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From One Generator to Loop Order on the Three-Dimensional Cube

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Abstract

We study the monoid $\mathcal{W}$ of based closed walks on the 1-skeleton of the three-dimensional cube Q_3. The fundamental group of this graph is the free group F_5. We prove that every additive one-step reading, given by a sum of edge weights in an abelian group, factors through the directed transition counts, and we exhibit two closed walks with equal transition counts and different reduced loop words. Hence commutative aggregation does not determine the reduced loop word. The reduced loop word gives a proper noncommutative recognition quotient. We also determine the shortest closed walk with trivial abelianization but nontrivial degree-two commutator information. Its minimum length in the cube edge metric is 14. We also obtain a separation between finite and unbounded memory. A two-state reading separates an order pair, while for every $k \geq 1$, there is an explicit factorial pair which no k-state reading separates. An unbounded stack recovers the reduced loop word on every walk. For integer-valued functions on the vertex set, potential readings vanish on closed walks, while occupation binding is not rectangular. More generally, every occupation-based constraint is rectangular on a class of histories if and only if the occupation vector is constant on that class. Finally, a declared quarter-turn quaternion clock gives a second proper order-sensitive congruence, incomparable with the reduced-word quotient. The obstruction to commutative recovery, and the minimum 14, remain valid on every hypercube Q_n, $n \geq 3$.

Keywords

Based Closed Walks, Reduced Loop Words, Recognition Quotients, Finite-State Memory, Hypercubes

1. Introduction

In [1], we studied the structure generated from an empty record 0 by a single re-

peatable step $t \mapsto St$, and showed that the orbit generated is a free additive monoid on one generator, isomorphic to $(\mathbb{N}, +, 0)$. A recognizer of a monoid M is a monoid homomorphism

$$r : M \to N,$$

where the monoid N need not be finite. This is more general than the finite monoid convention used in [2]. The relation

$$x \sim_r y \Leftrightarrow r(x) = r(y)$$

is the kernel congruence of r, and $M / \sim_r$ is its recognition quotient.

Under excluded middle, in [1], we classified the recognition quotients of this monogenic monoid: every recognizer is either injective or has a finite index-period quotient $M(i, p)$. Since the carrier is generated by $S0$, every monoid recognizer is determined by the image of this one generator, so there is no independent order information. We now consider a carrier (the monoid whose elements are to be recognized) with several independent loops, where order can matter.

We take as carrier the monoid $\mathcal{W}$ of based closed walks (we also call them histories) on the 1-skeleton of the three-dimensional cube Q_3. The restriction to based closed walks is a convention, not a consequence of recognition. We fix a spanning tree and obtain a homomorphism

$$\rho : \mathcal{W} \to F_5,$$

where F_5 is the free group of rank five, which assigns to each closed walk its reduced loop word. Thus Q_3 is the first hypercube whose reduced loop words can be noncommutative (Proposition 2.2). The cycle rank of Q_2 is one, so every reduced loop word on Q_2 is a power of one generator. The cycle rank of Q_3 is five, so the reduced loop words lie in the nonabelian group F_5. The abelianization a of ρ (see (1)) records the exponent sum of each generator, but it does not record the order in which the edges outside the spanning tree are crossed.

We also use Q_3 because its edges are divided into three coordinate directions. A walk is then a word on the alphabet $\{0,1,2\}$, and $\mathcal{W}$ is the set of those words in which each letter occurs an even number of times (Lemma 2.1). The value of an additive reading (Definition 1.1) depends only on the directed transition counts of the word (Proposition 3.2). A k-state reading has at most k states (Section 5), while the stack of Theorem 5.3 stores the current reduced word and cancels adjacent inverse pairs. Thus, these readings are applied to the same language of walks, but with different kinds of memory. The quarter-turn clock of Theorem 7.1 also uses the same three coordinate directions, with one quaternion factor assigned to each direction.

The tetrahedron K_4 has four vertices and six edges, hence cycle rank three and $\pi_1(K_4) \cong F_3$. The graph K_4 is smaller and also has a nonabelian fundamental group. Its edge set can be partitioned into three perfect matchings, so its walks can be encoded by words on three letters. But these three letters do not behave as independent coordinates: a walk on K_4 is closed when the three letter counts have the same parity, while on Q_3 each letter changes only its own coor-

dinate. Also K_4 contains triangles, while Q_3 is bipartite and contains no $K_{2,3}$. These latter properties of Q_3 enter the lower-bound argument in Theorem 4.4, and the same argument extends to every Q_n, $n \geq 3$ (Theorem 9.1).

Definition 1.1. A sequential reading consists of a state set S, an initial state $s_0 \in S$, update maps $T_e : S \to S$ for directed edges e, a set Y, and an output map $q : S \to Y$. On a walk whose directed edge sequence is $e_1, \cdots, e_n$, its value is

$$q\left(T_{e_n} \circ \cdots \circ T_{e_1}\left(s_0\right)\right).$$

The state set may be finite or infinite. For an additive reading, we take S to be an abelian group and

$$T_e\left(s\right) = s + g\left(e\right).$$

A sequential reading need not be a monoid recognizer. Here, we use the term *recognition quotient* only for monoid recognizers.

Let

$$\mathrm{ab} : F_5 \to \mathbb{Z}^5$$

be the abelianization map, and put

$$a = \mathrm{ab} \circ \rho. \tag{1}$$

The map ρ is the usual spanning-tree description of the elements of $\pi_1\left(Q_3, 000\right)$, and a is its abelianization, given by the cycle coordinates in $H_1\left(Q_3; \mathbb{Z}\right) \cong \mathbb{Z}^5$. A recognizer is a monoid homomorphism, and its recognition quotient is isomorphic to its image. An additive reading is a sum of edge weights in an abelian group. An antisymmetric one is the pairing of the oriented 1-chain of the walk with a 1-cochain (Section 3). A sequential reading is a deterministic automaton with output on the directed edges, and its state set may be infinite. The distinction between finite-state and stack readings is related to the classical results on regular and context-free word problems of groups [3] [4].

The carrier $\mathcal{W}$ is a monoid, not a group, and several readings studied below do not factor through $\pi_1\left(Q_3, 000\right)$: the transition counts τ (Remark 6.4), the quaternion recognizer c (Proposition 7.2), and the occupation vector. Hence these readings are not determined by the reduced loop word. We compare them on the same monoid of walks with additive readings, k-state readings, and free reduction by a stack. The separation results show which information is lost at each level: equal transition counts need not determine the reduced loop word, and no fixed k recovers it on all walks, while an unbounded stack does.

We construct two closed walks with the same directed transition counts. Every additive one-step reading takes the same value on these walks, while their reduced loop words are different. So, directed transition counts, and more generally any commutative aggregation, do not determine the reduced loop word (Section 3).

We next compare finite-state readings with unbounded memory. Let $B = \left(000; 1212\right)$. For every $k \geq 1$ we construct two closed walks

$$B^{k!},\ B^{2\cdot k!},$$

such that every k-state reading F (every sequential reading with at most k states) satisfies

$$F\left(B^{k!}\right)=F\left(B^{2\cdot k!}\right),$$

while

$$\rho\left(B^{k!}\right)\neq\rho\left(B^{2\cdot k!}\right).$$

This pair is already separated by the abelian reading a. What fails is any fixed finite-state bound (Section 5). An unbounded stack, which stores the current reduced word and cancels adjacent inverse pairs, performs free reduction and recovers $\rho(w)$ for every $w\in\mathcal{W}$.

The induced congruences satisfy

$$\Delta_{\mathcal{W}}\subsetneq\sim_{\rho}\subsetneq\sim_{a},$$

where $\Delta_{\mathcal{W}}$ is equality on $\mathcal{W}$. Thus ρ is not injective, but its kernel is strictly finer than that of a. In particular, ρ defines a proper noncommutative recognition quotient of $\mathcal{W}$ (Section 6).

