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Deormation Theory of Geometric Objects

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Abstract

About at the same time, E. Vessiot (1903) and E. Cartan (1904), the only two French heirs of S. Lie in competition, discovered structure equations that they used to study infinite groups of transformations of a manifold, now called Lie pseudogroups. Using exterior calculus on jet-bundles, Cartan could not quotient down his equations on the manifold, a result only obtained in 1970 by D.C. Spencer with the nonlinear Spencer sequences. On the contrary, Vessiot was able to quotient down his results on the manifold. This paper proves and illustrates through many examples the importance of the Vessiot structure equations totally ignored for one century, particularly by D.C. Spencer or E. Kolchin and their successors. It happens that, apart from the well-known Maurer-Cartan equations for Lie groups, the *only* Vessiot structure constant known today is that of the constant Riemannian curvature introduced by L. P. Eisenhart in 1918, confusing *integrability conditions* (IC) with *compatibility conditions* (CC) in the study of the Killing equations for a given nondegenerate metric but no reference can be found in the whole mathematical literature today. We shall prove that, in this case, *two* Vessiot structure constants indeed exist, one vanishing and the other being the only known one or both being equal. We also associate with the Vessiot structure constants a new algebraic cohomology generalizing the Chevalley-Eilenberg cohomology of finite dimensional Lie algebras, even if the Lie pseudogroup is infinite dimensional. This new tool is particularly useful for studying the famous equivalence problem for geometric objects and structures on manifolds while computing the finite co-dimension of a transitive Lie pseudogroup in its normalizer that can be used in the *Differential Galois Theory* (DGT). Most examples can be treated by using computer algebra.

Keywords

Lie Group, Lie Pseudogroup, Lie Algebroid, Differential Sequence, Geometric Object, Integrability Conditions, Compatibility Conditions, Differential Galois Theory, Structure Constants, Deformation Cohomology, Centralizer,

Normalizer

1. Introduction

In 1898, Jules Drach, a student of Ecole Normale Supérieure (ENS), published a thesis with a jury made by H. Poincaré, E. Picard and G. Darboux, the best French mathematicians at that time [1]. The thesis contains clever ideas on abstract differential fields and differential extensions, quite in advance at that time when people were more familiar with analysis. However, even if its extension to Lie pseudogroups, namely groups of transformations solutions of systems of OD or PD equations (called infinite groups at that time) was well understood, the definition of a kind of Galois group or Galois pseudogroup has been totally missed by Drach *and* the jury. Let us explain this problem by considering two elementary examples [2]-[5].

Let us consider the polynomial $P \equiv y^3 - 3y + 1 \in \mathbb{Q}[y]$ which is generating a prime ideal $\mathfrak{p} \subset \mathbb{Q}[y] = K[y]$ and the field extension L/K with $L = Q(K[y]/\mathfrak{p})$. A tedious substitution proves that if η is root of $P = 0$, then both $\sigma(\eta) = \eta^2 - 2$ and $\sigma^2(\eta) = -\eta^2 + \eta + 2$ are two other roots while $\sigma^3(\eta) = \eta$. It follows that the splitting field of L/K is just L/K which is therefore a Galois extension. Another way is to introduce the *discriminant* $\delta = (\eta_1 - \eta_2)(\eta_1 - \eta_3)(\eta_2 - \eta_3)$ as one can prove that $\delta^2 = 81 = 9^2 \neq 0$ and all the roots are different. We recall that:

$$\boxed{\delta^2 = -27(\omega^3)^2 + 18\omega^1\omega^2\omega^3 - 4(\omega^2)^3 - 4(\omega^1)^3\omega^3 + (\omega^1\omega^2)^2}$$

while introducing the *general* polynomial $P(y) = y^3 - \omega_1 y^2 + \omega_2 y - \omega_3 \in K[y]$.

Now, following Vessiot, we may introduce three copies (y^1, y^2, y^3) of y and consider the *general* system made by $P(y^1) = 0$, $P(y^2) = 0$, $P(y^3) = 0$. Introducing the polynomial $\Delta = (y^1 - y^2)(y^1 - y^3)(y^2 - y^3) \in K[y^1, y^2, y^3]$, we may consider the previous algebraic system as a linear system for $(\omega^1, \omega^2, \omega^3)$ and the corresponding determinant is known to be the Vandermonde determinant Δ. Accordingly, if $\Delta \neq 0$, the system is equivalent to:

$$\boxed{y^1 + y^2 + y^3 = \omega^1,\ y^1y^2 + y^1y^3 + y^2y^3 = \omega^2,\ y^1y^2y^3 = \omega^3}$$

This system, which is invariant by the permutation group S_3 with 6 elements, has been called by Vessiot an *automorphic system* because the search for all the solutions is equivalent to the search for a single one. Equivalently, the space of solutions is a *principal homogeneous space* (PHS) for the full group of permutation. However, for certain *special* $(\omega^1, \omega^2, \omega^3)$, the ideal generated in $K[y^1, y^2, y^3]$ is not prime but only perfect and thus an intersection of prime ideals. This is exactly the situation existing when $\omega^1 = 0$, $\omega^2 = -3$, $\omega^3 = -1$ because $\Delta^2 - 81 = (\Delta - 9)(\Delta + 9)$. It follows that, in this case, the system is not *irreducible* because one can add the compatible equation $\Delta - 9 = 0$ or the compatible equation $\Delta + 9 = 0$ as a way to reduce the group of invariance to the subgroup $A_3 \lhd S_3$ having only 3 elements.

Following again Vessiot but in the differential case by using a ground ordinary differential field K, let us look for all the solutions of the OD equation $y_{xx} - \omega^1 y_x + \omega^2 y = 0$ when $(\omega^1, \omega^2) \in K$. Introducing two copies (y^1, y^2) of y and proceeding similarly, we may consider the OD system:

$$\boxed{\left(y^1 y^2_{xx} - y^2 y^1_{xx}\right) / \left(y^1 y^2_x - y^2 y^1_x\right) = \omega^1,\ \left(y^1_x y^2_{xx} - y^2_x y^1_{xx}\right)\left(y^1 y^2_x - y^2 y^1_x\right) = \omega^2}$$

which is easily seen to be an automorphic system under the action of the linear group GL(2) with 4 elements made by 2×2 matrices with constant coefficients (a, b, c, d) that can be reduced to SL(2) with $ad - bc = 1$ by considering the new automorphic system with $w \in K$ as Wronskian determinant such that $\partial_x w = \omega^1 w$:

$$\boxed{\left(y^1 y^2_{xx} - y^2 y^1_{xx}\right) = w\omega^1,\ \left(y^1_x y^2_{xx} - y^2_x y^1_{xx}\right) = w\omega^2,\ y^1 y^2_x - y^2 y^1_x = w}$$

For the reader familiar with (differential) algebra, the concept of "irreducibility" is thus based on a deep confusion between *maximal* (differential) ideal $\mathfrak{m}$ and *prime* (differential) ideal $\mathfrak{p}$ because a maximal ideal is prime, but the converse may not be true. Surprisingly, as we shall now explain, the *same* confusion has been done by Drach and *all* his successors until... today.

Coming back to the historical framework, we may say for short that Lie has been caring about Lie groups of transformation around 1880 while providing his famous main three theorems, before discovering that they were only examples of Lie pseudogroups in 1890. Surprisingly, he had only to main French heirs who were in competition, namely E. Cartan [6] and E. Vessiot [7]. Cartan being an obscure provincial teacher before becoming famous for his relation with A. Einstein in 1930 [8] while Vessiot has been heading ENS during two periods of 10 years. Their common dream has been to introduce new mathematical methods in physics and mechanics, in particular exterior calculus for Cartan and geometric objects for Vessiot, both of them being fascinated by the work of E. Galois and trying to enlarge it in a kind of *Differential Galois Theory* (DGT) [9]. In 1909, the Italian Physicist Ugo Amaldi, in a prophetic comment, said that the two approaches were so different that only the future should show what is the best one. In fact, the structure equations of Cartan are using exterior calculus on jet spaces without *any* possibility to project them down on the base manifold, a possibility that has been done more than 60 years later by Spencer [10]-[12]. On the contrary, the structure equations of Vessiot are provided directly on the base manifold and can be considered as a kind of non-linear Janet sequence as we shall explain.

Shortly after the publication, Vessiot discovered that the main central result of the thesis on irreducibility was wrong. After he contacted Cartan, P. Painlevé and the jury, a "*mathematical affair*" started and became rather unpleasant because Drach refused to admit his mistake.

It is now time to explain how the author of this paper has been involved in this story. Indeed, I became a visiting student of D.C. Spencer in Princeton University for 6 months starting from the fall of 1969. I had in mind to use the new homological tools he had introduced in order to solve the very specific tool of diagram chasing for finding the number $n^2\left(n^2-1\right)\left(n-2\right)/24$ of components of the Bianchi identities without referring to any combinatoric argument. During nights in the mathematical library, I discovered the work of Drach (1898), Vessiot (1903, 1904) and M. Janet (1920) [13]. I also discovered that Spencer was strictly unable to understand them and had no knowledge of general relativity though being a close friend of J. Wheeler with about the same age, and I decided to give it up by coming back to France in 1970. As I had dedicated my first Gordon and Breach book of 1978 to Janet while discovering that his tabular could be a key for future computer algebra (See the Introduction of [11]) and was still unable to understand the thesis of Drach, I imagined that the family of Janet could have kept some of his personal papers. Surprisingly, I discovered he was still alive in Paris, a few blocks away from my parents. Being a former close friend of Vessiot, he was so pleased by the dedication that he offered me a few private documents that I gave to the main library of ENS in Paris where they can be now consulted with gloves [5]. As a summary of what he told me at that time before dying in 1983 at the age of 95:

- I lost three months trying vainly to understand the thesis but succeeded appreciating the work of Vessiot, in particular the Vessiot structure equations.
- He told me that the paper he wrote in 1920 had been used by Cartan without any reference to him, mainly because Cartan had not ben able to convince Einstein about the importance of exterior calculus.
- As a most important comment, he told me that the so-called *differential groups* introduced by J.F. Ritt in 1930 were useless for DGT because no non-trivial examples could be provided and that Ritt did miss Lie pseudogroups because he was mainly an analyst rather than a differential geometer [14] [15]. However, he did lay down the foundations of differential algebra by adding the word "*differential*" to all concepts initially only concerned with pure algebra like the definition of prime and perfect differential ideals, a great achievement indeed.

Surprisingly, *the same year*, after loosing more than three months for nothing, the author lectured during a month at the Columbia University in New York along an invitation of E. Kolchin [16], who, even after discovering these facts, refused to provide a preface for the DGT book of 1983 [4]. It is only two years later that the author understood such a reaction when the second book of Kolchin on "Differential Algebraic Groups" did appear in 1985 [17], a *fact* needing no more comment on the conceptual side as we said. As a byproduct, it is easy for any reader to check, along the list of the differential algebra community made by about hundred persons including the former students of Kolchin, that *not a single one did even quote once my DGT book of 1983 during forty years*, despite the fact that Hopf algebras had been used for the first time in this book [3] [4].

We now turn to the differential geometric background needed for the remaining of this paper.

If X is a manifold of dimension n with local coordinates $x=(x^1,\cdots,x^n)$, we denote as usual by $T=T(X)$ the *tangent bundle* of X, by $T^*=T^*(X)$ the *cotangent bundle*, by $\wedge^r T^*$ the *bundle of r-forms* and by S_qT^* the *bundle of q-symmetric covariant tensors*. More generally, let $\mathcal{E}$ be a *fibered manifold*, that is a manifold with local coordinates (x^i,y^k) for $i=1,\cdots,n$ and $k=1,\cdots,m$ simply denoted by (x,y), *projection* $\pi:\mathcal{E}\to X:(x,y)\to(x)$ and changes of local coordinates $\bar{x}=\varphi(x)$, $\bar{y}=\psi(x,y)$. If $\mathcal{E}$ and $\mathcal{F}$ are two fibered manifolds over X with respective local coordinates (x,y) and (x,z), we denote by $\mathcal{E}\times_X\mathcal{F}$ the *fibered product* of $\mathcal{E}$ and $\mathcal{F}$ over X as the new fibered manifold over X with local coordinates (x,y,z). We denote by $f:X\to\mathcal{E}:(x)\to(x,y=f(x))$ a global *section* of $\mathcal{E}$, that is a map such that $\pi\circ f=id_X$ but local sections over an open set $U\subset X$ may also be considered when needed. Under a change of coordinates, a section transforms like $\bar{f}(\varphi(x))=\psi(x,f(x))$ and the derivatives transform like:

$$\frac{\partial\bar{f}^l}{\partial\bar{x}^r}(\varphi(x))\partial_i\varphi^r(x)=\frac{\partial\psi^l}{\partial x^i}(x,f(x))+\frac{\partial\psi^l}{\partial y^k}(x,f(x))\partial_i f^k(x)$$

We may introduce new coordinates (x^i,y^k,y_i^k) transforming like:

$$\bar{y}_r^l\partial_i\varphi^r(x)=\frac{\partial\psi^l}{\partial x^i}(x,y)+\frac{\partial\psi^l}{\partial y^k}(x,y)y_i^k$$

in order to define the 1-*jet bunble* $J_1(\mathcal{E})$ of $\mathcal{E}$. Differentiating q times and proceeding similarly, we shall denote by $J_q(\mathcal{E})$ the *q-jet bundle* of $\mathcal{E}$ with local coordinates $(x^i,y^k,y_i^k,y_{ij}^k,\cdots)=(x,y_q)$ called *jet coordinates* and sections $f_q:(x)\to(x,f^k(x),f_i^k(x),f_{ij}^k(x),\cdots)=(x,f_q(x))$ transforming like the sections $j_q(f):(x)\to(x,f^k(x),\partial_if^k(x),\partial_{ij}f^k(x),\cdots)=(x,j_q(f)(x))$ where both f_q and $j_q(f)$ are over the section f of $\mathcal{E}$. Of course $J_q(\mathcal{E})$ is a fibered manifold over X with projection π_q while $J_{q+r}(\mathcal{E})$ is a fibered manifold over $J_q(\mathcal{E})$ with projection $\pi_q^{q+r},\forall r\geq 0$.

DEFINITION 1.1: A (non-linear) *system* of order q on $\mathcal{E}$ is a fibered submanifold $\mathcal{R}_q\subset J_q(\mathcal{E})$ and a *solution* of $\mathcal{R}_q$ is a section f of $\mathcal{E}$ such that $j_q(f)$ is a section of $\mathcal{R}_q$.

DEFINITION 1.2: When the changes of coordinates have the linear form $\bar{x}=\varphi(x)$, $\bar{y}=A(x)y$, we say that $\mathcal{E}$ is a *vector bundle* over X and denote for simplicity a vector bundle and its set of sections by the same capital letter E. When the changes of coordinates have the form $\bar{x}=\varphi(x)$, $\bar{y}=A(x)y+B(x)$ we say that $\mathcal{E}$ is an *affine bundle* over X and we define the *associated vector bundle* E over X by the local coordinates (x,v) changing like $\bar{x}=\varphi(x)$, $\bar{v}=A(x)v$.

DEFINITION 1.3: If the tangent bundle $T(\mathcal{E})$ has local coordinates

(x, y, u, v) changing like $\bar{u}^j = \partial_i \varphi^j(x) u^i$, $\bar{v}^l = \frac{\partial \psi^l}{\partial x^i}(x, y) u^i + \frac{\partial \psi^l}{\partial y^k}(x, y) v^k$, we may introduce the *vertical bundle* $V(\mathcal{E}) \subset T(\mathcal{E})$ as a vector bundle over $\mathcal{E}$ with local coordinates (x, y, v) obtained by setting $u = 0$ and changes $\bar{v}^l = \frac{\partial \psi^l}{\partial y^k}(x, y) v^k$. Of course, when $\mathcal{E}$ is an affine bundle over X with associated vector bundle E over X, we have $V(\mathcal{E}) = \mathcal{E} \times_X E$ with notation $E \cdots \mathcal{E} \xrightarrow{\pi} X$.

For a later use, if $\mathcal{E}$ is a fibered manifold over X and f is a section of $\mathcal{E}$, we denote by $f^{-1}(V(\mathcal{E}))$ the *reciprocal image* of $V(\mathcal{E})$ by f as the vector bundle over X obtained when replacing (x, y, v) by $(x, f(x), v)$ in each chart. A similar construction may also be done for any affine bundle over $\mathcal{E}$.

We now recall a few basic geometric concepts that will be constantly used through this paper. First of all, if $\xi, \eta \in T$, we define their *bracket* $[\xi, \eta] \in T$ by the local formula $([\xi, \eta])^i(x) = \xi^r(x) \partial_r \eta^i(x) - \eta^s(x) \partial_s \xi^i(x)$ leading to the *Jacobi identity* $[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0$, $\forall \xi, \eta, \zeta \in T$ allowing to define a *Lie algebra* and to the useful formula $[T(f)(\xi), T(f)(\eta)] = T(f)([\xi, \eta])$ where $T(f): T(X) \to T(Y)$ is the tangent mapping of a map $f: X \to Y$.

When $I = \{i_1 < \cdots < i_r\}$ is a multi-index, we may set $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_r}$ for describing $\wedge^r T^*$ and introduce the *exterior derivative* $d: \wedge^r T^* \to \wedge^{r+1} T^*: \omega = \omega_I dx^I \to d\omega = \partial_i \omega_I dx^i \wedge dx^I$ with $d^2 = d \circ d = \partial_{ij} \omega_I dx^i \wedge dx^j \wedge dx^I \equiv 0$ in the *Poincaré sequence*:

$$\wedge^0 T^* \xrightarrow{d} \wedge^1 T^* \xrightarrow{d} \wedge^2 T^* \xrightarrow{d} \cdots \xrightarrow{d} \wedge^n T^* \to 0$$

The *Lie derivative* of an r-form with respect to a vector field $\xi \in T$ is the linear first order operator $\mathcal{L}(\xi)$ linearly depending on $j_1(\xi)$ and uniquely defined by the following three properties:

1) $\mathcal{L}(\xi) f = \xi . f = \xi^i \partial_i f, \forall f \in \wedge^0 T^* = C^\infty(X)$.
2) $\mathcal{L}(\xi) d = d \mathcal{L}(\xi)$.
3) $\mathcal{L}(\xi)(\alpha \wedge \beta) = (\mathcal{L}(\xi)\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}(\xi)\beta), \forall \alpha, \beta \in \wedge T^*$.

It can be proved that $\mathcal{L}(\xi) = i(\xi) d + d i(\xi)$ where $i(\xi)$ is the *interior multiplication* $(i(\xi)\omega)_{i_1 \cdots i_r} = \xi^i \omega_{i i_1 \cdots i_r}$ and that $[\mathcal{L}(\xi), \mathcal{L}(\eta)] = \mathcal{L}(\xi) \circ \mathcal{L}(\eta) - \mathcal{L}(\eta) \circ \mathcal{L}(\xi) = \mathcal{L}([\xi, \eta])$, $\forall \xi, \eta \in T$.

