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Foundational Challenges in Proximal and Digital Topological Complexity: A Critical Review of Recent Developments

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Abstract

The theory of topological complexity is an important theory in the study of motion planning problems. Over the last decade, several attempts have been made to extend the theory of complexity (and its higher analogues) to proximal and digital topological settings. As a result, there are now digital versions of concepts like path spaces, motion planning algorithms, the Schwarz genus, the homotopic distance, digital fibrations, digital topological groups, and generalized topological complexity. Despite the number of publications on the topic of digital and proximal topological complexity, several mathematical issues within the area have yet to be resolved. For example, the theory of classical topological complexity suggests the construction of path spaces, product adjacencies, continuous maps, homotopy structures, lifting structures, and fibrations. Each of these concepts inherently relates to the notion of topology, which is a concept that applies to digital spaces only in a restricted manner. As a result, each of these concepts must be defined within digital spaces, and their mathematical consistency within those spaces must be established. In this paper, a collection of existing publications on the topic of proximal and digital topological complexity will be reviewed. Each of these publications has encountered issues related to adjacency structures, digital continuity, path spaces, digital fibrations, and generalized complexity. A focus will be placed upon the role that product adjacencies, digital coverings, the unique path lifting property, and the homotopy lifting property play in the definition of digital analogues of concepts from classical topology. Overall, while the current state of the known results related to digital and proximal topological complexity can be summarized, several challenges must be overcome in developing future results on the topic. Suggestions for future research in the field will be made and suggested challenges will be discussed. Specifically, it is clear from the published results that a theory based upon adjacencies in digital spaces is needed, as well as established results regarding digital continuity, digital covering maps, and lifting structures.

Keywords

Digital Topology, Proximal Topology, Topological Complexity, Digital Continuity, Product Adjacency, Digital Covering, Homotopy Lifting, Motion Planning

1. Introduction

Topological complexity (TC) was introduced by Farber and has since become one of the most influential invariants related to the study of motion planning problems [1]. There have been numerous developments related to the concept of topological complexity, including higher topological complexity, parametrized topological complexity, the topological complexity of maps, and the topological complexity of fibrations [2]-[5].

Related efforts have attempted to extend the concept of TC to discrete topologies, such as digital and proximal topologies. Digital topologies are often used to model problems related to digital images, image analysis, computer vision, medical imaging, and various other fields related to image analysis. Unlike classical topologies, digital topologies model concepts based upon adjacency relations between pixels within an image [6]-[16].

Attempts to extend the concept of TC to digital topologies have led to the introduction of the concept of digital analogues of path spaces, motion planning algorithms, the Schwarz genus, homotopic distance, digital fibrations, digital topological groups, and a generalized concept of topological complexity [17]-[29]. Each of these concepts has developed its own literature within the fields of digital image analysis, algebraic topology, and robotics. Each of these concepts, however, has a discussion regarding its mathematical foundations.

One of the main issues related to digital topologies is that digital images are not classical topological spaces. As a result, many of the concepts related to digital analogues must depend upon the adjacency relation for pixels within a digital image, the properties of product adjacencies, the definition of digital continuity, digital path spaces, and the existence of lifting properties for digital maps [6]-[16].

Furthermore, there are various studies that reveal that concepts related to digital topologies do not necessarily exhibit the same properties as the classical topological spaces [26] [28] [29]. For instance, digital homotopy invariants may exhibit different behaviors than classical homotopy invariants, cohomological techniques for finding lower bounds for classical topologies do not necessarily apply to digital images, and the choices of different adjacities for digital images can impact the homotopy type of the image and its related topological complexity [26] [28] [29]. As such, the mathematical consistency of digital topological complexity concepts should be examined in these contexts.

Related discussions of digital topologies also include topics like digital coverings, digital homotopy liftings, and digital fibrations [9] [11]. Digital covering and lifting theory provides a rigorous framework for discussing digital fibrations, digital path spaces, and generalized digital topological complexity. As a result, any digital analogue of the concept of topological complexity should be examined within this context.

The goal of this review is to investigate each of the mentioned areas of research in relation to the mathematical foundations of the concepts related to digital and proximal topologies. In particular, each of the issues related to product adjacencies, path spaces, digital continuity, digital fibrations, the Schwarz genus, homotopic distance, and generalized topological complexity will be examined. Where appropriate, remarks and counterexamples will be provided related to these topics to demonstrate the mathematical difficulties related to these concepts.

The main contribution of this review is threefold. First, it identifies several recurring foundational assumptions that appear throughout the current literature on proximal and digital topological complexity. Second, the paper analyzes the mathematical implications of these assumptions. Third, it proposes a collection of remarks and future directions for the field based on these counterexamples.

The remainder of the paper is organized as follows. Section 2 provides the necessary background in digital topology, digital continuity, homotopy theory, covering spaces, and lifting theory. Section 3 introduces the evaluation criteria that will be used throughout the paper to evaluate the developments in the field of digital and proximal topological complexity. Section 4 presents an assessment of the existing developments in the field. Section 5 contains the counterexamples that invalidate some of the assumptions in the field along with remarks that highlight these invalidations. Section 6 proposes a framework for future developments in the field of digital topological complexity. Finally, Section 7 contains the conclusions of the paper.

Review Methodology

In the following review, papers that have specifically investigated the topic of digital or analogues of topological complexity, higher topological complexity, digital fibrations, digital Schwarz genus, homotopic distance, and motion planning are reviewed. These papers were chosen primarily due to the fact that they are some of the bulk of the current literature on these specific topics, and that they represent the major directions of research within the field. The goal of this review is not to provide a complete list of the available publications on these topics, but to review some of the major contributions to the field through the common framework of adjacency, continuity, paths, product adjacencies, and lifting. As illustrated in **Figure 1**, the proposed evaluation framework is based on adjacency structures, digital continuity, path-space constructions, product adjacencies, and lifting mechanisms.