We denote by $|w|$ the number of steps of a walk w. The shortest based closed walk with trivial abelianization but nontrivial degree-two commutator information has length 14 (Section 4). More precisely,

$$\min\left\{|w|:\rho(w)\in\gamma_2(F_5)\setminus\gamma_3(F_5)\right\}=14.$$

Here $\gamma_1(F_5)=F_5$ and $\gamma_{n+1}(F_5)=\left[\gamma_n(F_5),F_5\right]$, so $\gamma_2(F_5)=\left[F_5,F_5\right]$. Thus $\rho(w)\in\gamma_2(F_5)$ means that its abelianization is trivial, while $\rho(w)\notin\gamma_3(F_5)$ means that its class in $\gamma_2(F_5)\setminus\gamma_3(F_5)$ is nontrivial. The value 14 is an exact minimum at degree two in the edge metric of Q_3, and is different from the word-metric and asymptotic questions considered in [5]-[7] (see Remark 4.6).

A declared quarter-turn transport, defined by assigning a quaternion factor to each of the three coordinate directions, gives a second proper order-sensitive congruence (see Theorem 7.1). This congruence is incomparable with $\sim_{\rho}$ (see Proposition 7.2).

2. The Three-Dimensional Cube

We now fix the vertices of the three-dimensional cube Q_3, the spanning tree, and the five generators associated with the cotree edges. Let

$$V=\{0,1\}^3$$

be the vertex set of Q_3. We number the coordinates by $0,1,2$. A step in direction i changes the i-th coordinate from 0 to 1, or from 1 to 0. We fix the basepoint $o=000$ and write a walk as

$$w=\left(o;a_1a_2\cdots a_n\right),\ a_k\in\{0,1,2\}.$$

Let $v_0=o,v_1,\cdots,v_n$ be the associated vertex sequence, where v_k is obtained from v_{k-1} by changing coordinate a_k. The walk w is closed when $v_n=o$. We

write

$$e_k = (v_{k-1}, v_k)$$

for the k-th directed edge. Thus, the same walk can also be written $w = e_1 \cdots e_n$ when its directed edges are used.

Lemma 2.1. *A walk* $w = (o; a_1 \cdots a_n)$ *is closed if and only if each direction* $0,1,2$ *occurs an even number of times among* $a_1, \cdots, a_n$.

Proof. Each step in direction i changes only the i-th coordinate. Hence the i-th coordinate returns to its initial value if and only if direction i is used an even number of times. Therefore $v_n = o$ if and only if each of the three directions occurs an even number of times.

Two walks are equal when their step sequences coincide. Concatenation of based closed walks is based and closed, and the empty walk $1 = (o;)$ is a neutral element. Thus $\mathcal{W}$ is a monoid. By Lemma 2.1, it is the submonoid of the free monoid on $\{0,1,2\}$ consisting of the words in which each letter occurs an even number of times.

We fix the Gray-code spanning tree (**Figure 1**)

$$000,\ 001,\ 011,\ 010,\ 110,\ 111,\ 101,\ 100,$$

where successive vertices differ in one coordinate. The remaining five edges are the cotree edges

$$(0,2),\ (0,4),\ (1,5),\ (3,7),\ (4,6),$$

where a vertex is written $b_2 b_1 b_0$, with b_i the i-th coordinate, and is read as the integer $4b_2 + 2b_1 + b_0$. Thus $000, \cdots, 111$ become $0, \cdots, 7$, and a step in direction i changes this integer by $\pm 2^i$. We orient each cotree edge from the smaller integer to the larger and denote them by $x_1, \cdots, x_5$.

For a based closed walk w, a forward crossing of the i-th cotree edge contributes x_i, a backward crossing contributes x_i^{-1}, and a tree edge contributes no letter. This gives a word $\ell(w)$. After free reduction, we obtain the homomorphism

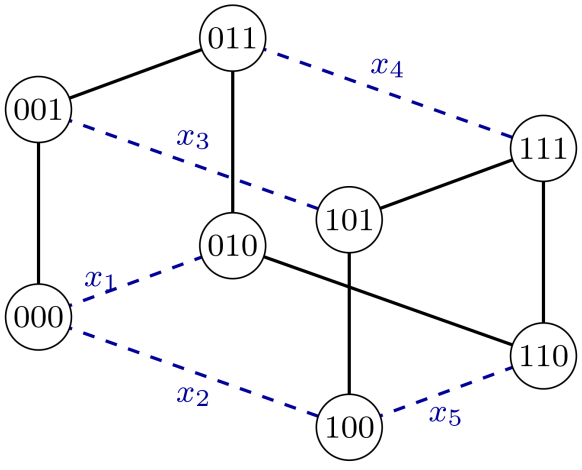


Figure 1. The 1-skeleton of Q_3. Solid edges form the Gray-code spanning tree. Dashed edges are the five cotree edges, labeled by $x_1, \cdots, x_5$.

$$\rho : \mathcal{W} \to F_5.$$

Since

$$\ell(uv) = \ell(u)\ell(v),$$

we have

$$\rho(uv) = \rho(u)\rho(v), \quad \rho(1) = 1.$$

Hence ρ is a monoid homomorphism. This is the spanning tree presentation of the fundamental group of a connected graph (see e.g. [8]-[10]). In the sense of Definition 1.1, free reduction is a sequential reading whose state is the reduced word.

One transcription fixes the convention. The square walk $(000;0101)$ passes through the vertices $0,1,3,2,0$. Its first three steps are tree edges and contribute nothing. The last step crosses the cotree edge $(0,2)$ from the larger integer to the smaller, so it contributes x_1^{-1}. Hence its reduced loop word is x_1^{-1}.

Proposition 2.2. *The cycle ranks of* Q_0, Q_1, Q_2, Q_3 *are*

$$0,\ 0,\ 1,\ 5.$$

Hence

$$\pi_1(Q_2) \cong \mathbb{Z}, \quad \pi_1(Q_3) \cong F_5.$$

Thus Q_3 is the first hypercube whose reduced loop words can be noncommutative.

Proof. For a connected graph, the cycle rank is

$$|E| - |V| + 1.$$

The graph Q_0 has one vertex and no edges. For $d \geq 1$, the d-cube has 2^d vertices and $d2^{d-1}$ edges. Hence its cycle rank is

$$d2^{d-1} - 2^d + 1,$$

which gives the corresponding values.

The fundamental group of a connected graph is free of rank equal to its cycle rank [8]-[10]. Thus

$$\pi_1(Q_2) \cong F_1 \cong \mathbb{Z}, \quad \pi_1(Q_3) \cong F_5.$$

Since F_1 is abelian and F_5 is nonabelian, the last statement follows. □

The Classical 7 + 5 Decomposition

We choose an orientation of the twelve edges and let $C_1(Q_3;\mathbb{R})$ denote the real edge space. Using the standard Euclidean identification of $C^1(Q_3;\mathbb{R})$ with $C_1(Q_3;\mathbb{R})$, let

$$d : C^0(Q_3;\mathbb{R}) \to C^1(Q_3;\mathbb{R})$$

be the coboundary map, so that for a vertex potential ϕ,

$$(d\phi)(u,v) = \phi(v) - \phi(u)$$

on an oriented edge $u \to v$. Since Q_3 is connected, $\ker d$ consists of the constant potentials. Hence

$$\dim(\operatorname{im} d) = 8 - 1 = 7.$$

Let

$$\partial : C_1(Q_3; \mathbb{R}) \to C_0(Q_3; \mathbb{R})$$

be the boundary map. Its kernel is the cycle space. Hence

$$\dim(\ker \partial) = 12 - 8 + 1 = 5.$$

For $\phi \in C^0(Q_3; \mathbb{R})$ and $z \in \ker \partial$, we have

$$\langle d\phi, z \rangle = \langle \phi, \partial z \rangle = 0.$$

Thus $\operatorname{im} d$ and $\ker \partial$ are orthogonal, and their dimensions add to twelve. Accordingly, the edge space admits the orthogonal decomposition

$$C_1(Q_3; \mathbb{R}) = \operatorname{im} d \oplus \ker \partial, \ 12 = 7 + 5.$$

The map $a = \mathrm{ab} \circ \rho$ given by (1) records the five cycle coordinates, *i.e.* the exponent sums of the five generators. More precisely, let C_i be the fundamental cycle associated with the i-th cotree edge, oriented so that this cotree edge has coefficient +1. Then $C_1, \cdots, C_5$ form a basis of the integral cycle space. Let z be an integral cycle and let b_i be its coefficient on the i-th cotree edge. Then

$$z - \sum_{i=1}^{5} b_i C_i$$

is a cycle supported only on the spanning tree. A tree has no nonzero cycles, so this difference is zero. Thus $C_1, \cdots, C_5$ span the integral cycle space. They are independent, since C_i contains the i-th cotree edge with coefficient +1, while the other C_j have coefficient zero on this edge.

