

Periodic Travelling Waves of a Generalized Curvature Flow Arising from Crystal Growth Models

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How to cite this paper: Li, H. (2026) Periodic Travelling Waves of a Generalized Curvature Flow Arising from Crystal Growth Models. *Engineering*, 18, 397-407. <https://doi.org/10.4236/eng.2026.1810022>

Received: August 23, 2026

Accepted: October 20, 2026

Published: October 23, 2026

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Abstract

In this paper, we consider a generalized curvature flow related to crystal growth in heterogeneous media with periodic oblique striations. The angle between the oblique striations and x -direction is α with $0 < \alpha < \pi/2$. We first prove the existence and uniqueness of periodic traveling waves that travel upward periodically with an average speed. Then, we derive the upper and lower bounds for the average speed of the periodic traveling wave.

Keywords

Periodic Traveling Wave, Generalized Curvature Flow, Average Speed

1. Introduction

Based on Herring's physical formula for anisotropic surface free energy, Taylor, Cahn, and Handwerker [1] reformulated crystal-interface dynamics into a sharp-interface geometric evolution model. They introduced the concept of crystalline curvature κ_γ , a generalization of mean curvature adapted to orientation-dependent surface energy $\gamma(n)$. The interface evolves according to the crystalline curvature flow $V = -\kappa_\gamma$, where V denotes the interface normal velocity. In the isotropic limit, κ_γ reduced to ordinary mean curvature κ . In 2007, Lou [2] investigated the following curvature flow

$$V = F(\kappa, n, x).$$

and showed the existence and uniqueness of the traveling waves. In this paper, we consider a generalized curvature flow

$$V = H(\kappa, \mathbf{n}, x \sin \alpha + y \cos \alpha) \quad (1)$$

in heterogeneous media with periodic oblique striations in the xy -plane, where α ($0 < \alpha < \pi/2$) is the inclination angle of the oblique striations, for any time t and any simple plane curve Γ_t and for any $P \in \Gamma_t$, κ denotes the curvature of Γ_t at P , $\mathbf{n}$ denotes the unit normal vector at P and V denotes the velocity along $\mathbf{n}$, $H \in C^1(\mathbb{R} \times S^1 \times \mathbb{R})$ and satisfying

(A) $H_\kappa(\kappa, \mathbf{n}, \eta) > 0$ in $\mathbb{R} \times S^1 \times \mathbb{R}$; $H(\kappa, \mathbf{n}, \eta)$ is 1-periodic in η ;
 $0 < H_m \leq H(0, \mathbf{n}, \eta) \leq H_M$.

Clearly, the typical example

$$V = a(x, \mathbf{n})\kappa + b(x, \mathbf{n}) \quad (2)$$

satisfies the condition (A) if a and b are continuous, positive and 1-periodic functions in x .

The local-in-time existence and the asymptotic behavior of solutions of (1) have been investigated by many authors (see [3] [4] and references therein). References [5] and [6] studied traveling wave solutions of the special case of (2):

$$V = a(\mathbf{n})\kappa + b(\mathbf{n}).$$

In 1993, Altschuler and Wu [7] considered the problem with $a(x, \mathbf{n}) \equiv A(\mathbf{n})$ and $b(x, \mathbf{n}) \equiv 0$ in (2) and proved that any global solution with some initial data converges to the unique (up to a shift) traveling wave solution. In 2006 and 2013, references [8] and [9] considered the problem (2) with $a(x, \mathbf{n}) \equiv A(\mathbf{n})$ and $b(x, \mathbf{n}) \equiv 0$ in a band domain with undulating boundaries. They constructed the periodic traveling wave solution and studied its uniqueness and asymptotic stability. Later, Cai and Lou [10] studied the analogue problem with the almost periodic boundaries and obtained the analogue results. In 1994, Altschuler and Wu [11] extended their result in [7] to the case of two dimensions. Recently, the results in [11]-[13] were further improved by Ma, Wang and Wei [14] to N -dimensional space ($N > 1$).