Indeed, if $\alpha = \alpha_i(x) dx^i \in T^*$, we have successively:

$$\begin{aligned} \mathcal{L}(\xi)\alpha &= \xi^r \partial_r \alpha_i dx^i + \alpha_i \mathcal{L}(\xi) dx^i \\ &= \xi^r \partial_r \alpha_i dx^i + \alpha_i \partial_r (\xi)^i dx^r \\ &= \xi^r (\partial_r \alpha_i - \partial_i \alpha_r) dx^i + \partial_r (\alpha_i \xi^i) dx^r. \end{aligned}$$

We now turn to group theory and start with two basic definitions:

Let G be a *Lie group*, that is another manifold with local coordinates $a=\left(a^1,\cdots,a^p\right)$ called *parameters*, a *composition* $G\times G\to G:(a,b)\to ab$, an *inverse* $G\to G:a\to a^{-1}$ and an *identity* $e\in G$ satisfying:

$$(ab)c=a(bc)=abc,\ aa^{-1}=a^{-1}a=e,\ ae=ea=a,\ \forall a,b,c\in G$$

DEFINITION 1.4: G is said to *act* on X if there is a map $X\times G\to X:(x,a)\to y=ax=f(x,a)$ such that $(ab)x=a(bx)=abx$, $\forall a,b\in G$, $\forall x\in X$ and we shall say that we have a *Lie group of transformations* of X. In order to simplify the notations, we shall use global notations even if only local actions are existing. The set $G_x=\{a\in G\,|\,ax=x\}$ is called the *isotropy subgroup* of G at $x\in X$ and the action is said to be *effective* if $ax=x,\forall x\in X\Rightarrow a=e$.

DEFINITION 1.5: A *Lie pseudogroup of transformations* $\Gamma\subset aut(X)$ is a group of transformations solutions of a system of OD or PD equations such that, if $y=f(x)$ and $z=g(y)$ are two solutions, called *finite transformations*, that can be composed, then $z=g\circ f(x)=h(x)$ and $x=f^{-1}(y)=g(y)$ are also solutions while $y=x$ is a solution and we shall set $id_q=j_q(id)$.

It is clear that Lie groups of transformations are particular examples of Lie pseudogroups of transformations as the system defining the finite transformations can be obtained by eliminating the parameters among the equations $y_q=j_q(f)(x,a)$ when q is large enough. The underlying system may be non-linear and of high order. For example, $y=ax^b\Rightarrow(y_{xx}/y_x)-(y_x/y)+\dfrac{1}{x}=0$ and $y=(ax+b)/(cx+d)\Rightarrow(y_{xxx}/y_x)-\dfrac{3}{2}(y_{xx}/y_x)^2=0$. Accordingly, we shall speak about an *algebraic pseudogroup* when the system is defined by *differential polynomials*, that is by polynomials in the derivatives. Looking for transformations " close " to the identity, that is setting $y=x+t\xi(x)+\cdots$ when $t\ll 1$ is a small constant parameter, dividing by t and passing to the limit $t\to 0$, we may linearize the above *non-linear system of finite lie equations* in order to obtain a *linear system of infinitesimal Lie equations* of the same order for vector fields. Such a system has the property that, if ξ and η are two solutions, then $[\xi,\eta]$ is also a solution. Accordingly, the set $\Theta\subset T$ of solutions of this new system satisfies $[\Theta,\Theta]\subset\Theta$ and can therefore be considered as the Lie algebra of Γ. In the two above examples, we obtain successively $\xi_{xx}-(1/x)\xi_x+\left(1/x^2\right)\xi=0$ and $\xi_{xxx}=0$. However, such a definition has no meaning *at all* in actual practice because, in general, the explicit solutions cannot be known as we hall see on examples.

Let me end this rather historical introduction with a short explanation of the reasons that pushed me to study the results of Vessiot with the aim to use them in physics, particularly in general relativity (GR). Being interested by the mathematical foundations of GR, I obtained a PhD diploma in GR at the university of Paris with top marks before going to Princeton. It is exactly at this moment that I discovered a few papers on the deformation theory of finite dimensional Lie algebras leading to the Chavalley-Eilenberg purely algebraic cohomology obtained by introducing a parametrization $c_t = c + tC + \cdots$ of the structure constants of a finite dimensional Lie algebra $\mathcal{G}$ when $t \ll 1$ is a small dimensionless parameter (See the section 2 of [18] for more details). In the meantime, I discovered the work of Spencer on the deformation of a geometric structure on a manifold, obtained through the parametrization $\omega_t = \omega + t\Omega + \cdots$ in the case of a geometric object ω like a metric on a manifold and t is again a small parameter. Comparing to GR, it became clear that the idea of Einstein had been to "*deform*" the Minkowski metric by introducing the very small dimensionless parameter $\phi/c^2 \ll 1$ where $\phi = Gm/r$ is the gravitational potential existing at the distance r of a central mass m and G is the gravitational constant while c is the speed of light in vacuum. When I discovered that the Vessiot structure equations obtained in the case of a Lie group G acting on a manifold X with $dim(G) = dim(X)$ were just the Maurer-Cartan equations, such a result led me to use the Vessiot structure constants in a similar way but for an arbitrary Lie pseudogroup $\Gamma \subset aut(X)$. The idea of studying the normalizer then came from the fact that the Poincaré group of space-time with 10 parameters is of codimension 1 in its normalizer which is the Weyl group of dimension 11 obtained by adding the only dilatation parameter. However, if the metric is not flat, that is has a non-vanishing scalar curvature, the corresponding group is self-normalizing as we shall see. It follows that all the structural results presented in this paper, namely the deformation cohomology, the deformation sequence and the study of the normalizer for a transitive Lie pseudogroup are new in the sense that no other reference can even be found elsewhere. In addition, all the motivating examples presented in section 5 are showing in an explicit way the role of the Vessiot structure constants.

In a quite more general homological framework, which is out of the frame of this paper, the work of Vessiot is fitting with the *Janet sequence* while the work of Cartan is fitting with the *Spencer sequence* (See the *fundamental diagram I* of section 3 and refer to [3] [10]-[12] for more details).

We finally insist on the fact that the confusion done by Eisenhart between IC and CC which is explained in example 5.4 by the corresponding technical lemma is still unknown and leads to revisiting the foundation of classical and conformal Riemannian geometry. Accordingly, and though striking it may be today, the mathematical foundations of GR must be revisited along these new structural results that question the origin and existence of gravitational waves and black holes.

2. Vessiot Structure Equations

We now turn to the clever theory proposed by Vessiot in 1903 [7] and sketch in a few successive steps the main results that we have presented through many books [4] [10] [11] [18] [19]. We invite the reader to follow all the definitions and results on the examples (5.2 + 5.3) provided by Vessiot himself in 1903 (!) that we revisit, by the example 5.4 which is by far the most striking one and by example 5.8 still not known though it is quite useful in analytical mechanics [3].

A) FINITE LIE EQUATIONS: If $\mathcal{E} = X \times X$, we shall denote by $\Pi_q = \Pi_q(X,X)$ the open subfibered manifold of $J_q(X \times X)$ defined independently of the coordinate system by $det(y_i^k) \neq 0$ with *source projection* $\alpha_q : \Pi_q \to X : (x, y_q) \to (x)$ and *target projection* $\beta_q : \Pi_q \to X : (x, y_q) \to (y)$. We shall sometimes introduce a copy Y of X with local coordinates (y) in order to avoid any confusion between the source and the target manifolds. Let us start with a Lie pseudogroup $\Gamma \subset aut(X)$ defined by a system $\mathcal{R}_q \subset \Pi_q$ of order q. In all the sequel we shall suppose that the system is *involutive* (see next section) and that Γ is *transitive* that is $\forall x, y \in X$, $\exists f \in \Gamma$, $y = f(x)$ or, equivalently, the map $(\alpha_q, \beta_q) : \mathcal{R}_q \to X \times X : (x, y_q) \to (x, y)$ is surjective.

B) INFINITESIMAL LIE EQUATIONS: The Lie algebra $\Theta \subset T$ of infinitesimal transformations is then obtained by linearization, that is to say setting $y = x + t\xi(x) + \cdots$, dividing by t and passing to the limit $t \to 0$ as we already said, in order to obtain the linear involutive system $R_q = id_q^{-1}(V(\mathcal{R}_q)) \subset J_q(T)$ by reciprocal image with $\Theta = \{\xi \in T \mid j_q(\xi) \in R_q\}$ while taking into account the fact that $T = id^{-1}(V(X \times X))$. From now on we shall suppose that R_q is *transitive*, that is to say the canonical projection $\pi_0^q : J_q(T) \to T$ induces an epimorphism $\pi_0^q : R_q \to T$ with kernel $R_q^0 \subset R_q$ and we have the useful short exact sequence $0 \to R_q^0 \to R_q \xrightarrow{\pi_0^q} T \to 0$.

C) DIFFERENTIAL INVARIANTS: Passing from source to target, we may *prolong* the vertical infinitesimal transformations $\eta = \eta^k(y)\frac{\partial}{\partial y^k}$ to the jet coordinates up to order q in order to obtain for any $\eta \in T(Y)$:

$$\eta^k(y)\frac{\partial}{\partial y^k} + \frac{\partial \eta^k}{\partial y^u} y_i^u \frac{\partial}{\partial y_i^k} + \left(\frac{\partial^2 \eta^k}{\partial y^u \partial y^v} y_i^u y_j^v + \frac{\partial \eta^k}{\partial y^u} y_{ij}^u\right)\frac{\partial}{\partial y_{ij}^k} + \cdots$$

where we have replaced $j_q(f)(x)$ by y_q, each component being the "formal" derivative of the previous one obtained by introducing $d_i = \partial_i + y_{\mu+1_i}^k \frac{\partial}{\partial y_\mu^k}$. Replacing the derivatives of η with respect to the target y by sections of $R_q(Y)$ over Y, we get for example:

$$\sharp(\eta_2) = \eta^k(y)\frac{\partial}{\partial y^k} + \eta_u^k(y) y_i^u \frac{\partial}{\partial y_i^k} + \left(\eta_{uv}^k(y) y_i^u y_j^v + \eta_u^k(y) y_{ij}^u\right)\frac{\partial}{\partial y_{ij}^k}$$

It is possible to prove that $[\sharp(R_q(Y)), \sharp(R_q(Y))] = \sharp([R_q(Y), R_q(Y)])$ where the *differential bracket* involved over the target will be defined independently over the source in Definition 4.2.

As $[\Theta, \Theta] \subset \Theta$ and thus $[R_q, R_q] \subset R_q$, we may use the Frobenius theorem in order to find a generating fundamental set of *differential invariants* $\{\Phi^\tau(y_q)\}$ up to order q which are such that $\Phi^\tau(\bar{y}_q) = \Phi^\tau(y_q)$ by using the chain rule for derivatives whenever $\bar{y} = g(y) \in \Gamma$ acting now on Y. Of course, in actual practice *one must use sections of* R_q *instead of solutions* but it is only in section 4 through Definition 4.2 that we shall see why the use of the Spencer operator will be crucial for this purpose. Specializing the Φ^τ at $id_q(x)$ we obtain the *Lie form* $\Phi^\tau(y_q)=\omega^\tau(x)$ of $\mathcal{R}_q$. Finally, if Φ^τ is any differential invariant at the order q, then $d_i\Phi^\tau$ is a differential invariant at order $q+1$, $\forall i=1,\cdots,n$ and so on.

D) GEOMETRIC OBJECTS: It has been the clever discovery of Vessiot in 1903 [7] to notice that a *natural bundle* $\mathcal{F}$ over X could be associated with any Lie pseudogroup $\Gamma\subset aut(X)$, both with a section ω of $\mathcal{F}$ called *geometric object* or *structure* on X, transforming the same way. Of course, as he also pointed out, different sections can provide the same Lie pseudogroup and this will be the starting point of *deformation theory*. Indeed, the prolongation $\flat\left(j_q(\xi)\right)$ at order q of any horizontal vector field $\xi=\xi^i(x)\frac{\partial}{\partial x^i}$ commutes with the prolongation at order q of any vertical vector field $\eta=\eta^k(y)\frac{\partial}{\partial y^k}$, exchanging therefore the differential invariants. Keeping in mind the well-known property of the Jacobian determinant while passing to the finite point of view, any (local) transformation $\bar{x}=\varphi(x)$ can be lifted to a (local) transformation of the differential invariants between themselves of the form $u\to\lambda\left(u,j_q(\varphi)(x)\right)$ allowing to introduce a *natural bundle* $\mathcal{F}$ over X by patching changes of coordinates $\bar{x}=\varphi(x)$, $\bar{u}=\lambda\left(u,j_q(\varphi)(x)\right)$. A section ω of $\mathcal{F}$ is called a *geometric object* or *structure* on X and transforms like $\bar{\omega}\left(f(x)\right)=\lambda\left(\omega(x),j_q(f)(x)\right)$ or simply $\bar{\omega}=j_q(f)(\omega)$ for any (local) transformation $y=f(x)$. This is a way to generalize vectors and tensors ($q=1$) or even connections ($q=2$). As a by-product we have $\Gamma=\left\{f\in aut(X)\mid \Phi_\omega\left(j_q(f)\right)\equiv j_q(f)^{-1}(\omega)=\omega\right\}$ as a new way to write out the Lie form and we may say that Γ *preserves* ω. Replacing $j_q(f)$ by f_q, we also obtain $\mathcal{R}_q=\left\{f_q\in\Pi_q\mid f_q^{-1}(\omega)=\omega\right\}$. Coming back to the infinitesimal point of view and setting $f_t=\exp(t\xi)\in aut(X)$, $\forall\xi\in T$, we may define the *ordinary Lie derivative* with value in $F=\omega^{-1}\left(V(\mathcal{F})\right)$ by the formula:

$$\mathcal{D}\xi=\mathcal{L}(\xi)\omega=\frac{\mathrm{d}}{\mathrm{d}t}j_q(f_t)^{-1}(\omega)\Big|_{t=0}\Rightarrow\Theta=\left\{\xi\in T\mid\mathcal{L}(\xi)\omega=0\right\}$$

along with the commutative diagram:

$$\begin{array}{ccc} F & \rightarrow & V(\mathcal{F}) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\omega} & \mathcal{F} \end{array}$$

We have $x\to\bar{x}=x+t\xi(x)+\cdots\Rightarrow u^\tau\to\bar{u}^\tau=u^\tau+t\partial_\mu\xi^k L_k^{\tau\mu}(u)+\cdots$ where $\mu=(\mu_1,\cdots,\mu_n)$ is a multi-index and we may write down the system of infinitesimal Lie equations in the *Medolaghi form* (care to signs):

$$\boxed{\Omega^\tau\equiv\left(\mathcal{L}(\xi)\omega\right)^\tau\equiv-L_k^{\tau\mu}\left(\omega(x)\right)\partial_\mu\xi^k+\xi^r\partial_r\omega^\tau(x)=0}$$

as a way to state the invariance of the section ω of $\mathcal{F}$, that is

$u^\tau - \omega^\tau(x) = 0 \Rightarrow \bar{u}^\tau - \bar{\omega}^\tau(\bar{x}) = 0$.

Finally, replacing $j_q(\xi)$ by a section $\xi_q \in J_q(T)$ over $\xi \in T$, we may define $R_q \subset J_q(T)$ *on sections* by the linear (non-differential) equations:

$$\boxed{\Omega^\tau \equiv \left(L(\xi_q)\omega\right)^\tau \equiv -L_k^{\tau\mu}(\omega(x))\xi_\mu^k + \xi^r \partial_r \omega^\tau(x) = 0}$$

and obtain the first prolongation $R_{q+1} \subset J_{q+1}(T)$ by adding:

$$\begin{aligned}\Omega_i^\tau &\equiv \left(L(\xi_{q+1})j_1(\omega)\right)_i^\tau \\ &\equiv -L_k^{\tau\mu}(\omega(x))\xi_{\mu+1_i}^k - \frac{\partial L_k^{\tau\mu}(\omega(x))}{\partial u^\sigma}\partial_i\omega^\sigma(x)\xi_\mu^k + \partial_r\omega^\tau(x)\xi_i^r + \xi^r\partial_r\left(\partial_i\omega^\tau(x)\right) \\ &= 0\end{aligned}$$

When ξ_{q+1} is replaced by $j_{q+1}(\xi)$, we obtain the PD *Medolaghi equations* of order $q+1$.

E) COMPATIBILITY CONDITIONS(CC): By analogy with "special" and "general" relativity, we shall call the given section *special* and any other arbitrary section *general*. The problem is now to study the formal properties of the linear system just obtained with coefficients only depending on $j_1(\omega)$, exactly like we shall proceed in the motivating examples later on. In particular, if any expression involving ω and its derivatives is a scalar object, it must reduce to a constant because Γ is assumed to be transitive and thus cannot be defined by any zero order equation. Now let us prove that the CC for $\bar{\omega}$, thus for ω too, only depend on the Φ and take the quasi-linear symbolic form $v \equiv I(u_1) \equiv A(u)u_x + B(u) = 0$ with $u_1 = (u, u_x)$, allowing to define an affine subfibered manifold $\mathcal{B}_1 \subset J_1(\mathcal{F})$ over $\mathcal{F}$. Indeed, if $\sum A(y_q)d_x\Phi$ is a minimum sum of formal derivatives of differential invariants of order q not containing any jet coordinate of strict order $q+1$, we may suppose by division that the first A in the sum is equal to 1. Applying the prolonged distribution of vector fields introduced in step 3 at order $q+1$, we obtain a new sum with less terms and a contradiction unless all the A are again differential invariants at order q and thus functions of the Φ because of the Frobenius theorem. A similar comment can be done for the B. Now, if one has two sections ω and $\bar{\omega}$ of $\mathcal{F}$, the *equivalence problem* is to look for $f \in aut(X)$ such that $j_q(f)^{-1}(\omega) = \bar{\omega}$. When the two sections satisfy the same CC, the problem is sometimes locally possible (Lie groups of transformations, Darboux problem in analytical mechanics, ...) but sometimes not ([10], p 333).