For every based closed walk w, its oriented 1-chain satisfies

$$z(w) = \sum_{i=1}^{5} a_i(w) C_i.$$

In particular, $a(w) = 0$ if and only if $z(w) = 0$.

3. Additive Readings

Let $\mathcal{A} = \mathbb{Z}^V$ be the group of integer functions on the vertex set V. For $\phi \in \mathcal{A}$ and a walk $w = (v_0, \cdots, v_n)$, define

$$R_\phi(w) = \sum_{k=1}^{n} \big(\phi(v_k) - \phi(v_{k-1})\big). \tag{2}$$

This is the pairing of the oriented 1-chain defined by the walk with the exact cochain $d\phi$.

Lemma 3.1. *For every $\phi \in \mathcal{A}$ and every walk $w = (v_0, \cdots, v_n)$, it holds*

$$R_\phi(w) = \phi(v_n) - \phi(v_0).$$

In particular, $R_\phi(w) = 0$ for every closed walk.

Proof. If we expand the sum, all intermediate terms cancel, so

$$\sum_{k=1}^{n}\left(\phi\left(v_k\right)-\phi\left(v_{k-1}\right)\right)=\phi\left(v_n\right)-\phi\left(v_0\right).$$

□

Thus an exact potential reading depends only on the endpoints of the walk. On $\mathcal{W}$, it is therefore the trivial recognizer, *i.e.* the zero homomorphism.

The potential reading (2) is a special additive one-step reading, with the weight of a directed edge $p\to q$ given by

$$g\left(p,q\right)=\phi\left(q\right)-\phi\left(p\right).$$

We now allow arbitrary weights on directed steps. Let G be an abelian group and let

$$g:V\times V\to G$$

be a weight function. For a walk $w=\left(v_0,\cdots,v_n\right)$, define

$$R_g\left(w\right)=\sum_{k=1}^{n}g\left(v_{k-1},v_k\right).$$

We call R_g *an additive one-step reading*. Thus $R_g\left(w\right)$ is the accumulated weight along the walk. We call g *antisymmetric* if

$$g\left(q,p\right)=-g\left(p,q\right)$$

for every edge $\left\{p,q\right\}$. Then opposite traversals cancel, so R_g depends only on the oriented 1-chain: $R_g\left(w\right)=\left\langle g,z\left(w\right)\right\rangle$, where g is a 1-cochain.

For $p,q\in V$, define the directed transition count by

$$\tau_w\left(p,q\right)=\#\left\{k:\left(v_{k-1},v_k\right)=\left(p,q\right)\right\}.$$

Proposition 3.2. *Every additive one-step reading factors through the directed transition counts:*

$$R_g\left(w\right)=\sum_{\left(p,q\right)\in V\times V}\tau_w\left(p,q\right)g\left(p,q\right).$$

Consequently, if $\tau_u=\tau_v$ then $R_g\left(u\right)=R_g\left(v\right)$ for every abelian group G and every weight function g.

Proof. Each directed step $\left(p,q\right)$ occurs exactly $\tau_w\left(p,q\right)$ times in the walk. By collecting equal terms, we obtain

$$R_g\left(w\right)=\sum_{\left(p,q\right)\in V\times V}\tau_w\left(p,q\right)g\left(p,q\right).$$

Hence $R_g\left(w\right)$ depends on the directed transition counts and not on their order.

□

Proposition 3.2 applies to one-step readings with values in an abelian group. It does not apply to arbitrary functions of the ordered sequence, nor to noncommutative updates.

Let

$$A=\left(000;0101\right),\ B=\left(000;1212\right).\tag{3}$$

be two square loops, and define

$$w_{AB} = AB,\ \ w_{BA} = BA.$$

Both walks are closed and have length eight.

Example 3.3. The walk A is the square loop in coordinates 0 and 1, and B is the square loop in coordinates 1 and 2, both based at 000. Hence AB and BA traverse the same two loops, but in different order (Figure 2). The corresponding vertex sequences are

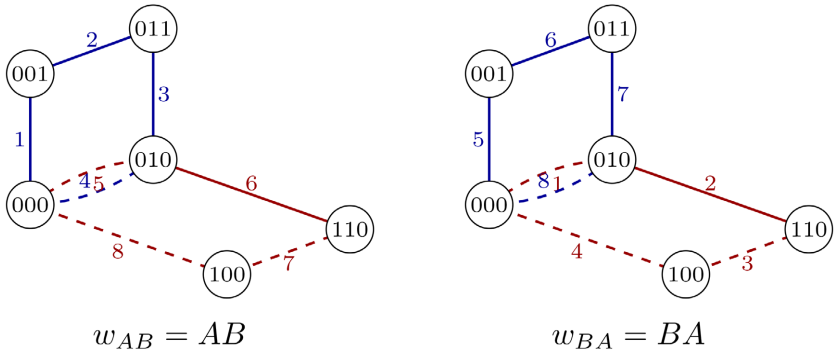


Figure 2. The square faces A and B concatenated in both orders, with the step order marked and cotree edges dashed. The two walks traverse the same directed edges, so $\tau_{w_{AB}} = \tau_{w_{BA}}$, while their reduced loop words differ.

$$0,1,3,2,0,2,6,4,0 \ \text{ and } \ 0,2,6,4,0,1,3,2,0.$$

Theorem 3.4. *The walks w_{AB} and w_{BA} have the same directed transition counts but different reduced loop words:*

$$\tau_{w_{AB}} = \tau_{w_{BA}},\ \ \rho(w_{AB}) \neq \rho(w_{BA}).$$

Proof. Transition counts are additive under concatenation, so we have

$$\tau_{AB} = \tau_A + \tau_B = \tau_B + \tau_A = \tau_{BA}.$$

We have

$$\rho(A) = x_1^{-1},\ \ \rho(B) = x_1 x_5^{-1} x_2^{-1}.$$

Therefore, we obtain

$$\rho(AB) = x_5^{-1} x_2^{-1},\ \ \rho(BA) = x_1 x_5^{-1} x_2^{-1} x_1^{-1}.$$

These are distinct reduced words in F_5. □

Thus no reading determined by the directed transition counts can recover the reduced loop word. Since a walk $w = e_1 \cdots e_n$ records the order of its directed edges, an arbitrary function of w may depend on this order. The claim is only that commutative aggregation does not.

4. The Fourteen-Step Witness

The pair w_{AB}, w_{BA} shows that additive readings do not detect order, but each walk has nonzero abelian loop content. We now give a stronger witness. Its five

signed cycle counts are all zero, but it has nontrivial noncommutative information.

Let w be a walk of length n. Write

$$a_i(w) \in \mathbb{Z}, \ 1 \le i \le 5,$$

for the signed traversal count of the i-th cotree edge. Let $s_i^{(k)} \in \{-1,0,1\}$ be the contribution of step k to this count, and set

$$a_i^{(0)} = 0, \ a_i^{(k)} = \sum_{\ell=1}^{k} s_i^{(\ell)}.$$

Thus $a_i^{(k-1)}$ is the signed count before step k, and $a_i^{(n)} = a_i(w)$. We define the antisymmetric degree two coordinate

$$\mu_{ij}(w) = \sum_{k=1}^{n} \left(a_i^{(k-1)} s_j^{(k)} - a_j^{(k-1)} s_i^{(k)} \right).$$

We write

$$\mu(w) = \left(\mu_{ij}(w) \right)_{1 \le i < j \le 5} \in \mathbb{Z}^{10}.$$

Thus μ_{ij} records the order interaction between the i-th and j-th cotree crossings: each j-crossing is weighted by the current i-count, and each i-crossing by the current j-count with the opposite sign.

We use the usual commutator convention $[x,y] = xyx^{-1}y^{-1}$.