Since the 1980s, various (mean) curvature flows, including the curvature flow $V = a(\mathbf{n})\kappa^\beta$ with $\beta > 0$ or even $\beta = -1$ (which corresponds to an expanding flow), have been extensively studied by geometers from a mathematical point of view (see [3] [15]). In 2023, Lou and Yuan [16] studied the generalized mean curvature flow $V = \kappa^\beta + b$ with $\beta > 0$, $b \in \mathbb{R}$ and proved the existence, uniqueness, and the asymptotic behavior of radial flows in a cylinder with constant boundary angles.

Our main purpose in this paper is to consider the generalized curvature flow (1) in the case that the curve Γ_t is the smooth graph of a function $y = u(x, t)$ for each t . To avoid sign confusion, the normal to the curve will always be chosen upward (toward y -direction), and the V and κ will be understood in accordance with this choice of the direction of the normal. Consequently,

$$V = \frac{u_t}{\sqrt{1+u_x^2}}, \quad \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \quad \mathbf{n} = \frac{(-u_x, 1)}{\sqrt{1+u_x^2}},$$

and (2) is equivalent to

$$\frac{u_t}{\sqrt{1+u_x^2}} = H \left(\frac{u_{xx}}{(1+u_x^2)^{3/2}}, \frac{(-u_x, 1)}{\sqrt{1+u_x^2}}, x \sin \alpha + u \cos \alpha \right), \quad x \in \mathbb{R}, t \in \mathbb{R}. \quad (3)$$

In the following, a solution $u(x, t)$ of (3) is called a periodic traveling wave if

$$u(x, t) = \varphi(x, t) + ct, \quad x \in \mathbb{R}, t \in \mathbb{R}, \quad (4)$$

where $\varphi(x, t)$ is X_α -periodic in x and T_α -periodic in t and $X_\alpha = 1/\sin \alpha$, T_α satisfies

$$u(x, t + T_\alpha) = u(x, t) + 1/\cos \alpha, \quad x \in \mathbb{R}, t \in \mathbb{R},$$

Note that the speed of the wave is time-periodic so that $c = 1/(T_\alpha \cos \alpha)$ represents the average speed over one period.

In Section 2, we transform Problems (3) and (4) into the boundary value problem for a first-order ordinary differential equation by means of several changes of variables. In Section 3, we first prove that the boundary value problem for a first-order ordinary differential equation has a unique solution. Then, we verify the existence and uniqueness of periodic traveling waves of (3) and derive the upper and lower bounds for the average speed of the wave. $\psi(0) = 0$ in (7) is used to give the uniqueness of the periodic traveling waves of (3). Our main result is the following Theorem.

Theorem 1. For any $\alpha \in (0, \pi/2)$ satisfying $(H_M^2 - H_m^2) \tan^2 \alpha \leq H_m^2$. Equation (3) has a unique periodic traveling wave $u^*(x, t) = \varphi^*(x, t) + c^*t$, $x \in \mathbb{R}$, $t \in \mathbb{R}$, and

$$H_m \leq c^* \leq H_M, \quad (5)$$

where the uniqueness holds under our chosen phase normalization condition $\psi(0) = 0$ in (7).

2. Transformation to a First-Order ODE

For fixed $\alpha \in (0, \pi/2)$ satisfying $(H_M^2 - H_m^2) \tan^2 \alpha \leq H_m^2$, in order to show the existence and uniqueness of periodic traveling wave solutions of (3) with inclination angle α , the method we adopt is to reduce (3) to a first-order ordinary differential equation.

Let $\eta(x, t) := x \sin \alpha + u \cos \alpha$, then (3) is rewritten as

$$\begin{aligned} & \frac{\eta_t}{\sqrt{1 - 2\eta_x \sin \alpha + \eta_x^2}} \\ &= H \left(\frac{\eta_{xx} \cos^2 \alpha}{(1 - 2\eta_x \sin \alpha + \eta_x^2)^{3/2}}, \frac{(\sin \alpha - \eta_x \cos \alpha)}{\sqrt{1 - 2\eta_x \sin \alpha + \eta_x^2}}, \eta \right), \quad x \in \mathbb{R}, t \in \mathbb{R}. \end{aligned} \quad (6)$$

