

The Pseudo Similarity Solutions of Laplace's Equation with Source Type and Doublet Singularities

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Abstract

The solution of boundary-value problem of Laplace's equation for a semi-infinite strip domain with Dirichlet boundary conditions was studied by applying the similarity method. For the B.C. $T(0, y) = f(y)$, where

$$f(y) = T_0 \left(1 - m \frac{y}{H} \right) \text{ with constant } T_0, \text{ positive integer } m, \text{ and } H \text{ in this}$$

study. The global similarity solution with $m = 0$ was found with similarity variable $\eta(x, y)$ fitted perfectly all the boundaries. Besides the global similarity solution, the new type of pseudo similarity solution of Laplace's equation with source type and doublet type singularities in the domain was also studied. For $m = 1$, and $m = 2$, the pseudo similarity solutions were found with similarity variable $\eta(x, y)$ fitted only part of the boundaries. And the source type singularity can be found analytical near some singular point in the domain. For the boundary condition with Dirac delta function $T(0, y) = \delta(y - a)$, $0 < a < H$, the closed form solution was given with doublet singularity near $x \rightarrow 0$, and $y \rightarrow a$. It is worth noting that a pseudo local doublet similarity solution, $T_D(x, y)$ is found which approaches the closed form solution perfectly near $x \rightarrow 0$, and $y \rightarrow a$. And the numerical comparison of these two solutions is also given.

Keywords

Laplace's Equation, Similarity Method, Similarity Solution, Pseudo Similarity Solution, Source Type Singularity, Doublet Type Singularity

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1. Introduction

Laplace's equation is one of the important partial differential equations (PDE) for studying the equilibrium problems in mathematical physics and applied mechanics problems [1]-[4]. Similarity method is one of the important mathematical methods for studying the PDE [5]-[7]. For partial differential equations at least a reduction in the number of independent variables is sought and in favorable case a reduction to ordinary differential equations with similarity variable perfectly fitting all the special boundary conditions [8].

In general, similarity solutions are often limiting or asymptotic solutions to giving problems. In other words, similarity solutions typically appear as local or far field solutions near some singular points [9]-[11]. However, the similarity solutions with special boundary conditions are valid everywhere in the domain that is called the Global Similarity Solutions.

A series of similarity analysis for Laplace's equation in two dimensions has been studied. The infinitesimal transformation group of similarity analysis is applied to Laplace's equation to classify the methods for finding the harmonic functions [12]. Some boundary value problems of Laplace's equation with Neumann and Dirichlet boundary condition are solved by using the Fourier transform method, the method of Green's function with images, and the solutions are in terms of the integral form. However, their solutions can be solved by using the similarity method and the similarity solutions are found analytically with source type singularity near some singular points in the region [13].

The traditional solution of Laplace's equation in a confined domain with discontinuous boundary conditions can be obtained by applying the method of separation of variables. Numerical computation is applied to find the infinite series solution in a very slowly convergent way accompanied by Gibb's phenomenon especially near certain singular part in the domain. However, by using the similarity method, the solution in a closed form and similarity solution shows no oscillation phenomenon near the singular points [14].

However, some similarity transforms were found from PDE, but the similarity variable fits only part of the boundary conditions. Therefore, the similarity transform is not complete, and we can only find the pseudo similarity solution. The pseudo similarity solution means that Laplace's equation in two dimensions can be transformed into a second order ordinary differential equation, but the similarity variable can fit only part of the boundary; in other words, we obtain only one boundary condition. Thus, the unknown boundary condition must be found in other ways.

In this paper, contrast to the "Global Similarity Solutions", the new type of pseudo similarity solutions of Laplace's equation with source type singularities such as the limits at boundary or corner points was studied. The solution of the Dirichlet boundary condition with Dirac delta function was also discussed and its pseudo local similarity solution near the singular point in the domain with doublet singularity was found.

2. Boundary-Value Problem of Laplace's Equation with Dirichlet Boundary Conditions

In the semi-infinite strip domain in **Figure 1**, Laplace's equation with Dirichlet boundary conditions is described as follows:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad 0 < x < \infty, \quad 0 < y < H \quad (1)$$

BCs

$$T(0, y) = f(y) \quad (2)$$

$$T(\infty, y) = 0 \quad (3)$$

$$T(x, 0) = 0, \quad T(x, H) = 0 \quad (4)$$

The Dirichlet boundary data are prescribed on open boundary segments, since the entrance data and the homogeneous horizontal-boundary data need not agree at the corner points.

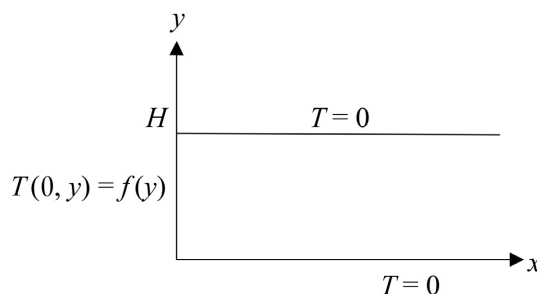


Figure 1. Laplace's equation in the semi-infinite strip domain.

The basic solution obtained from the method of separation of variables is

$$T(x, y) = \sum_{n=1}^{\infty} A_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y \quad (5)$$

the Fourier coefficient A_n of the (5) is

$$A_n = \frac{2}{H} \int_0^H f(y_0) \sin \frac{n\pi}{H}y_0 dy_0 \quad (6)$$

Finally, the infinite series solution of (5) is

$$T(x, y) = \int_0^H f(y_0) K(x, y; y_0) dy_0 \quad (7)$$

where the Kernel function $K(x, y; y_0)$ in (7) is

$$K(x, y; y_0) = \frac{2}{H} \sum_{n=1}^{\infty} e^{-\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y_0 \sin \frac{n\pi}{H}y, \quad 0 < x < \infty, \quad 0 < y, y_0 < H \quad (8)$$

2.1. Global Similarity Solution with Source-Type Singularity

From BCs (2), if the boundary condition is a constant T_0 , we have

$$T(0, y) = f(y) = T_0 \quad (9)$$

The infinite series solution from (7) is

$$T(x, y) = \frac{2T_0}{\pi} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n} e^{-\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y, \quad 0 < x < \infty, \quad 0 < y < H, \quad (10)$$

From [14], we have the closed-form relation

$$\sum_{n=\text{odd}}^{\infty} \frac{1}{n} e^{-\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y = \frac{1}{2} \tan^{-1} \frac{\sin \frac{\pi}{H}y}{\sinh \frac{\pi}{H}x}, \quad 0 < x < \infty, \quad 0 < y < H \quad (11)$$

From (11), the infinite series solution (10) converges to an exact solution as follows.

$$\begin{aligned} T(x, y) &= \frac{4T_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n} e^{-\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y = \frac{2T_0}{\pi} \tan^{-1} \frac{\sin \frac{\pi}{H}y}{\sinh \frac{\pi}{H}x} \\ &= \frac{2T_0}{\pi} \tan^{-1} \eta = g(\eta) \end{aligned} \quad (12)$$

where the similarity variable $\eta(x, y) = \frac{\sin \frac{\pi}{H}y}{\sinh \frac{\pi}{H}x}$ fits all the boundaries, *i.e.*

$$\eta(x, 0) = \eta(x, H) = \eta(\infty, y) = 0 \quad \text{and} \quad \eta(0, y) = \infty, \quad 0 < \eta < \infty.$$

The similarity function $g(\eta)$ in (12) satisfies the 2nd order ordinary differential equation such as shown in (13).

$$(1 + \eta^2) \frac{d^2 g}{d\eta^2} + 2\eta \frac{dg}{d\eta} = 0, \quad 0 < \eta < \infty \quad (13)$$

BCs

$$g(0) = 0, \quad g(\infty) = T_0 \quad (14)$$

The exact solution of (13) with BCs (14) is

$$g(\eta) = \frac{2T_0}{\pi} \tan^{-1} \eta, \quad 0 < \eta < \infty \quad (15)$$