F) INTEGRABILITY CONDITIONS (IC): Instead of the CC for the equivalence problem, let us look for the *integrability conditions* (IC) of the system of infinitesimal Lie equations and suppose that, for the given section, all the equations of order $q+r$ are obtained by differentiating r times only the equations of order q, then it was claimed by Vessiot in [7] but with no proof that such a property is held if and only if there is an equivariant section

$c: \mathcal{F} \to \mathcal{F}_1 : (x,u) \to (x,u,v = c(u))$ where $\mathcal{F}_1 = J_1(\mathcal{F})/\mathcal{B}_1$ is a natural vector bundle over $\mathcal{F}$ with local coordinates (x,u,v). Moreover, any such equivariant section depends linearly on a finite number of constants c called *structure* *constants* and the IC for the *Vessiot structure equations* $I(u_1)=c(u)$ are of a polynomial form $J(c)=0$. It is important to notice that, according to its construction, *the form of the Vessiot structure equations is invariant under any change of coordinate system*. In actual practice, this study can be divided into two parts according to the following commutative and exact diagram where R_{q+1} is the first prolongation of R_q and the *symbol* g_{q+1} of R_{q+1} is the kernel of the (not necessarily surjective) map $\pi_q^{q+1}:R_{q+1}\rightarrow R_q$ induced by the short exact sequence $0\rightarrow S_{q+1}T^*\otimes T\rightarrow J_{q+1}(T)\rightarrow J_q(T)\rightarrow 0$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & g_{q+1} & = & g_{q+1} & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R^0_{q+1} & \rightarrow & R_{q+1} & \rightarrow & T & & \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & R^0_q & \rightarrow & R_q & \rightarrow & T & \rightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Indeed, chasing in this diagram, we discover that $\pi_q^{q+1}:R_{q+1}\rightarrow R_q$ is an epimorphism if and only if $\pi_q^{q+1}:R^0_{q+1}\rightarrow R^0_q$ is an epimorphism *and* $\pi_0^{q+1}:R_{q+1}\rightarrow T$ is also an epimorphism. Looking at the form of the corresponding Medolaghi equations $L(\xi_q)\omega=0$ and $L(\xi_{q+1})j_1(\omega)=0$, these two conditions respectively bring the Vessiot structure equations of *first kind* $I_*(u_1)=0$ not invoving any structure constant and the Vessiot stucture equations of *second kind* $I_{**}(u_1)=c$ involving the same number of different structure constants. Such a study, only depending now on linear algebraic techniques, can be achieved by means of computer algebra, in particular when dealing with algebraic pseudogroups defined by differential polynomials.

Finally, looking at the formal integrability of the system $\mathcal{B}_1\subset J_1(\mathcal{F})$ defined by the equations $A(u)u_x+B(u)=0$ and their first prolongation

$A(u)u_{xx}+\dfrac{\partial A(u)}{\partial u}u_xu_x+\dfrac{\partial B(u)}{\partial u}u_x=0$, the symbols only depend on $A(u)$ and we may obtain equations of the form $a(u)u_xu_x+b(u)u_x=0$ by eliminating the jets of order 2. Using local coordinates $(x,u,v=A(u)u_x+B(u))$ for $\mathcal{F}_1$, and substituting (u,v) in place of (u,u_x), we obtain equations of the form $\alpha(u)vv+\beta(u)v+\gamma(u)=0$. As we may suppose that $c=0$ for the special section, we finally get equations of the form $\alpha(u)II+\beta(u)I=0$ and it only remains to set $I(u_1)=c(u)$ in order to get polynomial Jacobi conditions of degree ≤ 2 which may not depend on u anymore because these equations are invariant in form under any change of coordinates. Though Vessiot claimed that these results had been found by Lie, we have never seen them despite a close examination of his collected works.

When $q=1$, a close examination of the Medolaghi equations and their first prolongation show at once that we can choose $v=A(u)u_x$ and we get homogeneous Jacobi conditions of degree 2 but systems of order $q=2$ may even provide linear Jacobi relations as in [19]. Other examples will be provided in Section 5. The reader interested may find more details and explicit examples in [4] [10] [11]. Finally, we invite again the reader to look carefully at the example of Riemann structure in order to understand the deep confusion that has been done by L.P. Eisenhart and all successors between *integrability conditions* and *compatibility conditions* in the study of Killing equations that are only specific first order Medolaghi equations that may not provide an involutive system.

We now provide a survey of the work done by D.C. Spencer and coworkers in [20] [21] with the purpose to prove that the work of Cartan has been superseded by the work of Spencer while the work of Vessiot has been superseded by the work of Janet, namely that *curvature* + *torsion* is in the Spencer sequence while *curvature alone* is in the Janet sequence, a big gap indeed for physics, particularly for Einstein general relativity [22] [23].

3. Linear Janet and Spencer Sequences

Let $\mu=(\mu_1,\cdots,\mu_n)$ be a multi-index with *length* $|\mu|=\mu_1+\cdots+\mu_n$. We shall say that μ is of *class* i if $\mu_1=\cdots=\mu_{i-1}=0$, $\mu_i\neq 0$ and set $\mu+1_i=(\mu_1,\cdots,\mu_{i-1},\mu_i+1,\mu_{i+1},\cdots,\mu_n)$. We set $y_q=\{y^k_\mu \mid 1\le k\le m, 0\le|\mu|\le q\}$ with $y^k_\mu=y^k$ when $|\mu|=0$. If E is a vector bundle over X with local coordinates (x^i,y^k) for $i=1,\cdots,n$ and $k=1,\cdots,m$, we denote by $J_q(E)$ the q - *jet bundle* of E with local coordinates simply denoted by (x,y_q) and *sections* $f_q:(x)\to(x,f^k(x),f^k_i(x),f^k_{ij}(x),\cdots)$ transforming like the section $j_q(f):(x)\to(x,f^k(x),\partial_i f^k(x),\partial_{ij}f^k(x),\cdots)$ when f is an arbitrary section of E. Then both $f_q\in J_q(E)$ and $j_q(f)\in J_q(E)$ are over $f\in E$ and the *Spencer operator* just allows to distinguish them by introducing a kind of "*difference*" through the operator $d:J_{q+1}(E)\to T^*\otimes J_q(E):f_{q+1}\to j_1(f_q)-f_{q+1}$ with local components $(\partial_i f^k(x)-f^k_i(x),\partial_i f^k_j(x)-f^k_{ij}(x),\cdots)$ and more generally $(df_{q+1})^k_{\mu,i}(x)=\partial_i f^k_\mu(x)-f^k_{\mu+1_i}(x)$. Indeed, the composite map $J_{q+1}(E)\to J_q(E)\to J_1(J_q(E))$ and the canonical map $J_{q+1}(E)\to J_1(J_q(E))$ identifying $y_{\mu,i}$ with $y_{\mu+1_i}$ when $|\mu|=q$ are both over the identity map $J_q(E)=J_q(E)$ and their difference has therefore an image in $T^*\otimes J_q(E)$ because of the short exact sequence:

$$0\to T^*\otimes J_q(E)\to J_1(J_q(E))\xrightarrow{\pi^1_0} J_q(E)\to 0.$$

In a symbolic way, *when changes of coordinates are not involved*, it is sometimes useful to write down the components of d in the form $d_i=\partial_i-\delta_i$ and the restriction of d to the kernel $S_{q+1}T^*\otimes E$ of the canonical projection $\pi^{q+1}_q:J_{q+1}(E)\to J_q(E)$ is *minus* the *Spencer map* $\delta=dx^i\wedge\delta_i:S_{q+1}T^*\otimes E\to T^*\otimes S_qT^*\otimes E$ which can be extended to:

$$\delta : \wedge^r T^* \otimes S_{q+1} T^* \otimes E \to \wedge^{r+1} T^* \otimes S_q T^* \otimes E$$

with $\delta \circ \delta \equiv 0$ if we set $\omega = \left(\omega^k_{\mu,I} dx^I\right)$ and define $(\delta\omega)^k_\mu = dx^i \wedge \omega^k_{\mu+1_i}$ when $|\mu| = q$.

The kernel of d is made by sections such that $f_{q+1} = j_1(f_q) = j_2(f_{q-1}) = \cdots = j_{q+1}(f)$. Finally, if $R_q \subset J_q(E)$ is a *system* of order q on E locally defined by linear equations $\Phi^\tau(x, y_q) \equiv a^{\tau\mu}_k(x) y^k_\mu = 0$ and local coordinates (x,z) for the parametric jets up to order q, the r - *prolongation* $R_{q+r} = \rho_r(R_q) = J_r(R_q) \cap J_{q+r}(E) \subset J_r(J_q(E))$ is locally defined when $r = 1$ by the linear equations $\Phi^\tau(x, y_q) = 0 \Rightarrow d_i \Phi^\tau(x, y_{q+1}) \equiv a^{\tau\mu}_k(x) y^k_{\mu+1_i} + \partial_i a^{\tau\mu}_k(x) y^k_\mu = 0$ and has *symbol* $g_{q+r} = R_{q+r} \cap S_{q+r} T^* \otimes E \subset J_{q+r}(E)$ if one looks at the *top order terms*. If $f_{q+1} \in R_{q+1}$ is over $f_q \in R_q$, differentiating the identity $a^{\tau\mu}_k(x) f^k_\mu(x) \equiv 0$ with respect to x^i and subtracting the identity $a^{\tau\mu}_k(x) f^k_{\mu+1_i}(x) + \partial_i a^{\tau\mu}_k(x) f^k_\mu(x) \equiv 0$, we obtain the identity $a^{\tau\mu}_k(x)\left(\partial_i f^k_\mu(x) - f^k_{\mu+1_i}(x)\right) \equiv 0$ and thus the restriction $d : R_{q+1} \to T^* \otimes R_q$. This first order operator induces, up to sign, the purely algebraic monomorphism $0 \to g_{q+1} \xrightarrow{\delta} T^* \otimes g_q$ on the symbol level [10]-[12].

Though surprising it may look like when revisiting the mathematical foundations of physics, let us show how the Spencer operator may interfere with *gauge theory* (GT). For this, we notice that the movement of a rigid body can be described by a pair $\left(a^1(t), a^2(t)\right)$ where $a^1(t)$ is a time depending orthogonal matrix and $a^2(t)$ is a time depending on vector. Hence, we have transformed the constant parameters of the Lie group of rigid motions in space into time depending on parameters. More generally, if X is an arbitrary manifold and G is a Lie group *not necessarily acting on* X, we may look at maps $X \to G : (x) \to (a(x))$ and this is just the gauging principle on which GT is based. Now, if G *is acting on* X with an action law $x \to y = f(x,a)$, gauging again G in this new framework provides successively the following section of $\Pi_q(X,X)$:

$$f(x) = f(x,a)\big|_{a=a(x)}, \quad f_x(x) = \frac{\partial f}{\partial x}(x,a)\bigg|_{a=a(x)}, \quad f_{xx}(x) = \frac{\partial^2 f}{\partial x \partial x}(x,a)\bigg|_{a=a(x)}, \quad \cdots$$

We obtain therefore, using the chain rule for derivatives:

$$\partial_x f(x) - f_x(x) = \frac{\mathrm{d}f(x,a(x))}{\mathrm{d}x} - \frac{\partial f}{\partial x}(x,a)\bigg|_{a=a(x)} = \frac{\partial f}{\partial a}(x,a)\bigg|_{a=a(x)} \frac{\partial a(x)}{\partial x}, \cdots$$

and more generally:

$$f_{q+1}(x) = j_{q+1}(f(x,a))\big|_{a=a(x)} \Rightarrow df_{q+1}(x) = \frac{\partial j_q(f(x,a))}{\partial a}\bigg|_{a=a(x)} \frac{\partial a(x)}{\partial x}$$

As q is large enough in such a way that $rk \dfrac{\partial j_q(f(x,a))}{\partial a}\bigg|_{a=a(x)} = dim(G)$, we obtain therefore $\mathrm{d}f_{q+1}(x)=0 \Leftrightarrow \frac{\partial a(x)}{\partial x}=0 \Leftrightarrow a(x)=a=cst$. The future will decide whether the group must act on the manifold or not, in particular if electromagnetism has to do with $U(1)$ which is not acting on space-time as in classical gauge theory or with the conformal group of space-time as we have claimed in many recent books or papers, along with the idea of H. Weyl [24]-[26].

DEFINITION 3.1: R_q is said to be *formally integrable* when the restriction $\pi^{q+r+1}_{q+r}: R_{q+r+1} \to R_{q+r}$ is an epimorphism $\forall r \geq 0$ or, equivalently, when all the equations of order $q+r$ are obtained by r prolongations only, $\forall r \geq 0$. In that case, $R_{q+1} \subset J_1(R_q)$ is a canonical equivalent formally integrable first order system on R_q with no zero order equations, called the *Spencer form*.

DEFINITION 3.2: R_q is said to be *involutive* when it is formally integrable and the symbol g_q is *involutive*, that is all the sequences $\cdots \xrightarrow{\delta} \wedge^s T^* \otimes g_{q+r} \xrightarrow{\delta} \cdots$ are exact $\forall 0 \leq s \leq n$, $\forall r \geq 0$ or, equivalently, if the image $B^s_{q+r}(g_q)$ of the left δ is equal to the kernel $Z^s_{q+r}(g_q)$ of the right δ, that is to say if all the *Spencer* δ - *cohomology bundles* $H^s_{q+r}(g_q)=Z^s_{q+r}(g_q)/B^s_{q+r}(g_q)=$ vanish, $\forall r \geq 0$ and $\forall 0 \leq s \leq n$. With a few more details to help the reader for understanding the examples, using a linear change of local coordinates, if necessary, we may *successively* solve the maximum number $\beta^n_q, \beta^{n-1}_q, \cdots, \beta^1_q$ of equations with respect to the principal jet coordinates of strict order q and class $n, n-1, \cdots, 1$ in order to introduce the *characters* $\alpha^i_q = m\frac{(q+n-i-1)!}{(q-1)!(n-i)!}-\beta^i_q$ for $i=1,\cdots,n$ with $\alpha^n_q=\alpha$. Then R_q is involutive if R_{q+1} is obtained by only prolonging the β^i_q equations of class i with respect to $d_1,\cdots,d_i$ for $i=1,\cdots,n$. In that case $dim(g_{q+1})=\alpha^1_q+\cdots+\alpha^n_q$ and one can exhibit the *Hilbert polynomial* $dim(R_{q+r})$ in r with leading term $(\alpha/n!)r^n$ when $\alpha \neq 0$. Such a prolongation procedure allows to compute *in a unique way* the principal (pri) jets from the parametric (par) other ones along the way followed by Janet [10] [11].

REMARK 3.3: This definition may also be applied to nonlinear systems as well while using a generic linearization by means of vertical bundles. For example, with $m=1$, $n=2$ and $q=2$, the nonlinear system $y_{22}-\frac{1}{3}(y_{11})^3=0$, $y_{12}-\frac{1}{2}(y_{11})^2=0$ is involutive but the nonlinear system $y_{22}-\frac{1}{2}(y_{11})^2=0$, $y_{12}-y_{11}=0$ is not involutive.

When R_q is involutive, the linear differential operator $\mathcal{D}: E \xrightarrow{j_q} J_q(E) \xrightarrow{\Phi} J_q(E)/R_q = F_0$ of order q with space of solutions $\Theta \subset E$ is said to be *involutive* and one has the canonical *linear Janet sequence* ([10], p 144):

$$\boxed{0 \to \Theta \to T \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \cdots \xrightarrow{\mathcal{D}_n} F_n \to 0}$$

where each other operator is first order involutive and generates the *compatibility conditions* (CC) of the preceding one. As the Janet sequence can be cut at any place, *the numbering of the Janet bundles has nothing to do with that of the Poincaré sequence*, contrary to what many people believe.

Equivalently, we have the involutive *first Spencer operator*

$D_1 : C_0 = R_q \xrightarrow{j_1} J_1(R_q) \to J_1(R_q)/R_{q+1} \simeq T^* \otimes R_q/\delta(g_{q+1}) = C_1$ of order one induced by $d : R_{q+1} \to T^* \otimes R_q$. Introducing the *Spencer bundles*

$C_r = \wedge^r T^* \otimes R_q/\delta(\wedge^{r-1} T^* \otimes g_{q+1})$, the first order involutive ($r+1$)- *Spencer operator* $D_{r+1} : C_r \to C_{r+1}$ is induced by

$d : \wedge^r T^* \otimes R_{q+1} \to \wedge^{r+1} T^* \otimes R_q : \alpha \otimes \xi_{q+1} \to d\alpha \otimes \xi_q + (-1)^r \alpha \wedge d\xi_{q+1}$ and we obtain the canonical *linear Spencer sequence* ([10], p 150):

$$\boxed{0 \to \Theta \xrightarrow{j_q} C_0 \xrightarrow{D_1} C_1 \xrightarrow{D_2} C_2 \xrightarrow{D_3} \cdots \xrightarrow{D_n} C_n \to 0}$$

as the Janet sequence for the first order involutive system $R_{q+1} \subset J_1(R_q)$. Introducing the other Spencer bundles

$C_r(E) = \wedge^r T^* \otimes J_q(E)/\delta(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E)$ with $C_r \subset C_r(E)$, the linear Spencer sequence is induced by the *linear hybrid sequence*:

$$\boxed{0 \to E \xrightarrow{j_q} C_0(E) \xrightarrow{D_1} C_1(E) \xrightarrow{D_2} C_2 \xrightarrow{D_3} \cdots \xrightarrow{D_n} C_n \to 0}$$

which is at the same time the Janet sequence for j_q and the Spencer sequence for the first order system $J_{q+1}(E) \subset J_1(J_q(E))$ ([10], p 153):

We have the following commutative and exact diagram allowing to relate the Spencer bundles C_r and $C_r(E)$ to the *Janet bundles*

$F_r = \wedge^r T^* \otimes F_0/\delta(\wedge^{r-1} T^* \otimes h_1)$ if we start with the short exact sequence $0 \to g_{q+1} \to S_{q+1} T^* \otimes E \to h_1 \to 0$ where $h_1 \subset T^* \otimes F_0$.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \wedge^{r-1} T^* \otimes g_{q+1} & \xrightarrow{\delta} & \wedge^r T^* \otimes R_q & \to & C_r & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E & \xrightarrow{\delta} & \wedge^r T^* \otimes J_q(E) & \to & C_r(E) & \to & 0 \\
 & & \downarrow & & \downarrow \Phi & \searrow & \downarrow \Phi_r & & \\
 & & \wedge^{r-1} T^* \otimes h_1 & \xrightarrow{\delta} & \wedge^r T^* \otimes F_0 & \to & F_r & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and obtain the following crucial commutative *fundamental diagram I* with exact columns:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \Theta & \xrightarrow{j_q} & C_0 & \xrightarrow{D_1} & C_1 & \xrightarrow{D_2}\cdots\xrightarrow{D_n} & C_n & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & E & \xrightarrow{j_q} & C_0(E) & \xrightarrow{D_1} & C_1(E) & \xrightarrow{D_2}\cdots\xrightarrow{D_n} & C_n(E) & \to & 0 \\
 & & & & \| & & \downarrow\Phi_0 & & \downarrow\Phi_1 & & \downarrow\Phi_n & & \\
0 & \to & \Theta & \to & E & \xrightarrow{\mathcal{D}} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \xrightarrow{\mathcal{D}_2}\cdots\xrightarrow{\mathcal{D}_n} & F_n & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & &
\end{array}
$$

In this diagram, only depending on the linear differential operator $\mathcal{D} = \Phi \circ j_q$, the epimorhisms $\Phi_r : C_r(E) \to F_r$ for $0 \le r \le n$ are induced *step by step* by the canonical projection $\Phi = \Phi_0 : C_0(E) = J_q(E) \to J_q(E)/R_q = F_0$ if we start with the knowledge of $R_q \subset J_q(E)$ or from the knowledge of an epimorphism $\Phi : J_q(E) \to F_0$ if we set $R_q = ker(\Phi)$. It follows that the hybrid sequence projects onto the Janet sequence and that the kernel of this projection is the Spencer sequence. Also, chasing in the previous diagram, we may finally define the Janet bundles, up to an isomorphism, by the formula:

$$\boxed{F_r = \wedge^r T^* \otimes J_q(E) / \left(\wedge^r T^* \otimes R_q + \delta\left(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes E\right)\right)}$$

and $\mathcal{D}_r : F_{r-1} \to F_r$ is thus induced by $d : \wedge^{r-1} T^* \otimes J_{q+1} \to \wedge^r T^* \otimes J_q(E)$ ([10], p 391). This result will be crucially used in the next section dealing with the deformation theory of Lie equations when $E = T$, $R_q \subset J_q(T)$ is a transitive involutive system of infinitesimal Lie equations of order q and the operator $\mathcal{D}$ is a *Lie operator*, that is if $\mathcal{D}\xi = 0, \mathcal{D}\eta = 0 \Rightarrow \mathcal{D}[\xi,\eta] = 0$, $\forall \xi,\eta \in \Theta \in T$.

