Scientific Research Publishing

Advances in Pure Mathematics, 2026, 16(8), 535-570
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

A Heuristic Study on Goldbach Conjecture and Twin Prime Conjecture by Bertrand Theorem

Pingyuan Zhou

Beiyuan 35-210, Chengdu University of Technology, Chengdu, China
Email: pingyuanzhou49@163.com

How to cite this paper: Zhou, P.Y. (2026) A Heuristic Study on Goldbach Conjecture and Twin Prime Conjecture by Bertrand Theorem. *Advances in Pure Mathematics*, **16**, 535-570.
https://doi.org/10.4236/apm.2026.168029

Received: July 22, 2026
Accepted: August 23, 2026
Published: August 26, 2026

Abstract

Bertrand theorem states that there is at least one prime in $(x, 2x)$ for $x > 1$. Let P_n denote the n-th prime and take $x = P_n$. Then the theorem states that there is at least one prime in $(P_n, 2P_n)$. It shows $P_{n+1} - P_n < P_n$, which means that gap between primes is linearly controlled and supports infinitude of primes. Generalize the theorem into the prime index sequence $\{n\}$. Then the Bertrand-type theorem states there is at least one number $n + k$ in $(n, 2n)$ for $n > 1$ such that $n + k$ is prime p and P_p is called double prime. It is obvious that the Bertrand-type theorem implies infinitude of double primes. An even number L_n is defined as the largest strong Goldbach number generated by P_n if every even number from 4 to L_n is the sum of two primes not greater than P_n but $L_n + 2$ is not such a sum. Since $L_n \le L_{n+1}$ for all n, L_n is a non-decreasing function and there exist growth points of L_n. If $L_{n-1} < L_n$ then n is a growth point of L_n and corresponding prime P_n is called a nontrivial prime to structure a growth of L_n. It is clear that the infinitude of nontrivial primes implies Goldbach conjecture. Comparing counted number of nontrivial primes with counted number of double primes, we can conjecture that there is at least one number $n + k$ in $(n, 2n)$ for $n > 1$ such that P_{n+k} is a nontrivial prime. The Bertrand-type conjecture has been verified up to $n = 300{,}000$. If it is proven then Goldbach conjecture is true. Comparing counted number of twin primes with counted number of double primes, we can conjecture that there is at least one number $n + k$ in $(n, 2n)$ for $n > 1$ such that P_{n+k} is a twin prime. The Bertrand-type conjecture has been verified up to $n = 300{,}000$. If it is proven then twin prime conjecture is true.

Keywords

Bertrand Theorem, Bertrand-Type Theorem for Double Prime, The Largest Strong Goldbach Number, Nontrivial Prime, Bertrand-Type Conjecture for Nontrivial Prime, Bertrand-Type Conjecture for Twin Prime, Root of Prime,

Gap between Prime Roots, Goldbach Conjecture, Twin Prime Conjecture

1. Introduction

The Goldbach conjecture states that every even number greater than 2 is the sum of two primes and the conjecture remains unsolved to this day. Of studies on the conjecture, main research results arise from circle method, sieve method and exceptional set method [1]-[6]. The twin prime conjecture states that there are infinitely many primes p such that $p+2$ is also prime. The conjecture is a special case of Polignac conjecture, which states that there are infinitely many primes p such that $p+2k$ is also prime for every natural number k [7]. Hardy-Littlewood conjectured distribution of twin primes is asymptotically expressed as $2C_2x/(\log x)^2$, which is a special case of the first Hardy-Littlewood conjecture[1]. Although the conjecture has not been proven, it seems certain to be true. It presents a strong form for proving twin prime conjecture. An important research advance on the twin prime conjecture is that it was proven that there are infinitely many prime gaps with length bounded by $N=246$ in 2013 [8]. This is a weak result of Polignac conjecture. It is obvious that Goldbach conjecture and twin prime conjecture are not the same type of problem. In order to transform Goldbach conjecture into a problem about infinitude of a kind of special primes, we tightened concept of traditional Goldbach number, that is, an even number is called a Goldbach number generated by a prime if the even number is the sum of two primes not greater than this prime. The covering boundary of such Goldbach numbers to consecutive even numbers is defined as the largest strong Goldbach number generated by a prime [9]-[13]. Let P_n denote the n-th prime and L_n denote the largest strong Goldbach number generated by P_n. Then $L_{n+1} \geq L_n$ for all n. If $L_{n-1} < L_n$ then n is a growth point of L_n and corresponding P_n is definite as a nontrivial prime. It is clear that the infinitude of nontrivial primes implies Goldbach conjecture. Thus the Goldbach conjecture has been transformed into a problem about the infinitude of a kind of special primes as the twin prime conjecture does. What mathematical form can provide a common frame for studying the two conjectures? When we consider existence problems of nontrivial primes and twin primes in prime index interval, a Bertrand-type theorem for existence of double primes in prime index interval seems to be able to construct the frame, where P_n is called a double prime if n is prime p. Bertrand theorem states there is at least one prime in $(x, 2x)$ for $x > 1$. Taking $x = P_n$, the theorem states there is at least one prime in $(P_n, 2P_n)$. Generalize Bertrand theorem into the prime index sequence $\{n\}$. Then the Bertrand-type theorem states there is at least one number $n+k$ in $(n, 2n)$ for $n > 1$ such that $n+k$ is prime p. If n is defined as root of prime P_n then the Bertrand-type theorem is presented more clearly as follows. Let n denote root of P_n and $2n$ denote root of P_{2n}. Then there is at least one prime root $n+k$ in $(n, 2n)$ for $n > 1$ such that $n+k$ is double prime root, equivalently, there is at least one prime P_{n+k}

in (P_n, P_{2n}) for $n > 1$ such that P_{n+k} is a double prime. The Bertrand-type theorem has been verified up to $n = 300{,}000$. Write the i-th double prime root as $A(i)$. Then we have $A(i+1) - A(i) < A(i)$. It means that gap between double prime roots remains linearly controlled, which supports infinitude of double prime roots to show infinitude of double primes. If Bertrand theorem is established to show existence of primes in natural number interval, then Bertrand-type theorem is established to show existence of double primes in prime index interval. The Bertrand-type theorem for double prime is a standard system for solving existence problem of second order primes on n-axis. Comparing counted number of nontrivial primes on n-axis with counted number of double primes on n-axis, we have the following conjecture. Let n denote root of P_n and $2n$ denote root of P_{2n}. Then there is at least one prime root $n + k$ in $(n, 2n)$ for $n > 1$ such that $n + k$ is a nontrivial prime root, equivalently, there is at least one prime P_{n+k} in (P_n, P_{2n}) for $n > 1$ such that P_{n+k} is a nontrivial prime. Write the i-th nontrivial prime root as $B(i)$. Then we have $B(i+1) - B(i) < B(i)$, which means that gap between nontrivial prime roots remains linearly controlled. It supports infinitude of nontrivial prime roots to show infinitude of nontrivial primes and Goldbach conjecture is true. The Bertrand-type conjecture has been verified up to $n = 300{,}000$. Comparing counted number of twin primes on n-axis with counted number of double primes on n-axis, we have the following conjecture. Let n denote root of P_n and $2n$ denote root of P_{2n}. Then there is at least one prime root $n + k$ in $(n, 2n)$ for $n > 1$ such that $n + k$ is a twin prime root, equivalently, there is at least one prime P_{n+k} in (P_n, P_{2n}) for $n > 1$ such that P_{n+k} is a twin prime. Write the i-th twin prime root as $C(i)$. Then we have $C(i+1) - C(i) < C(i)$, which means gap between twin prime roots remains linearly controlled. It supports infinitude of twin prime roots to show infinitude of twin primes and twin prime conjecture is true. The Bertrand-type conjecture has been verified up to $n = 300{,}000$. If above two Bertrand-type conjectures are proven, then the Goldbach conjecture and the twin prime conjecture are true.

2. Bertrand-Type Theorem for Double Prime

2.1. Bertrand Theorem

Before prime number theorem was proven in 1896 [14] [15], Bertrand had made a conjecture while studying permutation groups in 1845, which is known as Bertrand's postulate. The conjecture states that there is at least one prime in $(x, 2x)$ for $x > 1$ [16]. Bertrand verified his conjecture up to $x = 3{,}000{,}000$. Chebyshev proved the conjecture using estimates of prime counting function and introducing Chebyshev functions in 1850 [17]. Thus, Bertrand theorem sometimes is called Bertrand-Chebyshev theorem. The first elementary proof of Bertrand's postulate was given by Ramanujan using the Gamma function and leading to the notion of Ramanujan primes in 1919 [18]. Later, the proof was improved by 19-year-old Erdös based on combinatorial arguments in 1932 [19]. Euclid proved there are infinitely many primes by definition of prime but did not address how far apart

consecutive primes can be. Take $x = P_n$. Then Bertrand's postulate states that there is at least one number $P_n + k$ in $(P_n, 2P_n)$ such that $P_n + k$ is a prime. Let $P_n + k$ be the first prime P_{n+1} in $(P_n, 2P_n)$. Then Bertrand's postulate was the first major theorem showing

$$P_{n+1} < 2P_n, \tag{2.1}$$

which yields the bound

$$g_n = P_{n+1} - P_n < P_n. \tag{2.2}$$

It implies that growth of primes is linearly controlled. The result was historically the first genuine control on prime gaps and also one of the earliest important results in the study of primes in intervals. Bertrand theorem actually controls primes in $(x, x + x)$, that is, interval length is x to be linear relative size. The theorem implies prime gaps cannot grow too large and provides a fundamental lower bound on prime density. Although Bertrand theorem is not yet a short interval result in the modern sense, the theorem is the starting point of the entire short interval theory because almost all later developments ask: Can the interval length x be reduced to x^θ for $\theta < 1$? A research result is that there is at least one prime in $(x, x + x^{0.547})$ [20] and the recent development is that there is at least one prime in $(x, x + x^{0.525})$ [21]. In this paper, we only consider Bertrand theorem shows an important result such that gap between primes is linearly controlled and such linear control may always be more easily handled than nonlinear control in technique.

2.2. Bertrand Theorem and the Infinitude of Primes

Since Euclid proved infinitude of primes using constructive method [22], there are over a hundred distinct proofs for the infinitude of primes across different areas of mathematics. Of these methods, Euclid's proof by using constructive method is the oldest, the simplest and the most perfect method as the following statement does. Suppose P_n is the largest prime. Then $P_1 \cdot P_2 \cdot P_3 \cdot \ldots \cdot P_n + 1$ has a prime divisor not among the assumed finite list. Thus, there is no the largest prime and primes are infinite. Representative methods to prove the infinitude of primes are as follows. Euler proved the infinitude of primes using analytic method in 1737 [23]. Dirichlet proved the infinitude of primes using arithmetic progressions in 1837 [24]. Erdös proved the infinitude of primes using elementary method in 1932 [19]. Furstenberg proved the infinitude of primes using topological method in 1955 [25]. Elsholtz proved the infinitude of primes using algebraic and combinatorial method in 2009 [26]. Meštrović proved the infinitude of primes using modern very short proofs in 2017 [27]. It should be emphasized that the prime number theorem is a very strong method to prove the infinitude of primes and one must ask: Can Bertrand theorem provide an independent proof for infinitude of primes? As we know, Bertrand's postulate itself does not contain infinitude of primes, Chebyshev's proof, Ramanujan's proof and Erdös's proof also do not contain infinitude of primes. Thus, if Bertrand's postulate can provide a proof for

infinitude of primes then the proof is independent and reliable. By Bertrand theorem, there is the following proof. Suppose P_n is the largest prime. Then there is a prime P_{n+1} greater than P_n such that $P_n < P_{n+1} < 2P_n$ by Bertrand theorem. Thus, there is no the largest prime and primes are infinite. It is similar to Euclid's proof and seems to be simpler than Euclid's method, however, three proofs of Bertrand theorem are more complex than definition of prime. Thus, one can conclude the proof for infinitude of primes given by Bertrand theorem is independent and reliable.

2.3. Generalization of Bertrand Theorem by Ramanujan Prime

Let P_n denote the n-th prime. Then natural numbers $\{x\}$ form a dense sequence and the prime sequence $\{P_n\}$ can be viewed as a "compressed natural number axis". Although the index n itself is just a natural number, it encodes the existence of the n-th prime. Therefore, one is naturally led to ask: Can Bertrand theorem be generalized from intervals on the natural number axis $(x, 2x)$ to structures indexed by primes? Historically, mathematicians did move in this direction, and the most important development is the theory of Ramanujan primes. As is known, Ramanujan primes can be viewed as the most important and natural higher-order version of Bertrand theorem, which is historically the most famous generalization. The central idea is: Instead of merely asking whether the interval $(x/2, x)$ contains at least one prime, one wants to know how many primes can be contained in $(x/2, x)$. It transforms Bertrand theorem from a mere existence problem into a local density problem. Let $\pi(x)$ be prime counting function and denote the number of primes not greater than x. Then Bertrand theorem is equivalent to

$$\pi(x)-\pi(x/2)\geq 1 \quad \text{for all} \quad x\geq 2. \tag{2.3}$$

When reproving Bertrand's postulate in 1919 [18], Ramanujan asked: Can one require that the interval $(x/2, x)$ contains not just at least one prime but at least n primes? It leads to the definition of the n-th Ramanujan prime R_n such that

$$\pi(x)-\pi(x/2)\geq n \quad \text{for all} \quad x\geq R_n. \tag{2.4}$$

Since $R_1 = 2$, Bertrand theorem becomes the first Ramanujan prime case. The first five Ramanujan primes are 2, 11, 17, 29, 41 to correspond to $n = 1, 2, 3, 4, 5$. Later Sondow systematically studied Ramanujan primes and got the core result such that $R_n \sim P_{2n}$ [28].

From above statements we see that definition of Ramanujan prime is yet established on interval of natural numbers $(x/2, x)$ but $\pi(x) - \pi(x/2) \geq 1$ for all $x \geq 2$ has been strengthened as $\pi(x) - \pi(x/2) \geq n$ for all $x \geq R_n$. It successfully transforms Bertrand theorem from a mere existence problem into a local density problem. Note that the second order natural number sequence $\{n\}$ to encode the existence of primes P_n did not be specially considered in Ramanujan's generalization of Bertrand theorem. Therefore, we feel there may be another research direction to generalize Bertrand theorem: If we directly generalize Bertrand theorem arising from the natural number sequence $\{x\}$ into the second order natural number sequence

$\{n\}$, then we will structure a new mathematical frame in interval $(n, 2n)$ on the second order natural number axis. It would not strengthen Bertrand theorem but can establish the Bertrand-type theorem or conjecture so that some open problems such as Goldbach conjecture and twin prime conjecture may be better studied.

2.4. Prime Number Theorem for Double Prime

Definition 2.1. Let P_n denote the n-th prime. Then P_n is called a *double prime* if n is a prime p, that is, $P_n = P_p$.