Let $z = x + ct \cos \alpha$, $\psi(z) = \eta(x, t) = x \sin \alpha + u \cos \alpha$, then (3) and (4) are rewritten as

$$\begin{cases} \frac{c\psi'(z)\cot\alpha}{\sqrt{1-2\psi'(z)\sin\alpha+(\psi'(z))^2}} \\ = H \left(\frac{\psi''(z)\cos^2\alpha}{\left(1-2\psi'(z)\sin\alpha+(\psi'(z))^2\right)^{3/2}}, \frac{(\sin\alpha-\psi'(z), \cos\alpha)}{\sqrt{1-2\psi'(z)\sin\alpha+(\psi'(z))^2}}, \psi(z) \right), z \in \mathbb{R}. \\ \psi(0)=0, \psi(X_\alpha)=1, \psi'(0)=\psi'(X_\alpha). \end{cases} \quad (7)$$

A pair $(c, \psi(z))$ solving (7) is called a solution of (7). Next, we prove the existence and uniqueness of a solution $(c, \psi(z))$ of (7); this implies the existence and uniqueness of a periodic traveling wave of (3). $\psi(0)=0$ in (7) is used to give the uniqueness of $\psi(z)$, without this condition, $\psi(z)$ is unique up to a translation.

We first prove the following result.

Lemma 1. *Let $(c, \psi(z))$ be a solution of (7); then the following properties hold.*

1) *The function $\psi(z) - z \sin \alpha$ is periodic with period X_α , namely,*

$$\psi(z + X_\alpha) - [z + X_\alpha] \sin \alpha = \psi(z) - z \sin \alpha, \quad x \in \mathbb{R} \quad (8)$$

2) *$\psi'(z)$ is periodic with period X_α .*

3) *$c > 0$ and $\psi'(z) > 0$, $z \in \mathbb{R}$.*

Proof. 1) Since $H(\kappa, n, \eta)$ is 1-periodic in η , the function $\phi(z) = \psi(z + X_\alpha) - 1$ satisfies the same differential equation in $\mathbb{R}$ as $\psi(z)$ does. Since $\phi(0) = \psi(0)$ and $\phi'(0) = \psi'(0)$, we easily obtain, by the uniqueness of the initial value problem for second-order ODEs, that $\phi(z) = \psi(z)$. This implies that (8) holds.

2) From (1), we have $\psi(z) = \psi(z + X_\alpha) - 1$. So, $\psi'(z) = \psi'(z + X_\alpha)$ in $\mathbb{R}$.

3) From (2), there exists $z^* \in [0, X_\alpha)$ such that $\psi'(z^*) = \max_{\mathbb{R}} \psi'(z) > 0$. As $\psi''(z^*) = 0$, we can get that $c > 0$ by (7).

Now, let $z_* \in [0, X_\alpha)$ such that $\psi'(z_*) = \min_{\mathbb{R}} \psi'(z)$. Then, $\psi''(z_*) = 0$ so that the differential equation in (7) yield $c\psi'(z_*) > 0$. Then, $\psi'(z) > 0$, $z \in \mathbb{R}$.

Lemma 2. *Let $s = \psi(z)$, $v(s) = \frac{\sin \alpha}{\psi'(z)}$, then $(c, \psi(z))$ is a solution of (7) if and only if $(c, v(s))$ be a solution of the following problem*

$$\begin{cases} \frac{c}{\sqrt{v^2 + (1-v)^2 \tan^2 \alpha}} = H \left(-\frac{\sin \alpha \cdot \tan \alpha \cdot v'(s)}{\left(\sqrt{v^2 + (1-v)^2 \tan^2 \alpha}\right)^3}, \frac{(\tan \alpha \cdot (v-1), v)}{\sqrt{v^2 + (1-v)^2 \tan^2 \alpha}}, s \right), s \in (0, 1), \\ v(0) = v(1), c > 0, v(s) > 0 \text{ for } s \in [0, 1], \int_0^1 v(s) ds = 1, \end{cases} \quad (9)$$

where the transformation from $(c, \psi(z))$ to $(c, v(s))$ is given by

$$\psi'(z) = \frac{\sin \alpha}{v(s)}, \quad s = \psi(z), \text{ and the transformation from } (c, v(s)) \text{ to}$$

$(c, \psi(z))$ is given by $s = \psi(z)$, $v(s) = \frac{\sin \alpha}{\psi'(z)}$.