2. Background and Foundational Setting

2.1. Classical Topological Complexity and Motion Planning

Topological complexity (TC), introduced by Farber, is a homotopy-theoretic

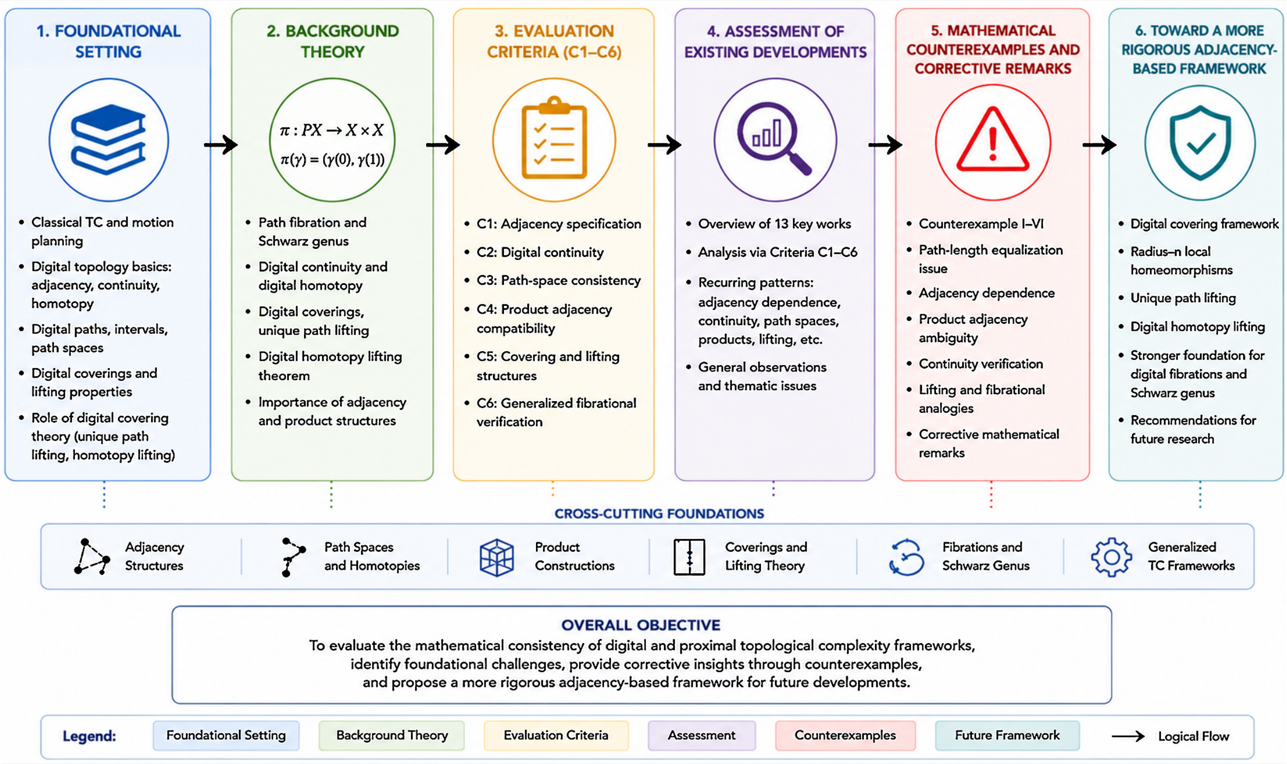

Figure 1. Overview of the evaluation framework used for assessing digital and proximal topological complexity.

invariant that measures the intrinsic complexity of constructing continuous motion-planning algorithms on a configuration space [1]. Let X be a path-connected topological space. The classical path fibration $\pi : PX \to X \times X$ is defined by $\pi(\gamma) = (\gamma(0), \gamma(1))$, where PX denotes the free path space of continuous paths in X. The topological complexity $TC(X)$ is defined as the Schwarz genus of the path fibration. Consequently, classical TC depends fundamentally on path spaces, continuity, fibrations, local sections, homotopy equivalence, Schwarz genus, and Lusternik–Schnirelmann category.

Because these constructions are deeply rooted in classical topology, their transfer into digital or proximal settings requires careful verification.

2.2. Digital Topology as an Adjacency-Based Framework

Unlike classical topology, digital topology studies discrete structures equipped with adjacency relations. A digital image is commonly represented by (X, κ), where $X \subset \mathbb{Z}^n$ and κ denotes a specified adjacency relation.

The key difference between classical and digital topology is that the notions of continuity, connectedness, homotopy and constructions involving paths in digital spaces depend on the adjacency used.

Consequently, every theorem that is proven about digital spaces will refer to the adjacency on the domain, the adjacency on the codomain, the adjacency on the product of two spaces, and will refer to the compatibility of maps with the adjacencies.

This dependence on adjacency is one of the most fundamental characteristics of digital topology, and it impacts every notion of topological complexity defined for digital spaces.

2.3. Digital Continuity

One of the most important concepts in digital topology is digital continuity. For digital images (X, k_1) and (Y, k_2), a map $f:(X, k_1) \to (Y, k_2)$ is digitally continuous if it preserves adjacency relations.

A more rigorous neighborhood-based characterization is given using digital neighborhoods. Let $N_k(x,1)$ denote the digital neighborhood of radius one around x. Then digital continuity may be expressed by the condition

$$f\left(N_{k_1}(x,1)\right) \subset N_{k_2}\left(f(x),1\right)$$

for every $x \in X$. This formulation provides a useful way to check digital continuity and plays an important role in digital covering theory and lifting constructions [6] [9].

The significance of digital continuity becomes especially apparent in digital topological complexity because motion-planning maps, evaluation maps, digital fibrations, and path assignments must all satisfy digital continuity requirements.

2.4. Digital Homotopy

Digital homotopy provides a discrete analogue of classical homotopy. Given digitally continuous maps $f, g:(X, k_1) \to (Y, k_2)$, a digital homotopy is a family of digitally continuous maps connecting f and g through a digital interval.

Unlike classical homotopy theory, digital homotopy depends on the adjacency on the domain, the adjacency on the codomain, the adjacency on the digital interval, and the product adjacency used in $X \times [0, m]_{\mathbb{Z}}$. Therefore, homotopy constructions cannot be transferred automatically from classical topology into digital settings.

2.5. Digital Paths and Digital Intervals

Digital path constructions play a central role in all versions of digital topological complexity. A digital interval is usually represented by $[a,b]_{\mathbb{Z}} = \{n \in \mathbb{Z} : a \le n \le b\}$, equipped with 2-adjacency.

A digital path is a digitally continuous map $\alpha:[0, m]_{\mathbb{Z}} \to (X, k)$. Digital path spaces form the foundation of digital motion-planning algorithms because every TC-type invariant requires assigning paths to pairs of points.

However, the construction of digital path spaces is considerably more delicate than in classical topology because the path space itself requires an adjacency structure. This issue becomes central in later sections when digital topological complexity and digital TC of maps are discussed.

