ldots,$$

$$26(43)=43+_{50}25(43)=43+_{50}25=18,$$

$$27(43)=43+_{50}26(43)=43+_{50}18=11, \ldots,$$

$$38(43)=43+_{60}37(43)=43+_{60}31=14,$$

$$39(43)=43+_{60}38(43)=43+_{60}14=57, \ldots,$$

$$59(43)=43+_{60}58(43)=43+_{60}34=17,$$

$$60(43)=43+_{60}59(43)=43+_{60}17=0=e$$

Thus $60(43) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(43)=60$

To determine $O(44)$:

$$1(44)=44, \quad 2(44)=44+_{60}44=28, \ldots,$$

$$14(44)=44+_{60}13(44)=44+_{60}32=16,$$

$$15(44)=44+_{60}14(44)=44+_{60}16=0=e$$

Thus $15(44) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(44)=15$

To determine $O(45)$:

$$1(45)=45,\ \ 2(45)=45+_{60}45=30,\ \ 3(45)=45+_{60}2(45)=45+_{60}30=15,$$

$$4(45)=45+_{60}3(45)=45+_{60}15=1=0=e$$

Thus 4(45) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(45)=4$

To determine $O(46)$:

$$1(46)=46,\ \ 2(46)=46+_{60}46=32,\ \ldots,$$

$$16(46)=46+_{60}15(46)=46+_{60}30=16,$$

$$17(46)=46+_{60}16(46)=46+_{60}16=2,\ \ldots,$$

$$29(46)=46+_{60}28(46)=46+_{60}28=14,$$

$$30(46)=46+_{60}29(46)=46+_{60}14=0=e$$

Thus 30(46) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(46)=30$

To determine $O(47)$:

$$1(47)=47,\ \ 2(47)=47+_{60}47=34,\ \ldots,$$

$$16(47)=47+_{60}15(47)=47+_{60}45=32,$$

$$17(47)=47+_{60}16(47)=47+_{60}32=19,\ \ldots,$$

$$26(47)=47+_{60}25(47)=47+_{60}35=22,$$

$$27(47)=47+_{60}26(47)=47+_{60}22=9,\ \ldots,$$

$$38(47)=47+_{60}37(47)=47+_{60}59=46,$$

$$39(47)=47+_{60}38(47)=47+_{60}46=33,\ \ldots,$$

$$59(47)=47+_{60}48(47)=47+_{60}26=13,$$

$$60(47)=47+_{60}59(47)=47+_{60}13=0=e$$

Thus 60(47) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(47)=60$

To determine $O(48)$:

$$1(48)=48,\ \ 2(48)=48+_{60}48=36,\ \ 3(48)=48+_{60}2(48)=48+_{60}36=24,$$

$$4(48)=48+_{60}3(48)=48+_{60}24=12,\ \ 5(48)=48+_{60}4(48)=48+_{60}12=0=e$$

Thus 5(48) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(48)=5$

To determine $O(49)$:

$$1(49)=49,\ \ 2(49)=49+_{60}49=38,\ \ldots,$$

$$16(49)=49+_{60}15(49)=49+_{60}15=4,$$

$$17(49)=49+_{60}16(49)=49+_{60}4=53,\ \ldots,$$

$$26(49)=49+_{60}25(49)=49+_{60}25=14,$$

$$27(49)=49+_{60}26(49)=49+_{60}14=3,\ \ldots,$$

$$38(49)=49+_{60}37(49)=49+_{60}13=2,$$

$$39(49) = 49 +_{60} 38(49) = 49 +_{60} 2 = 51, \ldots,$$

$$59(49) = 49 +_{60} 58(49) = 49 +_{60} 22 = 11,$$

$$60(49) = 49 +_{60} 59(49) = 49 +_{60} 11 = 0 = e$$

Thus 60(49) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(49) = 60$

To determine $O(50)$:

$$1(50) = 50, \;\; 2(50) = 50 +_{60} 50 = 40, \;\; 3(50) = 50 +_{60} 2(50) = 50 +_{60} 40 = 30,$$

$$4(50) = 50 +_{60} (50) = 50 +_{60} 30 = 20, \;\; 5(50) = 50 +_{60} 4(50) = 50 +_{60} 20 = 10$$

$$6(50) = 50 +_{60} 5(50) = 50 +_{60} 10 = 0 = e$$

Thus 6(50) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(50) = 6$

To determine $O(51)$:

$$1(51) = 51, \;\; 2(51) = 51 +_{60} 51 = 42, \ldots, \;\; 14(51) = 51 +_{60} 13(51) = 51 +_{60} 3 = 54,$$

$$15(51) = 51 +_{60} 14(51) = 51 +_{60} 54 = 45,$$

$$16(51) = 51 +_{60} 15(51) = 51 +_{60} 45 = 36, \ldots,$$

$$19(51) = 51 +_{60} 18(51) = 51 +_{60} 18 = 9, \;\; 20(51) = 51 +_{60} 19(51) = 51 +_{60} 9 = 0 = e$$