Proof. This Lemma is easily shown by direct verification and is omitted.

3. Periodic Traveling Wave

For fixed $\alpha \in (0, \pi/2)$ satisfying $(H_M^2 - H_m^2) \tan^2 \alpha \leq H_m^2$, we prove the existence and uniqueness of periodic traveling wave solutions of (3) with inclination angle α in this section. For this purpose, we first show the following Theorem.

Theorem 2. For any $\alpha \in (0, \pi/2)$ satisfying $(H_M^2 - H_m^2) \tan^2 \alpha \leq H_m^2$. Problem (9) has a unique solution $(c^*, v^*(s))$, and

$$H_m \leq c^* \leq H_M. \quad (10)$$

The proof of Theorem 2 easily follows from the following three lemmas.

Lemma 3. For any given $\alpha \in (0, \pi/2)$ satisfying $(H_M^2 - H_m^2) \tan^2 \alpha \leq H_m^2$ and for any given $p \geq 0$, the following problem

$$\begin{cases} \frac{c}{\sqrt{v^2 + (1-v)^2 \tan^2 \alpha}} = H \left(-\frac{\sin \alpha \cdot \tan \alpha \cdot v'(s)}{\left(\sqrt{v^2 + (1-v)^2 \tan^2 \alpha}\right)^3}, \frac{(\tan \alpha \cdot (v-1), v)}{\sqrt{v^2 + (1-v)^2 \tan^2 \alpha}}, s \right), s \in (0, 1), \\ v(0) = v(1) = p. \end{cases} \quad (11)$$

has a unique solution $(c(p, \alpha), v(s, p, \alpha))$. Moreover

$$H_m \max_{s \in [0, 1]} \sqrt{v^2(s, p, \alpha) + (1-v(s, p, \alpha))^2 \tan^2 \alpha} \leq c(p, \alpha) \leq H_M \min_{s \in [0, 1]} \sqrt{v^2(s, p, \alpha) + (1-v(s, p, \alpha))^2 \tan^2 \alpha}. \quad (12)$$

Proof. We first consider the initial value problem for the following ODE

$$\begin{cases} \frac{c}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}} = H \left(-\frac{\sin \alpha \cdot \tan \alpha \cdot V'(s)}{\left(\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}\right)^3}, \frac{(\tan \alpha \cdot (V-1), V)}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}}, s \right), s \geq 0, \\ V(0) = p. \end{cases} \quad (13)$$

Since $H_\alpha > 0$, for any given $c \geq 0$, by the implicit function theorem, the equation can be rewritten in (13) as $V' = G(V, s, \alpha, c)$, where G is a C^1 function. Thus (13) is equivalent to

$$V'(s) = G(V, s, \alpha, c), \quad V(0) = p. \quad (14)$$

By the theory of ODEs, (14) has a unique solution $V(s, p, \alpha, c)$ on intervals. Denote by $[0, s(c))$ the maximum existence interval.

1) When $c = c_1 := H_M \sqrt{(p+1)^2 + (p+2)^2 \tan^2 \alpha}$, we consider the existence interval of $V(s, p, \alpha, c_1)$. If $V(s, p, \alpha, c_1) \in (-p-1, p+1)$, then we have

$$H \left(-\frac{\sin \alpha \cdot \tan \alpha \cdot V'(s)}{\left(\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}\right)^3}, \frac{(\tan \alpha \cdot (V-1), V)}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}}, s \right) = \frac{c_1}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}} > H_M,$$

$$H_M \geq H \left(0, \frac{(\tan \alpha \cdot (V-1), V)}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}}, s \right).$$

Since $H_\kappa > 0$, we have $V'(s, p, \alpha, c_1) < 0$, as long as $V(s, p, \alpha, c_1) \in (-p-1, p+1)$.