2.6. Digital Coverings

Digital covering spaces provide one of the most important tools for studying digital homotopy theory. A digital covering map $p:(E, k_0) \to (B, k_1)$ is a digitally continuous surjection satisfying local digital homeomorphism conditions on suitable digital neighborhoods.

Digital coverings have a role similar to classical covering spaces but with additional requirements regarding the adjacency of the digital spaces. Digital coverings are important in that most concepts regarding digital fundamental groups, lifting, and digital fibrations employ covering spaces.

2.7. Unique Path Lifting

A fundamental result in digital covering theory is the unique path lifting theorem. Let $p: E \to B$ be a digital covering map. Then every digital path in B beginning at a specified base point admits a unique lifted digital path in E, provided the required digital covering assumptions are satisfied.

This theorem plays a crucial role in digital homotopy theory as it helps to transfer information along the paths in a covering space [9] [11]. The existence of a unique path lifting is relevant to the discussion of digital path spaces and digital motion planning.

2.8. Digital Homotopy Lifting

A second fundamental result is the digital homotopy lifting theorem. Given suitable conditions on digital covering maps, digital homotopies can be lifted through digital covering maps. This result establishes a stronger mathematical foundation for digital fibrations.

Because many of the notions utilized in the computation of digital topological complexity include the concepts of digital fibrations, the Schwarz genus, lifting properties, and motion-planning maps, the digital homotopy lifting theorem becomes an essential result within the study of digital topology.

2.9. Why Foundational Verification Matters

Motivated by the existence of these various concepts, numerous recent studies have introduced digital analogues of concepts such as path fibrations, Schwarz genus, homotopic distance, digital topological groups, higher topological complexity, and topological complexity of maps. However, in each of these studies, it is important to ensure that the concept of adjacency is made explicit, that the adjacencies are compatible with the product adjacency, that the digital continuity of certain maps is ensured, that digital path spaces are defined, and that lifting results are established.

These issues all motivate the present review, which forms the foundation for the criteria that will be established in the following section.

3. Evaluation Criteria for Digital and Proximal Topological Complexity

3.1. Complexity

3.1.1. Definitions and Scope of the Evaluation Framework

A foundational challenge refers to a mathematical issue that may affect the transfer of concepts from classical topology to digital or proximal settings. Such challenges do not necessarily imply that a result is incorrect; rather, they indicate that additional assumptions, verification, or structural justification may be required.

Mathematical consistency refers to the compatibility of definitions, continuity structures, adjacency relations, path-space constructions, and lifting mechanisms within a given framework.

Verification refers to the explicit mathematical confirmation that the hypotheses required by a construction are satisfied. In the context of digital and proximal topological complexity, verification includes the specification of adjacency structures, continuity conditions, path-space constructions, and lifting properties.

The criteria introduced below are intended as evaluative tools for identifying areas where additional mathematical clarification may be beneficial. They should not be interpreted as automatic tests of correctness or invalidity.

3.1.2. Motivation

The rapid development of the concepts of digital and proximal topological complexity has led to the creation of numerous constructions based on digital path spaces, digital fibrations, the digital Schwarz genus, the topological complexity of maps, and generalized topological complexity.

Many of these constructions have been inspired by the classical theories that have proven themselves to be valid throughout the years. Nevertheless, the validity of the digital analogues of these theories rests upon a few assumptions.

Before investigating the specific results that have been proven within the field of digital and proximal topological complexity, it is first necessary to establish a collection of criteria that can be used to assess the validity of the constructions of digital proximal topological complexity.

The purpose of this section, therefore, is not to determine whether various results presented within the field of digital and proximal topological complexity are correct or incorrect, but rather to reveal the assumptions that must be made regarding the validity of each of these theories when they are applied to the digital and proximal spaces.

3.2. Criterion C1: Explicit Adjacency Specification

The first requirement concerns adjacency structures.

Since digital topology is fundamentally adjacency-dependent, every digital construction should explicitly specify the adjacency relation used on:

$$(X,\kappa),$$

$$(Y,\lambda),$$

and all associated product spaces. In particular, constructions involving

$$X \times X,$$

$$X^n,$$

$$Y \times Z,$$

or

$$X \times [0,m]_{\mathbb{Z}}$$

should clearly identify the chosen product adjacency. Failure to specify adjacency structures may lead to ambiguity because connectedness, continuity, homotopy type, and topological complexity may depend on adjacency selection.

Remark 3.1

Two digital images possessing the same underlying point set may exhibit different topological complexity values under different adjacency relations.

Therefore, digital topological complexity should generally be interpreted as

$$TC(X,\kappa)$$

rather than simply

$$TC(X).$$

3.3. Criterion C2: Verification of Digital Continuity

The second requirement concerns digital continuity.

Whenever a construction introduces a map

$$f:(X,k_1)\to(Y,k_2),$$

the adjacency-preserving property should be explicitly verified.

In particular, continuity assumptions become crucial for:

- motion-planning maps,
- evaluation maps,
- digital fibrations,
- lifting maps,
- projection maps,
- multiplication maps in digital groups.

Using the neighborhood characterization of digital continuity,

$$f\left(N_{k_1}(x,1)\right)\subset N_{k_2}\left(f(x),1\right),$$

provides a useful criterion for verification.

Remark 3.2

Many digital constructions inherit maps directly from classical topology.

However, the notion of continuity does not automatically imply digital continuity.

Therefore, the notion of continuity has to be verified independently of classical topology.

3.4. Criterion C3: Path-Space Construction

A third requirement concerns digital path spaces.

Classical topological complexity is based on

$$PX.$$

Similarly, the notion of digital topological complexity typically relies on spaces of digital paths. However, unlike in the case of classical topology, the digital path space does not naturally admit an adjacency structure.

Therefore, every path-space construction should specify:

1) the digital interval used,

2) the path-space adjacency,

3) the continuity of the evaluation map,

4) compatibility with product adjacency.

Counterexample Motivation

Suppose two digital paths have identical lengths but different adjacency structures. This observation suggests that path-length equalization alone may be insufficient for defining a unique path-space structure. A formal example will be presented in Section 5.

3.5. Criterion C4: Product Adjacency Compatibility

Many digital TC constructions rely heavily on products.