The first 50 double primes are listed as follows

P_2, P_3, P_5, P_7, P_{11}, P_{13}, P_{17}, P_{19}, P_{23}, P_{29}, P_{31}, P_{37}, P_{41}, P_{43}, P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}, P_{83}, P_{89}, P_{97}, P_{101}, P_{103}, P_{107}, P_{109}, P_{113}, P_{127}, P_{131}, P_{137}, P_{139}, P_{149}, P_{151}, P_{157}, P_{163}, P_{167}, P_{173}, P_{179}, P_{181}, P_{191}, P_{193}, P_{197}, P_{199}, P_{211}, P_{223}, P_{227}, P_{229}.

It is obvious that the prime index sequence $\{n\}$ is the second order natural number sequence to encode the existence of the prime sequence $\{P_n\}$. First, the sequence $\{n\}$ itself can be viewed as a natural number sequence and some research results on primes among natural numbers are yet effective. Second, these results must be explained again using n to be a second order natural number encoding the existence of the n-th prime. In order to give a clear description for gap between indexes of primes, we have further definition.

Definition 2.2. Number n is called *root of prime* and also a *prime root* if P_n denotes the n-th prime.

By Definition 2.2, if P_n is a double prime P_p then $n = p$ is a double prime root.

Definition 2.3. Let $P_{A(i)}$ denote the i-th double prime. Then $g_{A(i)}$ is called the i-th *double prime root gap* if $g_{A(i)} = A(i+1) - A(i)$.

The first 50 double prime root gaps are listed as follows

1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2, 4, 14, 4, 6, 2, 10, 2, 6, 6, 4, 6, 6, 2, 10, 2, 4, 2, 12, 12, 4, 2, 4.

Definition 2.4 Let $P_{A(i)}$ denote the i-th double prime. Then $g_{A(i)}^{doub}$ is called the i-th *double prime gap* if $g_{A(i)}^{doub} = P_{A(i+1)} - P_{A(i)}$.

The first 50 double prime gaps are listed as follows

2, 6, 6, 14, 10, 18, 8, 16, 26, 18, 30, 22, 12, 20, 30, 36, 6, 48, 22, 14, 34, 30, 30, 48, 38, 16, 24, 12, 18, 92, 30, 34, 24, 62, 18, 42, 48, 24, 40, 32, 24, 66, 18, 30, 16, 80, 112, 24,14, 24.

Note that double prime is known as prime-indexed prime (PIP) in many studies. Let q_n denote prime-indexed prime P_p with $p = p_n$. Then it is shown that $q_n \sim n(\log n)^2$ using prime-indexed prime number theorem by Barnett-Boughan [29]. Its mathematical significance is: PIPs still obey highly regular PNT-type asymptotic form. Further, higher-order prime structures were also studied. Its central asymptotic law is: $P_n^{(k)} \sim n(\log n)^k$, corresponding to density: $\sim 1/(\log x)^k$ [30]. Thus, $P_n^{(2)} = q_n$ which is prime-indexed prime, corresponding to density: $\sim 1/(\log x)^2$. Although such deep studies on prime-indexed primes led to many developments, in this paper we only consider existence problem of double primes among primes and prime root is an universal and useful concept which can help

us to express many key mathematical objects such as gap between prime roots..

Let $\pi(n)$ denote the counted number of double primes among the first n primes. Then we have the following approximation.

$$\pi(n) \approx Li(n), \tag{2.5}$$

where

$$Li(n) = \int_2^n \frac{\mathrm{d}t}{\log t} \tag{2.6}$$

is the logarithmic integral and it has an equivalent asymptotic series such that

$$Li(n) \approx \frac{n}{\log n} \sum_{k=0}^{\infty} \frac{k!}{(\log n)^k}. \tag{2.7}$$

Let $n/\log n$ denote the weakest form of $Li(n)$. Then Table 1 gives counted number and predicted number by $n/\log n$ for double primes among the first n primes.

Table 1. Counted and predicted numbers for double primes among primes.

n	counted number	predicted number	$\pi(n) - n/\log n$	relative error
10^2	25	22	3	0.1200
10^3	168	145	23	0.1369
10^4	1229	1086	143	0.1163
10^5	9592	8686	906	0.0944
10^6	78498	72382	6116	0.0779
10^7	664579	620421	44185	0.0664
10^8	5761455	5428681	332774	0.0577
10^9	50847534	48254942	2592592	0.0509

Theorem 2.5. *Let $\pi(n)$ denote counted number of double primes among primes and $D^{doub}(n)$ denote average density of double primes among primes.*

If $\lim_{n\to\infty} \dfrac{\pi(n)}{\int_2^n \frac{\mathrm{d}t}{\log t}} = 1$, *then* $D^{doub}(n) \sim \dfrac{1}{\log n}$.

Proof. Using (2.7), $Li(n)$ can be written as

$$\frac{n}{\log n} \sum_{k=0}^{\infty} \frac{k!}{(\log n)^k}. \tag{2.8}$$

Considering asymptotic series (2.8), we see that the k-th term approaches higher order infinity than the $(k+1)$-th term as n grows without bound in the asymptotic series because there is the following limit.

$$\lim_{n\to\infty} \frac{\dfrac{(k+1)!n}{(\log n)^{k+2}}}{\dfrac{k!n}{(\log n)^{k+1}}} = \lim_{n\to\infty} \frac{k+1}{\log n} = 0. \tag{2.9}$$

Since it is assumed that

$$\lim_{n\to\infty}\frac{\pi(n)}{\int_2^n \frac{\mathrm{d}t}{\log t}}=1,$$

by (2.9) we have

$$\lim_{n\to\infty}\frac{\pi(n)}{\left(\frac{n}{\log n}\right)}=1. \tag{2.10}$$

The limit (2.10) means that

$$\pi(n)\sim\frac{n}{\log n}. \tag{2.11}$$

Since $D^{doub}(n)=\frac{\pi(n)}{n}$, we obtain

$$D^{doub}(n)\sim\frac{1}{\log n}. \tag{2.12}$$

Hence the theorem holds.

Corollary 2.6. *If* $\lim_{n\to\infty}\frac{\pi(n)}{\int_2^n \frac{\mathrm{d}t}{\log t}}=1$, *then there are infinitely many double primes.*

Proof. By Theorem 2.5, if $\lim_{n\to\infty}\frac{\pi(n)}{\int_2^n \frac{\mathrm{d}t}{\log t}}=1$ then we have

$$\pi(n)\sim\frac{n}{\log n}.$$

Since $n/\log n$ approaches infinity as n grows without bound, $\pi(n)$ approaches infinity as n grows without bound. It means there are infinitely many double primes among primes, that is, there are infinitely many double primes. Hence the corollary holds.

Remark 2.7 When every second order natural number n is viewed as a natural number, the prime number theorem for expressing asymptotic distribution law of primes among natural numbers can be used for expressing asymptotic distribution law of double primes among primes. Thus, the prime number theorem has become the prime number theorem for double prime if $\{n\}$ is considered as the second order natural number axis to generate double primes, that is, asymptotic expression (2.11) can be called prime number theorem for double prime. So, all above results can be thought as results which had been proven.

2.5. Bertrand-Type Theorem for Double Prime

We can generalize Bertrand theorem into the prime root sequence $\{n\}$ and have the following Bertrand-type theorem for double prime.

Theorem 2.8. *Let* n *denote root of* P_n *and* $2n$ *denote root of* P_{2n}. *Then there is at least one prime root* $n+k$ *in* $(n, 2n)$ *for* $n>1$ *such that* $n+k$ *is a double prime*

root, correspondingly, there is at least one prime P_{n+k} *in* (P_n, P_{2n}) *for* $n > 1$ *such that* P_{n+k} *is a double prime.*

Remark 2.9. The front part of Theorem 2.8 is a restatement of Bertrand's postulate on n-axis. If the prime root sequence $\{n\}$ is merely viewed as a natural number sequence, Chebyshev's proof, Ramanujan's proof and Erdös's proof are naturally reasonable proofs for the front part of the theorem. The latter part of Theorem 2.8 is a restatement for the front part by definition of prime root, thus, two parts of Theorem 2.8 are equivalent and the restatement naturally holds.

Corollary 2.10. *Let* $A(i)$ *denote the i-th double prime root. Then there are* $A(i+1) < 2A(i)$ *and* $A(i+1) - A(i) < A(i)$.

Proof. Taking $n = A(i)$, by Theorem 2.8, there is at least one prime root $A(i) + k$ in $(A(i), 2A(i))$ such that $A(i) + k$ is a double prime root. Let $A(i) + k = A(i+1)$ be the first double prime root in $(A(i), 2A(i))$. Then we have $A(i) < A(i+1) < 2A(i)$, that is, $A(i+1) < 2A(i)$ and $A(i+1) - A(i) < A(i)$. Hence the corollary holds.

Remark 2.11. Corollary 2.10 remains lineal control on gap between double prime roots. It is an important step for proving the infinitude of double prime roots.

Corollary 2.12. *Let* $P_{A(i)}$ *denote the i-th double prime. Then there are* $P_{A(i+1)} < P_{2A(i)}$ *and* $P_{A(i+1)} - P_{A(i)} < P_{2A(i)} - P_{A(i)}$.

Proof. Theorem 2.8 states there is at least one prime P_{n+k} in (P_n, P_{2n}) for $n > 1$ such that P_{n+k} is a double prime. Take $P_n = P_{A(i)}$. Let $P_{n+k} = P_{A(i+1)}$ be the first double prime in $(P_{A(i)}, P_{2A(i)})$. Then we have

$$P_{A(i+1)} < P_{2A(i)}, \tag{2.13}$$

equivalently,

$$P_{A(i+1)} - P_{A(i)} < P_{2A(i)} - P_{A(i)}. \tag{2.14}$$

Hence the corollary holds.

Remark 2.13. Corollary 2.12 means gap between double primes is nonlinearly controlled. It is different from gap between double prime roots which is linearly controlled. Note that $P_{2n} > 2P_n$ for $n > 1$ [17], thus, $P_{2n} - P_n > P_n$ for $n > 1$. Correspondingly, there is result such that $P_{2A(i)} - P_{A(i)} > P_{A(i)}$ for $i \geq 1$ and $P_{2A(i)} - P_{A(i)}$ yields the bound on double prime gap as (2.14) shows. It is different from (2.2) which shows prime gap to be linearly controlled.

Theorem 2.14. *There are infinitely many double primes.*

Proof. Suppose $A(i)$ is the largest double prime root. Then there is a double prime root $A(i+1)$ greater than $A(i)$ such that $A(i) < A(i+1) < 2A(i)$ by Corollary 2.10. Thus, there is no the largest double prime root and double prime roots are infinite. Since every double prime root $A(i)$ links with a double prime $P_{A(i)}$, there are infinitely many double primes among primes, that is, there are infinitely many double primes. Hence the theorem holds.

Note that Theorem 2.8, Corollary 2.10 and Corollary 2.12 form a Bertrand-type system for double prime, which is merely an existence problem of double prime

among primes.

2.6. Verification of Bertrand-Type Theorem for Double Prime

Verification of Bertrand-type theorem for double prime is divided into two parts: verification of Theorem 2.8 as well as verification of Corollary 2.10 and Corollary 2.12.

Table 2 verifies Theorem 2.8 using the number of double prime roots in $(n, 2n)$ greater than 0 and the number of double primes in (P_n, P_{2n}) greater than 0 for $1 < n \le 50$. Values of these numbers can be counted based on all double prime roots and all double primes given by the table. Figure 1 gives all numerical evidences to verify Theorem 2.8 for $1 < n \le 300{,}000$ because data curve in the figure shows the number of double primes in (P_n, P_{2n}) to be greater than 0 for $1 < n \le 300{,}000$, equivalently, the figure also verifies the number of double prime roots in $(n, 2n)$ to be greater than 0 for $1 < n \le 300{,}000$ because the number of double prime roots in $(n, 2n)$ is always equal to the number of double primes in (P_n, P_{2n}) for $n > 1$. Thus, Theorem 2.8 has been verified up to $n = 300{,}000$. All raw data to show counted number of double primes in (P_n, P_{2n}) for every n for $1 < n \le 300{,}000$, which have been drawn as curve in Figure 1, can be found in [31]. For example, the number of double primes in (P_{525}, P_{1050}) is counted as 77 and the number of double primes in (P_{936}, P_{1872}) is counted as 128 by raw data in [31]. The reference [31] includes 15105 pages to show raw data for double primes less than 10,000,000, which can cover any counting require for double primes up to $n = 300{,}000$.

Table 2. Double prime roots in $(n, 2n)$ and double primes in (P_n, P_{2n}) for $1 < n \le 50$.

n	all double prime roots	$2n$	P_n	all double primes	P_{2n}
2	3	4	P_2	P_3	P_4
3	5	6	P_3	P_5	P_6
4	5, 7	8	P_4	P_5, P_7	P_8
5	7	10	P_5	P_7	P_{10}
6	7, 11	12	P_6	P_7, P_{11}	P_{12}
7	11, 13	14	P_7	P_{11}, P_{13}	P_{14}
8	11, 13	16	P_8	P_{11}, P_{13}	P_{16}
9	11, 13, 17	18	P_9	P_{11}, P_{13}, P_{17}	P_{18}
10	11, 13, 17, 19	20	P_{10}	P_{11}, P_{13}, P_{17}, P_{19}	P_{20}
11	13, 17, 19	22	P_{11}	P_{13}, P_{17}, P_{19}	P_{22}
12	13, 17, 19, 23	24	P_{12}	P_{13}, P_{17}, P_{19}, P_{23}	P_{24}
13	17, 19, 23	26	P_{13}	P_{17}, P_{19}, P_{23}	P_{26}
14	17, 19, 23	28	P_{14}	P_{17}, P_{19}, P_{23}	P_{28}
15	17, 19, 23, 29	30	P_{15}	P_{17}, P_{19}, P_{23}, P_{29}	P_{30}
16	17, 19, 23, 29, 31	32	P_{16}	P_{17}, P_{19}, P_{23}, P_{29}, P_{31}	P_{32}