Finally, the global similarity solution of $T(x, y)$ with BCs (14) is

$$T(x, y) = g(\eta) = \frac{2T_0}{\pi} \tan^{-1} \eta = \frac{2T_0}{\pi} \tan^{-1} \frac{\sin \frac{\pi}{H}y}{\sinh \frac{\pi}{H}x}, \quad (16)$$

$$0 < x < \infty, \quad 0 < y < H$$

As $x \rightarrow 0$, $y \rightarrow 0$, there is a source type singularity in (16)

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} T(x, y) = \frac{2T_0}{\pi} \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \tan^{-1} \frac{\sin \frac{\pi}{H}y}{\sinh \frac{\pi}{H}x} \rightarrow \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \tan^{-1} \frac{y}{x} \quad (17)$$

and there is another source type singularity at $x \rightarrow 0$, $y \rightarrow H - y^*$, $y^* \rightarrow 0$ as

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow H - y^*}} T(x, y) = \frac{2T_0}{\pi} \lim_{\substack{x \rightarrow 0 \\ y \rightarrow H - y^*}} \tan^{-1} \frac{\sin \frac{\pi}{H} y}{\sinh \frac{\pi}{H} x} \rightarrow \lim_{\substack{x \rightarrow 0 \\ y^* \rightarrow 0}} \tan^{-1} \frac{y^*}{x} \quad (18)$$

The numerical calculation [15] of the infinite series solution (10) with $H = 1$, is shown in **Figure 2**. The global similarity solution (16) with $H = 1$, is shown in **Figure 3**.

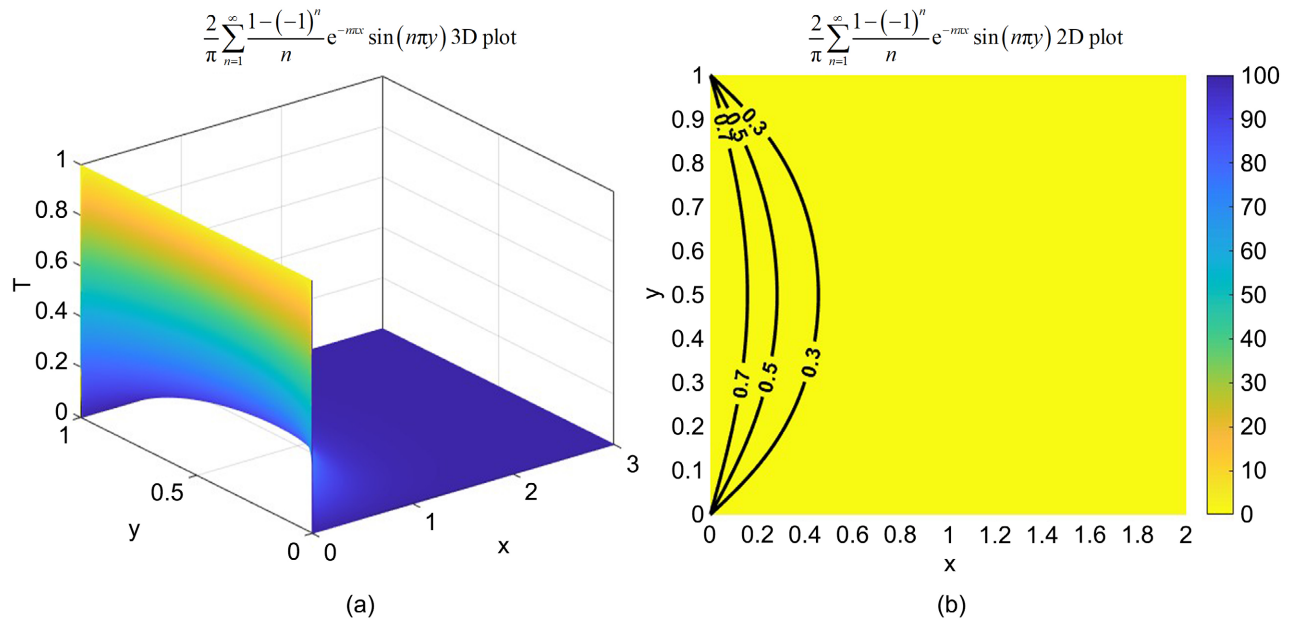


Figure 2. The infinite series solution of $T(x, y)$ in (a) 3D distribution, (b) 2D distribution in x - y plane.

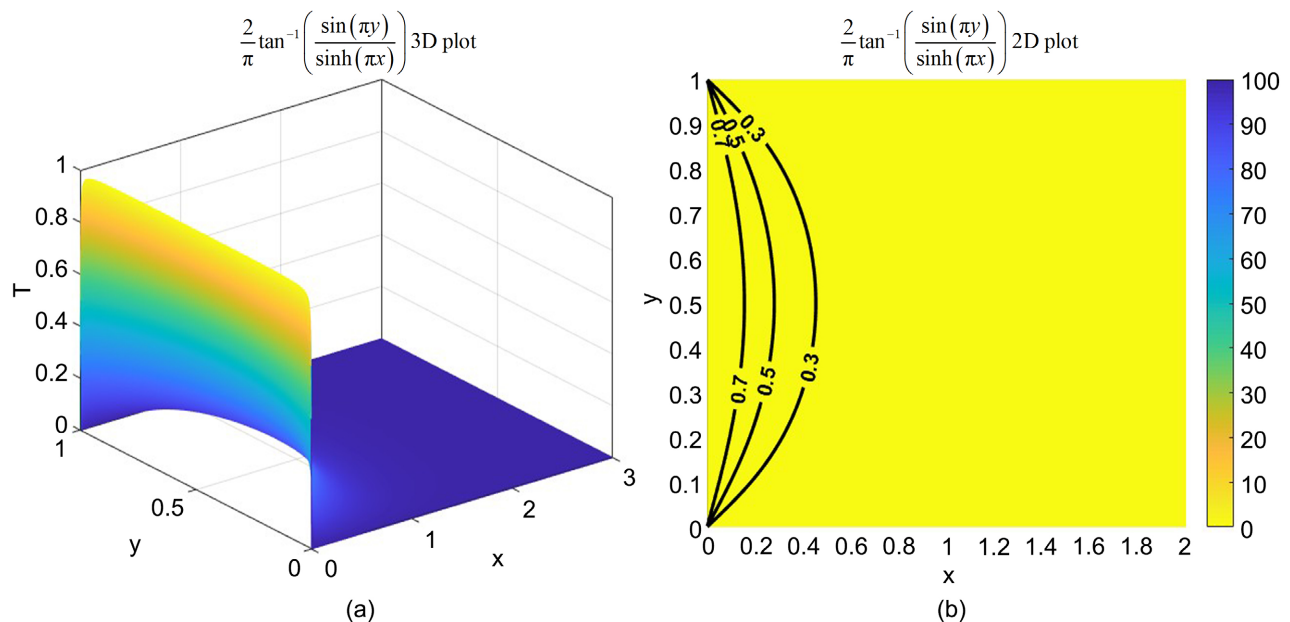


Figure 3. The global similarity solution of $T(x, y)$ in (a) 3D distribution, (b) 2D distribution in x - y plane.

2.2. Pseudo Similarity Solution with Source Type Singularity

From BC (2), for the case we have linear distribution of

$$T(0, y) = f(y) = T_0 \left(a + b \frac{y}{H} \right) \quad (19)$$

where a, b are arbitrary constants.

The Fourier coefficient A_n in (6) is expressed as in (20)

$$A_n = \int_0^H T_0 \left(a + b \frac{y}{H} \right) \sin \frac{n\pi}{H} y dy = \frac{2T_0}{\pi} \left[\frac{a - (a+b)(-1)^n}{n} \right], \quad n = 1, 2, 3, \dots \quad (20)$$

then we obtain the infinite series solution from (5) as

$$T(x, y) = \frac{2T_0}{\pi} \sum_{n=1}^{\infty} \left[\frac{a - (a+b)(-1)^n}{n} \right] e^{-\frac{n\pi}{H}x} \sin \frac{n\pi}{H} y, \quad (21)$$

$$0 < x < \infty, \quad 0 < y < H$$

According to reference [11], we have the general source type solution of Laplace's equation in (1),

$$T(x, y) = c_1 \tan^{-1} \frac{\beta_0 + \beta_1 V(x, y)}{\alpha_0 + \alpha_1 U(x, y)} + c_2 \quad (22)$$

where $U(x, y) = e^{-\frac{m\pi}{H}x} \cos \frac{m\pi}{H} y$, $V(x, y) = e^{-\frac{m\pi}{H}x} \sin \frac{m\pi}{H} y$, satisfying the Cauchy-Riemann equations, and the integer $m = 1, 2, 3, \dots$.