Examples include:

$$X \times X,$$

$$X^n,$$

$$E \times_B X,$$

and

$$X \times [0, m]_{\mathbb{Z}}.$$

The validity of homotopies, fibrations, and motion-planning maps may depend on the chosen product adjacency.

Therefore, product adjacency should be regarded as a foundational component rather than a technical detail.

3.6. Criterion C5: Covering and Lifting Structures

Digital covering theory provides one of the strongest available foundations for digital homotopy constructions.

Consequently, whenever digital fibrations or digital Schwarz genus are employed, the existence of suitable lifting structures should be examined.

Particular attention should be given to:

- unique path lifting,
- homotopy lifting,
- local digital homeomorphisms,
- covering compatibility.

Remark 3.3

The availability of lifting properties generally provides stronger mathematical support for digital fibrational constructions than informal analogies with classical path fibrations.

3.7. Criterion C6: Generalized Fibrational Frameworks

The final criterion concerns generalized constructions involving:

$$TC_n,$$

$$TC(f),$$

digital Schwarz genus, homotopic distance, and generalized fibrational models.

Such constructions often involve multiple layers of continuity, lifting, path-space, and product-adjacency assumptions.

Therefore, each layer should be verified independently before conclusions from classical homotopy theory are transferred into digital environments.

3.8. Summary of Evaluation Criteria

The six criteria developed in this section are summarized as follows:

Criterion	Description
C1	Explicit adjacency specification
C2	Verification of digital continuity
C3	Path-space consistency
C4	Product adjacency compatibility
C5	Covering and lifting structures
C6	Generalized fibrational verification

These criteria will be used throughout the remainder of the paper to evaluate existing developments in proximal and digital topological complexity.

4. Assessment of Existing Developments in Digital and Proximal Topological Complexity

4.1. Topological Complexity

4.1.1. Distinction between Digital and Proximal Frameworks

Although digital and proximal topological complexity are discussed together throughout this review, the underlying mathematical structures differ substantially.

Digital topology is fundamentally adjacency-based, whereas proximal topology is based on proximity relations. Consequently, Criteria C1-C4 are primarily directed toward digital constructions involving adjacency, continuity, path spaces, and product adjacencies.

In contrast, Criteria C5-C6 apply more broadly to both digital and proximal frameworks because lifting structures, generalized fibrational constructions, and homotopy-theoretic assumptions arise in both settings.

Accordingly, some observations in the following sections are specific to digital topology, whereas others apply to both digital and proximal topological complexity.

4.1.2. General Observations

The reviewed literature has significantly expanded the scope of topological complexity by introducing digital and proximal analogues of many classical constructions. These developments include digital topological complexity, higher digital topological complexity, digital topological groups, digital Schwarz genus, digital homotopic distance, generalized topological complexity, and topological complexity of maps.

The overall mathematical objective of these studies is both natural and important. Nevertheless, when evaluated through the criteria introduced in Section 3, several difficulties emerge in each of these studies.

These difficulties each involve the consideration of adjacency, digital continuity, constructions of paths, product adjacencies, and lifting.

The following subsections examine each of these difficulties.

4.2. Adjacency Dependence and Structural Ambiguity

The first of the issues to arise is that of adjacency dependence.

Since the theory of digital topology inherently depends upon the concept of adjacency, the validity of many of the constructions of homotopy theory depend upon the adjacency of the digital image under consideration.

While many of the constructions reviewed do consider digital images, they do not necessarily emphasize the relationship of those invariants to the adjacency of the digital image.

As a result, the complexity values can be different for the same digital image with different adjacency relations.

Thus, digital topological complexity should generally be interpreted in the form

$$TC(X,\kappa)$$

rather than $TC(X)$.

Remark 4.1

Adjacency selection is not merely a technical choice.

It directly affects:

- connectedness.
- contractibility.
- homotopy type.
- reducibility.
- digital continuity.
- and topological complexity.

Therefore, adjacency structures should be regarded as part of the mathematical object itself.

4.3. Digital Continuity Requirements

The second recurring issue concerns digital continuity.

Many constructions introduce:

- evaluation maps.
- projection maps.

- motion-planning maps.
- digital fibrations.
- group operations.

For every such map

$$f:(X,k_1)\rightarrow(Y,k_2),$$

digital continuity must be verified relative to the chosen adjacencies.

In several of the constructions reviewed in this paper, the concept of continuity is established through its relation to classical analogies, rather than through its verification within the context of the digital framework.

While this is not necessarily a way of invalidating the constructions presented, it does indicate that some form of verification of such concepts within the digital framework is required.

Remark 4.2

Classical continuity and digital continuity are fundamentally different notions.

Consequently, classical continuity arguments cannot automatically be transferred into adjacency-based environments.

4.4. Path-Space Constructions

The third issue that recurs in the analysis of the construction of digital topologies is the issue of digital path spaces.

Many constructions of digital topologies utilize digital paths and digital motion planners.

Unlike classical topology, however, digital path spaces do not have a canonical structure associated with them.

The validity of such constructions depends on:

- digital interval selection.
- path-space adjacency.
- endpoint evaluation maps.
- compatibility with product adjacencies.

These requirements become especially important in higher digital TC and TC of maps.

Counterexample 4.1

Consider two digital paths

$$\alpha=\langle a,b,c,d\rangle \quad \text{and}$$

$$\beta=\langle a,b,b,b\rangle.$$

Although both paths have equal length, their adjacency behavior differs substantially.

Consequently, path-space constructions based solely on path-length adjustments may fail to preserve the intended combinatorial structure.

This example illustrates why path-space adjacencies should be explicitly specified and justified.

4.5. Product Adjacency Issues

A fourth recurring issue concerns product spaces.

Many constructions involve

$$X \times X,$$

$$X^n,$$

$$Y \times Z,$$

or

$$E \times_B X.$$

However, product adjacencies are not uniquely determined in digital topology.

Different adjacencies can result in different continuities and homotopies.

Therefore, every product construction requires the specification of the chosen adjacency.

Remark 4.3

Product adjacency is not a secondary technical detail.

It forms part of the mathematical foundation of digital homotopy theory.

4.6. Digital Fibrations and Schwarz Genus

A fifth recurring issue concerns digital fibrational constructions.

Several reviewed works employ:

- digital fibrations.
- digital Schwarz genus.
- generalized motion-planning fibrations.
- TC of maps.

Such constructions require stronger structural assumptions than ordinary digital continuity. In particular, the existence of lifting mechanisms becomes important.