Continued

17	19, 23, 29, 31	34	P_{17}	P_{19}, P_{23}, P_{29}, P_{31}	P_{34}
18	19, 23, 29, 31	36	P_{18}	P_{19}, P_{23}, P_{29}, P_{31}	P_{36}
19	23, 29, 31, 37	38	P_{19}	P_{23}, P_{29}, P_{31}, P_{37}	P_{38}
20	23, 29, 31, 37	40	P_{20}	P_{23}, P_{29}, P_{31}, P_{37}	P_{40}
21	23, 29, 31, 37, 41	42	P_{21}	P_{23}, P_{29}, P_{31}, P_{37}, P_{41}	P_{42}
22	23, 29, 31, 37, 41, 43	44	P_{22}	P_{23}, P_{29}, P_{31}, P_{37}, P_{41}, P_{43}	P_{44}
23	29, 31, 37, 41, 43	46	P_{23}	P_{29}, P_{31}, P_{37}, P_{41}, P_{43}	P_{46}
24	29, 31, 37, 41, 43, 47	48	P_{24}	P_{29}, P_{31}, P_{37}, P_{41}, P_{43}, P_{47}	P_{48}
25	29, 31, 37, 41, 43, 47	50	P_{25}	P_{29}, P_{31}, P_{37}, P_{41}, P_{43}, P_{47}	P_{50}
26	29, 31, 37, 41, 43, 47	52	P_{26}	P_{29}, P_{31}, P_{37}, P_{41}, P_{43}, P_{47}	P_{52}
27	29, 31, 37, 41, 43, 47, 53	54	P_{27}	P_{29}, P_{31}, P_{37}, P_{41}, P_{43}, P_{47}, P_{53}	P_{54}
28	29, 31, 37, 41, 43, 47, 53	56	P_{28}	P_{29}, P_{31}, P_{37}, P_{41}, P_{43}, P_{47}, P_{53}	P_{56}
29	31, 37, 41, 43, 47, 53	58	P_{29}	P_{31}, P_{37}, P_{41}, P_{43}, P_{47}, P_{53}	P_{58}
30	31, 37, 41, 43, 47, 53, 59	60	P_{30}	P_{31}, P_{37}, P_{41}, P_{43}, P_{47}, P_{53}, P_{59}	P_{60}
31	37, 41, 43, 47, 53, 59, 61	62	P_{31}	P_{37}, P_{41}, P_{43}, P_{47}, P_{53}, P_{59}, P_{61}	P_{62}
32	37, 41, 43, 47, 53, 59, 61	64	P_{32}	P_{37}, P_{41}, P_{43}, P_{47}, P_{53}, P_{59}, P_{61}	P_{64}
33	37, 41, 43, 47, 53, 59, 61	66	P_{33}	P_{37}, P_{41}, P_{43}, P_{47}, P_{53}, P_{59}, P_{61}	P_{66}
34	37, 41, 43, 47, 53, 59, 61, 67	68	P_{34}	P_{37}, P_{41}, P_{43}, P_{47}, P_{53}, P_{59}, P_{61}, P_{67}	P_{68}
35	37, 41, 43, 47, 53, 59, 61, 67	70	P_{35}	P_{37}, P_{41}, P_{43}, P_{47}, P_{53}, P_{59}, P_{61}, P_{67}	P_{70}
36	37, 41, 43, 47, 53, 59, 61, 67, 71	72	P_{36}	P_{37}, P_{41}, P_{43}, P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}	P_{72}
37	41, 43, 47, 53, 59, 61, 67, 71, 73	74	P_{37}	P_{41}, P_{43}, P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}	P_{74}
38	41, 43, 47, 53, 59, 61, 67, 71, 73	76	P_{38}	P_{41}, P_{43}, P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}	P_{76}
39	41, 43, 47, 53, 59, 61, 67, 71, 73	78	P_{39}	P_{41}, P_{43}, P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}	P_{78}
40	41, 43, 47, 53, 59, 61, 67, 71, 73, 79	80	P_{40}	P_{41}, P_{43}, P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}	P_{80}
41	43, 47, 53, 59, 61, 67, 71, 73, 79	82	P_{41}	P_{43}, P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}	P_{82}
42	43, 47, 53, 59, 61, 67,71, 73, 79, 83	84	P_{42}	P_{43}, P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}, P_{83}	P_{84}
43	47, 53, 59, 61, 67, 71, 73, 79, 83	86	P_{43}	P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}, P_{83}	P_{86}
44	47, 53, 59, 61, 67, 71, 73, 79, 83	88	P_{44}	P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}, P_{83}	P_{88}
45	47, 53, 59, 61, 67, 71, 73, 79, 83, 89	90	P_{45}	P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}, P_{83}, P_{89}	P_{90}
46	47, 53, 59, 61, 67, 71, 73, 79, 83, 89	92	P_{46}	P_{47}, P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}, P_{83}, P_{89}	P_{92}
47	53, 59, 61, 67, 71, 73, 79, 83, 89	94	P_{47}	P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}, P_{83}, P_{89}	P_{94}
48	53, 59, 61, 67, 71, 73, 79, 83, 89	96	P_{48}	P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}, P_{83}, P_{89}	P_{96}
49	53, 59, 61, 67, 71, 73, 79, 83, 89, 97	98	P_{49}	P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}, P_{83}, P_{89}, P_{97}	P_{98}
50	53, 59, 61, 67, 71, 73, 79, 83, 89, 97	100	P_{50}	P_{53}, P_{59}, P_{61}, P_{67}, P_{71}, P_{73}, P_{79}, P_{83}, P_{89}, P_{97}	P_{100}

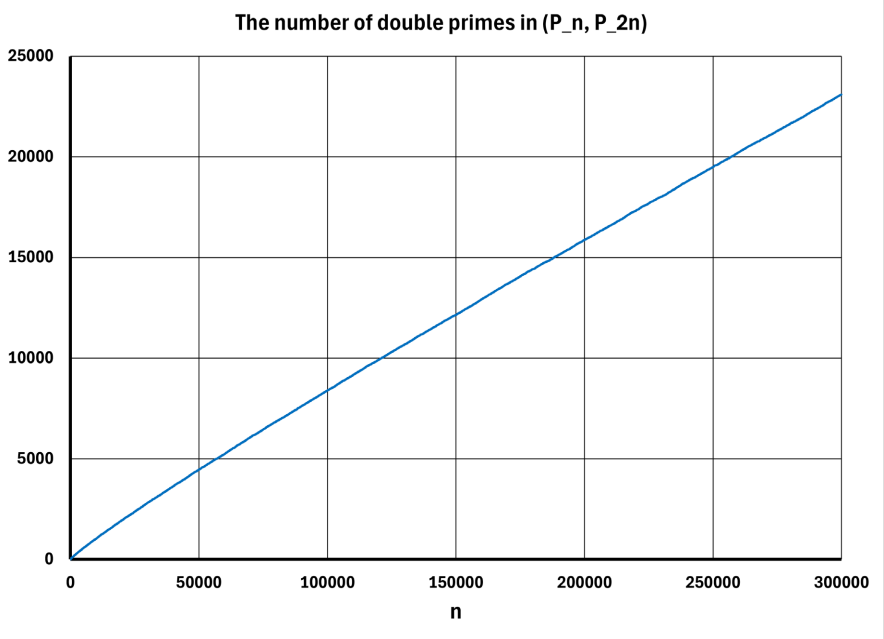

Figure 1. Counted number of double primes in (P_n, P_{2n}) for $1 < n \le 300{,}000$.

Table 3 verifies $A(i+1) - A(i) < A(i)$ and $P_{A(i+1)} - P_{A(i)} < P_{2A(i)} - P_{A(i)}$ for $i \le 50$. Because $A(i+1) - A(i) < A(i)$ and $P_{A(i+1)} - P_{A(i)} < P_{2A(i)} - P_{A(i)}$ are two corollaries of Theorem 2.8 and the theorem has been verified up to $n = 300{,}000$ in Figure 1, the two corollaries have also been verified up to $n = 300{,}000$.

Table 3. Numerical evidence verifying Corollary 2.10 and Corollary 2.12 for $i \le 50$.

i	$A(i)$	$A(i+1) - A(i)$	$P_{A(i)}$	$P_{A(i+1)} - P_{A(i)}$	$P_{2A(i)} - P_{A(i)}$
1	2	$3 - 2 = 1 < 2$	$P_2 = 3$	$5 - 3 = 2$	$7 - 3 = 4$
2	3	$5 - 3 = 2 < 3$	$P_3 = 5$	$11 - 5 = 6$	$13 - 5 = 8$
3	5	$7 - 5 = 2 < 5$	$P_5 = 11$	$17 - 11 = 6$	$29 - 11 = 18$
4	7	$11 - 7 = 4 < 7$	$P_7 = 17$	$31 - 17 = 14$	$43 - 17 = 26$
5	11	$13 - 11 = 2 < 11$	$P_{11} = 31$	$41 - 31 = 10$	$79 - 31 = 48$
6	13	$17 - 13 = 4 < 13$	$P_{13} = 41$	$59 - 41 = 18$	$101 - 41 = 60$
7	17	$19 - 17 = 2 < 17$	$P_{17} = 59$	$67 - 59 = 8$	$139 - 59 = 80$
8	19	$23 - 19 = 4 < 19$	$P_{19} = 67$	$83 - 67 = 16$	$163 - 67 = 96$
9	23	$29 - 23 = 6 < 23$	$P_{23} = 83$	$109 - 83 = 26$	$199 - 83 = 116$
10	29	$31 - 29 = 2 < 29$	$P_{29} = 109$	$127 - 109 = 18$	$271 - 109 = 162$
11	31	$37 - 31 = 6 < 31$	$P_{31} = 127$	$157 - 127 = 30$	$293 - 127 = 166$
12	37	$41 - 37 = 4 < 37$	$P_{37} = 157$	$179 - 157 = 22$	$373 - 157 = 216$
13	41	$43 - 41 = 2 < 41$	$P_{41} = 179$	$191 - 179 = 12$	$421 - 179 = 242$
14	43	$47 - 43 = 4 < 43$	$P_{43} = 191$	$211 - 191 = 20$	$443 - 191 = 252$
15	47	$53 - 47 = 6 < 47$	$P_{47} = 211$	$241 - 211 = 30$	$491 - 211 = 280$

Continued

16	53	$59-53=6<53$	$P_{53}=241$	$277-241=36$	$577-241=336$
17	59	$61-59=2<59$	$P_{59}=277$	$283-277=6$	$647-277=370$
18	61	$67-61=6<61$	$P_{61}=283$	$331-283=48$	$673-283=390$
19	67	$71-67=4<67$	$P_{67}=331$	$353-331=22$	$757-331=426$
20	71	$73-71=2<71$	$P_{71}=353$	$367-353=14$	$821-353=468$
21	73	$79-73=6<73$	$P_{73}=367$	$401-367=34$	$839-367=472$
22	79	$83-79=4<79$	$P_{79}=401$	$431-401=30$	$929-401=528$
23	83	$89-83=6<83$	$P_{83}=431$	$461-431=30$	$983-431=552$
24	89	$97-89=8<89$	$P_{89}=461$	$509-461=48$	$1061-461=600$
25	97	$101-97=8<97$	$P_{97}=509$	$547-509=38$	$1181-509=672$
26	101	$103-101=2<101$	$P_{101}=547$	$563-547=16$	$1231-547=684$
27	103	$107-103=4<103$	$P_{103}=563$	$587-563=24$	$1277-563=714$
28	107	$109-107=2<107$	$P_{107}=587$	$599-587=12$	$1307-587=720$
29	109	$113-109=4<109$	$P_{109}=599$	$617-599=18$	$1361-599=762$
30	113	$127-113=14<113$	$P_{113}=617$	$709-617=92$	$1429-617=812$
31	127	$131-127=4<127$	$P_{127}=709$	$739-709=30$	$1609-709=900$
32	131	$137-131=6<131$	$P_{131}=739$	$773-739=34$	$1667-739=928$
33	137	$139-137=2<137$	$P_{137}=773$	$797-773=24$	$1759-773=986$
34	139	$149-139=10<139$	$P_{139}=797$	$859-797=62$	$1789-797=992$
35	149	$151-149=2<149$	$P_{149}=859$	$877-859=18$	$1973-859=1114$
36	151	$157-151=6<151$	$P_{151}=877$	$919-877=42$	$1997-877=1120$
37	157	$163-157=6<157$	$P_{157}=919$	$967-919=48$	$2083-919=1164$
38	163	$167-163=4<163$	$P_{163}=967$	$991-967=24$	$2161-967=1194$
39	167	$173-167=6<167$	$P_{167}=991$	$1031-991=40$	$2243-991=1252$
40	173	$179-173=6<173$	$P_{173}=1031$	$1063-1031=32$	$2339-1031=1308$
41	179	$181-179=2<179$	$P_{179}=1063$	$1087-1063=24$	$2411-1063=1348$
42	181	$191-181=10<181$	$P_{181}=1087$	$1153-1087=66$	$2441-1087=1354$
43	191	$193-191=2<191$	$P_{191}=1153$	$1171-1153=18$	$2633-1153=1480$
44	193	$197-193=4<193$	$P_{193}=1171$	$1201-1171=30$	$2663-1171=1492$
45	197	$199-197=2<197$	$P_{197}=1201$	$1217-1201=16$	$2707-1201=1506$
46	199	$211-199=12<199$	$P_{199}=1217$	$1297-1217=80$	$2729-1217=1512$
47	211	$223-211=12<211$	$P_{211}=1297$	$1409-1297=112$	$2917-1297=1620$
48	223	$227-223=4<223$	$P_{223}=1409$	$1433-1409=24$	$3137-1409=1728$
49	227	$229-227=2<227$	$P_{227}=1433$	$1447-1433=14$	$3209-1433=1776$
50	229	$233-229=4<229$	$P_{229}=1447$	$1471-1447=24$	$3251-1447=1804$

2.7. A Standard Model for Generalizing Bertrand Theorem

All results of the Bertrand-type theorem for double prime are proved results and form a standard model for generalizing Bertrand theorem into other kinds of second order primes. There are three steps to be suitable for the model. Step 1. Set an enough large prime range to compare counted number of a kind of second order primes with counted number of double primes. We set the range is $n = 10^9$, that is, the first 1,000,000,000 primes. Step 2. If counted number of a kind of second order primes is greater than $\pi(n)$ for $n = 10^9$ then a Bertrand-Type Conjecture for the kind of second order primes can be proposed and corresponding corollaries can be established. Step 3. Verify the Bertrand-type conjecture and its corollaries up to $n = 300{,}000$ and the verification range accords with Bertrand's verification range for his conjecture (up to $x = 3{,}000{,}000$) because x expresses existence of natural number but n expresses existence of prime. If there is no counterexample in verifying results then the Bertrand-type conjecture may be true but it requires a rigorous proof. We discover that Goldbach conjecture, which can be transformed into a problem about infinitude of a kind of special primes, and twin prime conjecture are suitable for the standard model, thus, we have the following discussions.

3. Bertrand-type Conjecture for Nontrivial Prime

3.1. The Largest Strong Goldbach Number and Nontrivial Prime

Definition 3.1. An even number L_n is called *the largest strong Goldbach number generated by the n-th prime* P_n if

$$L_n = \max\left\{m : \{4, 6, \cdots, m\} \subset S_n\right\}, S_n = \left\{P_i + P_j : P_i, P_j \le P_n\right\}. \tag{3.1}$$

By Definition 3.1, the first 50 largest strong Goldbach numbers are listed as follows

4, 6, 10, 14, 18, 26, 30, 38, 42, 42, 54, 62, 74, 74, 90, 90, 90, 108, 114, 114, 134, 134, 146, 162, 172, 180, 186, 186, 218, 222, 230, 240, 240, 254, 258, 270, 270, 290, 290, 290, 330, 348, 348, 366, 366, 366, 398, 398, 410, 410.

More largest strong Goldbach numbers can be found in [31].

Remark 3.2. Definition 3.1 means every even number from 4 to L_n is the sum of two primes not greater than P_n but $L_n + 2$ is not such a sum. Thus, $L_n \le L_{n+1}$ for all n. There must exist growth points of L_n. If $L_{n-1} < L_n$, then n is a growth point of L_n and corresponding P_n is called a nontrivial prime as the following definition.