The unknown constants $c_1, c_2, \alpha_0, \alpha_1, \beta_0$ and β_1 in (22) can be fixed from the proper choice of BC in (19).

For $a = 1$, $b = -m$, with BC in (19) gives (23)

$$T(0, y) = f(y) = T_0 \left(1 - m \frac{y}{H} \right) \quad (23)$$

Then from (22), (23) with a constant T_0 , we have

$$T(0, y) = c_1 \tan^{-1} \frac{\beta_0 + \beta_1 \sin \frac{m\pi}{H} y}{\alpha_0 + \alpha_1 \cos \frac{m\pi}{H} y} + c_2 = T_0 \left(1 - m \frac{y}{H} \right) \quad (24)$$

And from trigonometric identity, the (25) can be expressed as follows.

$$\frac{\sin A}{1 - \cos A} = \tan \left(\frac{\pi}{2} - \frac{A}{2} \right) \quad (25)$$

From (24) and (25), the constants in (24) can be fixed as $c_2 = 0$, $\alpha_0 = 1$, $\alpha_1 = -1$, $\beta_0 = 0$, $\beta_1 = 1$, yielding

$$T(0, y) = c_1 \tan^{-1} \frac{\sin \frac{m\pi}{H} y}{1 - \cos \frac{m\pi}{H} y} = \frac{\pi}{2} c_1 \left(1 - m \frac{y}{H} \right) = T_0 \left(1 - m \frac{y}{H} \right) \quad (26)$$

From (26), the constant c_1 is found as follows.

$$c_1 = \frac{2T_0}{\pi} \quad (27)$$

Then the source type solution in (22) is expressed in (28)

$$T(x, y; m) = \frac{2T_0}{\pi} \tan^{-1} \frac{e^{\frac{m\pi}{H}x} \sin \frac{m\pi}{H}y}{1 - e^{\frac{m\pi}{H}x} \cos \frac{m\pi}{H}y}, \quad 0 < x < \infty, \quad 0 < y < H \quad (28)$$

The solution of (28) satisfies all the boundary conditions from (2) to (4). Then from (21) and (28), the infinite series solution (21) converges with a source type closed form solution as shown in (29).

$$\begin{aligned} T(x, y; m) &= \frac{2T_0}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - (1-m)(-1)^n}{n} \right] e^{\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y \\ &= \frac{2T_0}{\pi} \tan^{-1} \frac{e^{\frac{m\pi}{H}x} \sin \frac{m\pi}{H}y}{1 - e^{\frac{m\pi}{H}x} \cos \frac{m\pi}{H}y} \end{aligned} \quad (29)$$

Thus, it can be expressed in the following form (30),

$$T(x, y; m) = \frac{2T_0}{\pi} \tan^{-1} \eta = g(\eta) \quad (30)$$

where the similarity variable $\eta(x, y) = \frac{e^{\frac{m\pi}{H}x} \sin \frac{m\pi}{H}y}{1 - e^{\frac{m\pi}{H}x} \cos \frac{m\pi}{H}y}$ can only fit part of the

boundaries, that means $\eta(x, 0) = \eta(x, H) = \eta(\infty, y) = 0$, but $\eta(0, y)$ is not a constant.

Therefore, the similarity function $g(\eta)$ in (30) satisfies the second order ordinary differential equation, form (31) with only one boundary condition as shown in (32).

$$(1 + \eta^2) \frac{d^2 g}{d\eta^2} + 2\eta \frac{dg}{d\eta} = 0, \quad 0 < \eta \quad (31)$$

BC.

$$g(0) = 0 \quad (32)$$

The similarity transform of the Laplace's equation with the only one boundary condition (32) is not complete.

The solution of (31) with BC (32) is shown in (33).

$$\begin{aligned} T(x, y; m) &= g(\eta) = c_1 \tan^{-1} \eta = c_1 \tan^{-1} \frac{e^{\frac{m\pi}{H}x} \sin \frac{m\pi}{H}y}{1 - e^{\frac{m\pi}{H}x} \cos \frac{m\pi}{H}y}, \\ 0 < x < \infty, \quad 0 < y < H \end{aligned} \quad (33)$$

Therefore, the solution (33) is a pseudo similarity solution, and the constant

$c_1 = \frac{2T_0}{\pi}$ can be only found from (26) and (27).

For $m = 1$, from (33),

$$T(x, y; 1) = \frac{2T_0}{\pi} \tan^{-1} \frac{e^{\frac{\pi}{H}x} \sin \frac{\pi}{H}y}{1 - e^{\frac{\pi}{H}x} \cos \frac{\pi}{H}y}, \quad 0 < x < \infty, \quad 0 < y < H \quad (34)$$

For $m = 2$, we have

$$T(x, y; 2) = \frac{2T_0}{\pi} \tan^{-1} \frac{e^{\frac{2\pi}{H}x} \sin \frac{2\pi}{H}y}{1 - e^{\frac{2\pi}{H}x} \cos \frac{2\pi}{H}y}, \quad 0 < x < \infty, \quad 0 < y < H \quad (35)$$

On the other hand, $a = 1$, $b = -m = -2$, *i.e.* $m = 2$, the infinite series solution (21) gives

$$T(x, y; 2) = \frac{2T_0}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^{n+1}}{n} \right] e^{\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y, \quad 0 < x < \infty, \quad 0 < y < H \quad (36)$$

From (35), the infinite series solution (36) converges to a pseudo similarity solution as

$$\begin{aligned} T(x, y; 2) &= \frac{2T_0}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^{n+1}}{n} \right] e^{\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y \\ &= \frac{2T_0}{\pi} \tan^{-1} \frac{e^{\frac{2\pi}{H}x} \sin \frac{2\pi}{H}y}{1 - e^{\frac{2\pi}{H}x} \cos \frac{2\pi}{H}y} \end{aligned} \quad (37)$$

Finally, from (37), we obtain an important closed-form relation

$$\begin{aligned} &\sum_{n=1}^{\infty} \left[\frac{1 - (-1)^{n+1}}{n} e^{\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y \right] \\ &= e^{\frac{2\pi}{H}x} \sin \frac{2\pi}{H}y + \frac{1}{2} e^{\frac{4\pi}{H}x} \sin \frac{4\pi}{H}y + \frac{1}{3} e^{\frac{6\pi}{H}x} \sin \frac{6\pi}{H}y + \dots \\ &= \tan^{-1} \frac{e^{\frac{2\pi}{H}x} \sin \frac{2\pi}{H}y}{1 - e^{\frac{2\pi}{H}x} \cos \frac{2\pi}{H}y} \end{aligned} \quad (38)$$

For $a = 1$, $b = -m = -1$, *i.e.* $m = 1$, the infinite series solution (21) gives

$$\begin{aligned} T(x, y; 1) &= \frac{2T_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y \\ &= \frac{2T_0}{\pi} \left(e^{\frac{\pi}{H}x} \sin \frac{\pi}{H}y + \frac{1}{3} e^{\frac{3\pi}{H}x} \sin \frac{3\pi}{H}y + \frac{1}{5} e^{\frac{5\pi}{H}x} \sin \frac{5\pi}{H}y + \dots \right) \\ &\quad + \frac{T_0}{\pi} \left(e^{\frac{2\pi}{H}x} \sin \frac{2\pi}{H}y + \frac{1}{2} e^{\frac{4\pi}{H}x} \sin \frac{4\pi}{H}y + \frac{1}{3} e^{\frac{6\pi}{H}x} \sin \frac{6\pi}{H}y + \dots \right) \end{aligned} \quad (39)$$

From (11) and (38), the infinite series solution (39) becomes a closed form solution and converges to a pseudo similarity solution (40).