Thus 20(51) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(51) = 20$

To determine $O(52)$:

$$1(52) = 52, \;\; 2(52) = 52 +_{60} 52 = 44, \ldots, \;\; 14(52) = 52 +_{60} 13(52) = 52 +_{60} 16 = 8$$

$$15(52) = 52 +_{60} 14(52) = 52 +_{60} 8 = 0 = e$$

Thus 15(52) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(52) = 15$

To determine $O(53)$:

$$1(53) = 53, \;\; 2(53) = 53 +_{60} 53 = 46, \ldots, \;\; 16(53) = 53 +_{60} 15(53) = 53 +_{60} 15 = 8,$$

$$17(53) = 53 +_{60} 16(53) = 53 +_{60} 8 = 1, \ldots,$$

$$26(53) = 53 +_{60} 25(53) = 53 +_{60} 5 = 58,$$

$$27(53) = 53 +_{60} 26(53) = 53 +_{60} 58 = 51, \ldots,$$

$$38(53) = 53 +_{60} 37(53) = 53 +_{60} 41 = 34,$$

$$39(53) = 53 +_{60} 38(53) = 53 +_{60} 34 = 27, \ldots,$$

$$59(53) = 53 +_{60} 58(53) = 53 +_{60} 14 = 7,$$

$$60(53) = 53 +_{60} 59(53) = 53 +_{60} 7 = 0 = e$$

Thus 60(53) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(53) = 60$

To determine $O(54)$:

$$1(54) = 54, \;\; 2(54) = 54 +_{60} 54 = 48, \;\; 3(54) = 54 +_{60} 2(54) = 54 +_{60} 48 = 42, \ldots,$$

$$8(54) = 54 +_{60} 7(54) = 54 +_{60} 18 = 12, \;\; 9(54) = 54 +_{60} 8(54) = 54 +_{60} 12 = 6$$

$$10(54) = 54 +_{60} 9(54) = 54 +_{60} 6 = 0 = e$$

Thus 6(54) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(54) = 10$

To determine $O(55)$:

$$1(55) = 55,\ \ 2(55) = 55 +_{60} 55 = 50,\ \ 3(55) = 55 +_{60} 2(55) = 55 +_{60} 50 = 45, \ldots,$$

$$8(55) = 55 +_{60} 7(55) = 55 +_{60} 25 = 20,\ \ 9(55) = 55 +_{60} 8(55) = 55 +_{60} 20 = 15$$

$$10(55) = 55 +_{60} 9(55) = 55 +_{60} 15 = 10,\ \ 11(55) = 55 +_{60} 10(55) = 55 +_{60} 10 = 5$$

$$12(55) = 55 +_{60} 11(55) = 55 +_{60} 5 = 0 = e$$

Thus 12(55) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(55) = 12$

To determine $O(56)$:

$$1(56) = 56,\ \ 2(56) = 56 +_{60} 56 = 52, \ldots,\ \ 14(56) = 56 +_{60} 13(56) = 56 +_{60} 8 = 4,$$

$$15(56) = 56 +_{60} 14(56) = 56 +_{60} 4 = 0 = e.$$

Thus 15(56) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(56) = 15$

To determine $O(57)$:

$$1(57) = 57,\ \ 2(57) = 57 +_{60} 57 = 54,\ \ 3(57) = 57 +_{60} 2(57) = 57 +_{60} 54 = 51, \ldots,$$

$$19(57) = 57 +_{60} 18(57) = 57 +_{60} 6 = 3,$$

$$20(57) = 57 +_{60} 19(57) = 57 +_{60} 3 = 0 = e$$

Thus 20(57) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(57) = 20$

To determine $O(58)$:

$$1(58) = 58,\ \ 2(58) = 58 +_{60} 58 = 56, \ldots,\ \ 16(58) = 58 +_{60} 15(58) = 58 +_{60} = 30,$$

$$17(58) = 58 +_{60} 16(58) = 58 +_{60} 30 = 28, \ldots,$$

$$29(58) = 58 +_{60} 28(58) = 58 +_{60} 4 = 2,$$

$$30(58) = 58 +_{60} 29(58) = 58 +_{60} 2 = 0 = e$$

Thus 30(58) = e and $n(a) \neq e$ for $n < 60$. $\therefore O(58) = 30$

To determine $O(59)$:

$$1(59) = 59,\ \ 2(59) = 59 +_{60} 59 = 58, \ldots,$$

$$16(59) = 59 +_{60} 15(59) = 59 +_{60} 45 = 44,$$

$$17(59) = 59 +_{60} 16(59) = 59 +_{60} 44 = 43, \ldots,$$

$$26(59) = 59 +_{60} 25(59) = 59 +_{60} 35 = 34,$$

$$27(59) = 59 +_{60} 26(59) = 59 +_{60} 34 = 33, \ldots,$$

$$38(59) = 59 +_{60} 37(59) = 59 +_{60} 23 = 22,$$

$$39(59) = 59 +_{60} 38(59) = 59 +_{60} 22 = 21, \ldots,$$

$$59(59) = 59 +_{60} 58(59) = 59 +_{60} 2 = 1,$$

$$60(59) = 59 +_{60} 59(59) = 53 +_{60} 1 = 0 = e$$

Thus $60(59) = e$ and $n(a) \neq e$ for $n < 60$. $\therefore O(59) = 60$

5. Analysis of the Result

We discussed the result of order of every element for multiplication composition in the 60 and 61 orders of group for multiplication composition. In fact, we can use composition related theorem to evaluate order of group of different orders such as order $2, 3, 4, 5, \cdots, 20$ etc., *i.e.*, whose order is not so high (Not Higher Order Groups). As a result, we use multiplication related theorems to evaluate the order of groups of the higher order of group for composition. Here, to find orders of elements of a cyclic group. Thus, it has been found necessary and convenient to work or solve these structures in details.

6. Conclusion

In this work, we developed the higher order of elements of a group for various higher orders of a group. This result is very important for the order of every element of a group. For this reason, we can determine the orders of the elements in groups of different orders. In different situations, once it was found that a given solution satisfies the basic result of one structure, and having known the properties of that structure, it becomes extremely easy to forecast the behavior of the situation. This result in this paper will be advantageous for group theory related to subgroups and order of elements of a group. Thus, the demands of the work of mathematical problems as like as physical problems such as groups, number systems, vectors, matrices and so on.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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