Definition 3.3. P_n is called a *nontrivial prime* if P_n satisfies

$$L_{n-1} + 2 \in \left\{P_i + P_n : P_i \le P_n\right\}. \tag{3.2}$$

By Definition 3.3, the first 50 nontrivial primes are listed as follows

P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{11}, P_{12} P_{13}, P_{15}, P_{18}, P_{19}, P_{21}, P_{23}, P_{24}, P_{25}, P_{26}, P_{27}, P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}, P_{81}, P_{83}, P_{86}, P_{88}.

More nontrivial primes can be found in [31].

By Definition 2.2, the first 50 nontrivial prime root gaps are listed as follows

1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 3, 1, 2, 2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 2, 3, 1, 2, 3, 2, 2, 1, 2, 5, 2, 2, 1, 3, 2, 1, 1, 1, 5, 4, 2, 3, 2, 3.

More nontrivial prime root gaps can be found in [31].

The first 50 nontrivial prime gaps are listed as follows

2, 2, 4, 2, 4, 2, 4, 8, 6, 4, 6, 14, 6, 6, 10, 6, 8, 4, 2, 6, 4, 14, 4, 8, 10, 2, 12, 16, 2, 12, 18, 16, 6, 6, 12, 26, 6, 24, 4, 20, 16, 2, 4, 6, 30, 30, 12, 12, 14, 10.

More nontrivial prime gaps can be found in [31].

Theorem 3.4. *If there are infinitely many nontrivial primes, then Goldbach conjecture is true.*

Proof. By Definition 3.3, every nontrivial prime constructs a growth of L_n. It means that infinitude of nontrivial primes implies L_n approaches infinity as n grows without bound. Since Goldbach conjecture is equivalent to the result that L_n approaches infinity. Hence, if there are infinitely many nontrivial primes then Goldbach conjecture is true and the theorem holds.

Remark 3.5. Definition 3.1, Definition 3.3 and Theorem 3.4 have transformed Goldbach conjecture into a problem about infinitude of a kind of special primes. It has become a foundation, on which it seems to be possible to prove Goldbach conjecture by the Bertrand-type conjecture for nontrivial prime.

3.2. PNT-Type Conjecture System for Nontrivial Prime

Let $\Omega(n)$ denote counted number of nontrivial primes among the first n primes. Then we suppose there is an approximation for $\Omega(n)$ as follows

$$\Omega(n) \approx Li(n) + \frac{n}{\log n}\left(\frac{1}{\log\log n} + \frac{1}{4(\log\log n)^2}\right), \tag{3.3}$$

where

$$Li(n) = \int_2^n \frac{\mathrm{d}t}{\log t}$$

is the logarithmic integral and it has an equivalent form

$$Li(n) \approx \frac{n}{\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k}.$$

Take the weakest form of (3.3). Then we have

$$\Omega(n) \approx \frac{n}{\log n}\left(1 + \frac{1}{\log\log n} + \frac{1}{4(\log\log n)^2}\right). \tag{3.4}$$

Table 4 gives counted number of nontrivial primes and predicted number of nontrivial primes by (3.4), which shows the weakest form is a good approximation.

Proposition 3.6. *Let $\Omega(n)$ denote counted number of nontrivial primes among the first n primes and $D^{nont}(n)$ denote average density of nontrivial primes among primes.*

If $\lim_{n\to\infty}\dfrac{\Omega(n)}{\int_2^n \dfrac{\mathrm{d}t}{\log t}+\dfrac{n}{\log n}\left(\dfrac{1}{\log\log n}+\dfrac{1}{4(\log\log n)^2}\right)}=1$, *then* $D^{nont}(n)\sim\dfrac{1}{\log n}$.

Proof. Using (3.3), $Li(n)$ can be written as

$$\frac{n}{\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k}. \tag{3.5}$$

Considering asymptotic series (3.5), the k-th term approaches higher order infinity than the $(k+1)$-th term as n grows without bound in the asymptotic series because there is the following limit.

$$\lim_{n\to\infty}\frac{\dfrac{(k+1)!n}{(\log n)^{k+2}}}{\dfrac{k!n}{(\log n)^{k+1}}}=\lim_{n\to\infty}\frac{k+1}{\log n}=0. \tag{3.6}$$

Since the first term in the asymptotic series for $Li(n)$ is $n/\log n$, there are two limits for two additional terms as follows

$$\lim_{n\to\infty}\frac{\dfrac{n}{\log n}\dfrac{1}{\log\log n}}{\dfrac{n}{\log n}}=\lim_{n\to\infty}\frac{1}{\log\log n}=0, \tag{3.7}$$

$$\lim_{n\to\infty}\frac{\dfrac{n}{\log n}\dfrac{1}{4(\log\log n)^2}}{\dfrac{n}{\log n}}=\lim_{n\to\infty}\frac{1}{4(\log\log n)^2}=0. \tag{3.8}$$

Since it is assumed that

$$\lim_{n\to\infty}\frac{\Omega(n)}{\int_2^n \dfrac{\mathrm{d}t}{\log t}+\dfrac{n}{\log n}\left(\dfrac{1}{\log\log n}+\dfrac{1}{4(\log\log n)^2}\right)}=1,$$

by (3.6), (3.7) and (3.8) we have

$$\lim_{n\to\infty}\frac{\Omega(n)}{\left(\dfrac{n}{\log n}\right)}=1. \tag{3.9}$$

The limit (3.9) means that

$$\Omega(n)\sim\frac{n}{\log n}. \tag{3.10}$$

Since $D^{nont}(n)=\dfrac{\Omega(n)}{n}$, we obtain

$$D^{nont}(n)\sim\frac{1}{\log n}. \tag{3.11}$$

Hence the proposition holds.

Corollary 3.7. *If* $\lim_{n\to\infty}\frac{\Omega(n)}{\int_2^n \frac{\mathrm{d}t}{\log t}+\frac{n}{\log n}\left(\frac{1}{\log\log n}+\frac{1}{4(\log\log n)^2}\right)}=1$, *then there are infinitely many nontrivial primes and Goldbach conjecture is true.*

Proof. By Proposition 3.6, if $\lim_{n\to\infty}\frac{\Omega(n)}{\int_2^n \frac{\mathrm{d}t}{\log t}+\frac{n}{\log n}\left(\frac{1}{\log\log n}+\frac{1}{4(\log\log n)^2}\right)}=1$ then we have

$$\Omega(n)\sim\frac{n}{\log n}.$$

Since $n/\log n$ approaches infinity as n grows without bound, $\Omega(n)$ approaches infinity as n grows without bound. It means there are infinitely many nontrivial primes among primes. By Theorem 3.4, Goldbach conjecture is true. Hence the corollary holds.

Table 4. Comparison between counted and predicted numbers of nontrivial primes.

n	counted number	predicted number	relative error
100	54	38	0.2962
1000	276	229	0.1702
10000	1867	1628	0.1280
100000	13692	12594	0.0801
1000000	109564	102493	0.0645
10000000	912223	863005	0.0539
100000000	7819294	7448150	0.0474
1000000000	68459493	65433701	0.0441

Remark 3.8. Proposition 3.6 and Corollary 3.7 form a PNT-type conjecture system, which is a strong method to show infinitude of nontrivial primes for proving Goldbach conjecture. However, there is another method to imply Goldbach conjecture because counted number of nontrivial primes accords with standard model for generalizing Bertrand theorem into nontrivial primes and we have the following discussion.

3.3. Bertrand-Type Conjecture for Nontrivial Prime

Comparing Table 4 with Table 1, counted number of nontrivial primes to be 68,459,493 is greater than counted number of double primes to be 50,847,534 for $n = 10^9$. It accords with Step 1 in standard model. So, we make the following conjecture by Step 2 in the model.

Conjecture 3.9. *Let* n *denote root of* P_n *and* $2n$ *denote root of* P_{2n}. *Then there is at least one prime root* $n + k$ *in* $(n, 2n)$ *for* $n > 1$ *such that* $n + k$ *is a nontrivial*

prime root, correspondingly, there is at least one prime P_{n+k} *in* (P_n, P_{2n}) *for* $n > 1$ *such that* P_{n+k} *is a nontrivial prime.*

Remark 3.10. Conjecture 3.9 is Bertrand-type conjecture for nontrivial prime and is a new unproved hypothesis. This conjecture will lead to the infinitude of nontrivial primes as an existence problem of nontrivial prime on n-axis. It means the number of nontrivial prime roots in $(n, 2n)$ is equal to the number of nontrivial primes in (P_n, P_{2n}) for $n > 1$.

Corollary 3.11. *Let* $B(i)$ *denote the i-th nontrivial prime root. Then there are* $B(i+1) < 2B(i)$ *and* $B(i+1) - B(i) < B(i)$.

Proof. Taking $n = B(i)$, by Conjecture 3.9, there is at least one prime root $B(i) + k$ in $(B(i), 2B(i))$ such that $B(i) + k$ is a nontrivial prime root. Let $B(i) + k = B(i+1)$ be the first nontrivial prime root in $(B(i), 2B(i))$. Then we have $B(i) < B(i+1) < 2B(i)$, that is, $B(i+1) < 2B(i)$ and $B(i+1) - B(i) < B(i)$. Hence the corollary holds.

Corollary 3.12. *Let* $P_{B(i)}$ *denote the i-th nontrivial prime. Then there are* $P_{B(i+1)} < P_{2B(i)}$ *and* $P_{B(i+1)} - P_{B(i)} < P_{2B(i)} - P_{B(i)}$.

Proof. Conjecture 3.9 states there is at least one prime P_{n+k} in (P_n, P_{2n}) for $n > 1$ such that P_{n+k} is a nontrivial prime. Take $P_n = P_{B(i)}$. Let $P_{n+k} = P_{B(i+1)}$ be the first nontrivial prime in $(P_{B(i)}, P_{2B(i)})$. Then we have

$$P_{B(i+1)} < P_{2B(i)}, \tag{3.12}$$

equivalently,

$$P_{B(i+1)} - P_{B(i)} < P_{2B(i)} - P_{B(i)}. \tag{3.13}$$

Hence the corollary holds.

Corollary 3.13. *There are infinitely many nontrivial primes among primes.*

Proof. Suppose $B(i)$ is the largest nontrivial prime root. Then there is a nontrivial prime root $B(i+1)$ greater than $B(i)$ such that $B(i) < B(i+1) < 2B(i)$ by Corollary 3.11. Thus, there is no the largest nontrivial prime root and nontrivial prime roots are infinite. Since every nontrivial prime root $B(i)$ links with a nontrivial prime $P_{B(i)}$, there are infinitely many nontrivial primes among primes. Hence the corollary holds.

Corollary 3.14. *Goldbach conjecture is true.*

Proof. If Conjecture 3.9 is true, then, by Corollary 3.13 there are infinitely many nontrivial primes. By Theorem 3.4, Goldbach conjecture is true. Hence the corollary holds.

Remark 3.15. Corollary 3.14 means Goldbach conjecture may be proven as an existence problem of nontrivial primes.

All above results form a system of the Bertrand-type conjecture for nontrivial prime. The central content is: First, there is at least one prime root $n + k$ in $(n, 2n)$ for $n > 1$ such that $n + k$ is a nontrivial prime root, equivalently, there is at least one prime P_{n+k} in (P_n, P_{2n}) for $n > 1$ such that P_{n+k} is a nontrivial prime. Second, there are $B(i+1) < 2B(i)$ and $B(i+1) - B(i) < B(i)$. Third, there are $P_{B(i+1)} < P_{2B(i)}$, $P_{B(i+1)} - P_{B(i)} < P_{2B(i)} - P_{B(i)}$ and $P_{2B(i)} - P_{B(i)} > P_{B(i)}$ for $i \geq 1$.

3.4. Verification of Bertrand-Type Conjecture for Nontrivial Prime

Verification of Bertrand-type conjecture for nontrivial prime is divided into two parts: verification of Conjecture 3.9 as well as verification of Corollary 3.11 and Corollary 3.12.

Table 5 verifies Conjecture 3.9 for $1 < n \le 50$ using the number of nontrivial prime roots in $(n, 2n)$ greater than 0 and the number of nontrivial primes in (P_n, P_{2n}) greater than 0. Values of these numbers can be counted based on all nontrivial prime roots and all nontrivial primes given by the table. Figure 2 gives all numerical evidences to verify Conjecture 3.9 for $1 < n \le 300{,}000$ because data curve in the figure shows the number of nontrivial primes in (P_n, P_{2n}) to be greater than 0 for $1 < n \le 300{,}000$, equivalently, the figure also verifies the number of nontrivial prime roots in $(n, 2n)$ to be greater than 0 for $1 < n \le 300{,}000$ because the number of nontrivial prime roots in $(n, 2n)$ is always equal to the number of nontrivial primes in (P_n, P_{2n}). Thus, Conjecture 3.9 has been verified up to $n = 300{,}000$. All raw data to show counted number of nontrivial primes in (P_n, P_{2n}) for every n for $1 < n \le 300{,}000$, which have been drawn as curve in Figure 2, can be found in [31]. For example, the number of nontrivial primes in (P_{491}, P_{982}) is counted as 106 and the number of nontrivial primes in (P_{901}, P_{1802}) is counted as 180 by raw data in [31]. The reference [31] includes 15105 pages to show raw data for nontrivial primes less than 10,000,000, which can cover any counting require for nontrivial primes up to $n = 300{,}000$.