$$\begin{aligned}
 T(x, y; 1) &= \frac{2T_0}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} e^{-\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y \right] \\
 &= \frac{T_0}{\pi} \tan^{-1} \frac{\sin \frac{\pi}{H}y}{\sinh \frac{\pi}{H}x} + \frac{T_0}{\pi} \tan^{-1} \frac{e^{-\frac{2\pi}{H}x} \sin \frac{2\pi}{H}y}{1 - e^{-\frac{2\pi}{H}x} \cos \frac{2\pi}{H}y} \\
 &= \frac{2T_0}{\pi} \tan^{-1} \frac{e^{-\frac{\pi}{H}x} \sin \frac{\pi}{H}y}{1 - e^{-\frac{\pi}{H}x} \cos \frac{\pi}{H}y}, \quad 0 < x < \infty, \quad 0 < y < H
 \end{aligned} \tag{40}$$

In general, for $a = 1$, $b = -m$, and from (11), (38) and (40), we can prove that the infinite series solution (21) converges to an important pseudo similarity solution for an integer m as shown in (41).

$$\begin{aligned}
 T(x, y; m) &= \frac{2T_0}{\pi} \sum_{n=1}^{\infty} \left[\frac{1 - (1-m)(-1)^n}{n} e^{-\frac{n\pi}{H}x} \sin \frac{n\pi}{H}y \right] \\
 &= 2(2-m) \frac{T_0}{\pi} \left(e^{-\frac{\pi}{H}x} \sin \frac{\pi}{H}y + \frac{1}{3} e^{-\frac{3\pi}{H}x} \sin \frac{3\pi}{H}y + \frac{1}{5} e^{-\frac{5\pi}{H}x} \sin \frac{5\pi}{H}y + \dots \right) \\
 &\quad + m \frac{T_0}{\pi} \left(e^{-\frac{2\pi}{H}x} \sin \frac{2\pi}{H}y + \frac{1}{2} e^{-\frac{4\pi}{H}x} \sin \frac{4\pi}{H}y + \frac{1}{3} e^{-\frac{6\pi}{H}x} \sin \frac{6\pi}{H}y + \dots \right) \\
 &= (2-m) \frac{T_0}{\pi} \tan^{-1} \frac{\sin \frac{\pi}{H}y}{\sinh \frac{\pi}{H}x} + m \frac{T_0}{\pi} \tan^{-1} \frac{e^{-\frac{2\pi}{H}x} \sin \frac{2\pi}{H}y}{1 - e^{-\frac{2\pi}{H}x} \cos \frac{2\pi}{H}y} \\
 &= \frac{2T_0}{\pi} \tan^{-1} \frac{e^{-\frac{m\pi}{H}x} \sin \frac{m\pi}{H}y}{1 - e^{-\frac{m\pi}{H}x} \cos \frac{m\pi}{H}y}, \quad 0 < x < \infty, \quad 0 < y < H
 \end{aligned} \tag{41}$$

As $x \rightarrow 0$, $y \rightarrow 0$, the pseudo similarity solution $T(x, y; 1)$ in (41) has only one source type singularity as shown in (42).

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} T(x, y; 1) = \frac{2T_0}{\pi} \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \tan^{-1} \frac{e^{-\frac{\pi x}{H}} \sin \frac{\pi}{H}y}{1 - e^{-\frac{\pi x}{H}} \cos \frac{\pi}{H}y} \rightarrow \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \tan^{-1} \frac{y}{x} \tag{42}$$

With $T_0 = 1$, $H = 1$, the numerical calculation of the infinite series solution (29) with $m = 1$, as shown in **Figure 4**, and the Pseudo similarity solution in (41) with $m = 1$, as shown in **Figure 5**.

As $x \rightarrow 0$, $y \rightarrow 0$, the pseudo similarity solution $T(x, y; 2)$ in (41) has one source type singularity as (42) and there is another source type singularity at $x \rightarrow 0$, $y \rightarrow H + y^*$, $y^* \rightarrow 0$.

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow H+y^*}} T(x, y; 2) = \frac{2T_0}{\pi} \lim_{\substack{x \rightarrow 0 \\ y \rightarrow H+y^*}} \tan^{-1} \frac{e^{\frac{2\pi x}{H}} \sin \frac{2\pi}{H} y}{1 - e^{\frac{2\pi x}{H}} \cos \frac{2\pi}{H} y} = \lim_{\substack{x \rightarrow 0 \\ y^* \rightarrow 0}} \tan^{-1} \frac{y^*}{x} \quad (43)$$

For $m \geq 2$, there are only two source type singularities near $x \rightarrow 0$, $y \rightarrow 0$, and $x \rightarrow 0$, $y \rightarrow H$.

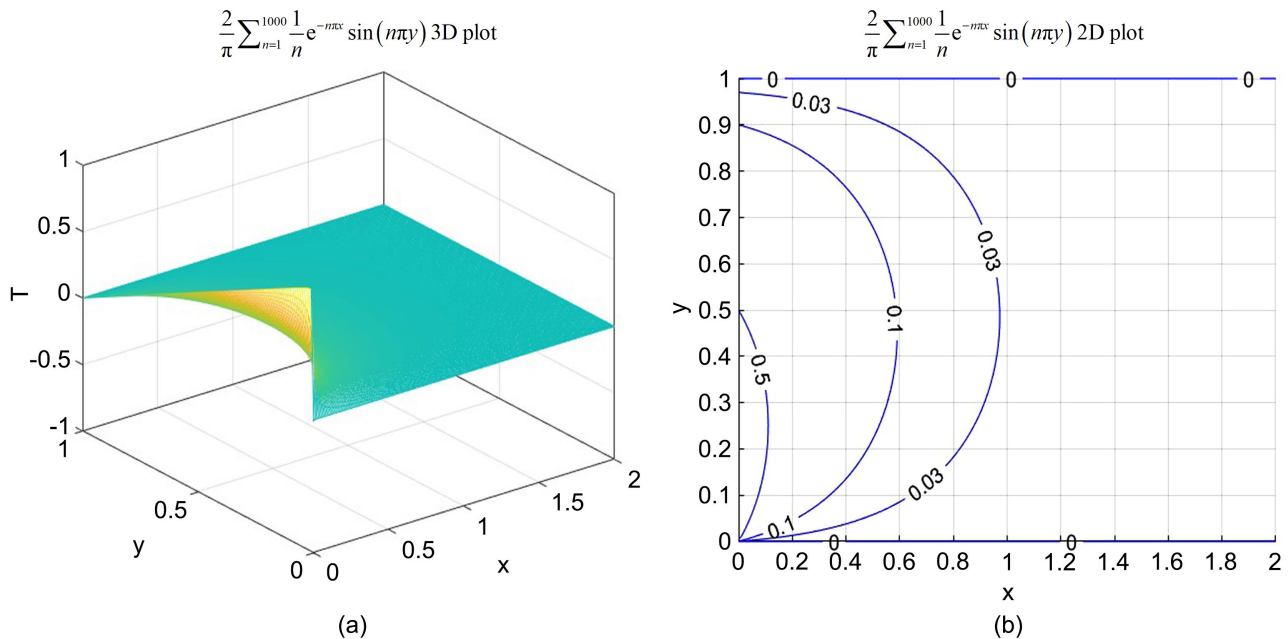


Figure 4. The numerical calculation of the infinite series solution (29) with $T_0=1$, $H=1$, $m=1$; (a) 3D distribution, (b) 2D distribution in x - y plane.