On the other hand, if $V(s, p, \alpha, c_1) < -V_0 := -\frac{c_1}{H_m}$, then

$$\begin{aligned} H & \left(-\frac{\sin \alpha \cdot \tan \alpha \cdot V'(s)}{\left(\sqrt{V^2 + (1-V)^2 \tan^2 \alpha} \right)^3}, \frac{(\tan \alpha \cdot (V-1), V)}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}}, s \right) \\ &= \frac{c_1}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}} < H_m, \\ H_m & \leq H \left(0, \frac{(\tan \alpha \cdot (V-1), V)}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}}, s \right). \end{aligned}$$

$H_\kappa > 0$ implies that $V'(s, p, \alpha, c_1) > 0$ for $V < -V_0$.

Therefore, $V(s, p, \alpha, c_1) \in (-V_0, p)$ and $s(c_1) = +\infty$. Especially, $V(1, p, \alpha, c_1) < p$.

2) $V(s, p, \alpha, c)$ is decreasing in c . In fact, using (14), $V_c := \frac{\partial V}{\partial c}$ and

$V_p := \frac{\partial V}{\partial p}$ satisfy

$$\begin{aligned} V_p' &= G_V \cdot V_p, \quad V_p|_{s=0} = 1, \\ V_c' &= G_V \cdot V_c + G_c, \quad V_c|_{s=0} = 0. \end{aligned}$$

So $V_p = \exp\left(\int_0^s G_V d\tau\right) > 0$. By using the variation of constants technique, we have for $\mu := \frac{V_c}{V_p}$,

$$\mu' = \frac{G_c}{V_p}, \quad \mu|_{s=0} = \frac{V_c}{V_p} \Big|_{s=0} = 0.$$

Substituting $V' = G(V, s, \alpha, c)$ into the equation of (13), we have

$$\frac{c}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}} = H \left(-\frac{\sin \alpha \cdot \tan \alpha \cdot G(V, s, \alpha, c)}{\left(\sqrt{V^2 + (1-V)^2 \tan^2 \alpha} \right)^3}, \frac{(\tan \alpha \cdot (V-1), V)}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}}, s \right)$$

Differentiating it with respect to c , we get that

$$-H_\kappa \cdot G_c \cdot \frac{\sin \alpha \cdot \tan \alpha}{\left(\sqrt{V^2 + (1-V)^2 \tan^2 \alpha} \right)^3} = \frac{1}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}} > 0.$$

Hence, $G_c < 0$ and we have $\mu' < 0$. This implies that $\mu < 0$ for $s > 0$. Therefore, $V_c < 0$ for any $s \in (0, s(c))$.

3) Set $Z := \{c \in [0, c_1] : s(c) > 1 \text{ and } V(1, p, \alpha, c) < p\}$. Since for $V(s, p, \alpha, 0)$, we have

$$H \left(-\frac{\sin \alpha \cdot \tan \alpha \cdot V'(s)}{\left(\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}\right)^3}, \frac{(\tan \alpha \cdot (V-1), V)}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}}, s \right) = 0$$

$$< H_m \leq H \left(0, \frac{(\tan \alpha \cdot (V-1), V)}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}}, s \right),$$

so $H_s > 0$ implies that $V'(s, p, \alpha, 0) > 0$. This implies that $0 \notin Z$. The above (1) means that $c_1 \in Z$ and (2) tells us that if $c_2 \in (0, c_1) \cap Z$, then $[c_2, c_1] \subset Z$.