Table 5. Nontrivial prime roots in $(n, 2n)$ and nontrivial primes in (P_n, P_{2n}).

n	all nontrivial prime roots	$2n$	P_n	all nontrivial primes	P_{2n}
2	3	4	P_2	P_3	P_4
3	4, 5	6	P_3	P_4, P_5	P_6
4	5, 6, 7	8	P_4	P_5, P_6, P_7	P_8
5	6, 7, 8, 9	10	P_5	P_6, P_7, P_8, P_9	P_{10}
6	7, 8, 9, 11	12	P_6	P_7, P_8, P_9, P_{11}	P_{12}
7	8, 9, 11, 12, 13	14	P_7	P_8, P_9, P_{11}, P_{12}, P_{13}	P_{14}
8	9, 11, 12, 13, 15	16	P_8	P_9, P_{11}, P_{12}, P_{13}, P_{15}	P_{16}
9	11, 12, 13, 15	18	P_9	P_{11}, P_{12}, P_{13}, P_{15}	P_{18}
10	11, 12, 13, 15, 18, 19	20	P_{10}	P_{11}, P_{12}, P_{13}, P_{15}, P_{18}, P_{19}	P_{20}
11	12, 13, 15, 18, 19, 21	22	P_{11}	P_{12}, P_{13}, P_{15}, P_{18}, P_{19}, P_{21}	P_{22}
12	13, 15, 18, 19, 21, 23	24	P_{12}	P_{13}, P_{15}, P_{18}, P_{19}, P_{21}, P_{23}	P_{24}
13	15, 18, 19, 21, 23, 24, 25	26	P_{13}	P_{15}, P_{18}, P_{19}, P_{21}, P_{23}, P_{24}, P_{25}	P_{26}
14	15, 18, 19, 21, 23, 24, 25, 26, 27	28	P_{14}	P_{15}, P_{18}, P_{19}, P_{21}, P_{23}, P_{24}, P_{25}, P_{26}, P_{27}	P_{28}
15	18, 19, 21, 23, 24, 25, 26, 27, 29	30	P_{15}	P_{18}, P_{19}, P_{21}, P_{23}, P_{24}, P_{25}, P_{26}, P_{27}, P_{29}	P_{30}
16	18, 19, 21, 23, 24, 25, 26, 27, 29, 30, 31	32	P_{16}	P_{18}, P_{19}, P_{21}, P_{23}, P_{24}, P_{25}, P_{26}, P_{27}, P_{29}, P_{30}, P_{31}	P_{32}

Continued

17	18, 19, 21, 23, 24, 25, 26, 27, 29, 30, 31, 32	34	P_{17}	P_{18}, P_{19}, P_{21}, P_{23}, P_{24}, P_{25}, P_{26}, P_{27}, P_{29}, P_{30}, P_{31}, P_{32}	P_{34}
18	19, 21, 23, 24, 25, 26, 27, 29, 30, 31, 32, 34, 35	36	P_{18}	P_{19}, P_{21}, P_{23}, P_{24}, P_{25}, P_{26}, P_{27}, P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}	P_{36}
19	21, 23, 24, 25, 26, 27, 29, 30, 31, 32, 34, 35, 36	38	P_{19}	P_{21}, P_{23}, P_{24}, P_{25}, P_{26}, P_{27}, P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}	P_{38}
20	21, 23, 24, 25, 26, 27, 29, 30, 31, 32, 34, 35, 36, 38	40	P_{20}	P_{21}, P_{23}, P_{24}, P_{25}, P_{26}, P_{27}, P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}	P_{40}
21	23, 24, 25, 26, 27, 29, 30, 31, 32, 34, 35, 36, 38, 41	42	P_{21}	P_{23}, P_{24}, P_{25}, P_{26}, P_{27}, P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}	P_{42}
22	23, 24, 25, 26, 27, 29, 30, 31, 32, 34, 35, 36, 38, 41, 42	44	P_{22}	P_{23}, P_{24}, P_{25}, P_{26}, P_{27}, P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}	P_{44}
23	24, 25, 26, 27, 29, 30, 31, 32, 34, 35, 36, 38, 41, 42, 44	46	P_{23}	P_{24}, P_{25}, P_{26}, P_{27}, P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}	P_{46}
24	25, 26, 27, 29, 30, 31, 32, 34, 35, 36, 38, 41, 42, 44, 47	48	P_{24}	P_{25}, P_{26}, P_{27}, P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}	P_{48}
25	26, 27, 29, 30, 31, 32, 34, 35, 36, 38, 41, 42, 44, 47, 49	50	P_{25}	P_{26}, P_{27}, P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}	P_{50}
26	27, 29, 30, 31, 32, 34, 35, 36, 38, 41, 42, 44, 47, 49, 51	52	P_{26}	P_{27}, P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}	P_{52}
27	29, 30, 31, 32, 34, 35, 36, 38, 41, 42, 44, 47, 49, 51, 52	54	P_{27}	P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}	P_{54}
28	29, 30, 31, 32, 34, 35, 36, 38, 41, 42, 44, 47, 49, 51, 52, 54	56	P_{28}	P_{29}, P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}	P_{56}
29	30, 31, 32, 34, 35, 36, 38, 41, 42, 44, 47, 49, 51, 52, 54	58	P_{29}	P_{30}, P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}	P_{58}
30	31, 32, 34, 35, 36, 38, 41, 42, 44, 47, 49, 51, 52, 54, 59	60	P_{30}	P_{31}, P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}	P_{60}
31	32, 34, 35, 36, 38, 41, 42, 44, 47, 49, 51, 52, 54, 59, 61	62	P_{31}	P_{32}, P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}	P_{62}
32	34, 35, 36, 38, 41, 42, 44, 47, 49, 51, 52, 54, 59, 61, 63	64	P_{32}	P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}	P_{64}
33	34, 35, 36, 38, 41, 42, 44, 47, 49, 51, 52, 54, 59, 61, 63, 64	66	P_{33}	P_{34}, P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}	P_{66}
34	35, 36, 38, 41, 42, 44, 47, 49, 51, 52, 54, 59, 61, 63, 64, 67	68	P_{34}	P_{35}, P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}	P_{68}
35	36, 38, 41, 42, 44, 47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69	70	P_{35}	P_{36}, P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}	P_{70}
36	38, 41, 42, 44, 47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71	72	P_{36}	P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}	P_{72}
37	38, 41, 42, 44, 47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72	74	P_{37}	P_{38}, P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}	P_{74}
38	41, 42, 44, 47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72	76	P_{38}	P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}	P_{76}
39	41, 42, 44, 47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77	78	P_{39}	P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}	P_{78}
40	41, 42, 44, 47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77	80	P_{40}	P_{41}, P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}	P_{80}
41	42, 44, 47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77, 81	82	P_{41}	P_{42}, P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}, P_{81}	P_{82}
42	44, 47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77, 81, 83	84	P_{42}	P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}, P_{81}, P_{83}	P_{84}

Continued

43	44, 47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77, 81, 83	86	P_{43}	P_{44}, P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}, P_{81}, P_{83} P_{86}
44	47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77, 81, 83, 86	88	P_{44}	P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}, P_{81}, P_{83}, P_{86} P_{88}
45	47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77, 81, 83, 86, 88	90	P_{45}	P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}, P_{81}, P_{83}, P_{86}, P_{88} P_{90}
46	47, 49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77, 81, 83, 86, 88, 91	92	P_{46}	P_{47}, P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}, P_{81}, P_{83}, P_{86}, P_{88}, P_{91} P_{92}
47	49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77, 81, 83, 86, 88, 91, 93	94	P_{47}	P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}, P_{81}, P_{83}, P_{86}, P_{88}, P_{91}, P_{93} P_{94}
48	49, 51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77, 81, 83, 86, 88, 91, 93, 95	96	P_{48}	P_{49}, P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}, P_{81}, P_{83}, P_{86}, P_{88}, P_{91}, P_{93}, P_{95} P_{96}
49	51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77, 81, 83, 86, 88, 91, 93, 95, 96	98	P_{49}	P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}, P_{81}, P_{83}, P_{86}, P_{88}, P_{91}, P_{93}, P_{95}, P_{96} P_{98}
50	51, 52, 54, 59, 61, 63, 64, 67, 69, 70, 71, 72, 77, 81, 83, 86, 88, 91, 93, 95, 96	100	P_{50}	P_{51}, P_{52}, P_{54}, P_{59}, P_{61}, P_{63}, P_{64}, P_{67}, P_{69}, P_{70}, P_{71}, P_{72}, P_{77}, P_{81}, P_{83}, P_{86}, P_{88}, P_{91}, P_{93}, P_{95}, P_{96} P_{100}

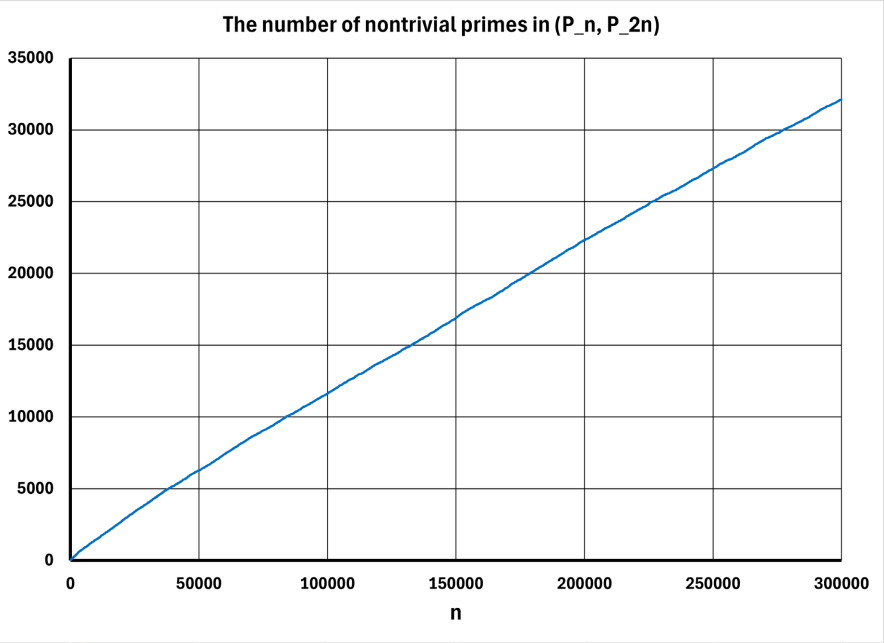

Figure 2. Counted number of nontrivial primes in (P_n, P_{2n}) for $1 < n \le 300{,}000$.

Table 6 verifies $B(i+1) - B(i) < B(i)$ and $P_{B(i+1)} - P_{B(i)} < P_{2B(i)} - P_{B(i)}$ for $i \le 50$. Because $B(i+1) - B(i) < B(i)$ and $P_{B(i+1)} - P_{B(i)} < P_{2B(i)} - P_{B(i)}$ are two corollaries of Conjecture 3.9 and the conjecture has been verified up to $n = 300{,}000$ in Figure 2, the two corollaries have also been verified up to $n = 300{,}000$.

Note that there is no counterexample in all verifying results up to $n = 300{,}000$ at Step 3 of our model. Thus, if Conjecture 3.9 is proven then Goldbach conjecture is true.

Table 6. Numerical evidence verifying Corollary 3.11 and Corollary 3.12 for $i \le 50$.

i	$B(i)$	$B(i+1)-B(i)$	$P_{B(i)}$	$P_{B(i+1)}-P_{B(i)}$	$P_{2B(i)}-P_{B(i)}$
1	2	$3-2=1<2$	$P_2=3$	$5-3=2$	$7-3=4$
2	3	$4-3=1<3$	$P_3=5$	$7-5=2$	$13-5=8$
3	4	$5-4=1<4$	$P_4=7$	$11-7=4$	$19-7=12$
4	5	$6-5=1<5$	$P_5=11$	$13-11=2$	$29-11=18$
5	6	$7-6=1<6$	$P_6=13$	$17-13=4$	$37-13=24$
6	7	$8-7=1<7$	$P_7=17$	$19-17=2$	$43-17=26$
7	8	$9-8=1<8$	$P_8=19$	$23-19=4$	$53-19=34$
8	9	$11-9=2<9$	$P_9=23$	$31-23=8$	$61-23=38$
9	11	$12-11=1<11$	$P_{11}=31$	$37-31=6$	$79-31=48$
10	12	$13-12=1<12$	$P_{12}=37$	$41-37=4$	$89-37=52$
11	13	$15-13=2<13$	$P_{13}=41$	$47-41=6$	$101-41=60$
12	15	$18-15=3<15$	$P_{15}=47$	$61-47=14$	$113-47=66$
13	18	$19-18=1<18$	$P_{18}=61$	$67-61=6$	$151-61=90$
14	19	$21-19=2<19$	$P_{19}=67$	$73-67=6$	$163-67=96$
15	21	$23-21=2<21$	$P_{21}=73$	$83-73=10$	$181-73=108$
16	23	$24-23=1<23$	$P_{23}=83$	$89-83=6$	$199-83=116$
17	24	$25-24=1<24$	$P_{24}=89$	$97-89=8$	$223-89=134$
18	25	$26-25=1<25$	$P_{25}=97$	$101-97=4$	$229-97=132$
19	26	$27-26=1<26$	$P_{26}=101$	$103-101=2$	$239-101=138$
20	27	$29-27=2<27$	$P_{27}=103$	$109-103=6$	$251-103=148$
21	29	$30-29=1<29$	$P_{29}=109$	$113-109=4$	$271-109=162$
22	30	$31-30=1<30$	$P_{30}=113$	$127-113=14$	$281-113=168$
23	31	$32-31=1<31$	$P_{31}=127$	$131-127=4$	$293-127=166$
24	32	$34-32=2<32$	$P_{32}=131$	$139-131=8$	$311-131=180$
25	34	$35-34=1<34$	$P_{34}=139$	$149-139=10$	$337-139=198$
26	35	$36-35=1<35$	$P_{35}=149$	$151-149=2$	$349-149=200$
27	36	$38-36=2<36$	$P_{36}=151$	$163-151=12$	$359-151=208$
28	38	$41-38=3<38$	$P_{38}=163$	$179-163=16$	$383-163=220$
29	41	$42-41=1<41$	$P_{41}=179$	$181-179=2$	$421-179=242$
30	42	$44-42=2<42$	$P_{42}=181$	$193-181=12$	$433-181=252$
31	44	$47-44=3<44$	$P_{44}=193$	$211-193=18$	$457-193=264$
32	47	$49-47=2<47$	$P_{47}=211$	$227-211=16$	$491-211=280$
33	49	$51-49=2<49$	$P_{49}=227$	$233-227=6$	$521-227=294$
34	51	$52-51=1<51$	$P_{51}=233$	$239-233=6$	$557-233=324$

Continued

35	52	$54-52=2<52$	$P_{52}=239$	$251-239=12$	$569-239=330$
36	54	$59-54=5<54$	$P_{54}=251$	$277-251=26$	$593-251=342$
37	59	$61-59=2<59$	$P_{59}=277$	$283-277=6$	$647-277=370$
38	61	$63-61=2<61$	$P_{61}=283$	$307-283=24$	$673-283=390$
39	63	$64-63=1<63$	$P_{63}=307$	$311-307=4$	$701-307=394$
40	64	$67-64=3<64$	$P_{64}=311$	$331-311=20$	$719-311=408$
41	67	$69-67=2<67$	$P_{67}=331$	$347-331=16$	$757-331=426$
42	69	$70-69=1<69$	$P_{69}=347$	$349-347=2$	$787-347=440$
43	70	$71-70=1<70$	$P_{70}=349$	$353-349=4$	$809-349=460$
44	71	$72-71=1<71$	$P_{71}=353$	$359-353=6$	$821-353=468$
45	72	$77-72=5<72$	$P_{72}=359$	$389-359=30$	$827-359=468$
46	77	$81-77=4<77$	$P_{77}=389$	$419-389=30$	$887-389=498$
47	81	$83-81=2<81$	$P_{81}=419$	$431-419=12$	$953-419=534$
48	83	$86-83=3<83$	$P_{83}=431$	$443-431=12$	$983-431=552$
49	86	$88-86=2<86$	$P_{86}=443$	$457-443=14$	$1021-443=578$
50	88	$91-88=3<88$	$P_{88}=457$	$467-457=10$	$1049-457=592$

4. Bertrand-Type Conjecture for Twin Prime

4.1. Twin Prime and the Twin Prime Conjecture

Definition 4.1. Let P_n denote the n-th prime. P_n is called a *twin prime* if $P_{n+1}-P_n=2$.

By Definition 4.1, the first 50 twin primes are listed as follows.

P_2, P_3, P_5, P_7, P_{10}, P_{13}, P_{17}, P_{20}, P_{26}, P_{28}, P_{33}, P_{35}, P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}, P_{81}, P_{83}, P_{89}, P_{98}, P_{104}, P_{109}, P_{113}, P_{116}, P_{120}, P_{140}, P_{142}, P_{144}, P_{148}, P_{152}, P_{171}, P_{173}, P_{176}, P_{178}, P_{182}, P_{190}, P_{201}, P_{206}, P_{209}, P_{212}, P_{215}, P_{225}, P_{230}, P_{234}, P_{236}.