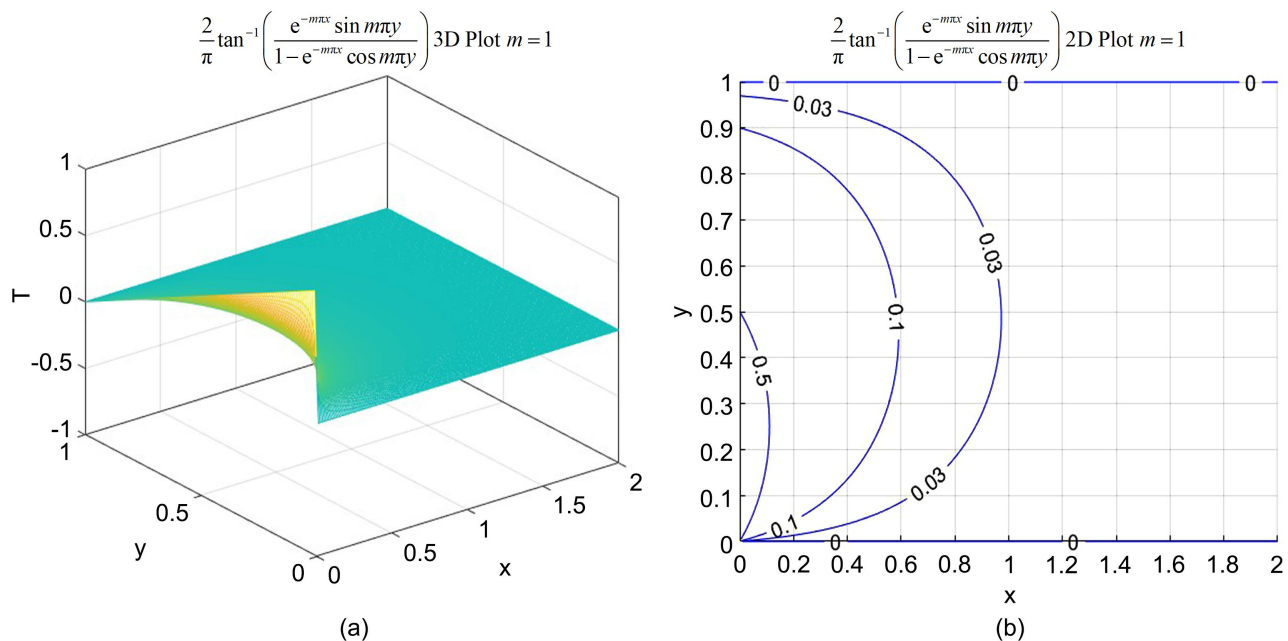


Figure 5. The Pseudo similarity solution in (41) with $T_0=1$, $H=1$, $m=1$; (a) 3D distribution, (b) 2D distribution in x - y plane.

With $T_0 = 1$, $H = 1$, numerical calculation of the infinite series solution (29) with $m = 2$ is shown in **Figure 6**, and the Pseudo similarity solution (41) with $m = 2$ is shown in **Figure 7**.

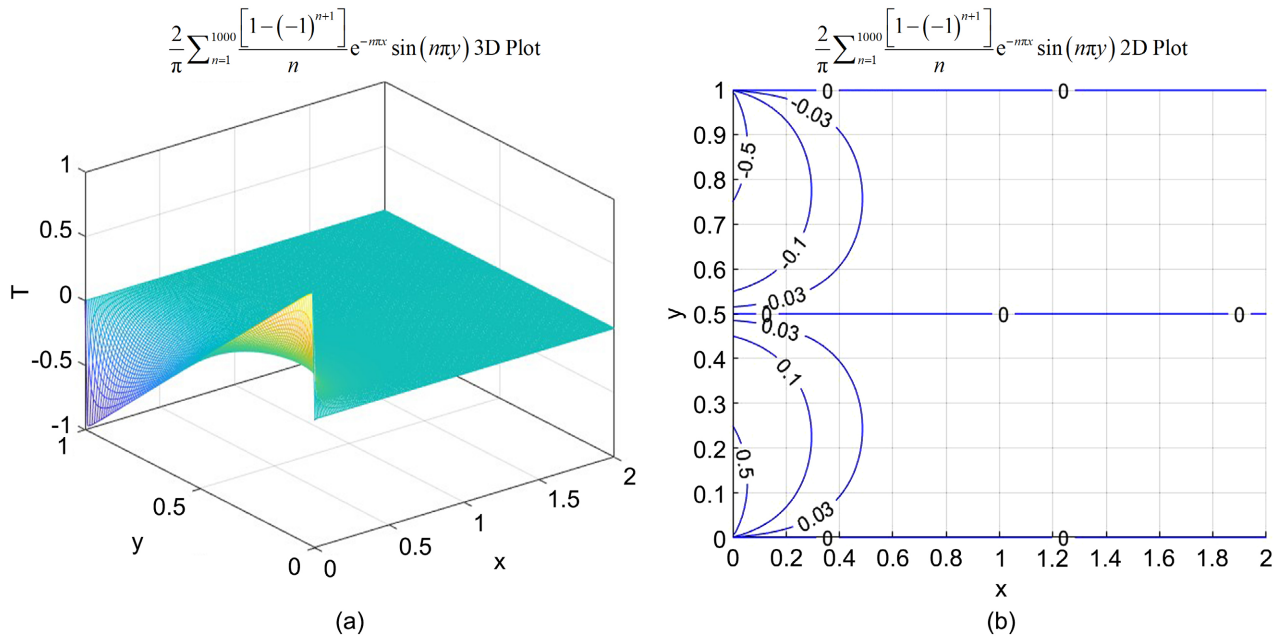


Figure 6. The numerical calculation of the infinite series solution (29) with $T_0 = 1$, $H = 1$, $m = 2$; (a) 3D distribution, (b) 2D distribution in x - y plane.

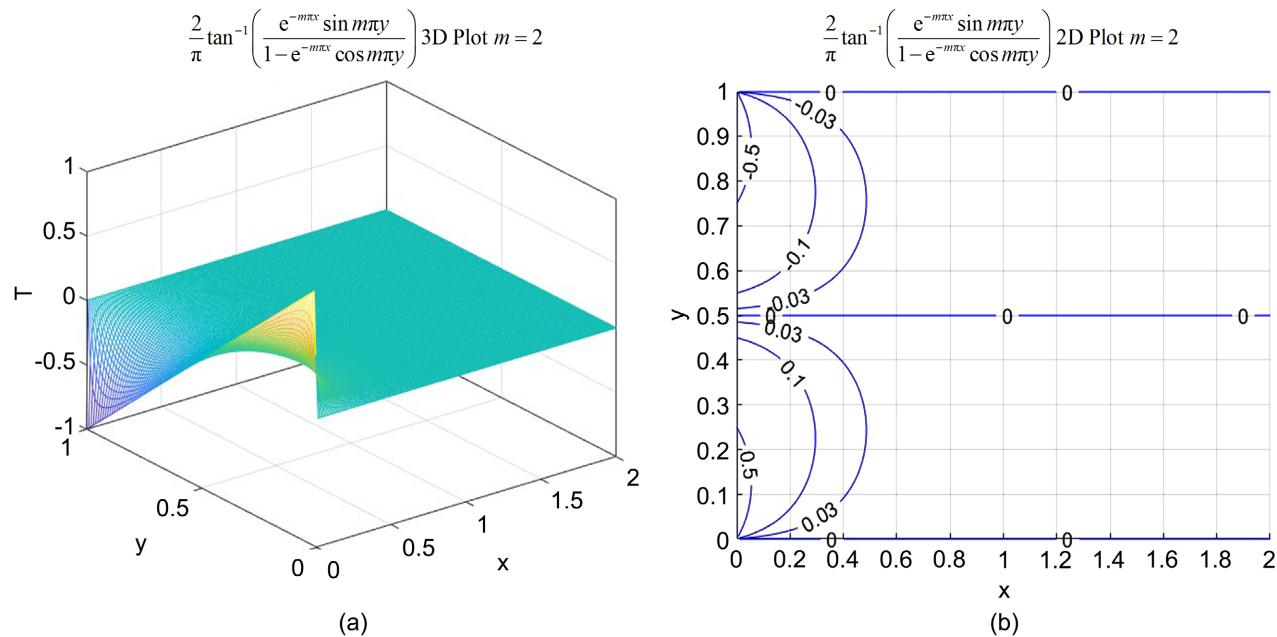


Figure 7. The Pseudo similarity solution in (41) with $T_0 = 1$, $H = 1$, $m = 2$; (a) 3D distribution, (b) 2D distribution in x - y plane.