Lemma 4.1. *The coordinates μ_{ij} are invariant under free reduction, so $\mu(w)$ depends only on the reduced loop word $\rho(w)$. If $a(w) = 0$, then each $\mu_{ij}(w)$ is even and, under the isomorphism*

$$\gamma_2(F_5)/\gamma_3(F_5) \cong \Lambda^2 \mathbb{Z}^5,$$

where $[x_i, x_j]$ corresponds to $x_i \wedge x_j$, the class of $\rho(w)$ has coefficient $\mu_{ij}(w)/2$ on $x_i \wedge x_j$ for $i < j$. In particular, if $a(w) = 0$, then

$$\rho(w) \in \gamma_3(F_5) \Leftrightarrow \mu(w) = 0.$$

Proof. For invariance, insert an adjacent pair $x_m x_m^{-1}$ into the cotree letter sequence. Fix $i < j$. If $m \notin \{i, j\}$, the inserted letters give no contribution to μ_{ij}. If $m = i$, the two contributions are

$$-a_j^{(k-1)} \ \text{and} \ +a_j^{(k-1)},$$

while if $m = j$, they are

$$+a_i^{(k-1)} \ \text{and} \ -a_i^{(k-1)}.$$

In both cases they cancel. After the inserted pair, all running counts have their previous values, so no later term is affected. The case $x_m^{-1} x_m$ is identical. Therefore μ is invariant under free reduction, and hence depends only on $\rho(w)$.

For the parity, we write

$$e_{ij} = \sum_{k=1}^{n} a_i^{(k-1)} s_j^{(k)}.$$

At step k, we have

$$a_i^{(k)} = a_i^{(k-1)} + s_i^{(k)}, \quad a_j^{(k)} = a_j^{(k-1)} + s_j^{(k)}.$$

Hence

$$a_i^{(k)} a_j^{(k)} - a_i^{(k-1)} a_j^{(k-1)} = a_i^{(k-1)} s_j^{(k)} + a_j^{(k-1)} s_i^{(k)} + s_i^{(k)} s_j^{(k)}.$$

One step crosses at most one cotree edge, so

$$s_i^{(k)} s_j^{(k)} = 0 \quad (i \neq j).$$

Summing over k, and using $a_i^{(0)} = a_j^{(0)} = 0$, we obtain

$$e_{ij} + e_{ji} = a_i(w) a_j(w).$$

From the definition, we also have

$$\mu_{ij} = e_{ij} - e_{ji}.$$

If $a(w) = 0$, then $e_{ji} = -e_{ij}$, and therefore

$$\mu_{ij} = 2e_{ij}.$$

In particular, each $\mu_{ij}(w)$ is even.

If $a(w) = 0$, then $\rho(w) \in \gamma_2(F_5)$. By the degree-two case of the Magnus expansion (see e.g. [11]-[14]), the coefficient of $x_i \wedge x_j$ in the class of $\rho(w)$ is $e_{ij} = \mu_{ij}(w)/2$. Hence this class is zero if and only if $\mu(w) = 0$. Therefore,

$$\rho(w) \in \gamma_3(F_5) \Leftrightarrow \mu(w) = 0.$$

□

We measure length in the Q_3 edge metric, before spanning-tree reduction. This is different from the word metric used in the results on the lower central series [5]-[7] (see Remark 4.6).

When $a(w) = 0$, geometrically $\mu_{ij}/2$ is a signed area coordinate. For smooth paths the same term appears in the path signature [15] [16]. For a discrete sequence of increments it is the degree-two coordinate of the iterated sums signature [17]. Let us consider the walk

$$W = (000; 01012020101202). \tag{4}$$

Its vertex sequence is

$$0,1,3,2,0,4,5,1,0,2,3,1,5,4,0.$$

Theorem 4.2. *The walk W defined by (4) is closed and satisfies $a(W) = 0$, while $\mu_{12}(W) = -2$. Its reduced loop word is*

$$\rho(W) = x_1^{-1} x_2 x_3^{-1} x_1 x_3 x_2^{-1}.$$

In particular, $\rho(W) \neq 1$ in F_5.

Proof. If we run the displayed fourteen steps (**Figure 3**), the walk returns to 0. Its cotree letter sequence is

$$x_1^{-1},\ x_2,\ x_3^{-1},\ x_1,\ x_3,\ x_2^{-1}.$$

We have that each exponent sum is zero, and the displayed word has no adjacent inverse pair. Hence it is freely reduced, and therefore

step	1	2	3	4	5	6	7	8	9	10	11	12	13	14
direction	0	1	0	1	2	0	2	0	1	0	1	2	0	2
vertex	1	3	2	0	4	5	1	0	2	3	1	5	4	0
cotree letter				x_1^{-1}	x_2		x_3^{-1}		x_1			x_3		x_2^{-1}
term in μ_{12}				0	-1		0		-1			0		0

Figure 3. The fourteen-step walk W of Theorem 4.2: for each step, the direction used, the vertex reached in decimal code, the cotree letter contributed, and its term in μ_{12}. The six letters spell $x_1^{-1}x_2x_3^{-1}x_1x_3x_2^{-1}$, all exponent sums vanish, so $a(W)=0$ while $\rho(W)\neq 1$ and $\mu_{12}(W)=-2$.

$$\rho(W)=x_1^{-1}x_2x_3^{-1}x_1x_3x_2^{-1}.$$

For the pair $(1,2)$, the six cotree letters contribute, in order,

$$0,-1,0,-1,0,0,$$

so we obtain $\mu_{12}(W)=-2$. □

Since W is closed and $a(W)=0$, Section 2 gives $z(W)=0$. Hence $R_g(W)=0$ for every antisymmetric weight g. Thus every antisymmetric additive reading, in particular every potential reading, takes the value 0 on W, as on the empty walk. Additive readings that are not antisymmetric can separate W from the empty walk: for $G=\mathbb{Z}$, the constant weight $g\equiv 1$ gives $R_g(W)=14$.

Lemma 4.3. *For every $n\geq 1$, the graph Q_n contains no subgraph isomorphic to $K_{2,3}$.*

Proof. Let p,q be distinct vertices of Q_n, and let r be a common neighbour of p and q. Then r differs from each of them in one coordinate, so p and q differ in exactly two coordinates, let us say i and j. In that case, r is obtained from p by changing coordinate i, or by changing coordinate j. Hence any two distinct vertices of Q_n have at most two common neighbours.

In $K_{2,3}$ the two vertices of the part of size two have three common neighbours. Therefore $K_{2,3}$ is not a subgraph of Q_n. □

Theorem 4.4. *Among based closed walks on Q_3, the shortest walk satisfying*

$$a(w)=0 \quad \textit{and} \quad \mu(w)\neq 0$$

has length 14. Equivalently,

$$\min\{|w|:\rho(w)\in\gamma_2(F_5)\setminus\gamma_3(F_5)\}=14.$$

The walk W realizes this minimum.

Proof. The walk W of Theorem 4.2 has length 14 and satisfies

$$a(W)=0,\ \mu(W)\neq 0.$$

Hence the minimum is at most 14.

We prove the lower bound. Let w be a based closed walk such that

$$a(w)=0,\ \mu(w)\neq 0.$$

Let H be the subgraph of Q_3 formed by all edges which occur in w. Since w is a walk, H is connected. Also, since $a(w)=0$, we have $z(w)=0$.

We first show that the cycle rank of H is at least two. Suppose first that the cycle rank is zero. Then H is a tree. Every closed walk in a tree is null-homotopic, so we have $\rho(w)=1$. Since $\mu(w)$ depends only on $\rho(w)$, it follows that $\mu(w)=0$, which is a contradiction.

Suppose now that the cycle rank of H is one. Then

$$\pi_1(H)\cong\mathbb{Z},\ H_1(H;\mathbb{Z})\cong\mathbb{Z},$$

and the abelianization map

$$\pi_1(H)\to H_1(H;\mathbb{Z})$$

is an isomorphism. The chain $z(w)$ is supported on H, and we have $z(w)=0$. Hence the class of w is zero in $H_1(H;\mathbb{Z})$. We obtain that w represents the trivial element of $\pi_1(H)$, and therefore also the trivial element of $\pi_1(Q_3,o)$. Thus $\rho(w)=1$, which again gives $\mu(w)=0$, a contradiction. Therefore the cycle rank of H is at least two.

Put

$$v=|V(H)|,\ e=|E(H)|.$$

Since H is connected, its cycle rank is $e-v+1$. Hence

$$e-v+1\geq 2,$$

and we obtain $e\geq v+1$.