Remark 4.4

The use of Schwarz genus presupposes the existence of sufficiently well-behaved local sections and lifting structures.

Consequently, digital Schwarz genus should be examined together with digital covering and lifting theory.

4.7. Covering Structures and Lifting Mechanisms

Digital covering theory provides one of the strongest available foundations for digital homotopy constructions.

The reviewed literature often relies on fibrational intuition derived from classical topology. However, digital covering theory offers more explicit mechanisms through:

- unique path lifting.
- digital homotopy lifting.
- local digital homeomorphisms.

These structures provide a mathematically rigorous framework for evaluating digital fibrational constructions.

Remark 4.5

Whenever a digital analogue of a classical fibration is proposed, the availability of lifting properties characteristic of covering spaces should be investigated.

This would provide a stronger foundation for such an analogue than would formal similarities with classical path fibrations.

4.8. Generalized Topological Complexity

The final recurring issue concerns generalized TC constructions.

These include:

$$TC_n,$$

$$TC(f),$$

generalized Schwarz genus, and generalized fibrational frameworks.

Because such theories simultaneously involve:

- path spaces.
- products.
- continuity.
- homotopies.
- fibrations.
- lifting properties.

Their mathematical consistency depends on all preceding criteria.

Consequently, generalized constructions require the most comprehensive verification.

Table 1 summarizes the primary verification considerations identified in the reviewed literature and relates each study to the evaluation criteria introduced in Section 3.

Table 1. Primary verification considerations identified in the reviewed literature.

Ref.	Paper	Primary Verification Consideration	Criterion
[17]	Topological Group Construction in Proximity and Descriptive Proximity Spaces	Proximity-compatible group structures may benefit from a more explicit axiomatic framework	C6
[18]	Some Notes and Comparisons on Topological Complexities	Comparisons between TC variants involve homotopy-theoretic assumptions that merit further clarification	C6
[19]	Proximal Motion Planning Algorithms	Motion-planning constructions may benefit from additional verification of section and lifting mechanisms	C5-C6
[20]	Digital Topological Complexity of Digital Maps	Digital continuity and product adjacency constitute key components requiring careful verification	C2-C4
[21]	Some Properties of Proximal Homotopy Theory	Proximal homotopy constructions may benefit from further structural justification	C3-C6
[22]	Discrete Topological Complexities of Simplicial Maps	The transfer of topological complexity concepts to simplicial settings requires independent verification of foundational assumptions	C6

Continued

[23]	Different Types of Topological Complexity Based on Higher Homotopic Distance	Higher homotopic distance frameworks involve fibrational assumptions that may require additional justification	C6
[24]	Certain Topological Methods for Computing Digital Topological Complexity	Computational approaches depend on continuity and adjacency assumptions that should be verified explicitly	C1-C4
[25]	Higher Topological Complexity for Fibrations	Fibrational constructions may benefit from explicit verification of lifting properties	C5-C6
[26]	Counterexamples for Topological Complexity in Digital Images	Highlights adjacency sensitivity and limitations of direct classical-to-digital transfer	C1-C4
[27]	Topological Complexities of Finite Digital Images	Theorem 3.5 and Corollary 3.7 involve topological complexity constructions whose dependence on adjacency selection merits additional clarification	C1
[28]	The Higher Topological Complexity in Digital Images	The path-space constructions appearing in Theorem 4.6 depend on product adjacencies and path-space assumptions that require explicit verification	C3-C4
[29]	Digital Topological Complexity Numbers	Foundational definitions depend on adjacency and continuity choices that should be specified explicitly	C1-C4

Note. The table identifies the mathematical components upon which the proofs in the included publications rely yet which appear to be missing or at least requiring further verification. Each of these components is a mathematical component that is generally required to establish the results of the proofs related to digital or topological complexity. While the existence of these gaps in and of itself does not indicate misconduct or that the results of the papers are invalid, their existence does suggest that the proofs as presented in the publications are lacking in certain cases in terms of establishing the mathematical components necessary to consider their proofs to be complete and verified.

4.9. Interim Conclusions

The reviewed literature contains many innovative and mathematically interesting ideas.

Nevertheless, the analysis performed using Criteria C1-C6 reveals several recurring foundational themes:

1) adjacency dependence.
2) continuity verification.
3) path-space consistency.
4) product adjacency compatibility.
5) lifting structures.
6) generalized fibrational assumptions.

These observations motivate the more detailed counterexamples and corrective remarks presented in the following section. The relationships among the recurring foundational challenges identified in digital and proximal topological complexity frameworks are summarized in Figure 2.

5. Mathematical Counterexamples and Corrective Remarks

5.1. Motivation

The purpose of this section is to provide explicit mathematical examples illustrating why adjacency structures, digital continuity, covering spaces, and lifting