2.3. Pseudo Local Similarity Solution with Doublet Singularity

From BC (2) with Dirac delta function

$$T(0, y) = f(y) = \delta(y - a) = \begin{cases} 0, & y \neq a \\ \infty, & y = a \end{cases}, \quad 0 < y, a < H \quad (44)$$

where a is any point between $[0, H]$ in y direction, then with the properties of the Dirac delta function

$$\int_0^H \delta(y - a) dy = 1 \quad (45)$$

$$\int_0^H f(y) \delta(y - a) dy = f(a) \quad (46)$$

Then the Fourier coefficient A_n of the infinite series solution (6) is

$$A_n = \frac{2}{H} \int_0^H f(y) \sin \frac{n\pi}{H} y dy = \frac{2}{H} \int_0^H \delta(y - a) \sin \frac{n\pi}{H} y dy = \frac{2}{H} \sin \frac{n\pi}{H} a \quad (47)$$

Finally, we obtain the infinite series solution (5) as

$$\begin{aligned} T(x, y) &= \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi}{H}x} \sin \frac{n\pi}{H} y \\ &= \frac{2}{H} \sum_{n=1}^{\infty} e^{-\frac{n\pi}{H}x} \sin \frac{n\pi}{H} a \sin \frac{n\pi}{H} y, \quad 0 < x < \infty, \quad 0 < y, a < H \end{aligned} \quad (48)$$

From (44), (48) we obtain

$$T(0, y) = \delta(y - a) = \frac{2}{H} \sum_{n=1}^{\infty} \sin \frac{n\pi}{H} a \sin \frac{n\pi}{H} y, \quad 0 < y, a < H \quad (49)$$

(49) is just a Dirac delta function $\delta(y - a)$ expressed in a Fourier sine series.

Rearrange (48) as follows,

$$\begin{aligned} T(x, y) &= \frac{2}{H} \sum_{n=1}^{\infty} e^{-\frac{n\pi}{H}x} \sin \frac{n\pi}{H} a \sin \frac{n\pi}{H} y \\ &= \frac{1}{H} \sum_{n=1}^{\infty} e^{-\frac{n\pi}{H}x} \cos \frac{n\pi}{H} (y - a) - \frac{1}{H} \sum_{n=1}^{\infty} e^{-\frac{n\pi}{H}x} \cos \frac{n\pi}{H} (y + a) \end{aligned} \quad (50)$$

From [16], the Poisson Kernel can be expressed as

$$\frac{1 - \rho^2}{1 - 2\rho \cos \theta + \rho^2} = 1 + 2(\rho \cos \theta + \rho^2 \cos 2\theta + \rho^3 \cos 3\theta + \dots) \quad (51)$$

(51) can be shown as

$$\sum_{n=1}^{\infty} \rho^n \cos n\theta = \frac{\rho \cos \theta - \rho^2}{1 - 2\rho \cos \theta + \rho^2}, \quad \rho^2 < 1 \quad (52)$$

By setting $\rho = e^{-\frac{\pi}{H}x}$, $\theta = \frac{\pi}{H}(y \pm a)$, then the infinite series solution (50) can be expressed in a closed form solution as follows,

$$T(x, y) = \frac{1}{H} \left[\frac{e^{-\frac{\pi}{H}x} \cos \frac{\pi}{H}(y - a) - e^{-\frac{2\pi}{H}x}}{1 - 2e^{-\frac{\pi}{H}x} \cos \frac{\pi}{H}(y - a) + e^{-\frac{2\pi}{H}x}} - \frac{e^{-\frac{\pi}{H}x} \cos \frac{\pi}{H}(y + a) - e^{-\frac{2\pi}{H}x}}{1 - 2e^{-\frac{\pi}{H}x} \cos \frac{\pi}{H}(y + a) + e^{-\frac{2\pi}{H}x}} \right] \quad (53)$$

By letting

$$T_1(x, y) = \frac{1}{H} \left[\frac{e^{\frac{\pi}{H}x} \cos \frac{\pi}{H}(y-a) - e^{\frac{2\pi}{H}x}}{1 - 2e^{\frac{\pi}{H}x} \cos \frac{\pi}{H}(y-a) + e^{\frac{2\pi}{H}x}} \right] \quad (54)$$

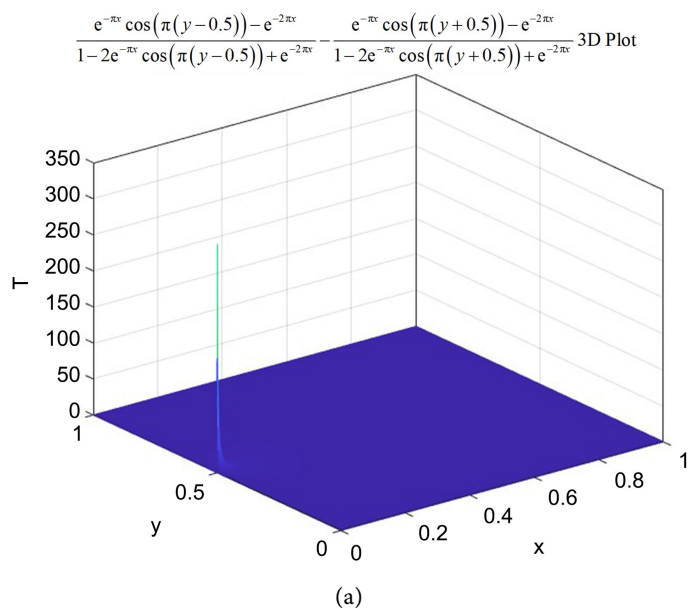
$$T_2(x, y) = \frac{1}{H} \left[\frac{e^{\frac{\pi}{H}x} \cos \frac{\pi}{H}(y+a) - e^{\frac{2\pi}{H}x}}{1 - 2e^{\frac{\pi}{H}x} \cos \frac{\pi}{H}(y+a) + e^{\frac{2\pi}{H}x}} \right] \quad (55)$$

For $H=1$, by setting $x \rightarrow x^*$, $y \rightarrow a + y^*$, now if $x^* \rightarrow 0$, $y^* \rightarrow 0$, yielding $T_2(x, y)$ is constant value. However, the function $T_1(x, y)$ becomes

$$\lim_{\substack{x \rightarrow x^* \\ y \rightarrow a+y^*}} T_1(x, y) = \lim_{\substack{x^* \rightarrow 0 \\ y^* \rightarrow 0}} \frac{e^{-\pi x^*} \cos \pi y^* - e^{-2\pi x^*}}{1 - 2e^{-\pi x^*} \cos \pi y^* + e^{-2\pi x^*}} \quad (56)$$

$$\begin{aligned} &= \lim_{\substack{x^* \rightarrow 0 \\ y^* \rightarrow 0}} \frac{\left(1 - \pi x^* + \frac{1}{2}(\pi x^*)^2 + \dots\right) \left(1 - \frac{1}{2}(\pi y^*)^2 + \dots\right) - (1 - 2\pi x^* + \dots)}{1 - 2 \left(1 - \pi x^* + \frac{1}{2}(\pi x^*)^2 + \dots\right) \left(1 - \frac{1}{2}(\pi y^*)^2 + \dots\right) + (1 - 2\pi x^* + \frac{1}{2}(2\pi x^*)^2 + \dots)} \\ &= \frac{1}{\pi} \lim_{\substack{x^* \rightarrow 0 \\ y^* \rightarrow 0}} \frac{x^*}{x^{*2} + y^{*2}} \end{aligned} \quad (57)$$

In other words, the closed form solution (53) gives the doublet singularity in (57) near the singular point $(0, a)$. The numerical calculation of the closed form solution (53) with $H=1$, $a=0.5$ is shown in **Figure 8(a)**. For $x=0$, $y \rightarrow 0.5$ there is a delta function in T - y plane as shown in **Figure 8(b)**. For $x \rightarrow 0$, $y \rightarrow 0.5$ the solution of the closed form (53) approaches a doublet solution in the x - y plane as shown in **Figure 8(c)**.