We show that $e\geq 7$. Suppose that $e\leq 6$. Then $v\leq 5$. The graph H is bipartite, since it is a subgraph of Q_3. If $v\leq 4$, then

$$e\leq\left\lfloor\frac{v^2}{4}\right\rfloor<v+1,$$

which contradicts $e\geq v+1$. Hence, it follows that

$$v=5,\ e=6.$$

A bipartite graph on five vertices with six edges must have bipartition sizes 2 and 3, and all six edges between the two parts are present. Hence it is $K_{2,3}$. This is excluded by Lemma 4.3. Hence

$$e\geq 7.$$

Finally, $z(w)=0$ means that every edge of H is traversed the same number of times in both directions. Since every edge of H occurs in w, every edge is traversed at least once in each direction. Therefore

$$|w|\geq 2e\geq 14.$$

Together with the walk W, we obtain that the minimum is 14.

For the equivalent formulation, we have

$$a(w)=0\Leftrightarrow\rho(w)\in\gamma_2(F_5).$$

Under this condition, Lemma 4.1 gives

$$\mu(w)\neq 0\Leftrightarrow\rho(w)\notin\gamma_3(F_5).$$

Finally, we obtain

$$\min\left\{|w| : \rho(w) \in \gamma_2(F_5) \setminus \gamma_3(F_5)\right\} = 14.$$

□

Remark 4.5. *The value* 14 *does not depend on the chosen spanning tree. For two spanning trees* T, T', *the corresponding free bases of* $\pi_1(Q_3, o)$ *differ by an automorphism* α *of* F_5. *Hence*

$$\rho_{T'} = \alpha \circ \rho_T.$$

Since the terms of the lower central series are characteristic, we have

$$\rho_T(w) \in \gamma_2(F_5) \setminus \gamma_3(F_5) \Leftrightarrow \rho_{T'}(w) \in \gamma_2(F_5) \setminus \gamma_3(F_5).$$

Also, the condition $a(w) = 0$ is tree-independent, since it is equivalent to $z(w) = 0$. Under this condition, Lemma 4.1 gives that $\mu(w) = 0$ is also tree-independent. The individual coordinates $\mu_{ij}(w)$ may depend on the tree. In particular,

$$\mu_{12}(W) = -2$$

refers to the tree fixed in Section 2.

Remark 4.6. For $g \in \pi_1(Q_3, 000)$, let $\ell_E(g)$ be the minimum length of a based closed walk representing g. It is the length of the unique closed walk without backtracks which represents g, so it does not depend on the spanning tree. After fixing the spanning tree, we identify $\pi_1(Q_3, 000)$ with F_5 and write $|g|$ for the reduced word length in the basis $x_1, \cdots, x_5$. The results in [5]-[7] concern word length, or related complexity, for elements deeper in the lower central series as the level varies. Here the problem is different: we determine the exact minimum at the fixed level $\gamma_2(F_5) \setminus \gamma_3(F_5)$, using the edge metric of Q_3. In the word metric this minimum is 4. Indeed, a nontrivial element of $\gamma_2(F_5)$ has exponent sum zero in every generator, so it contains at least two generators, each at least twice, and $[x_i, x_j] \in \gamma_2(F_5) \setminus \gamma_3(F_5)$ has word length 4. In the Q_3 edge metric, Theorem 4.4 gives the minimum 14. For the Gray-code spanning tree,

$$|g| \le \ell_E(g) \le 12|g|,$$

since each step gives at most one letter, and each letter $x_i^{\pm 1}$ is given by a based fundamental loop of length at most 12. Thus the two lengths are comparable up to a constant, but their exact minima are different. The witness W has $|\rho(W)| = 6$, so it is not shortest in the word metric. The lower bound 14 uses properties of the graph Q_3, that it is bipartite and contains no $K_{2,3}$, and not only the group F_5.

5. Memory and Order Resolution

The previous sections show what is lost by commutative summaries. We now consider finite sequential memory and an unbounded stack.

5.1. Two-State Separation from Additive Readings

A k *-state reading* is a sequential reading in the sense of Definition 1.1 whose

state set has at most k elements.

Theorem 5.1. *There is a two-state reading D such that*

$$D(w_{AB}) \neq D(w_{BA}).$$

Proof. The directed edges of A are

$$0 \to 1,\ 1 \to 3,\ 3 \to 2,\ 2 \to 0,$$

and the directed edges of B are

$$0 \to 2,\ 2 \to 6,\ 6 \to 4,\ 4 \to 0.$$

We set $e_A = (0,1)$ and $e_B = (0,2)$. Then e_A occurs in A only, and e_B in B only, each exactly once.

We take

$$S = \{0,1\},\ s_0 = 0,\ q(s) = s,$$

and define

$$T_{e_A}(s) = 0,\ T_{e_B}(s) = 1,\ T_e(s) = s$$

for every other directed edge e.

Thus the final state is determined by which of e_A and e_B occurs last. Therefore, we obtain

$$D(w_{AB}) = 1,\ D(w_{BA}) = 0.$$

□

We next show that no fixed finite-state bound recovers the whole reduced loop word. For B as in (3), let B^m denote its m-fold concatenation, a closed walk of length $4m$.

Theorem 5.2 *Let $k \geq 1$. For every k-state reading F,*

$$F(B^{k!}) = F(B^{2\cdot k!}),$$

while

$$\rho(B^{k!}) \neq \rho(B^{2\cdot k!}).$$

Proof. Let S be the state set of F, with $|S| \leq k$, and let $t : S \to S$ be the state update obtained by reading one copy of B. Starting from the initial state s_0, consider

$$s_0, t(s_0), \cdots, t^k(s_0).$$

Since $|S| \leq k$, there exist $0 \leq i < j \leq k$ such that

$$t^i(s_0) = t^j(s_0).$$

Set $p = j - i$. Then

$$t^{n+p}(s_0) = t^n(s_0) \text{ for every } n \geq i.$$

Since $1 \leq p \leq k$, we have $p \mid k!$, and since $i \leq k \leq k!$, it follows that

$$t^{k!}(s_0) = t^{2\cdot k!}(s_0).$$

Applying the output map gives

$$F\left(B^{k!}\right) = F\left(B^{2\cdot k!}\right).$$

On the other hand, we have

$$\rho(B) = x_1 x_5^{-1} x_2^{-1},$$

which is a nontrivial cyclically reduced word. Hence

$$\rho\left(B^m\right) = \rho(B)^m$$

is reduced for every $m \geq 1$. Therefore

$$\rho\left(B^{k!}\right) \neq \rho\left(B^{2\cdot k!}\right).$$

□

This factorial witness is not purely noncommutative, since

$$a(B) = (1,-1,0,0,-1) \neq 0, \quad a\left(B^m\right) = ma(B),$$

so abelianization already separates the two powers.

5.2. The Stack Recovers the Reduced Word

Theorem 5.3. *There is a sequential reading with an unbounded stack, whose state set is the set of finite reduced words, which recovers* $\rho(w)$ *for every* $w \in \mathcal{W}$.

Proof. We take the state set to be the set of finite reduced words in

$$x_1^{\pm 1}, \cdots, x_5^{\pm 1},$$

with the empty word as the initial state and the last letter as the top of the stack. The output map is the identity.

A tree edge leaves the state unchanged. If a cotree edge contributes a letter l, we delete the last letter when it is l^{-1}, and otherwise we append l.

By induction on the initial segments of w, after each step the stack contains the freely reduced cotree word of the segment already read. Therefore, after the whole walk is read, the stack contains $\rho(w)$. □

Corollary 5.4. *There is a pair of walks which is not distinguished by any additive reading, but is distinguished by a two-state reading. For every* $k \geq 1$, *the walks*

$$B^{k!}, \ B^{2\cdot k!}$$

are not distinguished by any k-state reading, while the stack distinguishes them. Hence no fixed finite-state bound recovers ρ on all walks, whereas the stack recovers $\rho(w)$ for every $w \in \mathcal{W}$.