By Definition 2.2, the first 50 twin prime root gaps are listed as follows.

1, 2, 2, 3, 3, 4, 3, 6, 2, 5, 2, 6, 2, 2, 4, 3, 5, 3, 4, 5, 12, 2, 6, 9, 6, 5, 4, 3, 4, 20, 2, 2, 4, 4, 19, 2, 3, 2, 4, 8, 11, 5, 3, 3, 3, 10, 5, 4, 2, 17.

The first 50 twin prime gaps are listed as follows.

2, 6, 6, 12, 12, 18, 12, 30, 6, 30, 12, 30, 12, 6, 30, 12, 30, 12, 30, 36, 72, 12, 30, 60, 48, 30, 18, 24, 18, 150, 12, 6, 30, 24, 138, 12, 18, 12, 30, 60, 78, 48, 12, 12, 18, 108, 24, 30, 6, 120.

Polignac conjecture states that there are infinitely many primes p such that $p+2k$ is also prime for every natural number k. Twin prime conjecture is the Polignac conjecture case for $k=1$. Let $\pi_2(x)$ denote counted number of primes $p \le x$ such that $p+2$ is also prime. Define twin prime constant C_2 as [32]

$$C_2=\prod_{p\ge 3}\left(1-\frac{1}{(p-1)^2}\right)=\prod_{p\ge 3}\frac{p(p-2)}{(p-1)^2}\approx 0.660161815\cdots. \tag{4.1}$$

Then there is a special case of the first Hardy-Littlewood conjecture such that

$$\pi_2(x) \approx 2C_2 \int_2^x \frac{\mathrm{d}t}{(\log t)^2},$$

$$\pi_2(x) \sim 2C_2 \frac{x}{(\log x)^2}, \tag{4.2}$$

in the sense that the quotient of the two expressions approaches 1 as x grows without bound [33]. Let $D^{twin}(x)$ denote average density of twin primes among natural numbers. Then from (4.2) we have

$$D^{twin}(x) \sim 2C_2 \frac{1}{(\log x)^2}, \tag{4.3}$$

The result is also a conjecture as asymptotic form (4.2) does.

4.2. PNT-Type Conjecture System for Twin Prime

Let $\pi_2(n)$ denote counted number of twin primes among the first n primes. Then it can be conjectured that there is an approximation for $\pi_2(n)$ as follows

$$\pi_2(n) \approx 2C_2 Li(n), \tag{4.4}$$

where C_2 is twin prime constant (4.1) and

$$Li(n) = \int_2^n \frac{\mathrm{d}t}{\log t} \approx \frac{n}{\log n} \sum_{k=0}^{\infty} \frac{k!}{(\log n)^k}.$$

Take the weakest form of (4.4). Then we have

$$\pi_2(n) \approx 2C_2 \frac{n}{\log n}. \tag{4.5}$$

Using the weakest form to predict the number of twin primes among primes, Table 7 gives counted number of twin primes among primes and predicted number of twin primes among primes by (4.5).

Proposition 4.2. *Let $\pi_2(n)$ denote counted number of twin primes among the first n primes and $D^{twin}(n)$ denote average density of twin primes among primes.*

If $\lim_{n\to\infty} \frac{\pi_2(n)}{2C_2 \int_2^n \frac{\mathrm{d}t}{\log t}} = 1$, *then* $D^{twin}(n) \sim 2C_2 \frac{1}{\log n}$.

Proof. Using (4.4), $2C_2 Li(n)$ can be written as

$$2C_2 \frac{n}{\log n} \sum_{k=0}^{\infty} \frac{k!}{(\log n)^k}. \tag{4.6}$$

Considering asymptotic series (4.6), the k-th term approaches higher order infinity than the $(k+1)$-th term as n grows without bound in the asymptotic series because there is the following limit.

$$\lim_{n\to\infty} \frac{2C_2 \frac{(k+1)!n}{(\log n)^{k+2}}}{2C_2 \frac{k!n}{(\log n)^{k+1}}} = \lim_{n\to\infty} \frac{k+1}{\log n} = 0. \tag{4.7}$$

Since it is assumed that

$$\lim_{n\to\infty}\frac{\pi_2(n)}{2C_2\int_2^n\frac{\mathrm{d}t}{\log t}}=1,$$

by (4.6) and (4.7) we have

$$\lim_{n\to\infty}\frac{\pi_2(n)}{2C_2\frac{n}{\log n}}=1. \tag{4.8}$$

The limit (4.8) means that

$$\pi_2(n)\sim 2C_2\frac{n}{\log n}. \tag{4.9}$$

Since $D^{twin}(n)=\frac{\pi_2(n)}{n}$, we obtain

$$D^{twin}(n)\sim 2C_2\frac{1}{\log n}. \tag{4.10}$$

Hence the proposition holds.

Table 7. Comparison between counted and predicted numbers of twin primes.

n	counted number	predicted number	relative error
100	25	28	0.1071
1000	174	191	0.0890
10000	1270	1433	0.1137
100000	10250	11468	0.1062
1000000	86027	95568	0.0998
10000000	738597	819156	0.0983
100000000	6497407	7167615	0.0935
1000000000	58047180	63712139	0.0889

Corollary 4.3. *If* $\lim_{n\to\infty}\frac{\pi_2(n)}{2C_2\int_2^n\frac{\mathrm{d}t}{\log t}}=1$, *then there are infinitely many twin primes.*

Proof. By Proposition 4.2, if $\lim_{n\to\infty}\frac{\pi_2(n)}{2C_2\int_2^n\frac{\mathrm{d}t}{\log t}}=1$ then we have

$$\pi_2(n)\sim 2C_2\frac{n}{\log n}.$$

Since $n/\log n$ approaches infinity as n grows without bound, $\pi_2(n)$ approaches infinity as n grows without bound. It means there are infinitely many twin primes

among primes, that is, there are infinitely many twin primes. Hence the corollary holds.

Remark 4.4. Corollary 4.3 implies twin prime conjecture by PNT-type system for twin prime, which is a strong form to show the infinitude of twin primes.

4.3. Bertrand-Type Conjecture for Twin Prime

Comparing Table 7 with Table 1, counted number of twin primes to be 58047180 is greater than counted number of double primes to be 50847534 for $n = 10^9$. It accords with Step 1 in standard model. So, we make the following conjecture by Step 2 in the model.

Conjecture 4.5. *Let n denote root of P_n and $2n$ denote root of P_{2n}. Then there is at least one prime root $n + k$ in $(n, 2n)$ for $n > 1$ such that $n + k$ is a twin prime root, correspondingly, there is at least one prime P_{n+k} in (P_n, P_{2n}) for $n > 1$ such that P_{n+k} is a twin prime.*

Remark 4.6. Conjecture 4.5 is Bertrand-type conjecture for twin prime and is a new unproved hypothesis. This conjecture will lead to the infinitude of twin primes as an existence problem of twin prime on n-axis. It means the number of twin prime roots in $(n, 2n)$ is equal to the number of twin primes in (P_n, P_{2n}) for $n > 1$.

Corollary 4.7. *Let $C(i)$ denote the i-th twin prime root. Then there are $C(i+1) < 2C(i)$ and $C(i+1) - C(i) < C(i)$.*

Proof. Taking $n = C(i)$, by Conjecture 4.5, there is at least one prime root $C(i) + k$ in $(C(i), 2C(i))$ such that $C(i) + k$ is a twin prime root. Let $C(i) + k = C(i+1)$ be the first twin prime root in $(C(i), 2C(i))$. Then we have $C(i) < C(i+1) < 2C(i)$, that is, $C(i+1) < 2C(i)$ and $C(i+1) - C(i) < C(i)$. Hence the corollary holds.

Corollary 4.8. *Let $P_{C(i)}$ denote the i-th twin prime. Then there are $P_{C(i+1)} < P_{2C(i)}$ and $P_{C(i+1)} - P_{C(i)} < P_{2C(i)} - P_{C(i)}$.*

Proof. Conjecture 4.5 states there is at least one prime P_{n+k} in (P_n, P_{2n}) for $n > 1$ such that P_{n+k} is a twin prime. Take $P_n = P_{C(i)}$. Let $P_{n+k} = P_{C(i+1)}$ be the first twin prime in $(P_{C(i)}, P_{2C(i)})$. Then we have

$$P_{C(i+1)} < P_{2C(i)}, \tag{4.11}$$

equivalently,

$$P_{C(i+1)} - P_{C(i)} < P_{2C(i)} - P_{C(i)}. \tag{4.12}$$

Hence the corollary holds.

Corollary 4.9. *There are infinitely many twin primes.*

Proof. Suppose $C(i)$ is the largest twin prime root. Then there is a twin prime root $C(i+1)$ greater than $C(i)$ such that $C(i) < C(i+1) < 2C(i)$ by Corollary 4.7. Thus, there is no the largest twin prime root and twin prime roots are infinite. Since every twin prime root $C(i)$ links with a twin prime $P_{C(i)}$, there are infinitely many twin primes among primes, that is, there are infinitely many twin primes. Hence the corollary holds.

All above results form a system of the Bertrand-type conjecture for twin prime. The central content is: First, there is at least one prime root $n + k$ in $(n, 2n)$ for

$n > 1$ such that $n + k$ is a twin prime root, equivalently, there is at least one prime P_{n+k} in (P_n, P_{2n}) for $n > 1$ such that P_{n+k} is a twin prime. Second, there are $C(i+1) < 2C(i)$ and $C(i+1) - C(i) < C(i)$. Third, there are $P_{C(i+1)} < P_{2C(i)}$, $P_{C(i+1)} - P_{C(i)} < P_{2C(i)} - P_{C(i)}$, $P_{2C(i)} - P_{C(i)} > P_{C(i)}$ for $i \geq 1$.

4.4. Verification of Bertrand-Type Conjecture for Twin Prime

Verification of Bertrand-type conjecture for twin prime is divided into two parts: verification of Conjecture 4.5 as well as verification of Corollary 4.7 and Corollary 4.8.

Table 8 verifies Conjecture 4.5 for $1 < n \leq 50$ using the number of twin prime roots in $(n, 2n)$ greater than 0 and the number of twin primes in (P_n, P_{2n}) greater than 0. Values of these numbers can be counted based on all twin prime roots and all twin primes given by the table. Figure 3 gives all numerical evidences to verify Conjecture 4.5 for $1 < n \leq 300{,}000$ because data curve in the figure shows the number of twin primes in (P_n, P_{2n}) to be greater than 0 for $1 < n \leq 300{,}000$, equivalently, the figure also verifies the number of twin prime roots in $(n, 2n)$ to be greater than 0 for $1 < n \leq 300{,}000$ because the number of twin prime roots in $(n, 2n)$ is always equal to the number of twin primes in (P_n, P_{2n}). Thus, Conjecture 4.5 has been verified up to $n = 300{,}000$. All raw data to show counted number of twin primes in (P_n, P_{2n}) for every n for $1 < n \leq 300{,}000$, which have been drawn as curve in Figure 3, can be found in [31]. For example, the number of twin primes in (P_{618}, P_{1236}) is counted as 89 and the number of twin primes in (P_{993}, P_{1986}) is counted as 127 by raw data in [31]. The reference [31] includes 15105 pages to show raw data for twin primes less than 10,000,000, which can cover any counting require for twin primes up to $n = 300{,}000$.

Table 8. Twin prime roots in $(n, 2n)$ and twin primes in (P_n, P_{2n}) for $1 < n \leq 50$.

n	all twin prime roots	$2n$	P_n	all twin primes	P_{2n}
2	3	4	P_2	P_3	P_4
3	5	6	P_3	P_5	P_6
4	5, 7	8	P_4	P_5, P_7	P_8
5	7	10	P_5	P_7	P_{10}
6	7, 10	12	P_6	P_7, P_{10}	P_{12}
7	10, 13	14	P_7	P_{10}, P_{13}	P_{14}
8	10, 13	16	P_8	P_{10}, P_{13}	P_{16}
9	10, 13, 17	18	P_9	P_{10}, P_{13}, P_{17}	P_{18}
10	13, 17	20	P_{10}	P_{13}, P_{17}	P_{20}
11	13, 17, 20	22	P_{11}	P_{13}, P_{17}, P_{20}	P_{22}
12	13, 17, 20	24	P_{12}	P_{13}, P_{17}, P_{20}	P_{24}
13	17, 20	26	P_{13}	P_{17}, P_{20}	P_{26}
14	17, 20, 26	28	P_{14}	P_{17}, P_{20}, P_{26}	P_{28}