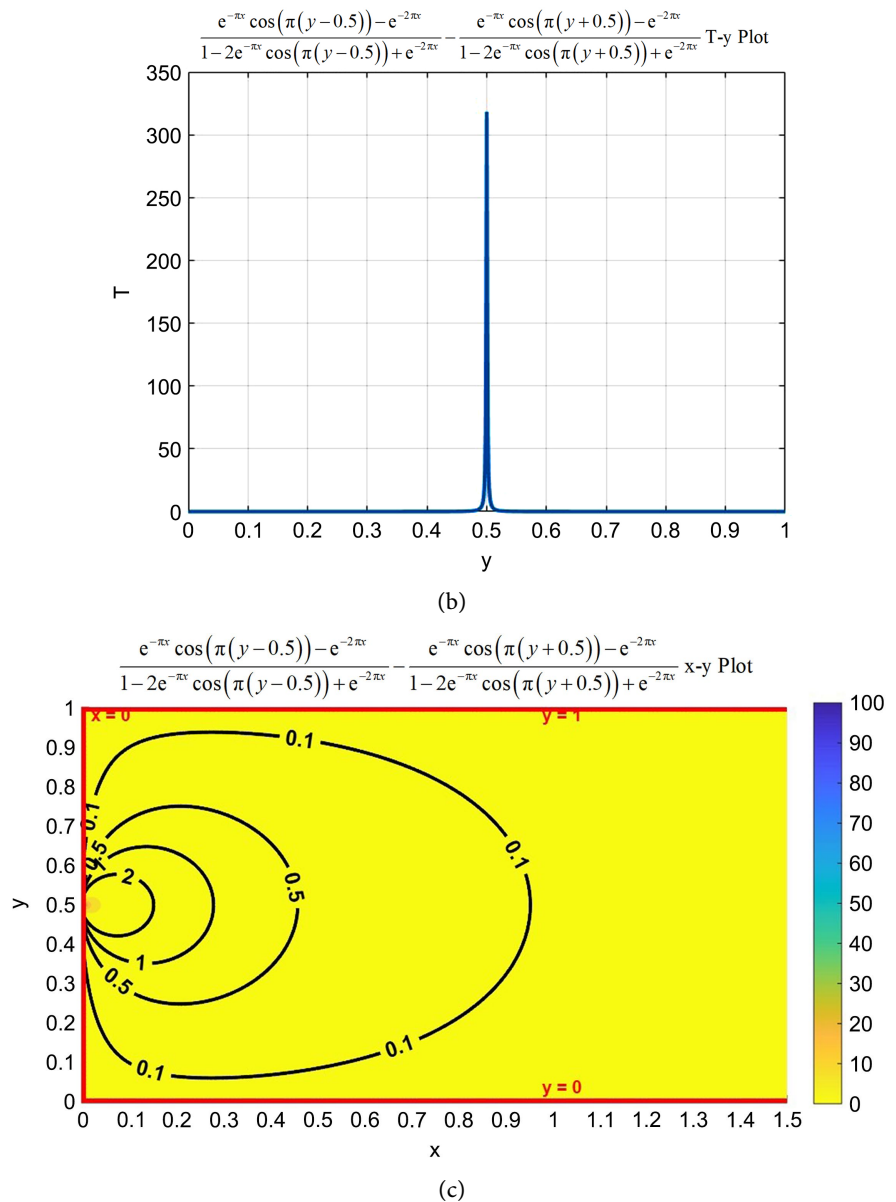


Figure 8. The numerical calculation of the closed form solution (53) with $H=1$, $a=0.5$; (a) 3D distribution, (b) 2D distribution in T - y plane, (c) 2D distribution in x - y plane.

For $H=1$, $x \rightarrow 0$, $y \rightarrow a$, from the closed form solution (53), the local solution of the infinite series form (48) can also approach a doublet solution $T_D(x, y)$ as

$$T(x, y) = 2 \sum_{n=1}^{\infty} e^{-n\pi x} \sin n\pi a \sin n\pi y \rightarrow T_D(x, y) \quad (58)$$

The pseudo local similarity solution $T_D(x, y)$ satisfying Laplace Equation (1) is of the similarity form as $x \rightarrow 0$, $y \rightarrow a$

$$T_D(x, y) = g(\eta) \quad (59)$$

where the similarity variable $\eta(x, y) = \frac{x}{x^2 + (y-a)^2}$ can fit only one boundary

condition $\eta(0, y) = 0$, and the similarity function $g(\eta)$ satisfying the 2nd order ordinary differential equation

$$\frac{d^2 g}{d\eta^2} = 0, \quad 0 < \eta \quad (60)$$

with only one boundary condition

$$g(0) = 0 \quad (61)$$

The solution of $T_D(x, y)$ in (60) with one unknown constant c_1 is

$$T_D(x, y) = c_1 \frac{x}{x^2 + (y - a)^2} \quad (62)$$

The unknown constant c_1 can be found from the definition of Dirac delta function (45) as

$$\int_0^1 \delta(y - a) dy = c_1 \int_0^1 \frac{x}{x^2 + (y - a)^2} dy = c_1 \tan^{-1} \left(\frac{y - a}{x} \right) \Big|_{a^-}^{a^+} = c_1 \pi = 1 \quad (63)$$

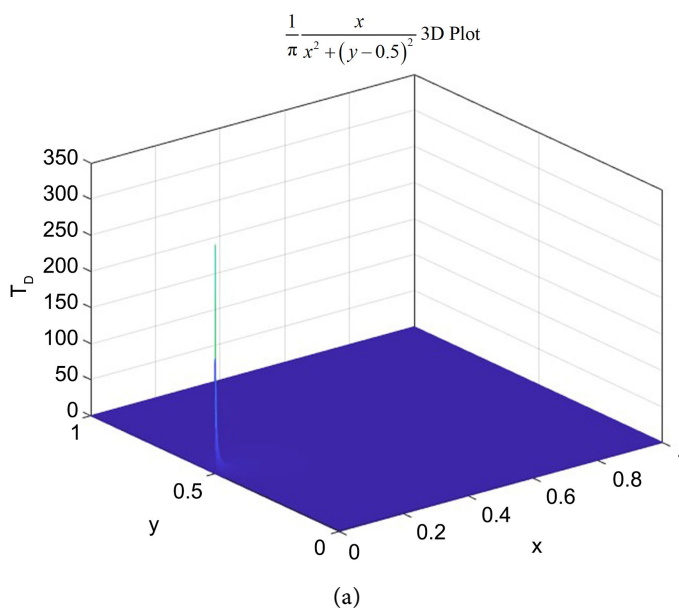
From (63), the unknown constant is

$$c_1 = \frac{1}{\pi}. \quad (64)$$

Finally, the pseudo local similarity solution with doublet singularity $T_D(x, y)$ is expressed as

$$T_D(x, y) = \frac{1}{\pi} \cdot \frac{x}{x^2 + (y - a)^2}, \quad x \rightarrow 0, y \rightarrow a \quad (65)$$

With $x \rightarrow 0$, $a \rightarrow 0.5$, the 3D plot of the local similarity solution $T_D(x, y)$ in (65), is shown in **Figure 9(a)**, and the T_D - y distribution of (65) is shown in **Figure 9(b)**. As $x \rightarrow 0$, $a \rightarrow 0.5$ the $T_D(x, y)$ solutions of (65) present a doublet in the x - y plane as shown in **Figure 9(c)**.