DEFINITION 3.4: The Janet sequence is said to be *locally exact* if any local section of F_r killed by $\mathcal{D}_{r+1}$ is the image by $\mathcal{D}_r$ of a local section of F_{r-1} over a convenient open subset for any $r = 0,\cdots,n$. The Poincaré sequence is locally exact, but counterexamples may exist ([10], p 202). As another useful definition for applications, we shall also say that a differential sequence is *formally exact* if each operator involved generates *all* the CC of the preceding one. The Poincaré, Janet and Spencer sequences are formally exact by construction.

In order to go further on, we need a few more concepts from differential geometry.

DEFINITION 3.5: A chain $\mathcal{E} \xrightarrow{\Phi} \mathcal{E}' \xrightarrow{\Psi} \mathcal{E}''$ of fibered manifolds is said to be a *sequence with respect to a section* f'' *of* $\mathcal{E}''$ if $im(\Phi) = ker_{f''}(\Psi)$, that is with local coordinates (x,y) on $\mathcal{E}$, (x,y') on $\mathcal{E}'$, (x,y'') on $\mathcal{E}''$ and $y' = \Phi(x,y)$, $y'' = \Psi(x,y')$, we have $\Psi(x,\Phi(x,y)) \equiv f''(x)$, $\forall (x,y) \in \mathcal{E}$. It is said to be an *exact sequence* if $im(\Phi) = ker_{f''}(\Psi)$ *and* the corresponding vertical sequence of vector bundles is exact.

Indeed, dfferentiating this identity with respect to y, we obtain:

$$\frac{\partial \Psi}{\partial y'}\left(x, \Phi(x,y)\right) \cdot \frac{\partial \Phi}{\partial y}(x,y) \equiv 0$$

and a sequence $V(\mathcal{E}) \xrightarrow{V(\Phi)} V(\mathcal{E}') \xrightarrow{V(\Psi)} V(\mathcal{E}'')$ of vector bundles pulled back over $\mathcal{E}$.

PROPOSITION 3.6: If $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ are affine bundles over X with corresponding model vector bundles E, E', E'' over X, a sequence of such affine bundles is exact if and only if the corresponding sequence of model vector bundles is exact. In that case, there is an exact sequence $\mathcal{E} \xrightarrow{\Phi} \mathcal{E}' \to E''$ which allows to avoid the use of a section of $\mathcal{E}''$ while replacing it by the zero section of E''. Finally, the map Φ is injective if and only if the map $V(\Phi)$ is injective and the map Ψ is surjective if and only if the map $V(\Psi)$ is surjective.

Proof: We have successively $y' = \Phi(x,y) = A(x)y + B(x)$,
$y'' = \Psi(x,y') = C(x)y' + D(x)$ and by composition
$y'' = C(x)A(x)y + C(x)B(x) + D(x) = f''(x)$. Accordingly, we *must* have
$C(x)A(x) \equiv 0$, $C(x)B(x) + D(x) = f''(x)$, $\forall x \in X$ and we obtain the following commutative diagram:

$$\begin{array}{ccccc}
E & \xrightarrow{V(\Phi)} & E' & \xrightarrow{V(\Psi)} & E'' \\
\vdots & & \vdots & \nearrow & \vdots \\
\mathcal{E} & \xrightarrow{\Phi} & \mathcal{E}' & \xrightarrow{\Psi} & \mathcal{E}'' \\
\pi \downarrow & & \pi' \downarrow & & \pi'' \downarrow \uparrow f'' \\
X & = & X & = & X
\end{array}$$

If $(x,y') \in \mathcal{E}'$ is such that $C(x)y' + D(x) = f''(x)$, we obtain by subtraction $C(x)(y' - B(x)) = 0 \Leftrightarrow C(x)(y' - \Phi(x,y)) = 0$ with $(x, y' - \Phi(x,y)) \in E'$. Supposing the model sequence exact, we may find $(x,v) \in E$ such that
$y' - (A(x)y + B(x)) = A(x)v \Leftrightarrow y' = A(x)(y+v) + B(x)$ and thus
$(x,y') \in im(\Phi)$. The converse can be proved by following the same procedure backwards.

Moreover, we notice that $(x,y'') \in \mathcal{E}'' \Leftrightarrow (x, y'' - f''(x)) \in E''$, $\forall x \in X$ with $y'' - f''(x) = C(x)(y' - B(x)) = C(x)(y' - \Phi(x,y))$ where $(x,y) \in \mathcal{E}$ is *any* point over $x \in X$. It follows that the upper diagonal arrow on the right may be defined by $C(x)y' - C(x)B(x) = v''$ and we have indeed
$v'' = 0 \Rightarrow \exists (x,y) \in \mathcal{E}, y' = A(x)y + B(x)$ as claimed.

Finally, if $(x,y_1),(x,y_2) \in \mathcal{E}$ and $V(\Phi)$ is injective, then
$y' = A(x)y_1 + B(x) = A(x)y_2 + B(x) \Rightarrow A(x)(y_1 - y_2) = 0 \Rightarrow y_1 = y_2$. Also, if $(x,y'') \in \mathcal{E}''$ and $V(\Psi)$ is surjective, then we can find $v' \in E'$ such that
$y'' - f''(x) = C(x)v' \in E'' \Rightarrow$

$$\begin{aligned} y'' &= f''(x) + C(x)v' = C(x)(B(x) + v') + D(x) \\ &= C(x)(A(x)y + B(x) + v') + D(x) = C(x)y' + D(x) \end{aligned}$$ for *any* $(x,y) \in \mathcal{E}$ and

we just need to set $y' = A(x)y + B(x) + v'$.

$\square$

Coming back to the construction of the Vessiot structure equations from the knowledge of the generating differential invariants at order q, in particular for the affine, projective and Rimann structures, we have already exhibited the system $\mathcal{B}_1 \subset J_1(\mathcal{F})$ locally defined by affine equations of the form $I(u_1) \equiv A(u)u_x + B(u) = 0$. The symbol $\mathcal{H}_1 \subset T^* \otimes V(\mathcal{F})$ of this system is defined by linear equations of the form $A(u)v_x = 0$ and the last proposition provides at once the following commutative and exact diagram of affine bundles and model vector bundles *over* $\mathcal{F}$ (care) where we decide, with a slight abuse of notations, that the left central arrow is injective (0 on the left) while right central arrow is surjective (0 on the right):

$$\begin{array}{ccccccccc} 0 \rightarrow & \mathcal{H}_1 & \rightarrow & T^* \otimes V(\mathcal{F}) & \rightarrow & \mathcal{F}_1 & \rightarrow 0 \\ & \vdots & & \vdots & \nearrow & \| & \\ 0 \rightarrow & \mathcal{B}_1 & \rightarrow & J_1(\mathcal{F}) & \rightarrow & \mathcal{F}_1 & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & \mathcal{F} & = & \mathcal{F} & = & \mathcal{F} & \end{array}$$

More generally, we may define the *nonlinear Janet bundles* $\mathcal{F}_r = \wedge^r T^* \otimes V(\mathcal{F}) / \delta(\wedge^{r-1} T^* \otimes \mathcal{H}_1)$ with $\mathcal{F}_0 = V(\mathcal{F})$ as a family of natural vector bundles *over* $\mathcal{F}$. The following commutative diagram of reciprocal images generalizes the one of Section 2.B:

$$\begin{array}{ccc} F_r & \rightarrow & \mathcal{F}_r \\ \downarrow & & \downarrow \\ X & \xrightarrow{\omega} & \mathcal{F} \end{array}$$

Adopting local coordinates (u, v) for $\mathcal{F}_r$, any infinitesimal change of source $\xi(x)\partial_i$ can be lifted to $\mathcal{F}_r$ with infinitesimal transformation rules of the formmay take the form:

$$\xi^i(x)\frac{\partial}{\partial x^i} + \partial_\mu \xi^k(x)\left(L_k^{\tau\mu}(u)\frac{\partial}{\partial u^\tau} + M_{\beta,k}^{\alpha,\mu}(u)v^\beta \frac{\partial}{\partial v^\alpha}\right)$$

where we may replace $\partial_\mu \xi^k(x)$ by $\xi_\mu^k(x)$ if we want to work formally. However, in the case of $\mathcal{F}_1$ for example, it is essential to notice that setting only $v = A(u)u_x$ is not sufficient for getting a natural vector bundle over $\mathcal{F}$ when $q \geq 2$ as one has to set $v = A(u)u_x + B(x)$ in general.

Omitting indices for simplicity and using the natural projection $T^* \otimes \mathcal{F}_1 \rightarrow \mathcal{F}_2 : v_x \rightarrow \gamma(u)v_x$ over $\mathcal{F}$ in such a way to have:

$$v_x = A(u)u_{xx} + \partial_u A(u)u_x u_x + \partial_u B(u)u_x \Rightarrow \gamma(u)v_x = a(u)u_{xx} + b(u)u_x$$

we may factor the dependence on u_x through a dependence on v as we are dealing with natural bundles and obtain expressions of the form:

$$\gamma(u)v_x = \alpha(u)vv + \beta(u)v$$

because $v = 0$ must be a solution for the special system of Lie equations considered. For the general system, it just remains to use the equivariant section $v = c(u)$ and substitute it in order to obtain polynomial conditions for the structure constants of degree ≥ 2. When $q = 1$, then $B(u), b(u), \beta(u)$ disappear and we get homogeneous Jacobi conditions of degree 2 exactly.

No classical technique could provide this result because all the known methods of computer algebra do construct the Janet sequence "*step by step*" and never" *as a whole*", that is from the Spencer operator ([10], p 391). It is also not evident to deal with high order natural bundles in this framework, even when $n = 1$. For example, in the affine case, we have:

$$y_x = y_{\bar{x}} \partial_x \varphi \Rightarrow y_{xx} = y_{\bar{x}\bar{x}} (\partial_x \varphi)^2 + y_{\bar{x}} \partial_{xx} \varphi$$

and obtain by division the transition rules of the second order natural bundles $\mathcal{F}$ and $J_1(\mathcal{F})$:

$$\bar{x} = \varphi(x), \; u = \bar{u} \partial_x \varphi + \frac{\partial_{xx}\varphi}{\partial_x \varphi}, \; u_x = \bar{u}_{\bar{x}} (\partial_x \varphi)^2 + \bar{u} \partial_{xx} \varphi + \left(\frac{\partial_{xxx}\varphi}{\partial_x \varphi} - \left(\frac{\partial_{xx}\varphi}{\partial_x \varphi} \right)^2 \right)$$

Hence, the transition rules for $V(\mathcal{F})$ are $v = \bar{v} \partial_x \varphi$ while the transition rules for $T^* \otimes V(\mathcal{F})$ are $v_x = \bar{v}_{\bar{x}} (\partial_x \varphi)^2$. The projective case can be treated similarly by using the formula:

$$y_{xxx} = y_{\bar{x}\bar{x}\bar{x}} (\partial_x \varphi)^3 + 3 y_{\bar{x}\bar{x}} \partial_x \varphi \partial_{xx} \varphi + y_{\bar{x}} \partial_{xxx} \varphi$$

EXAMPLE 3.7: When $n = 1$ and we consider the affine transformaions of the real line, we have the following Janet/Spencer diagram:

$$\begin{array}{cccccccccccc}
 & & & & & 0 & & 0 & & & \\
 & & & & & \downarrow & & \downarrow & & & \\
 & 0 & \rightarrow & \Theta & \xrightarrow{j_2} & 2 & \xrightarrow{D_1} & 2 & \rightarrow & 0 & \text{Spencer} \\
 & & & & & \downarrow & & \| & & & \\
 & 0 & \rightarrow & 1 & \xrightarrow{j_2} & 3 & \xrightarrow{D_1} & 2 & \rightarrow & 0 & \\
 & & & \| & & \downarrow \Phi & & \downarrow & & & \\
0 \rightarrow & \Theta & \rightarrow & 1 & \xrightarrow{\mathcal{D}} & 1 & \rightarrow & 0 & & & \text{Janet} \\
 & & & & & \downarrow & & & & & \\
 & & & & & 0 & & & & &
\end{array}$$

EXAMPLE 3.8: When $n = 2$ and we consider the situation of the Euclidean metric, we have $dim(g_1) = 1$, $g_2 = 0$ and the sequence $0 \rightarrow \wedge^2 T^* \otimes g_1 \rightarrow 0$ cannot be exact and therefore g_1 cannot be 2-acyclic, that is $H_1^2(g_1) \neq 0$. Accordingly, in order to construct the Janet sequence, we *must* start with R_2 such that $dim(R_2) = 3$ as we have ($2\,translations + 1\,rotation$) and the Spencer bundles becomes $C_r = \wedge^r T^* \otimes R_2$. The preceding diagram becomes:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & \Theta & \xrightarrow{j_2} & 3 & \xrightarrow{D_1} & 6 & \xrightarrow{D_2} & 3 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 2 & \xrightarrow{j_2} & 12 & \xrightarrow{D_1} & 16 & \xrightarrow{D_2} & 6 & \rightarrow & 0 \\
 & & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_2 & & \\
0 & \rightarrow & \Theta & \rightarrow & 2 & \xrightarrow{\mathcal{D}} & 9 & \xrightarrow{\mathcal{D}_1} & 10 & \xrightarrow{\mathcal{D}_2} & 3 & \rightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & 0 & &
\end{array}
$$

Increasing n, the bigger the group is involved, the bigger the dimensions of the Spencer bundles are, contrary to what happens in the Janet sequence where the first Janet bundle has only to do with differential invariants. This rather philosophical comment, namely, to replace the Janet sequence by the Spencer sequence, must be considered as the key for understanding the work of the brothers E. and F. Cosserat in 1909 for elasticity or of H. Weyl in 1916 for electromagnetism, *the best picture being that of two children playing at see-saw* [22]-[26].

4. Deformation Theory of Lie Equations

From this point on, $R_q \subset J_q(T)$ will be an involutive system of infinitesimal Lie equations. For example, the Killing system $R_1 \subset J_1(T)$ is a first order system of infinitesimal Lie equations which does not have an involutive symbol $g_1 \subset T^* \otimes T$ and may not even be formally integrable.

Bearing in mind the application of computer algebra to the local theory of Lie pseudogroups, we want first of all to insist on two points which have never been emphasized up to our knowledge. We sketch the *first point* on an example:

EXAMPLE 4.1: If we look for infinitesimal transformations preserving the 1-form $\alpha = \alpha_i(x)dx^i$, we have to cancel the Lie derivative as follows:

$$(\mathcal{L}(\xi)\alpha)_i \equiv \alpha_r(x)\partial_i\xi^r + \xi^r\partial_r\alpha_i(x) = 0$$

As a byproduct, we have a well-defined *Lie operator* $\mathcal{D}: T \rightarrow T^*: \xi \rightarrow \mathcal{D}\xi = \mathcal{L}(\xi)\alpha$ such that $\mathcal{D}\xi = 0, \mathcal{D}\eta = 0 \Rightarrow \mathcal{D}[\xi,\eta] = 0$ because of the well-known property of the Lie derivative $[\mathcal{L}(\xi),\mathcal{L}(\eta)] \equiv \mathcal{L}(\xi)\circ\mathcal{L}(\eta) - \mathcal{L}(\eta)\circ\mathcal{L}(\xi) = \mathcal{L}([\xi,\eta])$ and such a property can be extended to tensors or even any geometric object. Accordingly, it is usual to say that, if we have two solutions to the system, then their bracket is again a solution. However, such a result comes from mathematics and *cannot* be recognized by means of computer algebra, contrary to what is sometimes claimed. Surprisingly, the underlying reason has to do with formal integrability. Indeed, if we study the first derivative of the bracket $[\xi,\eta]$, it involves in fact the *second derivatives* of ξ and η and sometimes things may change a lot. For example, if $d\alpha \neq 0$, as $\mathcal{L}(\xi)d\alpha = di(\xi)d\alpha = d\mathcal{L}(\xi)\alpha = 0$, then the first order equations brought by $\mathcal{L}(\xi)d\alpha=0$ may not be linear combinations of the first order equations brought by $\mathcal{L}(\xi)\alpha=0$. In particular, if $n=2$ and $\alpha=x^2dx^1$, then $d\alpha=-dx^1\wedge dx^2$ and the new first order equation $\partial_1\xi^1+\partial_2\xi^2=0$, which is *automatically* satisfied by any solution of the system $R_1\subset J_1(T)$ defined by $\mathcal{L}(\xi)\alpha=0$, is not a linear combination of the equations $x^2\partial_1\xi^1+\xi^2=0$, $\partial_2\xi^1=0$ defining R_1. Accordingly, R_1 is not involutive as it is not even formally integrable because $R_1^{(1)}=\pi_1^2(R_2)\subset R_1$ with a strict inclusion. *Hence it becomes a challenge to define a kind of bracket for sections of* $R_1\subset J_1(T)$ *and not for solutions as usual, that is to say independently of formal integrability*. This idea, which is a crucial one indeed as it will lead to the concept of Lie algebroid, is to replace the *classical Lie derivative* $\mathcal{L}(\xi)$ for any $\xi\in T$ by a *formal Lie derivative* $L(\xi_1)$ for any $\xi_1\in J_1(T)$ over $\xi\in T$ in such a way that $\mathcal{L}(\xi)=L(j_1(\xi))$ and to compute the bracket $[L(\xi_1),L(\eta_1)]\equiv L(\xi_1)\circ L(\eta_1)-L(\eta_1)\circ L(\xi_1)$ in the operator sense in order to be sure that the new bracket on $J_1(T)$ will satisfy the desired Jacobi identity. We obtain successively:

$$(L(\xi_1)\alpha)_i\equiv\alpha_r(x)\xi_i^r+\xi^r\partial_r\alpha_i(x)=0,\ (L(\eta_1)\alpha)_j\equiv\alpha_s(x)\eta_j^s+\eta^s\partial_s\alpha_j(x)=0$$

$$L([\xi_1,\eta_1])\alpha\equiv\alpha_k(x)(\xi^r\partial_r\eta_i^k+\xi_i^r\eta_r^k-\eta_i^s\xi_s^k-\eta^s\partial_s\xi_i^k)+([\xi,\eta])^k\partial_k\alpha_i(x)=0$$

as a way to *define*:

$$([\xi_1,\eta_1])^k=([\xi,\eta])^k=\xi^r\partial_r\eta^k-\eta^s\partial_s\xi^k$$

$$([\xi_1,\eta_1])_i^k=\xi^r\partial_r\eta_i^k+\xi_i^r\eta_r^k-\eta_i^s\xi_s^k-\eta^s\partial_s\xi_i^k$$

The induced property $[R_1,R_1]\subset R_1$ *can therefore be checked linearly on sections and no longer on solutions*. In particular, we may exhibit the section $\{\xi^1=0,\xi^2=0,\xi_1^1=0,\xi_2^1=0,\xi_1^2=0,\xi_2^2=1\}$ of R_1, even if $\xi_1^1+\xi_2^2\neq0$.