4) For any fixed $c \in Z$. On the one hand, $V(s, p, \alpha, c) \geq V(s, p, \alpha, c_1) > -V_0$. On the other hand, if $v > p$ is a value of $V(s, p, \alpha, c)$ at some $s \in (0, 1)$, then there exists $s_1 \in (0, 1)$ such that $V(s, p, \alpha, c) = v$ and $V'(s_1, p, \alpha, c) \leq 0$, so at this point s_1 , we have

$$\frac{c}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}}$$

$$= H \left(-\frac{\sin \alpha \cdot \tan \alpha \cdot V'(s)}{\left(\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}\right)^3}, \frac{(\tan \alpha \cdot (V-1), V)}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}}, s \right)$$

$$\geq H \left(0, \frac{(\tan \alpha \cdot (V-1), V)}{\sqrt{V^2 + (1-V)^2 \tan^2 \alpha}}, s \right) \geq H_m.$$

i.e., $c_1 \geq c \geq H_m \sqrt{V^2 + (1-V)^2 \tan^2 \alpha}$. Since v can be chosen arbitrarily, as long as it is a value of V greater than p , we have $V^2 \leq c_1^2 / H_m^2$ when $V > p$. Hence, $V(s, p, \alpha, c) \in [-c_1 / H_m, p]$ or $[-c_1 / H_m, c_1 / H_m]$ for $s \in [0, 1]$.

From (14), there exists $M_1 = M_1(H, c_1, V_0, p) = M_1(H, p)$ such that, for any $c \in Z$, $V(s, p, \alpha, c)$ satisfies

$$\max_{s \in [0, 1]} |V| \leq M_1, \quad \max_{s \in [0, 1]} |V'| \leq M_1, \quad \max_{s \in [0, 1]} |V''| \leq M_1. \quad (15)$$

5) Denote $c_3 := \inf_Z c$, then (15) and (2) deduce that there exists a function $v(s, p, \alpha) \in C^1([0, 1])$ such that

$$\|V(s, p, \alpha, c) - v(s, p, \alpha)\|_{C^1} \rightarrow 0 \text{ as } c \rightarrow c_3 + 0.$$

Taking the limit $c \rightarrow c_3 + 0$ in (14), we have

$$v'(s, p, \alpha) = G(v(s, p, \alpha), s, c_3) \text{ for } s \in [0, 1], v(0) = p.$$

Therefore, $v(s, p, \alpha) \equiv V(s, p, \alpha, c_3)$ on $[0, 1]$. Clearly, $V(s, p, \alpha, c_3)$ satisfies (15) and $V(1, p, \alpha, c_3) \leq p$.

6) Next, we show that $V(1, p, \alpha, c_3) = p$. Otherwise, $V(1, p, \alpha, c_3) < p - 2\delta_1$ for some $\delta_1 \in (0, 1)$. Denote $M_2 := \max\{|G_V| + |G_c| : s \in [0, 1], V \in [-M_1 - 1, M_1 + 1], c \in [0, c_1]\}$. Consider (14) with $c = c_4 := c_3 - \delta_2$, where $\delta_2 < \min\{\delta_1 M_2^{-1} \exp(-M_2), c_3\}$.

First, we prove that

$$V(s, p, \alpha, c_3) \leq V(s, p, \alpha, c_4) \leq V(s, p, \alpha, c_3) + 1 \text{ for } s \in [0, 1]. \quad (16)$$

In fact, the first inequality is true on $[0, s(c_4)) \cap [0, 1]$. The second inequality holds on $[0, s_2]$ for some small s_2 . Set

$s_3 := \sup\{s : \text{the second inequality holds on } [0, s]\}$. If $s_3 \geq 1$, then (16) is proved. If $s_3 < 1$, then

$$V(s_3, p, \alpha, c_4) = V(s_3, p, \alpha, c_3) + 1. \quad (17)$$

Set $\tilde{V}(s) = V(s, p, \alpha, c_4) - V(s, p, \alpha, c_3) \geq 0$. Then, on $[0, s_3]$, there exists $\zeta(s)$ lies between $V(s, p, \alpha, c_3)$ and $V(s, p, \alpha, c_4)$, and $c_5 \in (c_4, c_3) \subset (0, c_1)$ such that

$$\tilde{V}'(s) = G_V(s, \zeta(s), \alpha, c_4)\tilde{V} + G_c(s, V(s, p, \alpha, c_3), \alpha, c_5)\delta_2.$$

Since $s_3 < 1$, we have $\tilde{V}(s_3) \leq \delta_2 M_2 \exp(M_2) \leq \delta_1 < 1$, in contradiction with (17). Therefore, (16) holds.