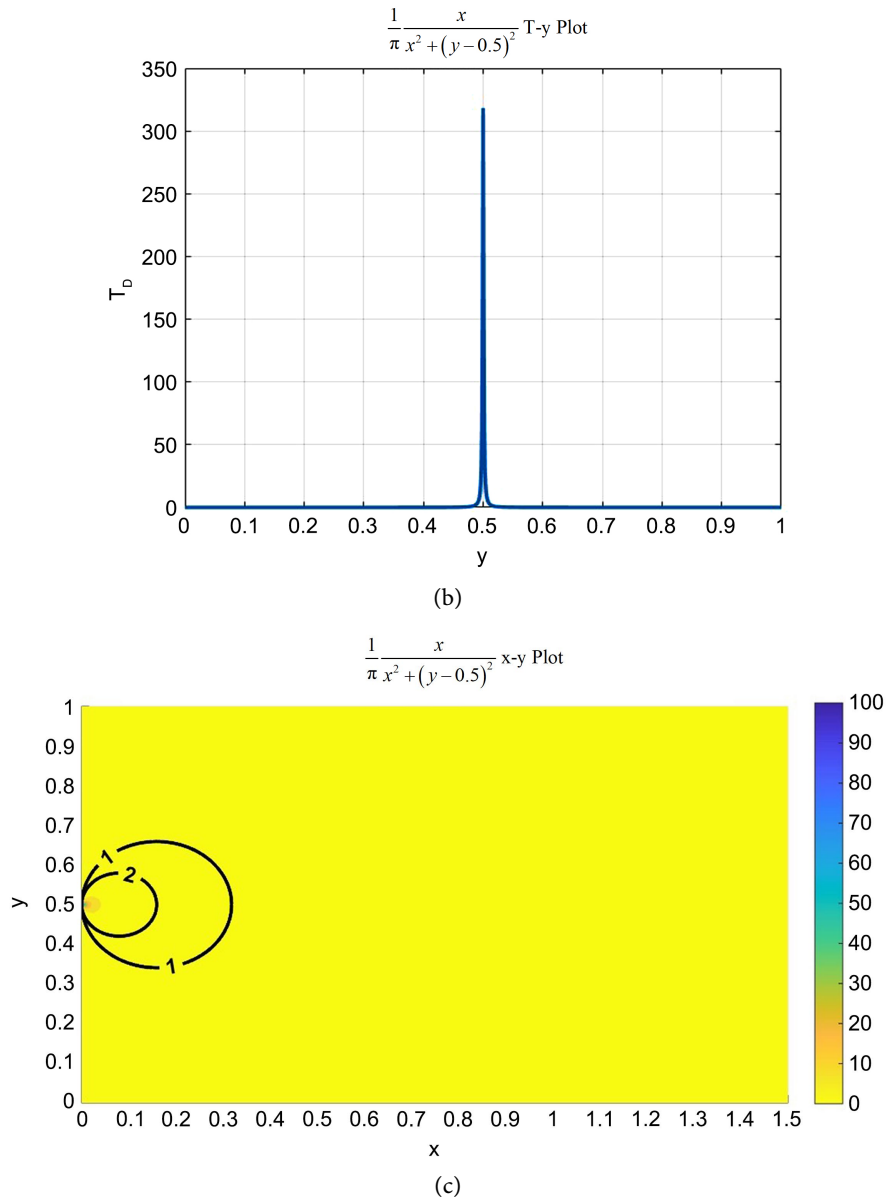


Figure 9. The numerical calculation of the closed form solution (65) with $H=1$, $a=0.5$; (a) 3D distribution, (b) 2D distribution in T - y plane, (c) 2D distribution in x - y plane.

Near the singular point $(0, a)$ i.e. $x \rightarrow x^*$ ($x^* \rightarrow 0$), $y \rightarrow a + y^*$ ($y^* \rightarrow 0$), (65) gives the same doublet singularity as (57), i.e.

$$\lim_{\substack{x \rightarrow x^* \\ y \rightarrow a + y^*}} T_D(x, y) = \frac{1}{\pi} \cdot \lim_{\substack{x^* \rightarrow 0 \\ y^* \rightarrow 0}} \frac{x^*}{x^{*2} + y^{*2}} \quad (66)$$

The solution with Dirac delta function boundary condition is an infinite series and it can be reduced to a closed form solution, but it is not a similarity solution, except near $x \rightarrow 0$, $y \rightarrow a$. In the limiting case, the asymptotic solution of the closed form solution is a pseudo local similarity solution with doublet singularity.

In other words, $x \rightarrow 0$, $y \rightarrow a$, the pseudo local similarity solution with dou-

blet singularity $T_D(x, y)$ in (65) approaches the local doublet solution of the closed form solution (53) with the boundary condition of Dirac delta function in (44), and the numerical comparison of local solutions in (53) and (65) is shown in **Figure 10**. The distribution of local solution fits perfectly near $x \rightarrow 0$, and $y \rightarrow a$.

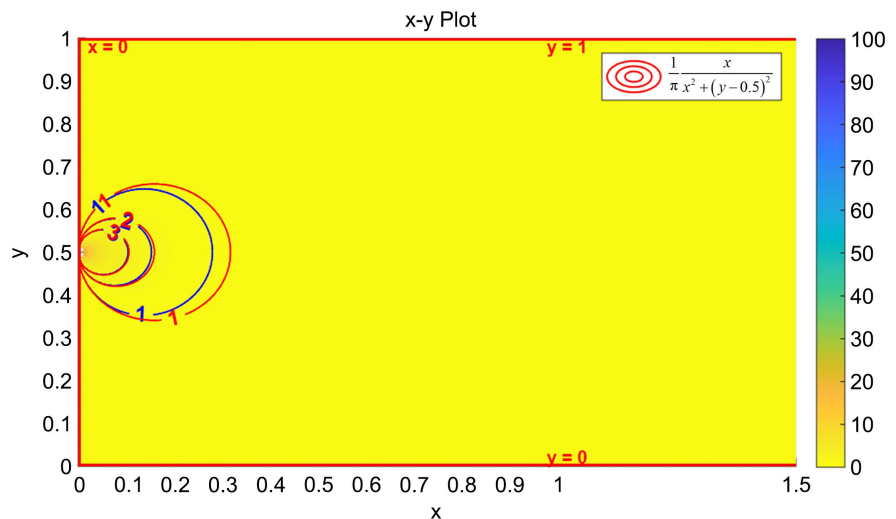


Figure 10. The numerical comparison of local solutions in (53) and (65) is shown and the distribution of local solution fits perfectly near $x \rightarrow 0$, and $y \rightarrow a$.

3. Discussion

The fundamental solution of Laplace's Equation (1) with Dirichlet boundary conditions can be found by applying the method of separation of variables and the solution is expressed in an infinite series. From B.C. (9), $T(0, y) = T_0$ (constant), by applying the similarity method, the infinite series solution (10) can converge to an analytical global similarity solution (16) with similarity variable $\eta(x, y)$ fits perfectly for all boundary conditions (2) to (4). We can also find some source-type singularities from the global similarity solution near the singular point $x \rightarrow 0$, $y \rightarrow 0$ and $x \rightarrow 0$, $y \rightarrow H$. However, from BC (23),

$$T(0, y) = T_0 \left(1 - m \frac{y}{H} \right), \text{ with integer } m, \text{ the infinite series solution for } m = 1 \text{ and}$$

$m = 2$ in (29) can only converge to pseudo similarity solutions with similarity variable $\eta(x, y)$ fits only part of the boundary conditions. The unknown boundary condition must be found in other ways as shown in (27). For $m = 0$, it will give global similarity solution with two source type singularities near $x \rightarrow 0$, $y \rightarrow 0$ and $x \rightarrow 0$, $y \rightarrow H$, which are shown in (16), (17) and (18). For $m = 1$, it gives a pseudo similarity solution with one source type singularity near $x \rightarrow 0$, $y \rightarrow 0$. For $m \geq 2$, it gives a pseudo similarity solution with only two source type singularities near $x \rightarrow 0$, $y \rightarrow 0$ and $x \rightarrow 0$, $y \rightarrow H$.

The boundary condition (44) with Dirac delta function, the infinite series solution (48) can converge to a closed form solution (53) with doublet singularity near

the singular point $x \rightarrow 0$ and $y \rightarrow a$ as shown in (57).

It is interesting to point out that a pseudo local similarity solution with doublet singularity $T_D(x, y)$ in (65) was found and it approaches the local solution (53) with Dirac delta boundary condition. The numerical comparison of local solutions in (53) and (65) are shown in **Figure 10**, in which isolines of T fit perfectly near $x \rightarrow 0$, and $y \rightarrow a$.

4. Conclusions

The fundamental solution of Laplace's equation with Dirichlet boundary conditions can be found by using the method of separation of variables and its solution expressed in an infinite series. By using similarity analysis, the infinite series solution of Laplace's equation can be proved to be an analytical global and pseudo similarity solution.

It is important for checking the numerical calculation for convergence of the infinite series solutions with the analytical similarity solution. The merit of this analysis is that the important singularity behavior of the boundary-value problem of Laplace's equation near some singular points in the domain, for example, the source type and doublet singularities near some singular point in the domain can be found analytically from the global similarity solution or the pseudo similarity solutions.

Author Contributions

Conceptualization, C. K. Feng; methodology, C. K. Feng; validation, J. H. Tang, C. K. Feng; formal analysis, C. K. Feng; investigation, C. K. Feng, J. H. Tang; resources, C. K. Feng; data curation, J. H. Tang; writing—original draft preparation, J. H. Tang; writing—review and editing, J. H. Tang.

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Data Availability Statement

Some or all data, formula or code that support the findings of this study are available upon reasonable request.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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