Proof. The first statement follows from Theorems 3.4, 5.1, and Proposition 3.2, and the second from Theorems 5.2 and 5.3. □

6. A Proper Order-Sensitive Congruence

We compare the kernel congruences induced by the maps ρ and $a = \mathrm{ab} \circ \rho$ of (1), *i.e.*

$$\rho : \mathcal{W} \to F_5 \quad \text{and} \quad a : \mathcal{W} \to \mathbb{Z}^5. \tag{5}$$

Recall that $a(w) = (a_1(w), \cdots, a_5(w))$ records the signed cotree traversal counts.

Theorem 6.1. *We have*

$$\Delta_{\mathcal{W}} \subsetneq \sim_\rho \subsetneq \sim_a .$$

Proof. Since $a = \mathrm{ab} \circ \rho$, we have

$$\rho(u) = \rho(v) \Rightarrow a(u) = a(v),$$

and therefore

$$\sim_\rho \subseteq \sim_a .$$

The inclusion is strict by Theorem 3.4, since

$$\rho(w_{AB}) \neq \rho(w_{BA}),$$

while

$$a(w_{AB}) = a(A) + a(B) = a(B) + a(A) = a(w_{BA}).$$

For the first inclusion, let

$$\beta = (000;00)$$

be the two-step backtrack along a tree edge. Since both steps are tree edges, we have

$$\rho(\beta) = 1 = \rho(1),$$

while $\beta \neq 1$ in $\mathcal{W}$, since their step sequences are different. Hence

$$\Delta_{\mathcal{W}} \subsetneq \sim_\rho .$$

□

Thus $\sim_\rho$ is a proper congruence on $\mathcal{W}$. By Theorem 3.4, it is order-sensitive: it separates w_{AB} from w_{BA}, although these walks have the same directed transition counts.

Corollary 6.2. *The recognition quotients determined by ρ and a given by (5) satisfy*

$$\mathcal{W}/\sim_\rho \cong F_5, \ \mathcal{W}/\sim_a \cong \mathbb{Z}^5.$$

Proof. By the first isomorphism theorem for monoids,

$$\mathcal{W}/\sim_\rho \cong \mathrm{im}\,\rho, \ \mathcal{W}/\sim_a \cong \mathrm{im}\, a.$$

For each cotree edge, the corresponding based fundamental loop maps to x_i, while its reverse maps to x_i^{-1}. Hence $\rho : \mathcal{W} \to F_5$ is surjective. Thus,

$$\mathcal{W}/\sim_\rho \cong F_5.$$

Since $a = \mathrm{ab} \circ \rho$ and both ρ and ab are surjective, $a : \mathcal{W} \to \mathbb{Z}^5$ is also surjective. Therefore

$$\mathcal{W}/\sim_a \cong \mathbb{Z}^5.$$

□

Proposition 6.3. *Let M be a commutative monoid and let $f:\mathcal{W}\to M$ be a homomorphism. Then*

$$f(uv)=f(vu)$$

for all $u,v\in\mathcal{W}$. In particular,

$$f(w_{AB})=f(w_{BA}).$$

Proof. Since M is commutative, we have

$$f(uv)=f(u)f(v)=f(v)f(u)=f(vu).$$

Setting $u=A$ and $v=B$ we obtain $f(w_{AB})=f(w_{BA})$. □

Thus a monoid recognizer separating w_{AB} from w_{BA} cannot take values in a commutative monoid. By Theorem 3.4, ρ is such a recognizer.

Remark 6.4. *The maps ρ and τ are incomparable. By Theorem 3.4, we have*

$$\tau_{w_{AB}}=\tau_{w_{BA}},\ \rho(w_{AB})\neq\rho(w_{BA}),$$

so τ does not determine ρ. Conversely, for $\beta=(000;00)$, we have

$$\rho(\beta)=\rho(1),\ \tau_\beta\neq\tau_1,$$

so ρ does not determine τ.

7. A Declared Quarter-Turn Clock

The stack of Theorem 5.3 is a sequential reading with an infinite state set, and it recovers $\rho(w)$ for every walk. It forgets backtracks, since tree edges give no letter and free reduction cancels inverse pairs. We now consider another order-sensitive recognizer, based on a quaternion product along the coordinate directions. In contrast with free reduction, this product does not cancel a backtrack; for example, $(1+\boldsymbol{i})^2=2\boldsymbol{i}\neq 1$. The resulting recognizer c keeps different order information from ρ, and the two induced congruences are incomparable by Proposition 7.2.

Let

$$\mathbb{H}_{\mathbb{Z}}=\{a+b\boldsymbol{i}+c\boldsymbol{j}+d\boldsymbol{k}:a,b,c,d\in\mathbb{Z}\}$$

be the multiplicative monoid of integer quaternions. The quaternion units satisfy

$$\boldsymbol{i}^2=\boldsymbol{j}^2=\boldsymbol{k}^2=-1,\ \boldsymbol{ij}=\boldsymbol{k},\ \boldsymbol{jk}=\boldsymbol{i},\ \boldsymbol{ki}=\boldsymbol{j}.$$

We assign to the three coordinate directions

$$q_0=1+\boldsymbol{i},\ q_1=1+\boldsymbol{j},\ q_2=1+\boldsymbol{k}.$$

The normalized elements

$$\frac{q_0}{\sqrt{2}},\ \frac{q_1}{\sqrt{2}},\ \frac{q_2}{\sqrt{2}}$$

are unit quaternions representing quarter-turns about the three coordinate axes.

For a walk

$$w = (o; a_1 \cdots a_n),$$

define

$$h : \mathcal{W} \to \mathbb{H}_{\mathbb{Z}},\ h(w) = q_{a_1} \cdots q_{a_n}.$$

The map h uses the integer elements q_a, not unit quaternions.

Let

$$\mathcal{T} = \{\tau_w : w \in \mathcal{W}\} \subseteq \mathbb{N}^{V \times V},$$

with pointwise addition. Since $\tau_{uv} = \tau_u + \tau_v$ and $\tau_1 = 0$, the set $\mathcal{T}$ is a submonoid of $\mathbb{N}^{V \times V}$.

Theorem 7.1. *The map*

$$c : \mathcal{W} \to \mathcal{T} \times \mathbb{H}_{\mathbb{Z}},\ c(w) = (\tau_w, h(w)),$$

is a monoid homomorphism and

$$\Delta_{\mathcal{W}} \subsetneq \sim_c \subsetneq \sim_\tau.$$

Proof. For $u, v \in \mathcal{W}$, we have

$$h(uv) = h(u)h(v),\ \tau_{uv} = \tau_u + \tau_v.$$

Hence c is a monoid homomorphism. Since the first component of c is τ, we have

$$\sim_c \subseteq \sim_\tau.$$

To show that this inclusion is strict, let us use

$$u = (000; 001122),\ v = (000; 002211).$$

The two walks have the same directed transition counts, so $\tau_u = \tau_v$. We have

$$h(u) = (1+\boldsymbol{i})^2 (1+\boldsymbol{j})^2 (1+\boldsymbol{k})^2 = 8\boldsymbol{ijk} = -8,$$
$$h(v) = (1+\boldsymbol{i})^2 (1+\boldsymbol{k})^2 (1+\boldsymbol{j})^2 = 8\boldsymbol{ikj} = 8.$$

Thus $c(u) \neq c(v)$, and therefore

$$\sim_c \subsetneq \sim_\tau.$$

For the first inclusion, take

$$r = (000; 001001),\ r' = (000; 100100).$$

These are distinct based closed walks. Their directed edge sequences are cyclic rotations, hence

$$\tau_r = \tau_{r'}.$$

Also,

$$h(r) = \left((1+\boldsymbol{i})^2 (1+\boldsymbol{j})\right)^2 = \left(2\boldsymbol{i}(1+\boldsymbol{j})\right)^2 = (2\boldsymbol{i} + 2\boldsymbol{k})^2 = -8,$$
$$h(r') = \left((1+\boldsymbol{j})(1+\boldsymbol{i})^2\right)^2 = \left((1+\boldsymbol{j})2\boldsymbol{i}\right)^2 = (2\boldsymbol{i} - 2\boldsymbol{k})^2 = -8.$$

Hence $c(r) = c(r')$, while $r \neq r'$. Therefore, we get $\Delta_{\mathcal{W}} \subsetneq \sim_c$. □

Thus $\sim_c$ is a proper congruence on $\mathcal{W}$. By Theorem 7.1, it is order-sensitive:

it separates u from v, although these walks have the same directed transition counts.