Repeating the above discussion on $[0, 1]$, we have $\tilde{V}(1) \leq \delta_1$, that is, $V(1, p, \alpha, c_4) \leq V(1, p, \alpha, c_3) + \delta_1 < p - \delta_1$. This implies that $c_4 \in Z$, which contradicts the definition of c_3 . Therefore, $V(1, p, \alpha, c_3) = p$, and so the pair $c(p, \alpha) := c_3$ and $v(s, p, \alpha) \equiv V(s, p, \alpha, c_3)$ is the unique solution of (11).

7) Now, we prove that (12) holds. Since $v(s, p, \alpha)$ is 1-periodic, for any $v \in [\min v(\cdot, p, \alpha), \max v(\cdot, p, \alpha)]$, there exists $s_4, s_5 \in [0, 1]$ such that $v(s_4, p, \alpha) = v(s_5, p, \alpha) = v$, $v'(s_4, p, \alpha) \geq 0 \geq v'(s_5, p, \alpha)$. Then, at this point $s = s_4$, we have

$$\begin{aligned} \frac{c(p, \alpha)}{\sqrt{v^2 + (1-v)^2 \tan^2 \alpha}} &= H \left(-\frac{\sin \alpha \cdot \tan \alpha \cdot v'(s)}{\left(\sqrt{v^2 + (1-v)^2 \tan^2 \alpha}\right)^3}, \frac{(\tan \alpha \cdot (v-1), v)}{\sqrt{v^2 + (1-v)^2 \tan^2 \alpha}}, s \right) \\ &\geq H \left(0, \frac{(\tan \alpha \cdot (v-1), v)}{\sqrt{v^2 + (1-v)^2 \tan^2 \alpha}}, s \right) \geq H_m, \end{aligned}$$

that is, $c(p, \alpha) \geq H_m \sqrt{v^2 + (1-v)^2 \tan^2 \alpha}$. Since

$v \in [\min v(\cdot, p, \alpha), \max v(\cdot, p, \alpha)]$ can be chosen arbitrarily, we have

$$c(p, \alpha) \geq H_m \max_{s \in [0, 1]} \sqrt{v^2(s, p, \alpha) + (1 - v(s, p, \alpha))^2 \tan^2 \alpha}.$$

Similarly, one can show that

$$c(p, \alpha) \leq H_M \min_{s \in [0,1]} \sqrt{v^2(s, p, \alpha) + (1 - v(s, p, \alpha))^2 \tan^2 \alpha}.$$

Lemma 4. For any given $\alpha \in (0, \pi/2)$ satisfying $(H_M^2 - H_m^2) \tan^2 \alpha \leq H_m^2$, there exists a unique $p = p_\alpha$ such that the corresponding unique solution of (11), that is, the pair

$$c^*(\alpha) := c(p_\alpha, \alpha) \quad \text{and} \quad v^*(s, \alpha) := v(s, p_\alpha, \alpha) = V(s, p_\alpha, \alpha, c(p_\alpha, \alpha)) \quad (18)$$

satisfies $v^*(s, \alpha) > 0$ for $s \in [0, 1]$, and $\int_0^1 v^*(s, \alpha) dx = 1$. Hence, $(c^*, v^*(s))$ is the unique solution of (9).

Proof. For any $p \geq 0$, recall the solution of (11) is

$v(s, p, \alpha) \equiv V(s, p, \alpha, c(p))$. Let $I(p, \alpha) = \int_0^1 v(s, p, \alpha) ds$, we need to prove that there exists a unique $p = p_\alpha$ such that $I(p_\alpha, \alpha) = 1$.

1) By (12), we have

$$H_m \sqrt{p^2 + (1-p)^2 \tan^2 \alpha} \leq \begin{cases} H_M \min_{s \in [0,1]} \sqrt{v^2(s, p, \alpha) + (1 - v(s, p, \alpha))^2 \tan^2 \alpha}, & \text{when } \min_{s \in [0,1]} v(s, p, \alpha) > 0, \\ H_M \tan \alpha, & \text{when } \min_{s \in [0,1]} v(s, p, \alpha) \leq 0. \end{cases}$$

The second inequality is impossible when $p > 0$. Hence, $v(s, p, \alpha) > 0$ and $\min v(\cdot, p, \alpha) \rightarrow +\infty$ when $p \rightarrow +\infty$ by the first inequality. This means that $I(p, \alpha) \rightarrow +\infty$ as $p \rightarrow +\infty$.