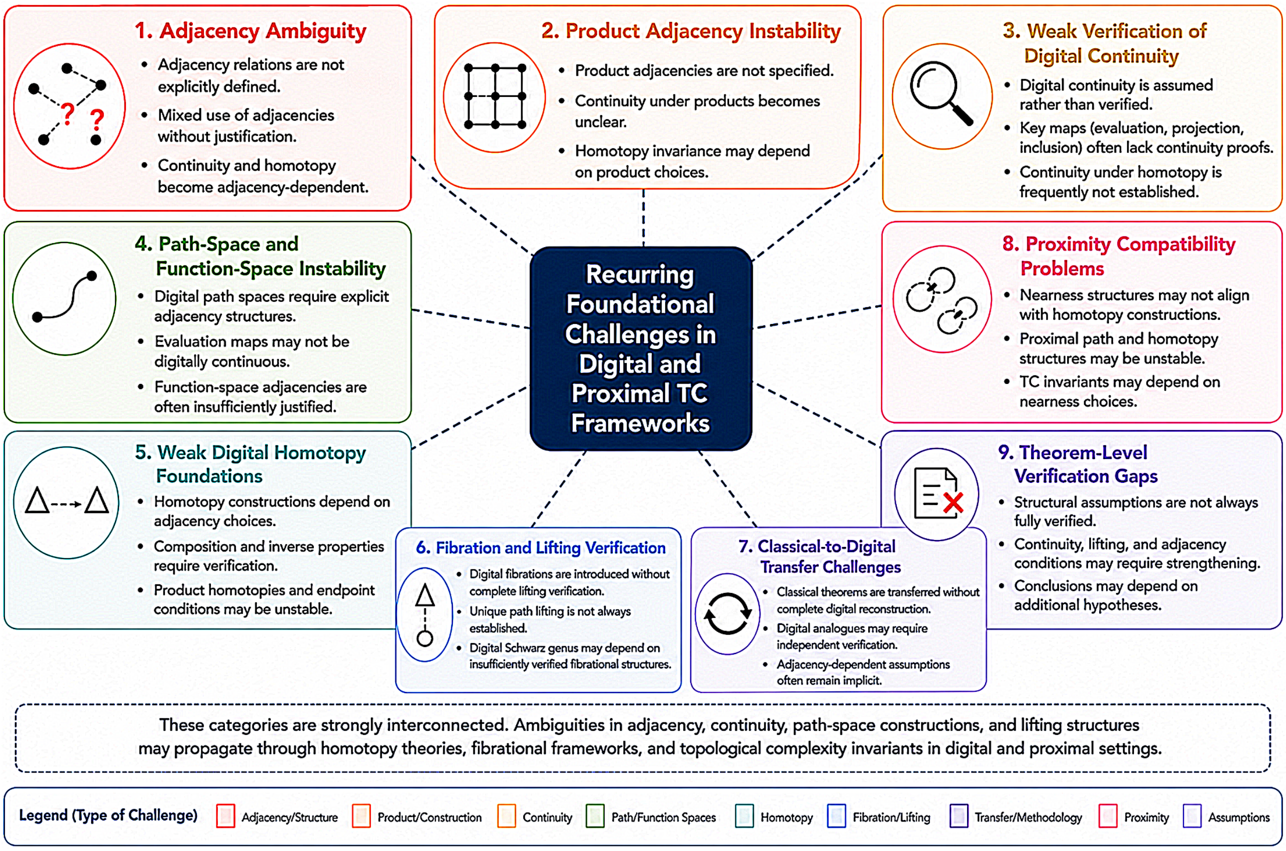

Figure 2. Relationships among adjacency structures, continuity, path spaces, and lifting mechanisms in digital topology.

properties should be verified carefully when extending classical topological complexity into digital settings.

The examples presented below are not intended to invalidate existing constructions. Rather, they demonstrate situations in which additional assumptions, clarifications, or reformulations may be necessary.

Consequently, these examples should be viewed as corrective observations aimed at strengthening future developments in digital and proximal topological complexity.

Counterexample 5.1. Path-Length Equalization Does Not Determine a Unique Path-Space Structure

Let $X=\{0,1,2,3\}\subset\mathbb{Z}$ be equipped with C_1-adjacency.

Define two digital paths $\alpha,\beta:[0,3]_{\mathbb{Z}}\to X$ by

$$\alpha(0)=0,\alpha(1)=1,\alpha(2)=2,\alpha(3)=3,$$

and

$$\beta(0)=0,\beta(1)=0,\beta(2)=1,\beta(3)=3.$$

Both paths have the same length. However, their internal combinatorial structures are different. The path α traverses the sequence $0\to1\to2\to3$, while β contains a repeated vertex and follows the sequence $0\to0\to1\to3$.

Consequently, equality of path lengths alone does not determine a unique adjacency relation on a digital path space. Additional assumptions regarding path-space adjacency and compatibility conditions are therefore required before path-length equalization can be used as a foundation for digital path-space constructions.

Corrective Remark 5.1

Whenever digital path spaces are introduced, the associated path-space adjacency should be specified explicitly and justified independently of path-length considerations.

Counterexample 5.2: Adjacency Dependence of Topological Complexity

Let

$$X = \{0,1,2\} \subset \mathbb{Z}.$$

Consider two different adjacency relations on the same underlying point set.

Under C_1-adjacency,

$$0 \leftrightarrow 1 \leftrightarrow 2,$$

so that X is connected.

Now consider a different adjacency relation c_2 defined by

$$0 \leftrightarrow 2,$$

while the point 1 is isolated.

In this case, the same set X becomes disconnected. Since connectedness, contractibility, homotopy type, and motion-planning constructions may depend on the chosen adjacency relation, the associated topological complexity invariant may also change.

Therefore, in general,

$$TC(X,c_1) \neq TC(X,c_2).$$

This example illustrates that digital topological complexity is fundamentally adjacency-dependent and should therefore be interpreted as an invariant of the pair (X,κ) rather than of the underlying point set X alone.

Corrective Remark 5.2

Digital topological complexity should generally be written as

$$TC(X,\kappa)$$

rather than

$$TC(X).$$

The adjacency relation forms part of the mathematical object itself.

Counterexample 5.3: Product Spaces Require Explicit Adjacency

Many digital TC constructions rely on

$$X \times X.$$

In classical topology, the product topology is canonical.

In digital topology, however, multiple product adjacencies may be used.

Consequently, a map

$$f : X \times X \to PX$$

may be digitally continuous under one product adjacency but fail to be digitally continuous under another.

Therefore, continuity claims involving products cannot be interpreted independently of the chosen adjacency structure.

Corrective Remark 5.3

Every theorem involving

$$X \times X,$$

$$X^n,$$

or

$$Y \times Z$$

should explicitly specify the associated product adjacency.

Counterexample 5.4: Classical Continuity Does Not Imply Digital Continuity

Consider a map

$$f : (X, k_1) \to (Y, k_2).$$

A map that is continuous in a classical topological sense may fail to preserve adjacency.

Digital continuity requires

$$f\left(N_{k_1}(x,1)\right) \subset N_{k_2}\left(f(x),1\right)$$

for all $x \in X$.

Therefore, continuity arguments inherited from classical topology cannot automatically be transferred into digital environments.

Corrective Remark 5.4

Digital continuity should always be verified directly within the chosen adjacency framework.

Counterexample 5.5: Fibrational Analogies Without Lifting Verification

Many generalized TC constructions employ digital fibrations or fibration-like structures.

Suppose

$$p : E \to B$$

is proposed as a digital analogue of a classical fibration.

The existence of a fibration-like diagram alone does not guarantee the availability of lifting mechanisms.

Without:

- unique path lifting.
- homotopy lifting.
- local digital homeomorphisms.

The analogy with classical fibrations remains incomplete.

Corrective Remark 5.5

Digital covering theory should be used whenever possible to verify lifting behavior before introducing digital analogues of Schwarz genus or generalized topological complexity.

Counterexample 5.6: Generalized TC Requires Simultaneous Verification

Higher TC constructions frequently involve:

$$TC_n,$$

$$TC(f),$$

and generalized Schwarz genus.