Continued

15	17, 20, 26, 28	30	P_{15}	P_{17}, P_{20}, P_{26}, P_{28}	P_{30}
16	17, 20, 26, 28	32	P_{16}	P_{17}, P_{20}, P_{26}, P_{28}	P_{32}
17	20, 26, 28, 33	34	P_{17}	P_{20}, P_{26}, P_{28}, P_{33}	P_{34}
18	20, 26, 28, 33, 35	36	P_{18}	P_{20}, P_{26}, P_{28}, P_{33}, P_{35}	P_{36}
19	20, 26, 28, 33, 35	38	P_{19}	P_{20}, P_{26}, P_{28}, P_{33}, P_{35}	P_{38}
20	26, 28, 33, 35	40	P_{20}	P_{26}, P_{28}, P_{33}, P_{35}	P_{40}
21	26, 28, 33, 35, 41	42	P_{21}	P_{26}, P_{28}, P_{33}, P_{35}, P_{41}	P_{42}
22	26, 28, 33, 35, 41, 43	44	P_{22}	P_{26}, P_{28}, P_{33}, P_{35}, P_{41}, P_{43}	P_{44}
23	26, 28, 33, 35, 41, 43, 45	46	P_{23}	P_{26}, P_{28}, P_{33}, P_{35}, P_{41}, P_{43}, P_{45}	P_{46}
24	26, 28, 33, 35, 41, 43, 45	48	P_{24}	P_{26}, P_{28}, P_{33}, P_{35}, P_{41}, P_{43}, P_{45}	P_{48}
25	26, 28, 33, 35, 41, 43, 45, 49	50	P_{25}	P_{26}, P_{28}, P_{33}, P_{35}, P_{41}, P_{43}, P_{45}, P_{49}	P_{50}
26	28, 33, 35, 41, 43, 45, 49	52	P_{26}	P_{28}, P_{33}, P_{35}, P_{41}, P_{43}, P_{45}, P_{49}	P_{52}
27	28, 33, 35, 41, 43, 45, 49, 52	54	P_{27}	P_{28}, P_{33}, P_{35}, P_{41}, P_{43}, P_{45}, P_{49}, P_{52}	P_{54}
28	33, 35, 41, 43, 45, 49, 52	56	P_{28}	P_{33}, P_{35}, P_{41}, P_{43}, P_{45}, P_{49}, P_{52}	P_{56}
29	33, 35, 41, 43, 45, 49, 52, 57	58	P_{29}	P_{33}, P_{35}, P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}	P_{58}
30	33, 35, 41, 43, 45, 49, 52, 57	60	P_{30}	P_{33}, P_{35}, P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}	P_{60}
31	33, 35, 41, 43, 45, 49, 52, 57, 60	62	P_{31}	P_{33}, P_{35}, P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}	P_{62}
32	33, 35, 41, 43, 45, 49, 52, 57, 60	64	P_{32}	P_{33}, P_{35}, P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}	P_{64}
33	35, 41, 43, 45, 49, 52, 57, 60, 64	66	P_{33}	P_{35}, P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}	P_{66}
34	35, 41, 43, 45, 49, 52, 57, 60, 64	68	P_{34}	P_{35}, P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}	P_{68}
35	41, 43, 45, 49, 52, 57, 60, 64, 69	70	P_{35}	P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}	P_{70}
36	41, 43, 45, 49, 52, 57, 60, 64, 69	72	P_{36}	P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}	P_{72}
37	41, 43, 45, 49, 52, 57, 60, 64, 69	74	P_{37}	P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}	P_{74}
38	41, 43, 45, 49, 52, 57, 60, 64, 69	76	P_{38}	P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}	P_{76}
39	41, 43, 45, 49, 52, 57, 60, 64, 69	78	P_{39}	P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}	P_{78}
40	41, 43, 45, 49, 52, 57, 60, 64, 69	80	P_{40}	P_{41}, P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}	P_{80}
41	43, 45, 49, 52, 57, 60, 64, 69, 81	82	P_{41}	P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}, P_{81}	P_{82}
42	43, 45, 49, 52, 57, 60, 64, 69, 81, 83	84	P_{42}	P_{43}, P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}, P_{81}, P_{83}	P_{84}
43	45, 49, 52, 57, 60, 64, 69, 81, 83	86	P_{43}	P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}, P_{81}, P_{83}	P_{86}
44	45, 49, 52, 57, 60, 64, 69, 81, 83	88	P_{44}	P_{45}, P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}, P_{81}, P_{83}	P_{88}
45	49, 52, 57, 60, 64, 69, 81, 83, 89	90	P_{45}	P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}, P_{81}, P_{83}, P_{89}	P_{90}
46	49, 52, 57, 60, 64, 69, 81, 83, 89	92	P_{46}	P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}, P_{81}, P_{83}, P_{89}	P_{92}
47	49, 52, 57, 60, 64, 69, 81, 83, 89	94	P_{47}	P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}, P_{81}, P_{83}, P_{89}	P_{94}
48	49, 52, 57, 60, 64, 69, 81, 83, 89	96	P_{48}	P_{49}, P_{52}, P_{57}, P_{60}, P_{64}, P_{69}, P_{81}, P_{83}, P_{89}	P_{96}
49	52, 57, 60, 64, 69, 81, 83, 89	98	P_{49}	P_{52}, P_{57}, P_{60}, P_{64}, P_{69}, P_{81}, P_{83}, P_{89}	P_{98}
50	52, 57, 60, 64, 69, 81, 83, 89, 98	100	P_{50}	P_{52}, P_{57}, P_{60}, P_{64}, P_{69}, P_{81}, P_{83}, P_{89}, P_{98}	P_{100}

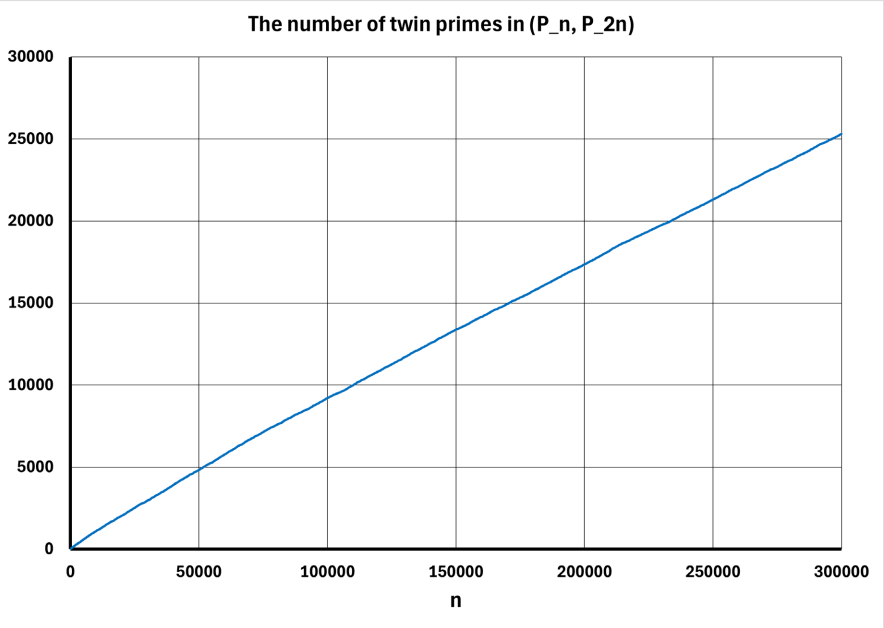

Figure 3. Counted number of twin primes in (P_n, P_{2n}) for $1 < n \leq 300{,}000$.

Table 9. Numerical evidence verifying Corollary 4.7 and Corollary 4.8 for $i \leq 50$.

i	$C(i)$	$C(i+1) - C(i)$	$P_{C(i)}$	$P_{C(i+1)} - P_{C(i)}$	$P_{2C(i)} - P_{C(i)}$
1	2	$3 - 2 = 1 < 2$	$P_2 = 3$	$5 - 3 = 2$	$7 - 3 = 4$
2	3	$5 - 3 = 2 < 3$	$P_3 = 5$	$11 - 5 = 6$	$13 - 5 = 8$
3	5	$7 - 5 = 2 < 5$	$P_5 = 11$	$17 - 11 = 6$	$29 - 11 = 18$
4	7	$10 - 7 = 3 < 7$	$P_7 = 17$	$29 - 17 = 12$	$43 - 17 = 26$
5	10	$13 - 10 = 3 < 10$	$P_{10} = 29$	$41 - 29 = 12$	$71 - 29 = 42$
6	13	$17 - 13 = 4 < 13$	$P_{13} = 41$	$59 - 41 = 18$	$101 - 41 = 60$
7	17	$20 - 17 = 3 < 17$	$P_{17} = 59$	$71 - 59 = 12$	$139 - 59 = 80$
8	20	$26 - 20 = 6 < 20$	$P_{20} = 71$	$101 - 71 = 30$	$173 - 71 = 102$
9	26	$28 - 26 = 2 < 26$	$P_{26} = 101$	$107 - 101 = 6$	$239 - 101 = 138$
10	28	$33 - 28 = 5 < 28$	$P_{28} = 107$	$137 - 107 = 30$	$263 - 107 = 156$
11	33	$35 - 33 = 2 < 33$	$P_{33} = 137$	$149 - 137 = 12$	$317 - 137 = 180$
12	35	$41 - 35 = 6 < 35$	$P_{35} = 149$	$179 - 149 = 30$	$349 - 149 = 200$
13	41	$43 - 41 = 2 < 41$	$P_{41} = 179$	$191 - 179 = 12$	$421 - 179 = 242$
14	43	$45 - 43 = 2 < 43$	$P_{43} = 191$	$197 - 191 = 6$	$443 - 191 = 252$
15	45	$49 - 45 = 4 < 45$	$P_{45} = 197$	$227 - 197 = 30$	$463 - 197 = 266$
16	49	$52 - 49 = 3 < 49$	$P_{49} = 227$	$239 - 227 = 12$	$521 - 227 = 294$
17	52	$57 - 52 = 5 < 52$	$P_{52} = 239$	$269 - 239 = 30$	$569 - 239 = 330$
18	57	$60 - 57 = 3 < 57$	$P_{57} = 269$	$281 - 269 = 12$	$619 - 269 = 350$
19	60	$64 - 60 = 4 < 60$	$P_{60} = 281$	$311 - 281 = 30$	$659 - 281 = 378$

Continued

20	64	$69-64=5<64$	$P_{64}=311$	$347-311=36$	$719-311=408$
21	69	$81-69=12<69$	$P_{69}=347$	$419-347=72$	$787-347=440$
22	81	$83-81=2<81$	$P_{81}=419$	$431-419=12$	$953-419=534$
23	83	$89-83=6<83$	$P_{83}=431$	$461-431=30$	$983-431=552$
24	89	$98-89=9<89$	$P_{89}=461$	$521-461=60$	$1061-461=600$
25	98	$104-98=6<98$	$P_{98}=521$	$569-521=48$	$1193-521=672$
26	104	$109-104=5<104$	$P_{104}=569$	$599-569=30$	$1283-569=714$
27	109	$113-109=4<109$	$P_{109}=599$	$617-599=18$	$1361-599=762$
28	113	$116-113=3<113$	$P_{113}=617$	$641-617=24$	$1429-617=812$
29	116	$120-116=4<116$	$P_{116}=641$	$659-641=18$	$1459-641=818$
30	120	$140-120=20<120$	$P_{120}=659$	$809-659=150$	$1511-659=852$
31	140	$142-140=2<140$	$P_{140}=809$	$821-809=12$	$1811-809=1002$
32	142	$144-142=2<142$	$P_{142}=821$	$827-821=6$	$1861-821=1040$
33	144	$148-144=4<144$	$P_{144}=827$	$857-827=30$	$1877-827=1050$
34	148	$152-148=4<148$	$P_{148}=857$	$881-857=24$	$1949-857=1092$
35	152	$171-152=19<152$	$P_{152}=881$	$1019-881=138$	$2003-881=1122$
36	171	$173-171=2<171$	$P_{171}=1019$	$1031-1019=12$	$2297-1019=1278$
37	173	$176-173=3<173$	$P_{173}=1031$	$1049-1031=18$	$2339-1031=1308$
38	176	$178-176=2<176$	$P_{176}=1049$	$1061-1049=12$	$2377-1049=1328$
39	178	$182-178=4<178$	$P_{178}=1061$	$1091-1061=30$	$2393-1061=1332$
40	182	$190-182=8<182$	$P_{182}=1091$	$1151-1091=60$	$2459-1091=1368$
41	190	$201-190=11<190$	$P_{190}=1151$	$1229-1151=78$	$2617-1151=1466$
42	201	$206-201=5<201$	$P_{201}=1229$	$1277-1229=48$	$2753-1229=1524$
43	206	$209-206=3<206$	$P_{206}=1277$	$1289-1277=12$	$2837-1277=1560$
44	209	$212-209=3<209$	$P_{209}=1289$	$1301-1289=12$	$2887-1289=1598$
45	212	$215-212=3<212$	$P_{212}=1301$	$1319-1301=18$	$2939-1301=1638$
46	215	$225-215=10<215$	$P_{215}=1319$	$1427-1319=108$	$2999-1319=1680$
47	225	$230-225=5<225$	$P_{225}=1427$	$1451-1427=24$	$3181-1427=1754$
48	230	$234-230=4<230$	$P_{230}=1451$	$1481-1451=30$	$3257-1451=1806$
49	234	$236-234=2<234$	$P_{234}=1481$	$1487-1481=6$	$3323-1481=1842$
50	236	$253-236=17<236$	$P_{236}=1487$	$1607-1487=120$	$3347-1487=1860$

Table 9 verifies $C(i+1)-C(i)<C(i)$ and $P_{C(i+1)}-P_{C(i)}<P_{2C(i)}-P_{C(i)}$ for $i\leq 50$. Because $C(i+1)-C(i)<C(i)$ and $P_{C(i+1)}-P_{C(i)}<P_{2C(i)}-P_{C(i)}$ are two corollaries of Conjecture 4.5 and the conjecture has been verified up to $n=300{,}000$ in Figure 3, the two corollaries have also been verified up to $n=300{,}000$.

Note that there is no counterexample in all verifying results up to $n = 300{,}000$ at Step 3 of our model. Thus, if Conjecture 4.5 is proven then twin prime conjecture is true.

5. Further Discussion on Bertrand-Type Problem

5.1. Density of Second Order Primes

As we know, double prime, nontrivial prime and twin prime are second order primes. Correspondingly, $\{x\}$ are natural numbers but $\{n\}$ are called second order natural numbers. Therefore, we can contrast asymptotic average density of second order primes on x-axis and n-axis. Let $D^{prim}(x)$ and $D^{prim}(n)$ denote density of primes on x-axis and n-axis. Let $D^{doub}(x)$ and $D^{doub}(n)$ denote density of double primes on x-axis and n-axis. Let $D^{nont}(x)$ and $D^{nont}(n)$ denote density of nontrivial primes on x-axis and n-axis. Let $D^{twin}(x)$ and $D^{twin}(n)$ denote density of twin primes on x-axis and n-axis. Then Table 10 gives all forms of these densities.

Table 10. Density of primes and second order primes on x-axis and n-axis.

type of prime	x-axis	n-axis
prime	$D^{prim}(x) \sim 1/\log x$	$D^{prim}(n) = 1$
double prime	$D^{doub}(x) \sim 1/(\log x)^2$	$D^{doub}(n) \sim 1/\log n$
nontrivial prime	$D^{nont}(x) \sim 1/(\log x)^2$	$D^{nont}(n) \sim 1/\log n$
twin prime	$D^{twin}(x) \sim 2C_2/(\log x)^2$	$D^{twin}(n) \sim 2C_2/\log n$

Table 10 means Bertrand theorem suitable for primes on x-axis is not suitable for above three kinds of second order primes because their density is too low to support the theorem for second order primes on x-axis. However, Density of the three kinds of second order primes on n-axis is enough high to generalize Bertrand theorem into the prime index sequence as Bertrand-type theorem or conjecture. It is clear that $P_{2n} > 2P_n$ for $n > 1$, which means $P_{2n} - 2P_n > 0$ for $n > 1$. By prime number theorem,

$$P_{2n} - 2P_n \sim 2n\log(2n) - 2n\log n = 2n\log 2. \tag{5.1}$$

In non-asymptotic case, we have the following approximation.

$$P_{2n} - 2P_n \approx 2n\log 2. \tag{5.2}$$

Thus, we obtain

$$P_{2n} - P_n \approx P_n + 2n\log 2. \tag{5.3}$$

Table 11 shows $P_n + 2n\log 2$ is a good approximation for $P_{2n} - P_n$ in non-asymptotic case. Because second order prime gap is bounded by $P_{2n} - P_n$, approximately, second order prime gap is bounded by $P_n + 2n\log 2$. Thus, we have $P_{X(i+1)} - P_{X(i)} < P_{X(i)} + 2X(i)\log 2$ for a given second order prime $P_{X(i)}$.