Proposition 7.2. *The congruences* $\sim_c$ *and* $\sim_\rho$ *are incomparable.*

Proof. Let u, v be the walks from Theorem 7.1. Both are concatenations of three two-step backtracks from the origin. Hence

$$\rho(u) = \rho(v) = 1,$$

while $c(u) \neq c(v)$. Therefore, we have $\sim_\rho \subsetneq \sim_c$.

For the converse, we use the walks A and B from (3). By Theorem 3.4, we have

$$\tau_{AB} = \tau_{BA}, \quad \rho(AB) \neq \rho(BA).$$

Also,

$$(1+\boldsymbol{i})(1+\boldsymbol{j}) = (1+\boldsymbol{j})(1+\boldsymbol{k}) = 1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}.$$

Hence

$$h(A) = \big((1+\boldsymbol{i})(1+\boldsymbol{j})\big)^2 = \big((1+\boldsymbol{j})(1+\boldsymbol{k})\big)^2 = h(B).$$

Therefore

$$h(AB) = h(A)h(B) = h(B)h(A) = h(BA).$$

Together with $\tau_{AB} = \tau_{BA}$, we obtain

$$c(AB) = c(BA).$$

Since

$$\rho(AB) \neq \rho(BA),$$

it follows that $\sim_c \subsetneq \sim_\rho$. □

The map h is not determined by the graph Q_3. It depends on the three coordinate directions and on the choice of the quaternion units $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ for these directions. A different choice can give a different homomorphism. Thus h is an additional structure on $\mathcal{W}$.

8. Rectangular Constraints on Histories

In the previous sections, we studied readings of a history $w \in \mathcal{W}$. We now consider constraints on pairs $(\alpha, w) \in \mathcal{A} \times \mathcal{W}$. We determine when such constraints are rectangular and when they are not.

Let $\mathcal{U} = \mathcal{A} \times \mathcal{W}$ be the product carrier. A constraint $P \subseteq \mathcal{U}$ is called *rectangular* if there exist subsets $P_{\mathcal{A}} \subseteq \mathcal{A}$ and $P_{\mathcal{W}} \subseteq \mathcal{W}$ such that

$$(\alpha, w) \in P \Leftrightarrow \alpha \in P_{\mathcal{A}} \text{ and } w \in P_{\mathcal{W}}.$$

Equivalently,

$$P = P_{\mathcal{A}} \times P_{\mathcal{W}}.$$

We first work on the full group $\mathcal{A} = \mathbb{Z}^V$. Write

$$\mathcal{A}_0 = \left\{\alpha \in \mathcal{A} : \sum_{p \in V} \alpha(p) = 0\right\}$$

for the kernel of the sum map

$$\alpha \mapsto \sum_{p \in V} \alpha(p).$$

Since every $w \in \mathcal{W}$ is closed, Lemma 3.1 gives $R_\phi(w) = 0$ for every potential reading. Hence every constraint depending only on potential readings is constant on $\mathcal{U}$, and consequently rectangular.

For $w = (v_0, \cdots, v_n) \in \mathcal{W}$, define $\mathrm{occ} : \mathcal{W} \to \mathcal{A}$ by

$$\mathrm{occ}(w)(p) = \#\{k : 0 \le k < n, v_k = p\}.$$

Thus the closing vertex $v_n = v_0$ is not counted. The occupation vector is an additive one-step reading with values in $\mathcal{A}$. Indeed, if $\delta_p \in \mathcal{A}$ is the indicator of p, then

$$\mathrm{occ}(w) = \sum_{(p,q) \in V \times V} \tau_w(p,q)\delta_p.$$

Thus occ factors through τ by Proposition 3.2. In particular, $\mathrm{occ}(w_{AB}) = \mathrm{occ}(w_{BA})$. This section returns to the commutative readings of Section 3: we ask which constraints on $\mathcal{A} \times \mathcal{W}$ can be written using potential readings and the occupation vector.

Define $\mathcal{O} \subseteq \mathcal{U}$ by

$$(\alpha, w) \in \mathcal{O} \Leftrightarrow \alpha = \mathrm{occ}(w).$$

We call $\mathcal{O}$ the *occupation-binding constraint*.

Proposition 8.1. *The occupation-binding constraint $\mathcal{O}$ is not rectangular on $\mathcal{U}$.*

Proof. Choose $u, v \in \mathcal{W}$ with

$$\mathrm{occ}(u) \neq \mathrm{occ}(v),$$

for example the empty walk $u = 1$ and the square $v = (o; 0101)$. Then

$$(\mathrm{occ}(u), u) \in \mathcal{O}, \ (\mathrm{occ}(v), v) \in \mathcal{O},$$

while

$$(\mathrm{occ}(u), v) \notin \mathcal{O}.$$

Suppose that $\mathcal{O}$ is rectangular. Then

$$\mathcal{O} = P_{\mathcal{A}} \times P_{\mathcal{W}}$$

for some $P_{\mathcal{A}} \subseteq \mathcal{A}$ and $P_{\mathcal{W}} \subseteq \mathcal{W}$. From $(\mathrm{occ}(u), u) \in \mathcal{O}$ we obtain $\mathrm{occ}(u) \in P_{\mathcal{A}}$, and from $(\mathrm{occ}(v), v) \in \mathcal{O}$ we obtain $v \in P_{\mathcal{W}}$. Hence

$$(\mathrm{occ}(u), v) \in \mathcal{O},$$

which is a contradiction. □

Proposition 8.2. *For $(\alpha, w) \in \mathcal{A}_0 \times \mathcal{W}$,*

$$(\alpha, w) \in \mathcal{O} \Leftrightarrow \alpha = 0 \ \text{ and } \ w = 1.$$

Proof. For every $w \in \mathcal{W}$, it holds

$$\sum_{p\in V}\mathrm{occ}(w)(p)=|w|.$$

If $(\alpha,w)\in\mathcal{O}$ and $\alpha\in\mathcal{A}_0$, then $\alpha=\mathrm{occ}(w)$, so

$$|w|=\sum_{p\in V}\mathrm{occ}(w)(p)=\sum_{p\in V}\alpha(p)=0.$$

Hence $w=1$ and $\alpha=0$. The converse is immediate. □

Remark 8.3. *Thus*

$$\mathcal{O}\cap(\mathcal{A}_0\times\mathcal{W})=\{0\}\times\{1\}.$$

Hence $\mathcal{O}$ is rectangular on $\mathcal{A}_0\times\mathcal{W}$; in this case the restriction consists of a single point.

A Criterion for Rectangularity

A constraint $P\subseteq\mathcal{U}$ is called *occupation-based* if

$$(\alpha,w)\in P\Leftrightarrow\Psi(\alpha,\mathrm{occ}(w))$$

for some predicate

$$\Psi:\mathcal{A}\times\mathcal{A}\to\{\mathrm{false},\mathrm{true}\}.$$

Let $\mathcal{S}\subseteq\mathcal{W}$ be an arbitrary subset.

Here a subset $P\subseteq\mathcal{U}$ is called rectangular on $\mathcal{A}\times\mathcal{S}$ if

$$P\cap(\mathcal{A}\times\mathcal{S})=P_{\mathcal{A}}\times P_{\mathcal{S}}$$

for some $P_{\mathcal{A}}\subseteq\mathcal{A}$ and $P_{\mathcal{S}}\subseteq\mathcal{S}$.

Theorem 8.4. *The following statements are equivalent:*

(i) There is $c_0\in\mathcal{A}$ such that $\mathrm{occ}(w)=c_0$ for every $w\in\mathcal{S}$.

(ii) Every occupation-based constraint is rectangular on $\mathcal{A}\times\mathcal{S}$.