Now, using the *algebraic bracket* $\{j_{q+1}(\xi),j_{q+1}(\eta)\}=j_q([\xi,\eta])$, $\forall\xi,\eta\in T$, we may obtain by bilinearity a *differential bracket* on $J_q(T)$ extending the bracket on T:

$$\boxed{[\xi_q,\eta_q]=\{\xi_{q+1},\eta_{q+1}\}+i(\xi)d\eta_{q+1}-i(\eta)d\xi_{q+1},\ \forall\xi_q,\eta_q\in J_q(T)}$$

which does not depend on the respective lifts ξ_{q+1} and η_{q+1} of ξ_q and η_q in $J_{q+1}(T)$. Applying j_q to the Jacobi identity for the ordinary bracket, we obtain:

$$\boxed{\{\xi_{q+1},\{\eta_{q+2},\zeta_{q+2}\}\}+\{\eta_{q+1},\{\zeta_{q+2},\xi_{q+2}\}\}+\{\zeta_{q+1},\{\xi_{q+2},\eta_{q+2}\}\}\equiv0}$$

As we shall see later on, this bracket on sections satisfies the Jacobi identity and *the following definition is the only one that can be tested by means of computer algebra*:

DEFINITION 4.2: We say that a vector subbundle $R_q\subset J_q(T)$ is a *system of infinitesimal Lie equations* or a *Lie algebroid* if $[R_q,R_q]\subset R_q$, that is to say $[\xi_q,\eta_q]\in R_q$, $\forall\xi_q,\eta_q\in R_q$. The kernel R_q^0 of the projection $\pi_0^q:R_q\to T$ is the *isotropy Lie algebra bundle* of $\mathcal{R}_q^0=id^{-1}(\mathcal{R}_q)$ and $[R_q^0,R_q^0]\subset R_q^0$ does not contain derivatives, being thus defined fiber by fiber.

Of course, another difficulty to overcome in this new setting, is that we have no longer an identity like $d\mathcal{L}(\xi)\alpha-\mathcal{L}(\xi)d\alpha=0$ but it is easy to check in local coordinates that:

$$\left(dL(\xi_1)\alpha-L(\xi_1)d\alpha\right)_{ij}=\alpha_r(x)\left(\partial_i\xi_j^r-\partial_j\xi_i^r\right)+\left(\partial_i\xi^r-\xi_i^r\right)\partial_r\alpha_j(x)$$
$$-\left(\partial_j\xi^r-\xi_j^r\right)\partial_r\alpha_i(x)$$

and the Spencer operator allows to factorize the formula if we notice that:

$$\left(\partial_i\xi_j^r-\partial_j\xi_i^r\right)=\left(\partial_i\xi_j^r-\xi_{ij}^r\right)-\left(\partial_j\xi_i^r-\xi_{ij}^r\right)$$

We finally obtain:

$$i\left(\zeta_{(1)}\right)i\left(\zeta_{(2)}\right)\left(dL(\xi_1)\alpha-L(\xi_1)d\alpha\right)$$
$$=i\left(\zeta_{(2)}\right)L\left(i\left(\zeta_{(1)}\right)d\xi_2\right)\alpha-i(\zeta_1)L\left(i\left(\zeta_{(2)}\right)d\xi_2\right)\alpha$$

and more generally:

LEMMA 4.3: When $\alpha\in\wedge^{r-1}T^*$ we have the formula:

$$i\left(\zeta_{(1)}\right)\cdots i\left(\zeta_{(r)}\right)\left(dL(\xi_1)\alpha-L(\xi_1)d\alpha\right)$$
$$=\sum_{s=1}^{r}(-1)^{s+1}i\left(\zeta_{(1)}\right)\cdots i\left(\hat{\zeta}_{(s)}\right)\cdots i\left(\zeta_{(r)}\right)L\left(i\left(\zeta_{(s)}\right)D\xi_2\right)\alpha$$

which does not depend on the lift $\xi_2\in J_2(T)$ of $\xi_1\in J_1(T)$.

In order to understand the *second point*, we have to revisit the work of Vessiot. Indeed, if we have a geometric object, that is a section ω of a natural bundle $\mathcal{F}$ of order q, then we may consider the system $\mathcal{R}_q=\left\{f_q\in\Pi_q \mid f_q^{-1}(\omega)=\omega\right\}$ of finite Lie equations and the corresponding linearized system $R_q=\left\{\xi_q\in J_q(T)\mid L\left(\xi_q\right)\omega=0\right\}$ of infinitesimal Lie equations, both with the particular way to write them out, namely the Lie form and the Medolaghi form respectively. As a byproduct, when constructing the Janet sequence, we can write $F_0=J_q(T)/R_q$ but we can also use the isomorphic definition $F_0=\omega^{-1}\left(V(\mathcal{F})\right)$ *depending on whether we want to pay attention to the system or to the object*. The main idea of deformation theory will be to begin with the second point of view and finish with the first. Starting with a system $R_q\subset J_q(T)$, we shall suppose that R_q is transitive with a short exact sequence $0\rightarrow R_q^0\rightarrow R_q\rightarrow T\rightarrow 0$ and, whatever is the definition of F_0, introduce an epimorphism $\Phi: J_q(T)\rightarrow F_0$ along the following commutative and exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_q^0 & \rightarrow & J_q^0(T) & \rightarrow & F_0 & \rightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & R_q & \rightarrow & J_q(T) & \xrightarrow{\Phi} & F_0 & \rightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & T & = & T & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The next definition will also be crucial for our purpose and generalizes the standard definition:

$$\mathcal{L}(\xi)\omega = \frac{\mathrm{d}}{\mathrm{d}t} j_q\left(exp\, t\xi\right)^{-1}(\omega)\Big|_{t=0} .$$

DEFINITION 4.4: We say that a vector bundle F is *associated* with R_q if there exists a first order differential operator $L(\xi_q): F \to F$ called *formal Lie derivative* and such that:

1) $L(\xi_q + \eta_q) = L(\xi_q) + L(\eta_q) \quad \forall \xi_q, \eta_q \in R_q$.

2) $L(f\xi_q) = fL(\xi_q) \quad \forall \xi_q \in R_q, \forall f \in C^\infty(X)$.

3) $\left[L(\xi_q), L(\eta_q)\right] \equiv L(\xi_q)\circ L(\eta_q) - L(\eta_q)\circ L(\xi_q) = L(\left[\xi_q, \eta_q\right])$
$\forall \xi_q, \eta_q \in R_q$.

4) $L(\xi_q)(f\eta) = fL(\xi_q)\eta + (\xi\cdot f)\eta \quad \forall \xi_q \in R_q, \forall f \in C^\infty(X), \forall \eta \in F$ where $\xi \cdot f = i(\xi)df$.

As a byproduct, if E and F are associated with R_q, we may set on $E\otimes F$:

$$L(\xi_q)(\eta\otimes\zeta) = L(\xi_q)\eta\otimes\zeta + \eta\otimes L(\xi_q)\zeta \quad \forall \xi_q \in R_q, \forall \eta \in E, \forall \zeta \in F$$

REMARK 4.5: If $\Theta \subset T$ denotes the solutions of R_q, then $\mathcal{L}(\xi) = L(j_q(\xi))$ is simply called the *classical Lie derivative* but cannot be used in actual practice as we already said because Θ may be infinite dimensional as in Examples 1.3, 1.4, and 1.5. We obtain at once:

1) $\mathcal{L}(\xi+\eta) = \mathcal{L}(\xi) + \mathcal{L}(\eta) \quad \forall \xi, \eta \in \Theta$.

3) $\left[\mathcal{L}(\xi), \mathcal{L}(\eta)\right] = \mathcal{L}([\xi,\eta]) \quad \forall \xi, \eta \in \Theta$.

4) $\mathcal{L}(\xi)(f\eta) = f\mathcal{L}(\xi)\eta + (\xi\cdot f)\eta \quad \forall \xi \in \Theta, \forall f \in C^\infty(X), \forall \eta \in F$.

The following technical proposition and its corollary will be of constant use later on:

PROPOSITION 4.6: We have:

$$i(\zeta)d\left\{\xi_{q+1}, \eta_{q+1}\right\} = \left\{i(\zeta)d\xi_{q+1}, \eta_q\right\} + \left\{\xi_q, i(\zeta)d\eta_{q+1}\right\}$$

Proof: We have:

$$\left(\left\{\xi_{q+1}, \eta_{q+1}\right\}\right)^k_\nu = \sum_{\lambda+\mu=\nu}\left(\xi^r_\lambda \eta^k_{\mu+1_r} - \eta^s_\lambda \xi^k_{\mu+1_s}\right)$$

Now, caring only about ξ_{q+1}, we get:

$$\begin{aligned}&\partial_i\left(\left\{\xi_{q+1}, \eta_{q+1}\right\}\right)^k_\nu - \left(\left\{\xi_{q+1}, \eta_{q+1}\right\}\right)^k_{\nu+1_i}\\ &= \sum_{\lambda+\mu=\nu}\left(\partial_i\xi^r_\lambda - \xi^r_{\lambda+1_i}\right)\eta^k_{\mu+1_r} - \left(\partial_i\xi^k_{\mu+1_s} - \xi^k_{\mu+1_s+1_i}\right)\eta^s_\lambda + \cdots\end{aligned}$$

and the Proposition follows by bilinearity.

□

The proof of the following proposition is similar (See [10], p 296 for details):

PROPOSITION 4.7: We have the formula:

$$i(\zeta)d\left[\xi_{q+1},\eta_{q+1}\right]=\left[i(\zeta)d\xi_{q+1},\eta_q\right]+\left[\xi_q,i(\zeta)d\eta_{q+1}\right]$$
$$+i\left(L(\eta_1)\zeta\right)d\xi_{q+1}-i\left(L(\xi_1)\zeta\right)d\eta_{q+1}$$

COROLLARY 4.8: If $R_q\subset J_q(T)$ is such that $\left[R_q,R_q\right]\subset R_q$, then $R_{q+1}\subset J_{q+1}(T)$ satisfies $\left[R_{q+1},R_{q+1}\right]\subset R_{q+1}$ even if R_q is not formally integrable.

EXAMPLE 4.9: T and T^* both with any tensor bundle are associated with $J_1(T)$. The case of T^* has been treated at the beginning of this section while for T we may define $L(\xi_1)\eta=[\xi,\eta]+i(\eta)d\xi_1=\{\xi_1,j_1(\eta)\}$. We have indeed $\xi^r\partial_r\eta^k-\eta^s\partial_s\xi^k+\eta^s\left(\partial_s\xi^k-\xi_s^k\right)=-\eta^s\xi_s^k+\xi^r\partial_r\eta^k$ and the four properties of the formal Lie derivative can be checked directly as we did for T^*. Of course, we find back $\mathcal{L}(\xi)\eta=[\xi,\eta],\forall\xi,\eta\in T$.

More generally, we have in a coherent way:

PROPOSITION 4.10: $J_q(T)$ is associated with $J_{q+1}(T)$ if we define:

$$L\left(\xi_{q+1}\right)\eta_q=\left\{\xi_{q+1},\eta_{q+1}\right\}+i(\xi)d\eta_{q+1}=\left[\xi_q,\eta_q\right]+i(\eta)d\xi_{q+1}$$

and thus R_q is associated with R_{q+1}. *Proof*: It is easy to check the properties 1, 2, 4 and it only remains to prove property 3 as follows.

$$\begin{aligned}\left[L\left(\xi_{q+1}\right),L\left(\eta_{q+1}\right)\right]\zeta_q&=L\left(\xi_{q+1}\right)\left(\left\{\eta_{q+1},\zeta_{q+1}\right\}+i(\eta)d\zeta_{q+1}\right)\\&\quad-L\left(\eta_{q+1}\right)\left(\left\{\xi_{q+1},\zeta_{q+1}\right\}+i(\xi)d\zeta_{q+1}\right)\\&=\left\{\xi_{q+1},\left\{\eta_{q+2},\zeta_{q+2}\right\}\right\}-\left\{\eta_{q+1},\left\{\xi_{q+2},\zeta_{q+2}\right\}\right\}\\&\quad+\left\{\xi_{q+1},i(\eta)d\zeta_{q+2}\right\}-\left\{\eta_{q+1},i(\xi)d\zeta_{q+2}\right\}\\&\quad+i(\xi)d\left\{\eta_{q+2},\zeta_{q+2}\right\}-i(\eta)d\left\{\xi_{q+2},\zeta_{q+2}\right\}\\&\quad+i(\xi)d\left(i(\eta)d\zeta_{q+2}\right)-i(\eta)d\left(i(\xi)d\zeta_{q+2}\right)\\&=\left\{\left\{\xi_{q+2},\eta_{q+2}\right\},\zeta_{q+1}\right\}+\left\{i(\xi)d\eta_{q+2},\zeta_{q+1}\right\}\\&\quad-\left\{i(\eta)d\xi_{q+2},\zeta_{q+1}\right\}+i([\xi,\eta])d\zeta_{q+1}\\&=\left\{\left[\xi_{q+1},\eta_{q+1}\right],\zeta_{q+1}\right\}+i([\xi,\eta])d\zeta_{q+1}\end{aligned}$$

by using successively the Jacobi identity for the algebraic bracket and the last proposition.

□

COROLLARY 4.11: The differential bracket satisfies the Jacobi identity:

$$\boxed{\left[\xi_q,\left[\eta_q,\zeta_q\right]\right]+\left[\eta_q,\left[\zeta_q,\xi_q\right]\right]+\left[\zeta_q,\left[\xi_q,\eta_q\right]\right]\equiv 0\quad\forall\xi_q,\eta_q,\zeta_q\in J_q(T)}$$

PROPOSITION 4.12: We have the formula:

$$\boxed{i(\zeta)\left(dL\left(\xi_{q+2}\right)\eta_{q+1}-L\left(\xi_{q+1}\right)d\eta_{q+1}\right)=L\left(i(\zeta)d\xi_{q+2}\right)\eta_q}$$

]

Proof: Using Proposition 4.6, we have:

$$i(\zeta)dL(\xi_{q+2})\eta_{q+1} = i(\zeta)d\{\xi_{q+2},\eta_{q+2}\} + i(\zeta)di(\xi)D\eta_{q+2}$$
$$= \{i(\zeta)d\xi_{q+2},\eta_{q+1}\} + \{\xi_{q+1},i(\zeta)d\eta_{q+2}\} + i(\zeta)di(\xi)d\eta_{q+2}$$

and we must subtract:

$$i(\zeta)L(\xi_{q+1})d\eta_{q+1} = L(\xi_{q+1})(i(\zeta)d\eta_{q+1}) - i(L(\xi_1)\zeta)d\eta_{q+1}$$
$$= \{\xi_{q+1},i(\zeta)d\eta_{q+2}\} + i(\xi)Di(\zeta)d\eta_{q+2} - i(L(\xi_1)\zeta)d\eta_{q+1}$$

in order to obtain for the difference:

$$L(i(\zeta)d\xi_{q+2})\eta_q - i(i(\zeta)d\xi_1)d\eta_{q+1} + i(L(\xi_1)\zeta)d\eta_{q+1}$$
$$+ i(\zeta)di(\xi)d\eta_{q+2} - i(\xi)di(\zeta)d\eta_{q+2}$$

Finally, the last four terms vanish because $L(\xi_1)\zeta - i(\zeta)d\xi_1 = [\xi,\zeta]$ and:

$$i(\zeta)di(\xi)d\eta_{q+2} - i(\xi)di(\zeta)d\eta_{q+2} = -i([\xi,\zeta])d\eta_{q+1}.$$

□

Combining this proposition and Lemma 4.3, we obtain:

PROPOSITION 4.13: When $A_{q+1}^{r-1} \in \wedge^{r-1}T^* \otimes J_{q+1}(T)$, we have the formula:

$$i(\zeta_{(1)})\cdots i(\zeta_{(r)})(dL(\xi_{q+2}) - L(\xi_{q+1})d)A_{q+1}^{r-1}$$
$$= \sum_{s=1}^{r}(-1)^{s+1} i(\zeta_{(1)})\cdots i(\hat{\zeta}_{(s)})\cdots i(\zeta_{(r)})L(i(\zeta_{(s)})d\xi_{q+2})A_q^{r-1}$$

where $A_q^{r-1} \in \wedge^{r-1}T^* \otimes J_q(T)$ is the projection of A_{q+1}^{r-1}.

Proof: With $\alpha \in \wedge^{r-1}T^*$ and $\eta_{q+1} \in J_{q+1}(T)$, we obtain successively:

$$dL(\xi_{q+2})(\alpha \otimes \eta_{q+1}) = d(L(\xi_1)\alpha \otimes \eta_{q+1} + \alpha \otimes L(\xi_{q+2})\eta_{q+1})$$
$$= dL(\xi_1)\alpha \otimes \eta_q + (-1)^{r-1}(L(\xi_1)\alpha) \wedge d\eta_{q+1}$$
$$+ d\alpha \otimes L(\xi_{q+1})\eta_q + (-1)^{r-1}\alpha \wedge DL(\xi_{q+2})\eta_{q+1}$$

$$L(\xi_{q+1})d(\alpha \otimes \eta_{q+1}) = L(\xi_{q+1})(d\alpha \otimes \eta_q + (-1)^{r-1}\alpha \wedge d\eta_{q+1})$$
$$= L(\xi_1)d\alpha \otimes \eta_q + d\alpha \otimes L(\xi_{q+1})\eta_q$$
$$+ (-1)^{r-1}L(\xi_1)\alpha \wedge d\eta_{q+1} + (-1)^{r-1}\alpha \wedge L(\xi_{q+1})d\eta_{q+1}$$

and obtain y subtraction:

$$(dL(\xi_{q+2}) - L(\xi_{q+1})d)(\alpha \otimes \eta_{q+1}) = (dL(\xi_1) - L(\xi_1)d)\alpha \otimes \eta_q$$
$$+ (-1)^{r-1}\alpha \wedge (dL(\xi_{q+2}) - L(\xi_{q+1})d)\eta_{q+1}$$

and the proposition follows by skew linearity.