When $p = 0$, by (12) we know that

$$H_m \sqrt{\max_{s \in [0,1]} v^2(s, p, \alpha) + \tan^2 \alpha} \leq c(p, \alpha) \leq H_M \tan \alpha.$$

Since $(H_M^2 - H_m^2) \tan^2 \alpha \leq H_m^2$, it follows that $\max_{s \in [0,1]} v(s, p, \alpha) \leq 1$. Clearly, $v(s, \alpha) \neq 1$, hence $I(0, \alpha) < 1$. In fact, if $v(s, \alpha) \equiv 1$, then $\psi(z) = z \sin \alpha$. So $u(x, t) = ct \sin \alpha$ is the periodic traveling wave of (3), but $u(x, t) = ct \sin \alpha$ does not satisfy (4).

2) Denote $\bar{V}(p, \alpha, c(p, \alpha)) := V(1, p, \alpha, c(p, \alpha))$, then by

$\bar{V}(p, \alpha, c(p, \alpha)) = p$, we have $\bar{V}_p + \bar{V}_c \cdot c_p = 1$, and so $c_p = (1 - \bar{V}_p) / \bar{V}_c$. Using the notation μ in the proof of Lemma 2, we have

$$\begin{aligned} I_p &= \int_0^1 (V_p + V_c \cdot c_p) ds = \int_0^1 \frac{V_p \bar{V}_c + V_c - V_c \bar{V}_p}{\bar{V}_c} ds \\ &= \int_0^1 \frac{\mu(1) V_p \bar{V}_p + \mu(s) V_p - \mu(s) V_p \bar{V}_p}{\mu(1) \bar{V}_p} dx > 0. \end{aligned}$$

The above two assertions imply that there exists a unique $p_\alpha > 0$ such that $I(p_\alpha, \alpha) = 1$. Thus, $(c^*(\alpha), v^*(s, \alpha))$ defined by (18) is the unique solution of (9) and $v^*(s, \alpha)$ satisfies that $\int_0^1 v^*(s, \alpha) dx = 1$ and $v^*(s, \alpha) > 0$ for $s \in [0, 1]$.

Lemma 5. Let $(c^*(\alpha), v^*(s, \alpha))$ be the unique solution obtained in Lemma 4, then (10) holds.

Proof. $v^*(s, \alpha) > 0$ for $s \in [0, 1]$ and $\int_0^1 v^*(s, \alpha) dx = 1$ imply that $0 < \min_{s \in [0, 1]} v^*(s, \alpha) \leq 1 \leq \max_{s \in [0, 1]} v^*(s, \alpha)$. Hence, (10) follows from (12).

Proof of Theorem 1. From Theorem 2 and Lemma 2, we easily obtain that Problem (9) has a unique solution $(c^*, \psi^*(z))$, and (10) holds. Let

$$u^*(x, t) = \frac{\psi^*(x + c^* t \cos \alpha) - x \sin \alpha}{\cos \alpha}, \quad \phi^*(x, t) = u^*(x, t) - c^* t, \quad \forall x \in \mathbb{R}, t \in \mathbb{R}.$$

By using Lemma 1, it is easy to check that

$$\phi^*(x, t) = \frac{\psi^*(x + c^* t \cos \alpha) - x \sin \alpha}{\cos \alpha} - c^* t$$

is X_α -periodic in x and T_α -periodic in t . From the discussion in Section 2, we follow that Equation (3) has a unique periodic traveling wave

$$u^*(x, t) = \phi^*(x, t) + c^* t, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \quad \text{and} \quad H_m \leq c^* \leq H_M.$$

Acknowledgements

The author is grateful to the referees for helpful suggestions.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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