Proof. First, we assume (i). If

$$(\alpha,w)\in P\Leftrightarrow\Psi(\alpha,\mathrm{occ}(w)),$$

then on $\mathcal{A}\times\mathcal{S}$, we have

$$(\alpha,w)\in P\Leftrightarrow\Psi(\alpha,c_0).$$

Hence

$$P\cap(\mathcal{A}\times\mathcal{S})=P_{\mathcal{A}}\times\mathcal{S},\quad P_{\mathcal{A}}=\{\alpha\in\mathcal{A}:\Psi(\alpha,c_0)\}.$$

Conversely, suppose (i) is not valid. Then there exist $u,v\in\mathcal{S}$ with $\mathrm{occ}(u)\neq\mathrm{occ}(v)$. The constraint $\mathcal{O}$ is occupation-based, and the cross-pair argument of Proposition 8.1 shows that $\mathcal{O}\cap(\mathcal{A}\times\mathcal{S})$ is not rectangular in $\mathcal{A}\times\mathcal{S}$. □

For instance, let $\mathcal{S}$ be the class of based closed walks $(v_0,\cdots,v_8)$ with $v_0,\cdots,v_7$ pairwise distinct. The Gray cycle shows $\mathcal{S}\neq\varnothing$, and $\mathrm{occ}(w)(p)=1$ for every $w\in\mathcal{S}$ and every $p\in V$, so every occupation-based constraint is rectangular there.

9. Higher-Dimensional Cubes

The 1-skeleton of Q_n has 2^n vertices and $n2^{n-1}$ edges. Hence its cycle rank is

$$r_n = n2^{n-1} - 2^n + 1 = (n-2)2^{n-1} + 1.$$

As in the proof of Proposition 2.2, the fundamental group of a connected graph is free of rank equal to its cycle rank. Hence

$$\pi_1(Q_n) \cong F_{r_n}.$$

For every $n \geq 3$, a coordinate copy of Q_3 is contained in Q_n. We extend the Gray spanning tree of this subcube to a spanning tree T_n of Q_n. Since the Gray tree already spans the eight vertices of the subcube, no other edge of the subcube belongs to T_n. Thus its five cotree edges remain cotree edges for T_n. By ρ_{T_n} we denote the reduced loop word map of Q_n determined by T_n, defined as in Section 2.

For a closed walk in this subcube, the cotree word calculated with T_n is obtained from its Q_3 cotree word by the homomorphism

$$\iota : F_5 \to F_{r_n}$$

which maps the five generators to distinct members of a free basis of F_{r_n}. This homomorphism is injective, since a subset of a free basis freely generates a free subgroup. In particular, w_{AB} and w_{BA} have different reduced loop words in Q_n, while their directed transition counts agree. Therefore, commutative aggregation does not determine the reduced loop word on Q_n for any $n \geq 3$.

There is also a retraction

$$\pi : F_{r_n} \to F_5$$

which maps these five basis elements to the corresponding generators of F_5, and every remaining basis element to 1. Hence $\pi \circ \iota = id_{F_5}$.

The minimum of Theorem 4.4 is also independent of n.

Theorem 9.1. *Let $n \geq 3$. Among based closed walks on Q_n,*

$$\min\left\{|w| : \rho_{T_n}(w) \in \gamma_2(F_{r_n}) \setminus \gamma_3(F_{r_n})\right\} = 14.$$

The walk W of (4), read in the coordinate copy of Q_3 fixed above, realizes this minimum.

Proof. For the upper bound, all steps of W use edges of the subcube, so

$$\rho_{T_n}(W) = \iota(\rho(W)).$$

By Theorem 4.4, we have

$$\rho(W) \in \gamma_2(F_5) \setminus \gamma_3(F_5).$$

Since ι is a homomorphism, we have $\iota(\rho(W)) \in \gamma_2(F_{r_n})$. Suppose that $\iota(\rho(W)) \in \gamma_3(F_{r_n})$. Then applying π gives $\rho(W) \in \gamma_3(F_5)$, which is a contradiction. Hence

$$\rho_{T_n}(W) \in \gamma_2(F_{r_n}) \setminus \gamma_3(F_{r_n}),$$

and the minimum is at most 14.

For the lower bound, let w be a based closed walk on Q_n with

$$\rho_{T_n}(w) \in \gamma_2\left(F_{r_n}\right) \setminus \gamma_3\left(F_{r_n}\right).$$

Since $\rho_{T_n}(w) \in \gamma_2\left(F_{r_n}\right)$, all signed cotree traversal counts of w vanish, so as in Section 2, applied to T_n, we have $z(w)=0$. Let H be the subgraph of Q_n formed by all edges which occur in w. Since w is a walk, H is connected.

The cycle rank of H is at least two. Indeed, for cycle rank zero or one, the two arguments in the proof of Theorem 4.4 use only $z(w)=0$ and the fact that $\pi_1(H)$ is free of that rank. So, we have $\rho_{T_n}(w)=1$, hence $\rho_{T_n}(w) \in \gamma_3\left(F_{r_n}\right)$, which is a contradiction. Putting

$$v=\left|V(H)\right|, \ e=\left|E(H)\right|,$$

we obtain $e \geq v+1$.

The graph H is bipartite, since it is a subgraph of Q_n. If $e \leq 6$, then $v \leq 5$, and the count in the proof of Theorem 4.4 implies $v=5$, $e=6$, so H is $K_{2,3}$. This is excluded by Lemma 4.3. Hence $e \geq 7$.

Finally, $z(w)=0$ means that every edge of H is traversed the same number of times in both directions, and every edge of H occurs in w. Therefore

$$|w| \geq 2e \geq 14.$$

□

For $n=3$ we have $r_3=5$, and T_3 is the Gray spanning tree of Section 2. Thus Theorem 4.4 is the case $n=3$.

As in Remark 4.5, the statement does not depend on the choice of T_n, since the terms of the lower central series are characteristic.

10. Conclusions

The passage from one repeatable generator to several independent loops introduces order information. Every additive one-step reading factors through the directed transition counts. Therefore, commutative aggregation does not recover the reduced loop word. The reduced-word recognizer gives a proper noncommutative recognition quotient

$$\Delta_{\mathcal{W}} \subsetneq \sim_{\rho} \subsetneq \sim_{a}.$$

Thus the free-group reading retains order information which is lost by the abelian reading.

We also determined nontrivial degree-two commutator information in the cube edge metric. We proved that

$$\min\left\{|w| : \rho(w) \in \gamma_2\left(F_5\right) \setminus \gamma_3\left(F_5\right)\right\}=14.$$

Thus the fourteen-step witness is minimal. Its abelianization is trivial, while its degree-two commutator information is nonzero.

We also obtain a separation between finite and unbounded sequential memory.

A two-state reading separates one order pair. For every $k \geq 1$, there are two walks which no k-state reading separates, although their reduced loop words are different. No fixed finite-state bound therefore recovers ρ on all histories, while an unbounded stack recovers $\rho(w)$ for every $w \in \mathcal{W}$.

The declared quarter-turn quaternion clock gives another proper order-sensitive congruence,

$$\Delta_{\mathcal{W}} \subsetneq \sim_c \subsetneq \sim_\tau .$$

The congruences $\sim_c$ and $\sim_\rho$ are incomparable. Thus different noncommutative recognizers can retain different order information. The quaternion map is an additional structure on $\mathcal{W}$ and is not determined by the graph Q_3.

For constraints on $\mathcal{A} \times \mathcal{W}$, potential readings are constant on closed histories, while the occupation-binding constraint is not rectangular on $\mathcal{A} \times \mathcal{W}$. On $\mathcal{A}_0 \times \mathcal{W}$, its restriction is

$$\{0\} \times \{1\}.$$

More generally, for a class $\mathcal{S} \subseteq \mathcal{W}$, every occupation-based constraint is rectangular on $\mathcal{A} \times \mathcal{S}$ if and only if the occupation vector is constant on $\mathcal{S}$. Hence this condition characterizes when occupation-based constraints are rectangular.

Finally, the obstruction to commutative recovery is not specific to Q_3. For every $n \geq 3$, a coordinate copy of Q_3 in Q_n gives closed walks with equal directed transition counts and different reduced loop words. Thus the same obstruction to commutative recovery occurs in every hypercube Q_n, $n \geq 3$. The minimum is also independent of n. On every Q_n with $n \geq 3$, the shortest based closed walk with vanishing abelianization and nontrivial degree-two commutator information has length 14 (Theorem 9.1).

Author Contributions

Conceptualization, J.W.; Methodology, J.W. and M.Z.; Validation, J.W. and M.Z.; Formal Analysis, M.Z. and J.W.; Investigation, J.W. and M.Z.; Resources, J.W.; Writing—Original Draft Preparation, J.W.; Writing—Review and Editing, M.Z. and J.W.; Funding Acquisition, J.W.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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