□

PROPOSITION 4.14: We have the formula:

$$L(\xi_q)\{\eta_q,\zeta_q\} = \{L(\xi_{q+1})\eta_q,\zeta_q\} + \{\eta_q,L(\xi_{q+1})\zeta_q\}$$

which does not depend on the lift $\xi_{q+1} \in J_{q+1}(T)$ of $\xi_q \in J_q(T)$.

Similarly, we have the formula:

$$L(\xi_{q+1})[\eta_q,\zeta_q]=[L(\xi_{q+1})\eta_q,\zeta_q]+[\eta_q,L(\xi_{q+1})\zeta_q]$$
$$+L(i(\zeta)d\xi_{q+2})\eta_q-L(i(\eta)d\xi_{q+2})\zeta_q$$

which does not depend on the lift $\xi_{q+2}\in J_{q+2}(T)$ of $\xi_{q+1}\in J_{q+1}(T)$.

Proof: Using Jacobi identity for the algebraic bracket and proposition 5.6, we obtain:

$$\begin{aligned}L(\xi_q)\{\eta_q,\zeta_q\}&=\{\xi_q,\{\eta_{q+1},\zeta_{q+1}\}\}+i(\xi)d\{\eta_{q+1},\zeta_{q+1}\}\\&=\{\{\xi_{q+1},\eta_{q+1}\},\zeta_q\}+\{\eta_q,\{\xi_{q+1},\zeta_{q+1}\}\}\\&\quad+\{i(\xi)d\eta_{q+1},\zeta_q\}+\{\eta_q,i(\xi)d\zeta_{q+1}\}\\&=\{L(\xi_{q+1})\eta_q,\zeta_q\}+\{\eta_q,L(\xi_{q+1})\zeta_q\}\end{aligned}$$

□

THEOREM 4.15: In order to use all the previous results in a more natural way, we shall, for simplicity, restrict to the classical Lie derivative $\mathcal{L}(\xi)=L(j_q(\xi))$ through the following formulas:

$\mathcal{L}(\xi)\eta_q=[j_q(\xi),\eta_q]$	$\forall\xi\in T,\forall\eta_q\in J_q(T)$
$\mathcal{L}(\xi)\{\eta_q,\zeta_q\}=\{\mathcal{L}(\xi)\eta_q,\zeta_q\}+\{\eta_q,\mathcal{L}(\xi)\zeta_q\}$	$\forall\eta_q,\zeta_q\in J_q(T)$
$\mathcal{L}(\xi)[\eta_q,\zeta_q]=[\mathcal{L}(\xi)\eta_q,\zeta_q]+[\eta_q,\mathcal{L}(\xi)\zeta_q]$	$\forall\eta_q,\zeta_q\in J_q(T)$
$d\mathcal{L}(\xi)-\mathcal{L}(\xi)d=0$	$\forall\xi\in T$
$\mathcal{L}(\xi)\mathcal{D}-\mathcal{D}\mathcal{L}(\xi)=0$	$\forall\xi\in\Theta$
$\mathcal{L}(\xi)\mathcal{D}_r-\mathcal{D}_r\mathcal{L}(\xi)=0$	$\forall\xi\in\Theta,\forall r=1,\cdots,n$

Proof: Most of these formulas have already been proved for the formal Lie derivative and one has just to replace ξ_q by $j_q(\xi)$ while using $j_{q+1}(\xi)$ as a lift. Accordingly, when restricting to an involutive system $R_q\subset J_q(T)$ of infinitesimal Lie equations, we have to preserve the section ω of the natural bundle $\mathcal{F}$ in order to construct the Janet sequence and we obtain at once:

$$\begin{aligned}(\mathcal{L}(\xi)\mathcal{D}-\mathcal{D}\mathcal{L}(\xi))\eta&=\mathcal{L}(\xi)\mathcal{L}(\eta)\omega-\mathcal{D}[\xi,\eta]\\&=(\mathcal{L}(\xi)\mathcal{L}(\eta)-\mathcal{L}([\xi,\eta]))\omega\\&=\mathcal{L}(\eta)\mathcal{L}(\xi)\omega=\mathcal{L}(\eta)\mathcal{D}\xi=0\end{aligned}$$

Such a result may be applied at once to all the known structures.

□

Before going ahead, let us stop for a moment and wonder how we could proceed for generalizing the deformation theory of Lie algebras by using the Vessiot structure equations even though we know that the structure constants have nothing to do in general with any Lie algebra. Of course, we could start similarly from the Jacobi relations but, if we do want to exhibit a kind of cohomology, we should be able to define a trivial deformation, that is the analogue of a change of basis of the underlying vector space V of the Lie algebra $\mathcal{G}$ in such a natural way that it could induce a change of the structure constants which is surely not of a tensorial nature anymore.

The following "trick", already known to Vessiot in 1903 ([7], p 445), is still ignored today. For this, assuming that the natural bundle $\mathcal{F}$ is known, let us consider two sections ω and $\bar{\omega}$ giving rise respectively to the systems R_q and $\bar{R}_q$ of infinitesimal Lie equations:

$$R_q \qquad \Omega^\tau \equiv -L_k^{\tau\mu}\left(\omega(x)\right)\xi_\mu^k + \xi^r \partial_r \omega^\tau(x) = 0$$
$$\bar{R}_q \qquad \bar{\Omega}^\tau \equiv -L_k^{\tau\mu}\left(\bar{\omega}(x)\right)\xi_\mu^k + \xi^r \partial_r \bar{\omega}^\tau(x) = 0$$

and define the following equivalence relation:

DEFINITION 4.16: $\bar{\omega} \sim \omega \Leftrightarrow \bar{R}_q = R_q$

The study of such an equivalence relation is not evident at all, and we improve earlier presentations (Compare to [10], p 336). First of all, we shall use a solved form of the system obtained by choosing principal jets or, equivalently, choosing a square submatrix $M = \left(M(u)\right)$ of rank $dim(F_0)$ in the matrix $L = \left(L(u)\right)$ defining R_q^0 with $dim\left(J_q^0(T)\right)$ columns which describe the infinitesimal generators of prolongations of changes of coordinates on X acting on the fibers of $\mathcal{F}$ and $dim(F_0) = m$ rows. Of course, a major problem will be to obtain intrinsic results not depending on this choice. The columns of L are thus made by vector fields $L_k^\mu = L_k^{\tau\mu}(u)\dfrac{\partial}{\partial u^\tau}$ that we can therefore separate into two parts, namely the vectors $Ł_\sigma = M_\sigma^\tau(u)\dfrac{\partial}{\partial u^\tau}$ for $\sigma = 1,\cdots,dim(F_0)$ and the vectors $L_{m+r} = \mathcal{E}_{m+r}^\sigma(u)L_\sigma$ for $r = 1,\cdots,dim\left(R_q^0\right)$ obtained by introducing the *stationary functions* $\mathcal{E}(u)$, also called *Grassmann determinants*, while describing the matrix $M^{-1}L = \left(id_{F_0}, \mathcal{E}(u)\right)$. We are therefore led to look for transformations $\bar{u} = g(u)$ of the fibers of $\mathcal{F}$ such that:

$$\left(M^{-1}\right)_\tau^\sigma(\bar{u})du^\tau = \left(M^{-1}\right)_\tau^\sigma(u)du^\tau, \ \mathcal{E}_{m+r}^\sigma(\bar{u}) = \mathcal{E}_\tau^\sigma(u)$$

In order to study such a system and to prove that it is defining a Lie pseudogroup of transformations, let us notice that the first conditions are equivalent to saying that the transformations $\bar{u} = g(u)$ preserve the vector fields L_σ and also the vector fields L_{m+r} according to the second conditions. It follows that the transformations $\bar{u} = g(u)$ preserves the vector fields L_k^μ, a property thus not depending on the choice of the principal jets. In addition, we have:

PROPOSITION 4.17: The Lie pseudogroup of transformations of the fibers of $\mathcal{F}$ that we have exhibited is in fact a Lie group of transformations, namely the *reciprocal* of the lie group of transformations describing the natural structure of $\mathcal{F}$.

Proof: The defining system is finite type with a zero first order symbol. If $W = W^\tau(u)\dfrac{\partial}{\partial u^\tau}$ is an infinitesimal transformation, we obtain therefore the Lie operator $\left[W, L_k^\mu\right] = 0$, $\forall 1 \le |\mu| \le q$, $\forall k = 1,\cdots,m$. Indeed, if W_1 and W_2 are two solutions, then $[W_1,W_2]$ is also a solution because of the Jacobi identity for the bracket and there are at most $dim(F_0)$ linearly independent such vector fields denoted by W_α. It follows that $[W_1,W_2]=\rho_{12}^\alpha(u)W_\alpha$ and we deduce from the Jacobi identity again that $L_k^\mu\cdot\rho_{12}^\alpha(u)=0\Rightarrow\rho_{12}^\alpha(u)=\rho_{12}^\alpha=cst$ because $rk(L_k^\mu)=dim(F_0)$. Accordingly, the W_α are the infinitesimal generators of a Lie group of transformations of the fibers of $\mathcal{F}$ and the effective action does not depend on the coordinate system.

□

DEFINITION 4.18: These finite transformations will be called *label transformations* and will be noted $\bar{u}=g(u,a)$ where the number of parameters a is $\leq dim(F_0)$.

If R_q is formally integrable/involutive, then $\bar{R}_q=R_q$ is also formally integrable/involutive and thus $I(j_1(\omega))=c(\omega)\Leftrightarrow I(j_1(\bar{\omega}))=\bar{c}(\bar{\omega})$ with eventually different structure constants.

COROLLARY 4.19: Any finite label transformation $\bar{u}=g(u,a)$ induces a finite transformation $\bar{c}=h(c,a)$ of the structure constants which is not effective in general and we may set $\bar{\omega}\sim\omega\Rightarrow\bar{c}\sim c$.

It now remains to exhibit a deformation cohomology coherent with the above results.

DEFINITION 4.20: When F is a vector bundle associated with R_q, we may define $\Upsilon=\Upsilon(F)=\{\eta\in F \mid L(\xi_q)\eta=0,\forall\xi_q\in R_q\}$ and the sub-vector bundle $E=\{\eta\in F\mid L(\xi_q^0)\eta=0,\forall\xi_q^0\in R_q^0\}\subseteq F$ in such a way that $\Upsilon\subset E\subseteq F$.

In order to look for Υ in general, we shall decompose this study into two parts, exactly as we did in section 3.F, by using a splitting of the short exact sequence $0\to R_q^0\to R_q\xrightarrow{\pi_0^q}T\to 0$ called R_q- *connection*, namely a map $\chi_q:T\to R_q$ such that $\pi_0^q\circ\chi_q=id_T$, in order to have $R_q\simeq R_q^0\oplus\chi_q(T)$. Such a procedure does not depend on the choice of χ_q because, if $\bar{\chi}_q$ is another R_q-connection, then $(\bar{\chi}_q-\chi_q)(T)\in R_q^0$. An R_q-connection may also be considered as a section $\chi_q\in T^*\otimes R_q$ over $id_T\in T^*\otimes T$ and $\bar{\chi}_q-\chi_q\in T^*\otimes R_q^0$ in this case. It follows that we have $\Upsilon=\{\eta\in E\mid L(\chi_q(\xi))\eta=0,\forall\xi\in T\}$ and we define a first order operator $\nabla:E\to T^*\otimes E$ with zero symbol, called *covariant derivative*, by the formula $(\nabla\cdot\eta)(\xi)=\nabla_\xi\eta=L(\chi_q(\xi))\eta$. We may finally extend ∇ to a first order operator $\nabla=\wedge^rT^*\otimes E\to\wedge^{r+1}T^*\otimes E$ by the formula:

$$\nabla(\alpha\otimes\eta)=d\alpha\otimes\eta+(-1)^r\alpha\wedge\nabla\eta,\ \forall\alpha\in\wedge^rT^*,\forall\eta\in E$$

LEMMA 4.21: With $\nabla^2=\nabla\circ\nabla$, we have:

$$\boxed{(\nabla^2\eta)(\xi,\bar{\xi})=L([\chi_q(\xi),\chi_q(\bar{\xi})]-\chi_q([\xi,\bar{\xi}]))\eta=0}$$

Proof: We have:

$$\nabla^2(\alpha\otimes\eta)=\nabla\left(d\alpha\otimes\eta+(-1)^r\alpha\wedge\nabla\eta\right)$$
$$=d^2\alpha\otimes\eta+(-1)^{r+1}d\alpha\wedge\nabla\eta+(-1)^r d\alpha\wedge\nabla\eta+(-1)^r\alpha\wedge\nabla^2\eta$$
$$=(-1)^r\alpha\wedge\nabla^2\eta$$

Setting $\nabla\eta=dx^i\nabla_i\eta$, we get:

$$\nabla^2\eta=\nabla\left(dx^i\nabla_i\eta\right)=dx^i\wedge dx^j\nabla_j\nabla_i\eta=-\frac{1}{2}dx^i\wedge dx^j\left(\nabla_i\nabla_j-\nabla_j\nabla_i\right)\eta$$

and we have just to use the fact that $\nabla_i=L\left(\chi_q\left(\partial_i\right)\right)$ with $\left[\partial_i,\partial_j\right]=0$. Indeed, we have:

$$\left[\chi_q(\xi),\chi_q(\bar{\xi})\right]-\chi_q\left([\xi,\bar{\xi}]\right)$$
$$=\left\{\chi_{q+1}(\xi),\chi_{q+1}(\bar{\xi})\right\}+i(\xi)d\chi_{q+1}(\bar{\xi})-i(\bar{\xi})d\chi_{q+1}(\xi)-\chi_q\left([\xi,\bar{\xi}]\right)$$

The first term in the right member is linear in ξ and $\bar{\xi}$ while the sum of the three others becomes:

$$\xi^i\left(\partial_i\left(\chi^k_{\mu,j}\bar{\xi}^j\right)-\chi^k_{\mu+1_i,j}\bar{\xi}^j\right)-\bar{\xi}^j\left(\partial_j\left(\chi^k_{\mu,i}\xi^i\right)-\chi^k_{\mu+1_j,i}\xi^i\right)-\chi^k_{\mu,r}\left(\xi^i\partial_i\bar{\xi}^r-\bar{\xi}^j\partial_j\xi^r\right)$$
$$=\left(\partial_i\chi^k_{\mu,j}-\partial_j\chi^k_{\mu,i}+\chi^k_{\mu+1_j,i}-\chi^k_{\mu+1_i,j}\right)\xi^i\bar{\xi}^j$$

The proposition follows from the fact that $\chi_0=id_T$ and E is R^0_q-invariant.

□

Hence, we obtain by linearity the ∇ - *sequence*:

$$\boxed{0\rightarrow\Upsilon\rightarrow E\xrightarrow{\nabla}T^*\otimes E\xrightarrow{\nabla}\wedge^2T^*\otimes E\xrightarrow{\nabla}\cdots\xrightarrow{\nabla}\wedge^nT^*\otimes E\rightarrow 0}$$

which does not depend on the choice of the connection and is made by first order involutive operators. It follows that Υ can be locally described by a linear combination with constant coefficients of certain sections of $E\subset F$ and we may therefore set $dim(\Upsilon)=dim(E)\leq dim(F)$.

We now provide a few definitions:

DEFINITION 4.22: When $\Theta\subset T$ is given, we may define:

Centralizer	$C(\Theta)$	=	$\{\eta\in T\mid[\xi,\eta]=0,\forall\xi\in\Theta\}$
Center	$Z(\Theta)$	=	$\{\eta\in\Theta\mid[\xi,\eta]=0,\forall\xi\in\Theta\}$
Normalizer	$N(\Theta)$	=	$\{\eta\in T\mid[\xi,\eta]\in\Theta,\forall\xi\in\Theta\}$

It is essential to notice that these definitions are not very useful at all in actual practice when Θ is infinite dimensional but are necessary in order to construct new Lie groups or Lie pseudogroups of transformations through their systems of defining finite or infinitesimal Lie equations.

PROPOSITION 4.23: $C(\Theta)=\Upsilon(T)$.

Proof: If $R_1=\pi^q_1\left(R_q\right)\subset J_1(T)$, it follows from Example 4.9 that $\Upsilon(T)=\left\{\eta\in T\mid L\left(\xi_1\right)\eta=0,\forall\xi_1\in R_1\right\}$, that is to say $\Upsilon(T)=\left\{\eta\in T\mid\left\{\xi_1,j_1(\eta)\right\}=0,\forall\xi_1\in R_1\right\}$ if we choose $j_1(\eta)$ as a lift of η in

$J_1(T)$. In particular, if $\xi \in \Theta$ and thus $j_1(\xi) \in R_1$, we have $\{j_1(\xi), j_1(\eta)\} = [\xi, \eta]$ and thus $\Upsilon(T) \subseteq C(\Theta)$.

Now, $j_{q-1}([\xi,\eta]) = \{j_q(\xi), j_q(\eta)\}$ and thus $C(\Theta) = \{\eta \in T \mid \{\xi_q, j_q(\eta)\} = 0, \forall \xi_q \in R_q\}$, providing by projection $\{\xi_1, j_1(\eta)\} = 0$, that is $C(\Theta) \subseteq \Upsilon(T)$ and thus $C(\Theta) = \Upsilon(T)$.

□

It follows that $Z(\Theta) = \Theta \cap C(\Theta) \Rightarrow Z(\Theta) = \{\eta \in T \mid \mathcal{D}\eta = 0, L(\xi_1)\eta = 0, \forall \xi_1 \in R_1\}$ and $Z(\Theta)$ is made by sections of $\Upsilon(T)$ killed by $\mathcal{D}$. The study of $N(\Theta)$ is much more delicate and we first need the next proposition where we notice the importance of involution or at least formal integrability.

PROPOSITION 4.24: The Lie operator $\mathcal{D}: T \to F_0$ induces a homomorphism of Lie algebras $\mathcal{D}: \Upsilon(T) \to \Upsilon(F_0)$ where the bracket on $\Upsilon(T)$ is induced by the ordinary bracket on T and the bracket on $\Upsilon(F_0)$ is induced by the differential bracket on $J_q(T)$.

Proof: T has a structure of Lie algebra on sections from the ordinary bracket of vector fields and the situation is similar for $J_q(T)$ with the differential bracket. Now, if $L(\xi_1)\eta = 0$ and $L(\xi_1)\zeta = 0$, then $L(\xi_1)[\eta,\zeta] = 0$ and $[\Upsilon(T), \Upsilon(T)] \subset \Upsilon(T)$.