Such constructions combine:

- path spaces.
- products.
- homotopies.
- continuity.
- fibrations.
- lifting properties.

Failure of any one component may affect the validity of the entire construction.

Corrective Remark 5.6

Generalized topological complexity should be regarded as the final stage of a hierarchy whose lower-level assumptions must first be verified.

5.2. Summary of Mathematical Difficulties

The examples presented above reveal six recurring sources of mathematical difficulty:

Issue	Description
D1	Path-space ambiguity
D2	Adjacency dependence
D3	Product adjacency ambiguity
D4	Digital continuity verification
D5	Lifting and covering structures
D6	Generalized fibrational assumptions

These observations do not imply that existing constructions are necessarily incorrect. Rather, they indicate that additional mathematical verification may be required before classical topological machinery can be transferred reliably into digital and proximal settings.

6. Toward a More Rigorous Adjacency-Based Framework for Digital and Proximal Topological Complexity

6.1. Motivation

As discussed in the previous sections, it is clear that many of the difficulties inherent to the concept of digital topological complexity are rooted in issues related to the theory itself, rather than in technical mistakes made in formulating those concepts. Such issues include, but are not limited to, problems related to adjacency, products, path spaces, digital continuity, homotopy, and lifting.

In response to these issues, then, future developments in the field of digital topological complexity are likely to be best served by first establishing a solid foundation for the field itself. Accordingly, this section proposes a framework for the field that is founded upon adjacencies, digital continuity, path spaces, and lifting.

The proposed foundational framework and its interconnections are summarized in Figure 3.

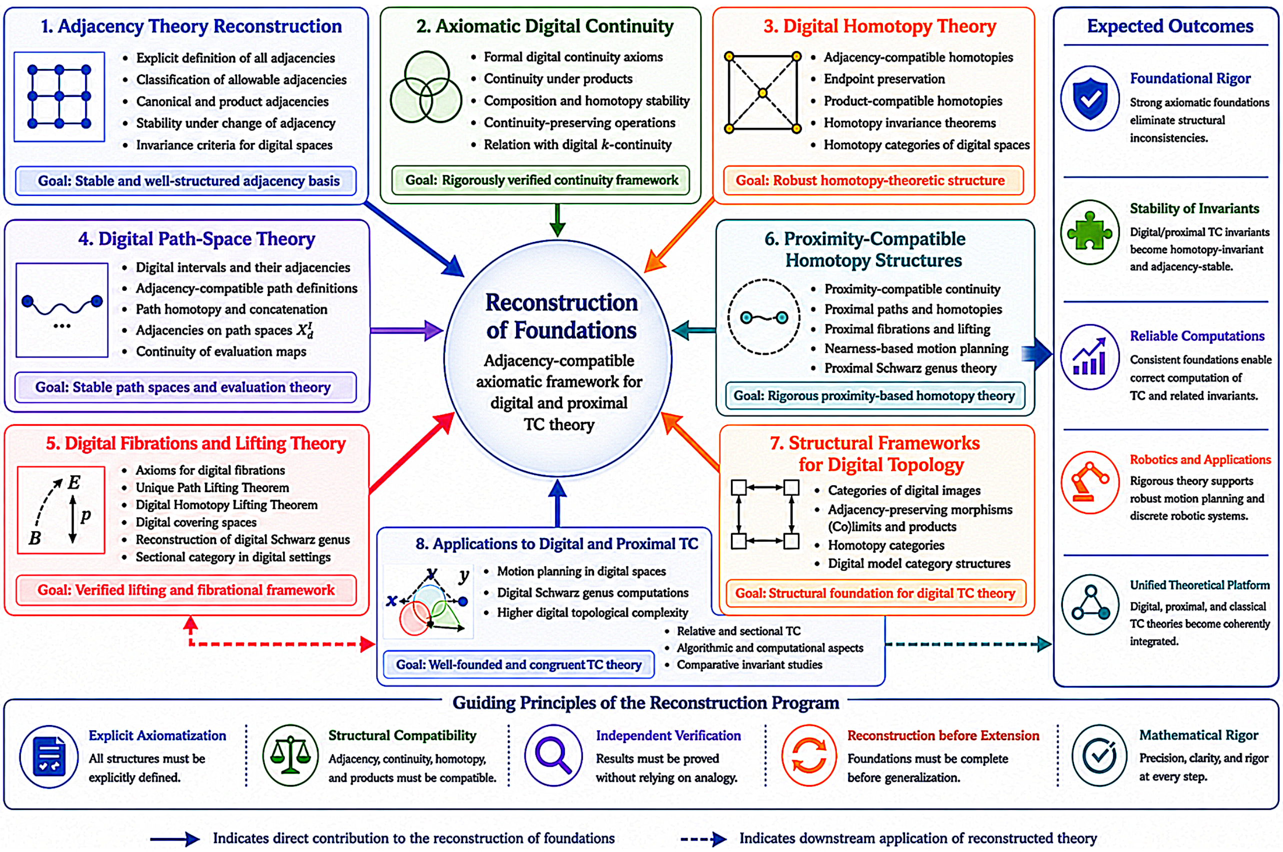

Figure 3. Proposed framework for developing mathematically rigorous digital and proximal topological complexity theories.

6.2. Explicit Axiomatization of Adjacency Structures

The first requirement for a rigorous theory is the explicit specification of adjacency structures.

Since digital topology is fundamentally adjacency-dependent, every construction should clearly identify:

- the digital image (X,κ).
- the adjacency relation κ.
- the adjacency relations used on codomain spaces.
- the product adjacencies employed in subsequent constructions.

Many homotopy-theoretic properties depend directly on adjacency choices.

Consequently, future research should aim to classify admissible adjacencies and identify conditions under which topological complexity invariants remain stable.

Recommendation R1

Every theorem should explicitly state the adjacency structures used throughout the construction.

6.3. Axiomatic Digital Continuity

A second requirement concerns digital continuity.

Many of the constructions that have been made so far have used evaluation maps, projection maps, inclusion maps, motion-planning maps, and digital fibrations. Each of these concepts requires some discussion of the notion of continuity.

A neighborhood-based characterization of digital continuity may be expressed through

$$f\left(N_{k_1}\left(x,1\right)\right) \subseteq N_{k_2}\left(f\left(x\right),1\right),$$

for every $x \in X$.