Let $P_{X(i)}$ be a double prime $P_{A(i)}$. Then we have the following approximate bound for double prime gap.

$$P_{A(i+1)} - P_{A(i)} < P_{A(i)} + 2A(i)\log 2, \tag{5.4}$$

$$P_{A(i+1)} < 2P_{A(i)} + 2A(i)\log 2. \tag{5.5}$$

Let $P_{X(i)}$ be a nontrivial prime $P_{B(i)}$. Then we have the following approximate bound for nontrivial prime gap.

$$P_{B(i+1)} - P_{B(i)} < P_{B(i)} + 2B(i)\log 2, \tag{5.6}$$

$$P_{B(i+1)} < 2P_{B(i)} + 2B(i)\log 2. \tag{5.7}$$

Let $P_{X(i)}$ be a twin prime $P_{C(i)}$. Then we have the following approximate bound for twin prime gap.

$$P_{C(i+1)} - P_{C(i)} < P_{C(i)} + 2C(i)\log 2, \tag{5.8}$$

$$P_{C(i+1)} < 2P_{C(i)} + 2C(i)\log 2. \tag{5.9}$$

Table 11. Comparison between accurate and approximate values for $P_{2n} - P_n$.

n	$P_{2n} - P_n$	$P_n + 2n\log 2$	relative error
2	4	5.772	0.3069
3	8	9.158	0.1264
5	18	17.93	0.0038
7	26	26.70	0.0262
10	42	42.86	0.0200
100	682	679	0.0043
1000	9470	9305	0.0174
10000	120008	118589	0.0118
100000	1450450	1438309	0.0083
1000000	16966980	16871863	0.0056
10000000	194163210	193284673	0.0045

Comparing with corollary of Bertrand theorem such that $P_{n+1} - P_n < P_n$, approximations (5.4), (5.6) and (5.8) expanded the interval length so that such intervals are suitable for density of second order primes on natural number axis as Table 10 shows. Suppose $P_{A(i+1)} - P_{A(i)} < P_{A(i)}$. Then we discover $P_{A(3)} - P_{A(2)} = 11 - 5 = 6 > P_{A(2)} = 5$. It means there is no double prime in $(P_{A(2)}, 2P_{A(2)})$. The counterexample for $i = 2$ negated this hypothesis. However, using (5.4), we have $P_{A(3)} - P_{A(2)} = 6 < P_{A(2)} + 6\log 2 = 9.158$. Thus, the bound (5.4) is reasonable for controlling double prime gap on x-axis. Suppose $P_{C(i+1)} - P_{C(i)} < P_{C(i)}$. Then we discover $P_{C(3)} - P_{C(2)} = 11 - 5 = 6 > P_{C(2)} = 5$. It means there is no twin prime in $(P_{C(2)}, 2P_{C(2)})$. The counterexample for $i = 2$ negated this hypothesis. However, using (5.8), we have $P_{C(3)} - P_{C(2)} = 6 < P_{C(2)} + 6\log 2 = 9.158$. Thus, the bound (5.8) is reasonable for controlling twin prime gap on x-axis.

5.2. Ramanujan-Type Second Order Prime Root

Can one require that the second order natural number interval ($n/2$, n) contains not just at least one second order prime root but at least m second order prime roots? It leads to some results similar to Ramanujan primes.

Let $\pi(n)$ denote the number of double prime roots among the first n prime roots. Then R_m^{doub} is called the m-th Ramanujan-type double prime root if R_m^{doub} satisfies

$$\pi(n)-\pi(n/2)\geq m \quad \text{for all} \quad n\geq R_m^{doub}. \tag{5.10}$$

Since $R_1^{doub}=2$, Bertrand-type theorem for double prime is the first Ramanujan-type double prime root case. By calculating $\pi(n)-\pi(n/2)$ for $n>1$, we have found the first five Ramanujan-type double prime roots 2, 11, 17, 29, 41 to correspond to $m=1, 2, 3, 4, 5$. They are just the first five Ramanujan primes 2, 11, 17, 29, 41.

Let $\Omega(n)$ denote the number of nontrivial prime roots among the first n prime roots. Then R_m^{nont} is called the m-th Ramanujan-type nontrivial prime root if R_m^{nont} satisfies

$$\Omega(n)-\Omega(n/2)\geq m \quad \text{for all} \quad n\geq R_m^{nont}. \tag{5.11}$$

Since $R_1^{nont}=2$, Bertrand-type conjecture for nontrivial prime is the first Ramanujan-type nontrivial prime root case. By calculating $\Omega(n)-\Omega(n/2)$ for $n>1$, we have found the first five Ramanujan-type nontrivial prime roots 2, 3, 5, 7, 11 to correspond to $m=1, 2, 3, 4, 5$. One can expect there are infinitely many Ramanujan-type nontrivial prime roots to show infinitude of Ramanujan-type nontrivial primes.

Let $\pi_2(n)$ denote the number of twin prime roots among the first n prime roots. Then R_m^{twin} is called the m-th Ramanujan-type twin prime root if R_m^{twin} satisfies

$$\pi_2(n)-\pi_2(n/2)\geq m \quad \text{for all} \quad n\geq R_m^{twin}. \tag{5.12}$$

Since $R_1^{twin}=2$, Bertrand-type conjecture for twin prime is the first Ramanujan-type twin prime root case. By calculating $\pi_2(n)-\pi_2(n/2)$ for $n>1$, we have found the first five Ramanujan-type twin prime roots 2, 7, 17, 28, 41 to correspond to $m=1, 2, 3, 4, 5$. One can expect there are infinitely many Ramanujan-type twin prime roots to show infinitude of Ramanujan-type twin primes.

These results mean an existence problem of second order prime can be transformed into a problem of local density for second order prime because every second order prime root must link with a second order prime.

5.3. Asymptotic Form of Bertrand-Type System for Second Order Prime

We have known that, in non-asymptotic case, there are three corollaries which suggest $P_{X(i+1)}$ is bounded by $P_{2X(i)}$ and gap $P_{X(i+1)}-P_{X(i)}$ is bounded by $P_{2X(i)}-P_{X(i)}$ for a given second order prime $P_{X(i)}$. However, in asymptotic case, we have

$$\frac{P_{2n}}{P_n} \sim \frac{2n\log(2n)}{n\log n} = 2 + \frac{2\log 2}{\log n}.$$

Since $\frac{2\log 2}{\log n} \sim 0$, we obtain $\frac{P_{2n}}{P_n} \sim 2$. Thus, we get

$$P_{2n} \sim 2P_n. \tag{5.13}$$

Result (5.13) means there are the following asymptotic forms.

For Bertrand-type theorem for double prime, we have

$$P_{A(i+1)} < P_{2A(i)} \sim P_{A(i+1)} < 2P_{A(i)}, \tag{5.14}$$

$$P_{A(i+1)} - P_{A(i)} < P_{2A(i)} - P_{A(i)} \sim P_{A(i+1)} - P_{A(i)} < P_{A(i)}. \tag{5.15}$$

For Bertrand-type conjecture for nontrivial prime, we have

$$P_{B(i+1)} < P_{2B(i)} \sim P_{B(i+1)} < 2P_{B(i)}, \tag{5.16}$$

$$P_{B(i+1)} - P_{B(i)} < P_{2B(i)} - P_{B(i)} \sim P_{B(i+1)} - P_{B(i)} < P_{B(i)}. \tag{5.17}$$

For Bertrand-type conjecture for twin prime, we have

$$P_{C(i+1)} < P_{2C(i)} \sim P_{C(i+1)} < 2P_{C(i)}, \tag{5.18}$$

$$P_{C(i+1)} - P_{C(i)} < P_{2C(i)} - P_{C(i)} \sim P_{C(i+1)} - P_{C(i)} < P_{C(i)}. \tag{5.19}$$

These asymptotic forms mean the relative error between $P_{2X(i)}$ and $2P_{X(i)}$ approaches 0 as i grows without bound, but we cannot use such asymptotic forms in non-asymptotic case as approximations.

6. Conclusion

This paper presents a standard model such that if counted number of a kind of second order primes is greater than counted number of double primes for $n = 10^9$ then a Bertrand-type conjecture for the kind of second order primes can be established and the conjecture may imply the infinitude of the kind of second order primes. Based on the model, nontrivial primes and twin primes are considered. Goldbach conjecture has been transformed into an existence problem of nontrivial prime on n-axis by establishing Bertrand-type conjecture for nontrivial prime. The Bertrand-type conjecture can give proof of Goldbach conjecture. By establishing Bertrand-type conjecture for twin prime, twin prime conjecture has also become an existence problem of twin prime on n-axis. The Bertrand-type conjecture can also give proof of twin prime conjecture. In Bertrand-type system, it is a basic step that prime index n is defined as root of prime, which leads to the core significance: Bertrand-type form on n-axis remains linear control on second order prime root gap. The key result is: $P_{A(i+1)} - P_{A(i)}$ is bounded by $P_{2A(i)} - P_{A(i)}$, $P_{B(i+1)} - P_{B(i)}$ is bounded by $P_{2B(i)} - P_{B(i)}$ and $P_{C(i+1)} - P_{C(i)}$ is bounded by $P_{2C(i)} - P_{C(i)}$ on x-axis and these controls on second order prime gap are nonlinear, which is obviously different from linear control on prime gap as Bertrand theorem itself shows on natural number axis.

Acknowledgements

The author would like to acknowledge Rong Ao for his careful and helpful calculation and verification of the data.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Hardy, G.H. and Littlewood, J.E. (1923) Some Problems of 'Partitio Numerorum'; III: On the Expression of a Number as a Sum of Primes. *Acta Mathematica*, **44**, 1-70. https://doi.org/10.1007/bf02403921

[2] Vaughan, R.C. (1997) The Hardy-Littlewood Method. 2nd Edition, Cambridge University Press. https://doi.org/10.1017/cbo9780511470929

[3] Chen, J.R. (1973) On the Representation of a Large Even Number as the Sum of a Prime and the Product of at Most Two Primes. *Scientia Sinica*, **16**, 157-176.

[4] Ross, P.M. (1975) On Chen's Theorem That Each Large Even Number Has the Form $(p_1 + p_2)$ or $(p_1 + p_2p_3)$. *Journal of the London Mathematical Society*, **10**, 500-506. https://doi.org/10.1112/jlms/s2-10.4.500

[5] Montgomery, H. and Vaughan, R. (1975) The Exceptional Set of Goldbach's Problem. *Acta Arithmetica*, **27**, 353-370. https://doi.org/10.4064/aa-27-1-353-370

[6] Kaczorowski, J., Perelli, A. and Pintz, J. (1993) A Note on the Exceptional Set for Goldbach's Problem in Short Intervals. *Monatshefte Für Mathematik*, **116**, 275-282. https://doi.org/10.1007/bf01301533

[7] de Polignac, A. (1849) Recherches nouvelles sur les nombres premiers. *Comptes Rendus de l'Académie des Sciences* (*Paris*), **29**, 397-401.

[8] Broughan, K. (2021) Bounded Gaps between Primes: The Epic Breakthroughs of the Early 21st Century. Cambridge University Press, 1-34. https://doi.org/10.1017/9781108872201

[9] Zhou, P. (2017) Strong Goldbach Number in Goldbach's Problem. *Journal of Mathematics Research*, **9**, 95-105. https://doi.org/10.5539/jmr.v9n6p95

[10] Zhou, P. and Ao, R. (2018) Distribution of the Largest Strong Goldbach Numbers Generated by Primes. *Journal of Mathematics Research*, **10**, 1-8. https://doi.org/10.5539/jmr.v10n5p1

[11] Zhou, P. (2019) A Weak Method to Come Close to Solution of Goldbach Conjecture. *International Mathematical Forum*, **14**, 247-252. https://doi.org/10.12988/imf.2019.9938

[12] Zhou, P. (2024) Prime Number Theorem and Goldbach Conjecture. *Journal of Mathematics Research*, **16**, 1-27. https://doi.org/10.5539/jmr.v16n3p1

[13] Zhou, P. (2025) On the Connections between Goldbach Conjecture and Prime Number Theorem. *Advances in Pure Mathematics*, **15**, 412-457. https://doi.org/10.4236/apm.2025.156020

[14] Hadamard, J. (1896) Sur la distribution des zéros de la fonction ζ(s) et ses conséquences arithmétiques. *Bulletin de la Société Mathématique de France*, **24**, 199-220. https://doi.org/10.24033/bsmf.545

[15] de la Vallée Poussin, C.J. (1896) Recherches analytiques sur la théorie des nombres premiers. *Annales de la Société Scientifique de Bruxelles*, **20**, 183-256.

[16] Bertrand, J. (1845) Mémoire sur le nombre de valeurs que peut prendre une fonction quand on y permute les lettres qu'elle renferme. *Journal de l'École Royale Polytechnique*, **17**, 123-140.

[17] Chebyshev, P.L. (1850) Mémoire sur les nombres premiers. *Mémoires de l'Académie Impériale des Sciences de St.-Pétersbourg*, **7**, 17-33.

[18] Ramanujan, S. (1919) A Proof of Bertrand's Postulate. *Journal of the Indian Mathematical Society*, **11**, 181-182.

[19] Erdös, P. (1932) Beweis eines Satzes von Tschebychef. *Acta Scientiarum Mathematicarum* (*Szeged*), **5**, 194-198.

[20] Iwaniec, H. and Pintz, J. (1984) Primes in Short Intervals. *Monatshefte für Mathematik*, **98**, 115-143. https://doi.org/10.1007/bf01637280

[21] Baker, R.C., Harman, G. and Pintz, J. (2001) The Difference between Consecutive Primes, II. *Proceedings of the London Mathematical Society*, **83**, 532-562. https://doi.org/10.1112/plms/83.3.532

[22] Heath, T.L. (1926) The Thirteen Books of Euclid's Elements. Cambridge University Press.

[23] Euler, L. (1737) Variae observations circa series infinitas. Commentarii academiae scientiarum Petropolitanae.

[24] Dirichlet, P.G.L. (1837) Beweis des Satzes, dass jede unbegrenzte arithmetische Progression. Abhandlungen der Königlichen PreuBischen Akademie.

[25] Furstenberg, H. (1955) On the Infinitude of Primes. *The American Mathematical Monthly*, **62**, 353-354. https://doi.org/10.2307/2307043

[26] Elsholtz, C. (2009) Fermat's Last Theorem Implies Euclid's Infinitude of Primes. arXiv: 2009.06722.

[27] Meštrović, R. (2017) A Very Short Proof of the Infinitude of Primes. *The American Mathematical Monthly*, **124**, 562-562. https://doi.org/10.4169/amer.math.monthly.124.6.562

[28] Sondow, J. (2009) Ramanujan Primes and Bertrand's Postulate. *The American Mathematical Monthly*, **116**, 630-635. https://doi.org/10.1080/00029890.2009.11920980

[29] Broughan, K.A. and Barnett, A.R. (2009) On the Subsequence of Primes Having Prime Subscripts. *Journal of Integer Sequences*, **12**, 1-10.

[30] Bayless, J., Klyve, D. and Oliveira e Silva, T. (2013) New Bounds and Computations on Prime-Indexed Primes. *Integers*, **13**, 1-21.

[31] Zhou, P. (2018) Numerical Evidence for Primes Less than 10000000 in An Experimental Mathematical Method to Prove Goldbach's Conjecture. https://doi.org/10.13140/RG.2.2.31869.05608

[32] Sloane, N.J.A. (2019) Sequence A005597 (Decimal Expansion of the Twin Prime Constant). The Online Encyclopedia of Integer Sequences. OEIS Foundation.

[33] Bateman, P.T. and Diamond, H.G. (2004) Analytic Number Theory. World Scientific. https://doi.org/10.1142/5605