Similarly, as $F_0 = J_q(T)/R_q = J_q^0(T)/R_q^0$, if $L(\xi_q)\eta_q^0 \equiv [\xi_q, \eta_q^0] \in R_q^0$ and $L(\xi_q)\zeta_q^0 \equiv [\xi_q, \zeta_q^0] \in R_q^0$, then $L(\xi_q)[\eta_q^0, \zeta_q^0] \in R_q^0$ according to the Jacobi identity for the bracket on $J_q^0(T)$ and the fact that $[R_q^0, R_q^0] \subset R_q^0$. Also, if $L(\xi_{q+1})\eta_q \in R_q$ and $L(\xi_{q+1})\zeta_q \in R_q$, then it is less evident to prove that $L(\xi_{q+1})[\eta_q, \zeta_q] \in R_q$. For this, using Proposition 4.14, if we set $L(\xi_{q+1})\eta_q = \theta_q \in R_q$, it is sufficient to notice that $[\theta_q, \zeta_q] = L(\theta_{q+1})\zeta_q - i(\zeta)d\theta_{q+1} \in R_q$ because both terms do belong to R_q. Thus, in any case, we obtain $[\Upsilon(F_0), \Upsilon(F_0)] \subset \Upsilon(F_0)$.

Finally, we have $j_q([\eta,\zeta]) = \{j_{q+1}(\eta), j_{q+1}(\zeta)\} = [j_q(\eta), j_q(\zeta)]$ and we may take $j_q(\eta)$ as a representative of $\mathcal{D}\eta$ in $J_q(T)$. We shall prove that, if $\eta \in \Upsilon(T)$, that is if $L(\xi_1)\eta = 0$, $\forall \xi_1 \in R_1$, then $j_q(\eta) \in J_q(T)$ is such that $L(\xi_{q+1})j_q(\eta) = 0$. Introducing $R_2 = \pi_2^q(R_q) \subseteq \rho_1(R_1)$ and choosing any $\xi_2 \in R_2$ over $\xi_1 \in R_1$, we have $\pi_0^1(L(\xi_2)j_1(\eta)) = L(\xi_1)\eta = 0 \Rightarrow L(\xi_2)j_1(\eta) \in T^* \otimes T$. However, using the fact that $dj_q(\eta) = 0$, we obtain from Proposition 4.12:

$$i(\zeta)d(L(\xi_2)j_1(\eta)) = L(i(\zeta)d\xi_2)\eta = 0$$

because $R_2 \subset \rho_1(R_1) \Rightarrow dR_2 \subset T^* \otimes R_1$ and thus $L(\xi_2)j_1(\eta) = 0$ because there is a monomorphism (even an isomorphism) $0 \to T^* \otimes T \xrightarrow{\delta} T^* \otimes T$. Supposing by induction that $L(\xi_q)j_{q-1}(\eta) = 0$, we should obtain in the same way:

$$i(\zeta)d\left(L(\xi_{q+1})j_q(\eta)\right)=L\left(i(\zeta)d\xi_{q+1}\right)j_{q-1}(\eta)=0$$

because $R_{q+1}\subseteq\rho_1(R_q)\Rightarrow dR_{q+1}\subset T^*\otimes R_q$ and thus $L(\xi_{q+1})j_q(\eta)=0$ because there is a monomorphism $0\to S_{q+1}T^*\otimes T\xrightarrow{\delta}T^*\otimes S_qT^*\otimes T$ and the restriction of d to a symbol is $-\delta$.

In actual practice, using Theorem 4.15, we obtain
$\mathcal{L}(\xi)\mathcal{D}\eta=\mathcal{D}\mathcal{L}(\xi)\eta=\mathcal{D}[\xi,\eta]=0$, $\forall\xi\in\Theta$.

□

The reader will discover in the last section that *the study of the normalizer is much more delicate*.

DEFINITION 4.25: The *normalizer* $\tilde{\Gamma}=N(\Gamma)$ of Γ in $aut(X)$ is the biggest Lie pseudogroup in which Γ is *normal*, that is (roughly)
$N(\Gamma)=\tilde{\Gamma}=\left\{\tilde{f}\in aut(X)\,|\,\tilde{f}\circ f\circ\tilde{f}^{-1}\in\Gamma,\forall f\in\Gamma\right\}$ and we write
$\Gamma\lhd N(\Gamma)\subset aut(X)$.

Of course, $N(\Theta)$ will play the part of a Lie algebra for $N(\Gamma)$ exactly like Θ did for Γ. However, we shall see that $N(\Gamma)$ may have many components different from the connected component of the identity, for example two in the case of the algebraic Lie pseudogroup of contact transformations where $N(\Gamma)/\Gamma$ is isomorphic to the permutation group of two objects, a result not evident at first sight. Passing to the jets, we get
$j_q\left(\tilde{f}\circ f\circ\tilde{f}^{-1}\right)^{-1}(\omega)=j_q\left(\tilde{f}\right)\circ j_q(f)^{-1}\circ j_q\left(\tilde{f}\right)^{-1}(\omega)=\omega\Leftrightarrow$
$j_q(f)^{-1}\left(j_q\left(\tilde{f}\right)^{-1}(\omega)\right)=j_q\left(\tilde{f}\right)^{-1}(\omega)$, that is to say $j_q(f)^{-1}(\varpi)=\varpi$ if we set $j_q\left(\tilde{f}\right)^{-1}(\omega)=\varpi$ and we find back the equivalence relation of Definition 5.16. It follows that $\tilde{\Gamma}=\left\{\tilde{f}\in aut(X)\,|\,j_q\left(\tilde{f}\right)^{-1}(\omega)=g(\omega,a),h(c,a)=c\right\}$ is defined by the system $\tilde{\mathcal{R}}_{q+1}=\left\{\tilde{f}_{q+1}\in\Pi_{q+1}\,|\,\tilde{f}_{q+1}(R_q)=R_q\right\}$ with linearization
$\tilde{R}_{q+1}=\left\{\tilde{\xi}_{q+1}\,|\,L\left(\tilde{\xi}_{q+1}\right)\eta_q\in R_q,\forall\eta_q\in R_q\right\}$, that is to say
$\left\{\tilde{\xi}_{q+1},\eta_{q+1}\right\}+i\left(\tilde{\xi}\right)d\eta_{q+1}\in R_q\Leftrightarrow\left\{\tilde{\xi}_{q+1},\eta_{q+1}\right\}\in R_q$. Accordingly, the system of infinitesimal Lie equations defining $\tilde{\Theta}=N(\Theta)$ can be obtained by purely algebraic techniques from the system defining Θ. In particular, we notice that
$\pi_0^{q+1}:\tilde{R}_{q+1}\to T$ is an epimorphism because $\pi_0^{q+1}:R_{q+1}\to T$ is an epimorphisme by assumption and $R_{q+1}\subseteq\tilde{R}_{q+1}$. We obtain on the symbol level
$\left\{\tilde{g}_{q+1},\eta_{q+1}\right\}\subset g_q$ and thus $\delta\tilde{g}_{q+1}\subset T^*\otimes g_q$ leading to $\tilde{g}_{q+1}\subseteq g_{q+1}=\rho_1(g_q)$ and thus $\tilde{g}_{q+1}=g_{q+1}$ because $R_{q+1}\subseteq\tilde{R}_{q+1}\Rightarrow g_{q+1}\subseteq\tilde{g}_{q+1}$. Using arguments from δ-cohomology, it can be proved that $\tilde{R}_{q+1}$ is involutive when R_q is involutive ([10], p 351, 390). Another proof will be given later on.

With more details, using the result of Proposition 4.17, we get the following important local result:

PROPOSITION 4.26: $\Upsilon_0 = \Upsilon(F_0) = \{\Omega^\tau(x) = A^\alpha W_\alpha^\tau(\omega(x)) \mid A = cst\}$

Proof: Recalling that $F_0 = \omega^{-1}(V(\mathcal{F}))$, we shall first study the natural bundle $\mathcal{F}_0 = V(\mathcal{F})$ of order q. Adopting local coordinates (x,u,v), any infinitesimal change of source $\bar{x} = x + t\xi(x) + \cdots$ can be lifted to $\mathcal{F}_0$ with $\bar{u}^\tau = u^\tau + t\xi_\mu^k(x)L_k^{\tau\mu}(u) + \cdots$, $\bar{v}^\tau = v^\tau + t\dfrac{\partial L_k^{\tau\mu}(u)}{\partial u^\sigma}v^\sigma + \cdots$, according to the definition of a vertical bundle provided by Definition 3.3. The corresponding infinitesimal generators on $\mathcal{F}_0$ will be:

$$\xi^i(x)\frac{\partial}{\partial x^i} + \xi_\mu^k(x)\left(L_k^{\tau\mu}\frac{\partial}{\partial u^\tau} + \frac{\partial L_k^{\tau\mu}(u)}{\partial u^\sigma}v^\sigma\frac{\partial}{\partial v^\tau}\right),\ 1 \le |\mu| \le q, \forall \xi_q \in J_q(T)$$

It follows that a section $\epsilon : \mathcal{F} \to \mathcal{F}_0 : (x,u) \to (x,u,v = \epsilon(x,u))$ will be equivariant, that is $v - \epsilon(x,u) = 0 \Rightarrow \bar{v} - \epsilon(\bar{x},\bar{u}) = 0$ if and only if $v = \epsilon(u)$ satisfies:

$$L_k^{\tau\mu}(u)\frac{\partial \epsilon^\sigma(u)}{\partial u^\tau} - \frac{\partial L_k^{\tau\mu}(u)}{\partial u^\sigma}\epsilon^\sigma(u) = 0,\ 1 \le |\mu| \le q$$

Hence, we have $[L_k^\mu, \epsilon] = 0 \Rightarrow \epsilon^\tau(u) = A^\alpha W_\alpha^\tau(u)$ with $A = cst$.

As another approach, working directly with F_0, we may consider the invariance of the section $u - \omega(x) = 0$, $v - \Omega(x) = 0$ and get (care to the sign):

$$-L_k^{\tau\mu}(\omega(x))\xi_\mu^k + \xi^r\partial_r\omega^\tau(x) = 0,\ -\frac{\partial L_k^{\tau\mu}(\omega(x))}{\partial u^\sigma}\Omega^\sigma(x)\xi_\mu^k + \xi^r\partial_r\Omega^\tau(x) = 0$$

The first condition brings at once $\xi_q \in R_q$ and we obtain therefore the following central local result for F_0:

$$\Upsilon_0 = \left\{\Omega \in F_0 \mid -\frac{\partial L_k^{\tau\mu}(\omega(x))}{\partial u^\sigma}\Omega^\sigma\xi_\mu^k + \xi^r\partial_r\Omega^\tau = 0, \forall \xi_q \in R_q\right\}$$

As usual, the study of this system can be cut into two parts. First of all, we have to look for:

$$E_0 = \left\{\Omega \in F_0 \mid \frac{\partial L_k^{\tau\mu}(\omega(x))}{\partial u^\sigma}\Omega^\sigma\xi_\mu^k = 0, \forall \xi_q^0 \in R_q^0\right\} \subseteq F_0$$

In a symbolic way with $pri(R_q^0) = \{\xi^\sigma\}$ and $par(R_q^0) = \{\xi^{m+r}\}$, we get:

$$\frac{\partial L_\sigma}{\partial u}\Omega\xi^\sigma + \frac{\partial(\mathcal{E}_{m+r}^\sigma L_\sigma)}{\partial u}\Omega\xi^{m+r} = 0$$

whenever $\xi^\sigma + \mathcal{E}_{m+r}^\sigma\xi^{m+r} = 0$ and thus $\dfrac{\partial \mathcal{E}}{\partial u}\Omega = 0$ in agrement with Proposition 4.17.

Then we have to introduce an R_q-connection, that is a section $\chi_q \in T^* \otimes R_q$ over $id_T \in T^* \otimes T$ such that:

$$-L_k^{\tau\mu}(\omega(x))\chi_{\mu,i}^k + \partial_i\omega^\tau(x) = 0 \Rightarrow -\frac{\partial L_k^{\tau\mu}(\omega(x))}{\partial u^\sigma}\chi_{\mu,i}^k\Omega^\sigma + \partial_i\Omega^\tau = 0$$

and it just remains to study this last system for $\Omega^\tau(x) = A^\alpha(x) W_\alpha^\tau(\omega(x))$ as it does not depend on the choice of the connection χ_q. Substituting, we obtain successively:

$$\left(-\frac{\partial L_k^{\tau\mu}}{\partial u^\sigma}\chi_{\mu,i}^k W_\alpha^\sigma + \frac{\partial W_\alpha^\tau}{\partial u^\sigma}\partial_i\omega^\sigma\right)A^\alpha + \left(\partial_i A^\alpha\right)W_\alpha^\tau = 0$$

$$\left(\left[L_k^\mu, W_\alpha\right]\right)^\tau \chi_{\mu,i}^k A^\alpha + \left(\partial_i A^\alpha\right)W_\alpha^\tau = 0$$

As $\left[L_k^\mu, W_\alpha\right] = 0$ and the action is effective, we finally obtain $\partial_i A^\alpha = 0$ that is $A = cst$.

□

The following corollary is a direct consequence of the above proposition and explains many classical results as we shall see through the motivating examples.

COROLLARY 4.27: We have the relation:

$$\boxed{\Theta = \left\{\xi \in T \mid \mathcal{D}\xi \equiv \mathcal{L}(\xi)\omega = 0\right\} \Rightarrow N(\Theta) = \left\{\mathcal{D}\xi \equiv \mathcal{L}(\xi)\omega = AW(\omega) \in \Upsilon_0\right\}}$$

THEOREM 4.28: If a Lie algebroid $R_q \subset J_q(T)$ is such that $\pi_q^{q+1}: R_{q+1} \to R_q$ is an epimorphism and its symbol g_q is 2-acyclic, then its normalizer is the Lie algebroid $\tilde{R}_{q+1} \subset J_{q+1}(T)$ defined by the purely algebraic condition $\left\{\tilde{R}_{q+1}, R_{q+1}\right\} \subset R_q$ in such a way that $\pi_{q+1}^{q+2}: \tilde{R}_{q+2} \to \tilde{R}_{q+1}$ defined similarly by $\left\{\tilde{R}_{q+2}, R_{q+2}\right\} \subset R_{q+1}$ is also an epimorphism and its symbol is $\tilde{g}_{q+1} = g_{q+1}$ is also 2-acyclic while $\tilde{R}_{q+2}$ is the first prolongation of $\tilde{R}_{q+1}$.

Proof: First of all, we notice that the normalizer is defined by the condition:

$$L\left(\tilde{\xi}_{q+1}\right)\xi_q = \left\{\tilde{\xi}_{q+1}, \xi_{q+1}\right\} + i\left(\tilde{\xi}\right)d\xi_{q+1} \in R_q \Rightarrow \left\{\tilde{\xi}_{q+1}, \xi_{q+1}\right\} \in R_q$$

Let us introduce splitting maps $\alpha: T \to R_q$, $\beta: T \to R_{q+1}$, $\gamma: T \to R_{q+2}$, $\varphi: R_q \to R_{q+1}$.

We may define $Z \in \wedge^2 T^* \otimes R_q$ by the formula:

$$Z(\xi,\eta) = \left\{\tilde{\xi}_{q+1}, \left\{\gamma(\xi), \gamma(\eta)\right\}\right\} - \left\{\varphi\left(\left\{\tilde{\xi}_{q+1}, \beta(\xi)\right\}\right), \beta(\eta)\right\}$$
$$+\left\{\varphi\left(\left\{\tilde{\xi}_{q+1}, \beta(\eta)\right\}\right), \beta(\xi)\right\}$$

First of all, using the Jacobi identity while introducing $\tilde{\xi}_{q+2} \in J_{q+2}(T)$ over $\tilde{\xi}_{q+1}$, we have:

$$\left\{\tilde{\xi}_{q+1}, \left\{\gamma(\xi), \gamma(\eta)\right\}\right\} + \left\{\gamma(\eta), \left\{\tilde{\xi}_{q+2}, \gamma(\xi)\right\}\right\} + \left\{\beta(\xi), \left\{\gamma(\eta), \tilde{\xi}_{q+2}\right\}\right\} = 0$$

and the projection at order $q-1$ is thus:

$$-\left(\left\{\alpha(\eta), \left\{\tilde{\xi}_{q+1}, \beta(\xi)\right\}\right\} + \left\{\alpha(\xi), \left\{\beta(\eta), \tilde{\xi}_{q+1}\right\}\right\}\right)$$
$$-\left\{\left\{\tilde{\xi}_{q+1}, \beta(\xi)\right\}, \alpha(\eta)\right\} + \left\{\left\{\tilde{\xi}_{q+1}, \beta(\eta)\right\}, \alpha(\xi)\right\} = 0$$

a result showing that $Z \in \wedge^2 T^* \otimes g_q$ indeed.

Using a cyclic summation, we have $\delta Z(\xi,\eta,\zeta) = \mathcal{C}(\xi,\eta,\zeta)\left\{Z(\xi,\eta), \alpha(\zeta)\right\}$ and, using the previous Jacobi identity, we obtain:

$$\mathcal{C}(\xi,\eta,\zeta)\left\{\left\{\tilde{\xi}_{q+1},\left\{\gamma(\xi),\gamma(\eta)\right\}\right\},\alpha(\zeta)\right\}=\mathcal{C}(\xi,\eta,\zeta)\left\{\left\{\tilde{\xi}_{q+1},\beta(\xi)\right\},\left\{\beta(\eta),\beta(\zeta)\right\}\right\}$$

because of the Jacobi identity $\mathcal{C}(\xi,\eta,\zeta)\left\{\left\{\left\{\gamma(\xi),\gamma(\eta)\right\},\beta(\zeta)\right\},\tilde{\xi}_q\right\}=0$.

in which $\tilde{\xi}_q$ is the projection of $\tilde{\xi}_{q+1}$ in $J_q(T)$.