Such formulations provide explicit criteria for verifying continuity and can serve as foundational tools in future developments.

Recommendation R2

Digital continuity should be verified before introducing homotopy-theoretic or fibrational constructions.

6.4. Reconstruction of Digital Homotopy Theory

Digital homotopy constitutes the homotopy-theoretic foundation of digital topological complexity.

Future developments should establish:

- adjacency-compatible homotopies.
- endpoint-preserving homotopies.
- product-compatible homotopies.
- homotopy invariance results.
- compatibility with digital continuity.

Particular attention should be given to the interaction between homotopies and product adjacencies, since many TC constructions rely on products of digital images.

The proposed framework is not intended to replace existing theories. Rather, it seeks to provide a common foundational structure within which current digital and proximal topological complexity constructions can be reformulated, verified, and further developed.

Recommendation R3

Homotopy constructions should be verified relative to explicitly specified adjacency and product-adjacency structures.

6.5. Development of Digital Path-Space Theory

Digital path spaces remain one of the most important foundational challenges in current digital TC theory.

A rigorous path-space framework should include:

- digital intervals.
- adjacency structures on path spaces.
- path concatenation operations.
- evaluation maps.
- endpoint preservation properties.
- compatibility with homotopy constructions.

The examples discussed in Section 5 demonstrate that path-length equalization alone is insufficient to guarantee path-space compatibility.

Recommendation R4

Digital path-space theory should be developed as an independent mathematical structure rather than being transferred directly from classical path-space theory.

6.6. Digital Coverings, Fibrations, and Lifting Theory

One of the strongest available foundations for digital homotopy theory is digital covering theory.

Future developments involving digital fibrations, digital Schwarz genus, and generalized topological complexity should be supported by:

- digital covering spaces.
- local digital homeomorphisms.
- unique path lifting properties.
- digital homotopy lifting properties.

These structures provide rigorous mechanisms for transferring paths and homotopies through adjacency-based environments.

Recommendation R5

Whenever possible, digital fibrational constructions should be justified through covering and lifting theory.

6.7. Product Adjacency Compatibility

Many of the difficulties identified in this review originate from product constructions.

Future theories should explicitly address spaces such as:

- $X \times X$,
- X^n ,
- $Y \times Z$,
- $X \times [0, m]_{\mathbb{Z}}$,

through a systematic product-adjacency framework.

Without such compatibility, continuity, homotopy invariance, and motion-planning constructions may become unstable.

Recommendation R6

Product adjacencies should be regarded as foundational components of digital topological complexity theory rather than secondary technical choices.

6.8. Generalized Topological Complexity Frameworks

Generalized constructions such as:

- TC_n,
- $TC(f)$,
- digital Schwarz genus,
- generalized sectional category,
- higher digital topological complexity,

should be viewed as higher-level structures built upon more fundamental concepts.

Their validity depends on the successful verification of:

- adjacency structures.
- digital continuity.
- homotopy theory.
- path-space constructions.
- lifting properties.

Consequently, generalized theories should only be developed after these foundational components have been rigorously established.

Recommendation R7

Reconstruction should precede generalization.

6.9. Expected Outcomes

The proposed framework may offer several benefits.

First, it can enhance the mathematical rigor of the theory.

Second, it can improve the stability of digital and proximal topological invariants under changes of adjacency and homotopy framework.

Third, it can provide a more reliable basis for the computation of TC-type invariants.

Fourth, it may enable a better relationship to be established between digital topological spaces, proximal topological spaces, and classical algebraic topological spaces.

Finally, it can be applied to several important areas such as robotics, image analysis, computer vision, and computer-aided motion planning.

6.10. Concluding Perspective

The message of this paper is not that current developments in digital and proximal topology should be abandoned. Instead, the paper suggests that the future developments of the field will benefit from reconstructing the current developments with adjacency structures, continuity, path spaces, and covering spaces.

In other words, the future development of the field may depend less upon the discovery of new invariants, and more upon the establishment of solid foundations for the invariants that already exist.

6.11. Limitations of the Present Review

The evaluation criteria proposed in this paper are intended as heuristic tools for assessing mathematical rigor and structural completeness. They are not designed as automatic criteria for determining the correctness or incorrectness of published results. A construction may satisfy some criteria while requiring additional verification in others. Consequently, the observations presented in this review should be interpreted as constructive suggestions aimed at strengthening future developments rather than as definitive judgments concerning existing theories.

7. Conclusions

Topological complexity has recently become one of the most important invariants used to study motion planning problems. The extension of topological complexity theory to the digital and proximal settings has led to a growing body of literature on digital analogues of concepts like digital path spaces, digital homotopies, digital fibrations, Schwarz genus, homotopic distance, and generalized topological complexity.

Through reviewing the literature on digital and proximal topological complexity, we have identified several common themes and issues regarding digital spaces and their generalizations of classical topology concepts. The main themes identified include the dependence of many concepts upon adjacency structures and choices, the need for continuity to be proven for digital spaces, complexities introduced by path spaces and product adjacencies, and the need for covering and lifting structures within fibrations to be more thoroughly investigated.

To address these various issues in the literature, we propose a new framework for generalizing topological complexity concepts. Our framework intends to specifically address issues like the need to axiomatize the adjacencies of digital spaces, to verify that digital maps are continuous, to develop a theory of path spaces for digital spaces, to define lifting structures for digital spaces based upon the covering structures of maps, and to develop appropriate adjacencies for digital spaces that ensure compatibility with the requirements of topological complexity.

The main principle that underpins the proposed framework is that before generalizing and extending topological complexity to new settings, those foundations and structures should first be established.

The objective of this review is not to reject the existing results of the current theories on digital and proximal topological complexity, but to suggest that they may benefit from being more rigorously founded mathematically. Each of the current theories contains some valuable ideas and insights; however, they may benefit from further verification of the foundations of those theories.

Future research in this area may be beneficial to create unified frameworks that relate digital topologies, proximal topologies, coverings, homotopies, and topological complexities. Such frameworks may produce both solid mathematical justifications of the existing theories, as well as enable their current applications in robotics, computer vision, and related fields.

Thus, the future advancement of digital and proximal topological complexity is likely to depend upon both the creation of new invariants of digital and proximal spaces, but also upon the reconstruction of the mathematical foundations of those fields.

The objective of this review is constructive rather than corrective; namely, to identify common foundational requirements that may contribute to a more unified and mathematically rigorous theory.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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