In addition, we have the new Jacobi identity:

$$\left\{\left\{\varphi\left(\left\{\tilde{\xi}_{q+1},\beta(\xi)\right\}\right),\beta(\eta)\right\},\alpha(\zeta)\right\}+\left\{\left\{\beta(\eta),\beta(\zeta)\right\},\left\{\tilde{\xi}_{q+1},\beta(\xi)\right\}\right\}$$
$$+\left\{\beta(\zeta),\varphi\left(\left\{\tilde{\xi}_{q+1},\beta(\xi)\right\}\right),\alpha(\eta)\right\}=0$$

and obtain finally $\delta Z=0$. It follows that $Z=\delta B$ with $B\in T^*\otimes g_{q+1}$ because g_q is 2-acyclic, that is:

$$Z(\xi,\eta)=\left\{B(\xi),\beta(\eta)\right\}-\left\{B(\eta),\beta(\xi)\right\}$$

Using again the previous Jacobi identity in which we introduced $\tilde{\xi}_{q+2}\in J_{q+2}(T)$ over $\tilde{\xi}_{q+1}\in J_{q+1}(T)$ while defining $C\in T^*\otimes S_{q+1}T^*\otimes T$ by the formula:

$$C(\xi)=\left\{\tilde{\xi}_{q+2},\gamma(\xi)\right\}-\varphi\left(\left\{\tilde{\xi}_{q+1},\beta(\xi)\right\}\right)-B(\xi)$$

we obtain $\left\{C(\xi),\beta(\eta)\right\}-\left\{C(\beta),\beta(\xi)\right\}=0\Rightarrow\delta C=0$. As we have the exact δ-sequence:

$$0\rightarrow S_{q+2}T^*\otimes T\xrightarrow{\delta}T^*\otimes S_{q+1}T^*\otimes T\xrightarrow{\delta}\wedge^2T^*\otimes S_qT^*\otimes T$$

there exists $A\in S_2T^*\otimes T$ such that $C=\delta A$ and thus $C(\xi)=\left\{A,,\gamma(\xi)\right\}$. It follows that we can modify $\tilde{\xi}_{q+2}$ over $\tilde{\xi}_{q+1}$ in such a way that:

$$\boxed{\left\{\tilde{\xi}_{q+2},\gamma(\xi)\right\}=\varphi\left(\left\{\tilde{\xi}_{q+1},\beta(\xi)\right\}\right)+B(\xi)\in R_{q+1}}$$

With now an arbitrary section $\xi_{q+2}\in J_{q+2}(T)$, we have the Jacobi identity:

$$\left\{\left\{\tilde{\xi}_{q+2},\xi_{q+2}\right\},\beta(\eta)\right\}=\left\{\tilde{\xi}_{q+1},\left\{\xi_{q+2},\gamma(\xi)\right\}\right\}+\left\{\left\{\tilde{\xi}_{q+2},\gamma(\xi)\right\},\xi_{q+1}\right\}\in R_q$$

by using the previous framed result. Hence, if we define $M\in T^*\otimes S_qT^*\otimes T$ by the formula:

$$M(\eta)=\left\{\left\{\tilde{\xi}_{q+2},\xi_{q+2}\right\}-\varphi\left(\left\{\tilde{\xi}_{q+1},\xi_{q+1}\right\}\right),\beta(\xi)\right\}$$

we have thus $M\in T^*\otimes g_q$ with $\delta M=0$ and thus $M=\delta N$ with $N\in g_{q+1}$, that is $M(\eta)=\left\{N,\beta(\eta)\right\}$. Accordingly, we have:

$$\boxed{\left\{\tilde{\xi}_{q+2},\xi_{q+2}\right\}=\varphi\left(\left\{\tilde{\xi}_{q+1},\xi_{q+1}\right\}\right)+N\in R_{q+1}}$$

Finally, we have:

$$\left\{i(\eta)d\tilde{\xi}_{q+2},\xi_{q+1}\right\}=i(\eta)d\left\{\tilde{\xi}_{q+2},\xi_{q+2}\right\}-\left\{\tilde{\xi}_{q+1},i(\eta)d\xi_{q+2}\right\}\in R_q$$

and $\tilde{R}_{q+2}=\rho_1\left(\tilde{R}_{q+1}\right)$ with symbol $\tilde{g}_{q+2}=g_{q+2}$ prolongation of $\tilde{g}_{q+1}=g_{q+1}$. It follows that:

$$\tilde{R}_{q+2}/R_{q+2} \simeq \tilde{R}_{q+1}/R_{q+1}$$

and so on at any higher order as we shall see in any example later on.

□

According to the definition of the Janet bundles at the end of section 3, we may use the inclusion $\wedge^r T^* \otimes S_q T^* \otimes T \subset \wedge^r T^* \otimes J_q^0(T) \subset \wedge^r T^* \otimes J_q(T)$. In the case of an involutive system $R_q \subset J_q(T)$ of infinitesimal transitive Lie equations, we have thus:

$$\boxed{F_r = \wedge^r T^* \otimes J_q^0(T) \Big/ \left(\wedge^r T^* \otimes R_q^0 + \delta\left(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes T\right)\right)}$$

As R_q^0 is associated with R_q and $J_q^0(T)$ is associated with $R_q \subset J_q(T)$, we obtain:

LEMMA 4.29: The Janet bundles are associated with R_q and we set $\Upsilon_r = \Upsilon(F_r)$.

From this lemma and Theorem 4.15, we shall deduce the following important theorem:

THEOREM 4.30: The first order operators $\mathcal{D}_r : F_{r-1} \to F_r$ induce maps $\mathcal{D}_r : \Upsilon_{r-1} \to \Upsilon_r$ in the *deformation sequence*:

$$\boxed{0 \to Z(\Theta) \to C(\Theta) \xrightarrow{\mathcal{D}} \Upsilon_0 \xrightarrow{\mathcal{D}_1} \Upsilon_1 \xrightarrow{\mathcal{D}_2} \cdots \xrightarrow{\mathcal{D}_n} \Upsilon_n \to 0}$$

which is locally described by finite dimensional vector spaces and linear maps.

Proof: Any section of F_{r-1} can be lifted to a section $A_q^{r-1} \in \wedge^{r-1} T^* \otimes J_q(T)$ modulo $\wedge^{r-1} T^* \otimes R_q + \delta\left(\wedge^{r-2} T^* \otimes S_{q+1} T^* \otimes T\right)$ where the second component is in the image of d. Then we can lift again this section to a section $A_{q+1}^{r-1} \in \wedge^{r-1} T^* \otimes J_{q+1}(T)$, modulo a section of $\wedge^{r-1} T^* \otimes R_{q+1} + d\left(\wedge^{r-2} T^* \otimes J_{q+2}^q(T)\right) + \wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes T$ where J_{q+2}^q is the kernel of the projection $\pi_q^{q+2} : J_{q+2}(T) \to J_q(T)$. The image by $\mathcal{D}_r$ is obtained by applying $d : \wedge^{r-1} T^* \otimes J_{q+1}(T) \to \wedge^r T^* \otimes J_q(T)$ and projecting $dA_{q+1}^{r-1} \in \wedge^r T^* \otimes J_q(T)$ thus obtained to F_r while taking into account successively the restriction $d : \wedge^{r-1} T^* \otimes R_{q+1} \to \wedge^r T^* \otimes R_q$, the fact that $d \circ d = d^2 = 0$ and the restriction δ providing an element in $\delta\left(\wedge^{r-1} T^* \otimes S_{q+1} T^* \otimes T\right)$. Of course, with such a choice we need to use associations with R_{q+1} *even though finally only* R_q *is involved* because of the above lemma.

□

DEFINITION 4.31: The deformation sequence is not necessarily exact and only depends on R_q. We can therefore define as usual *coboundaries* $B_r(R_q)$, *cocycles* $Z_r(R_q)$ and *cohomology groups* $H_r(R_q) = Z_r(R_q)/B_r(R_q)$ with $B_r(R_q) \subseteq Z_r(R_q) \subseteq \Upsilon_r$ for $r = 0, 1, \cdots, n$.

PROPOSITION 4.32: We have $B_0(R_q) = C(\Theta)/Z(\Theta), Z_0(R_q) = N(\Theta)/\Theta$ and the short exact sequence:

$$\boxed{0 \to C(\Theta)/Z(\Theta) \to N(\Theta)/\Theta \to H_0(R_q) \to 0}$$

Proof: The monomorphism on the left is induced by the inclusion $C(\Theta) \subset N(\Theta)$ because $Z(\Theta) = \Theta \cap C(\Theta)$. Then $B_0(R_q)$ is the image of $\mathcal{D}$ in Υ_0 because of Proposition 4.23 and Proposition 4.24. Finally, $Z_0(R_q) = N(\Theta)/\Theta$ is just a way to rewrite Corollary 4.27 because $\mathcal{D}_1 \circ \mathcal{D} = 0$. The cocycle condition just tells that the label transformations induced by the normalizer do not change the structure constants because the Vessiot structure equations are invariant under *any* natural transformation.

□

5. Motivating Examples

The eight following examples will show how group theoretical concepts and differential extension modules may depend on the Vessiot structure constants, but references cannot be found either in mathematics or in the physical literature [24]-[26]. Following the advice of a reviewer that we thank, the general point illustrated by each example is first presented at the beginning of the example.

EXAMPLE 5.1: The determination of the normalizer may be tricky even when $n=1$ and no Vessiot structure equation may exist. Indeed, let us consider the dilatation group $y = ax$ and consider it as a Lie pseudogroup Γ. The only generating differential invariant is known to be $\Phi \equiv \frac{y_x}{y}$ and the Lie form $\mathcal{R}_1$ becomes $\Phi(y, y_x) = \omega(x)$ with $\omega(x) = \frac{1}{x}$. The corresponding Medolaghi equation is easily seen to be $\Omega \equiv \omega(x)\xi_x + \xi\partial_x\omega(x) = 0$. Dividing by ω, we discover that $\omega, \bar{\omega} \in T^*$ provide the same system $R_1 \subset J_1(T)$ if and only if $\frac{1}{\bar{\omega}}\partial_x\bar{\omega} = \frac{1}{\omega}\partial_x\omega$ that is if and only if $\partial_x\left(\frac{\bar{\omega}}{\omega}\right) = 0$ and thus $\bar{\omega} = b\omega$ for a constant b. The first idea is to consider the new second order geometric object $\sigma = \frac{1}{\omega}\partial_x\omega$ and to introduce the new Lie form $\Psi(y, y_x, y_{xx}) = \frac{1}{\Phi}d_x\Phi = \sigma$, that is to say $\frac{y_{xx}}{y_x} - \frac{y_x}{y} = -\frac{1}{x}$ in the present situation. Doing the change of coordinates $y = f(x)$ and extending it to T by setting $\eta(f(x)) = \partial_x f(x)\xi(x)$, we may extend the transformation to $J_1(T)$ by introducing $j_2(f)$ in order to discover that the new Lie pseudogroup defined by the geometric object σ is just the normalizer of Γ ([10], p 344). However, we shall recover the corresponding Medolaghi equations $\tilde{R}_2 \subset J_2(T)$ defining now $N(\Theta)$ by means of the new technique discovered by Vessiot. For this, we just need to consider $\Omega \equiv \omega\xi_x + \xi\partial_x\omega = B\omega$ by linearization where now B is obtained by setting $b = 1 + tB + \cdots$. Dividing by ω, we obtain $\xi_x + \xi\sigma = B = cst$ with $\sigma = -\frac{1}{x}$ and, differentiating, we obtain the second order Medolaghi equation

$\xi_{xx} - \frac{1}{x}\xi_x + \frac{1}{x^2}\xi = 0$ allowing to define $N(\Theta)$ with of course $dim(N(\Theta)/\Theta) = 1$. As the normalizer becomes $y = ax^b$, it is not evident to discover that $N(\Theta)$ just admits the two generators $\{x\partial_x, x\log(x)\partial_x\}$, which was the aim of this example. Of course, this result is coherent with the fact that $\log(y) = b\log(x) + \log(a)$ but we shall find examples of finite or infinitesimal Lie equations that cannot be integrated.

EXAMPLE 5.2: (*Vessiot I*) This second example has been presented by Vessiot himself in [7] and can be considered as the simplest one allowing us to understand the part plaid by the Vessiot structure constants in the study of differential duality and double duality. The difficulty is that it can be presented by a first order system of Lie equation with a corresponding 1-form α but such a system is *not* involutive and the corresponding geometric object must be completed by a 2-form β with $\alpha \wedge \beta \neq 0$.

With $n = 2$, let us consider the geometric object $\omega = (\alpha, \beta) \in \wedge^1 T^* \times_X \wedge^2 T^*$ made by the combination of a 1-form $\alpha = \alpha_i(x)dx^i \neq 0$ and a 2-form $\beta = \beta(x)dx^1 \wedge dx^2 \neq 0$ leading to the Medolaghi equations $R_1 \subset J_1(T)$:

$$A_i \equiv \alpha_r(x)\xi_i^r + \xi^r\partial_r\alpha_i(x) = 0,\ B \equiv \beta(x)(\xi_1^1 + \xi_2^2) + \xi^r\partial_r\beta(x) = 0$$

When $\alpha = x^2dx^1, \beta = dx^1 \wedge dx^2$, we obtain the three PD equations $x^2\xi_1^1 + \xi^2 = 0$, $\xi_2^1 = 0$, $\xi_1^1 + \xi_2^2 = 0$ and the following equivalent involutive system with corresponding Janet tabular:

$$\left\{\begin{array}{lcl} \xi_2^2 - \frac{1}{x^2}\xi^2 & = & 0 \\ \xi_2^1 & = & 0 \\ \xi_1^1 + \frac{1}{x^2}\xi^2 & = & 0 \end{array}\right. \quad \boxed{\begin{array}{ll} 1 & 2 \\ 1 & 2 \\ 1 & \bullet \end{array}}$$

There is thus only 1 CC and the intrinsic Janet sequence becomes:

$$\boxed{0 \to \Theta \to T \xrightarrow{\mathcal{D}} \wedge^1 T^* \times_X \wedge^2 T^* \xrightarrow{\mathcal{D}_1} \wedge^2 T^* \to 0}$$

Introducing the standard Lie derivative, we have:

$$\mathcal{D} : \xi \to \Omega = (\mathcal{L}(\xi)\alpha = A, \mathcal{L}(\xi)\beta = B), \mathcal{D}_1 : (A, B) \to dA + B$$

More generally, as both $d\alpha$ and β are 2-forms, we obtain the only Vessiot structure equation $d\alpha = c\beta$ with the single Vessiot structure constant $c = -1$. If $\alpha = dx^1$ we get $d\alpha = 0 \Rightarrow c = 0$ while, if $\alpha = x^2dx^1 - x^1dx^2$, we get $d\alpha = -2dx^1 \wedge dx^2 \Rightarrow c = -2$. Otherwise, if $c = c(x)$, the system contains the zero order equation $\xi^r\partial_r c(x) = 0$ and R_1 is not a transitive algebroid.

As c is easily seen not to depend on the coordinate system, it is clear that no change of variables may allow to pass from from one case to the other in the sense of the equivalence problem, that is if $\omega \to c$, $\bar{\omega} \to \bar{c}$ then we cannot find $f \in aut(X)$ with $j_1(f)(\omega) = \bar{\omega}$ unless $\bar{c} = c$.

At the same time, we notice that the corresponding Lie pseudogroups may be quite different.

Indeed, in the first case we get the explicit formulas $y^1 = f(x^1)$, $y^2 = x^2/f'(x^1)$ with notation $f'(x^1) = \partial_1 f(x^1)$ while in the second, we get transformations $y^1 = x^1 + a$, $y^2 = x^2 + g(x^1)$ with one arbitrary constants a and one arbitrary function $g(x^1)$. On the contrary, in the third case there is no explicit description of the corresponding pseudogroup by means of arbitrary functions or constants, even though we have well defined finite and infinitesimal Lie equations.

However, the main interest of this elementary example first provided by Vessiot in... 1903 (!) is that it is of a great help for answering about structural questions about Θ and, in particular about the inclusions $Z(\Theta) \subseteq C(\Theta) \subseteq N(\Theta)$. For example, in order to look for $C(\Theta)$, we have to solve the PD equations $[\xi^1\partial_1 + \xi^2\partial_2, f(x^1)\partial_1 - x^2 f'(x^1)\partial_2] = 0$ for any function $f(x^1)$. Developping,, we obtain successively $f'' \rightarrow \xi^1 = 0, f' \rightarrow x^2\partial_2\eta^2 - \eta^2 = 0, f \rightarrow \partial_1\eta^2 = 0$ giving $C(\Theta) = \{x^2\partial_2\}$ and thus $Z(\Theta) = C(\Theta) \cap \Theta = 0$, a result leading to $N(\Theta) = \{f(x^1)\partial_1 - x^2 f'(x^1)\partial_2, x^2\partial_2\}$ *in this particular situation* because the determination of $N(\Theta)$ *must* involve second order PD equations.

In order to recover this result independently, let us look for the Lie group of *label* transformations. For this, if $\beta = \beta(x)dx^1 \wedge dx^2$, we notice that R_1^0 only depend on $(\alpha_1(x), \alpha_2(x), \beta(x))$ that cannot be all vanishing and thus only on $(\alpha_2(x)/\alpha_1(x), \beta(x))$. It follows that $\bar{\omega} \simeq \omega \Leftrightarrow (\bar{\alpha} = a\alpha, \bar{\beta} = b\beta)$ for two constants $(a, b \neq 0)$ and we obtain $\bar{c} = (a/b)c$, a result leading only to two possible situations:

• $c \neq 0$: In order to look for $N(\Theta)$, we *must* have $\bar{c} = c \Rightarrow a = b$, a result showing that $dim(N(\Theta)/\Theta) = 1$. Setting $a_t = 1 + tA + \cdots$ we obtain $c_t = c + tC + \cdots$ and thus $C = cA$. It follows that *exactly like in the situation of the Weyl group which is the normalizer of the Poincaré group as we shall see*, $N(\Theta)$ *must* be defined by the Medolaghi equations obtained by eliminating the constant A in the infinitesimal Lie equations in order to obtain $\tilde{R}_2$ with symbol $\tilde{g}_2 = g_2$ as follows:

$$x^2\xi_1^1 + \xi^2 = Ax^2, \xi_2^1 = 0, \xi_1^1 + \xi_2^2 = A \Rightarrow \xi_2^2 - \frac{1}{x^2}\xi^2 = 0, \xi_2^1 = 0$$

$$\xi_{22}^2 = 0, \xi_{22}^1 = 0, \xi_{12}^2 - \frac{1}{x^2}\xi_1^2 = 0, \xi_{12}^1 = 0, \xi_{11}^1 + \frac{1}{x^2}\xi_1^2 = 0$$

and a completely different way to find back $N(\Theta) = \{f(x^1)\partial_1 - x^2 f'(x^1)\partial_2, x^2\partial_2\}$.

• $c = 0$: We may have $a \neq b$ and thus $dim(N(\Theta)/\Theta) = 2$ for example, in the situation already considered, we let the reader check that we had $\Theta = \{\partial_1, g(x^1)\partial_2\}$ while we obtain now $N(\Theta) = \{\partial_1, g(x^1)\partial_2, x^1\partial_1, x^2\partial_2\}$ and thus $dim\left(N(\Theta)/\Theta\right)=2$.

We study the effect of the Vessiot structure constant on the differential extension modules.

For this, multiplying (A_1,A_2,B) respectively by (μ^1,μ^2,μ^3), we obtain $ad(\mathcal{D})$ in the form:

$$-\alpha_1\left(\partial_1\mu^1+\partial_2\mu^2\right)-\beta\left(\partial_1\mu^3-c\mu^2\right)=\nu^1,-\alpha_2\left(\partial_1\mu^1+\partial_2\mu^2\right)-\beta\left(\partial_2\mu^3+c\mu^1\right)=\nu^2$$

and we shall set $U=\partial_1\mu^1+\partial_2\mu^2$, $V_1=\partial_1\mu^3-c\mu^2$, $V_2=\partial_2\mu^3+c\mu^1$ and $ker\left(ad(\mathcal{D})\right)$, being defined by $\alpha_1U+\beta V_1=0$, $\alpha_2U+\beta V_2=0$ is thus generated by U because $\beta\neq 0$.

Then, multiplying $\partial_1A_2-\partial_2A_1-cB$ by λ, we obtain $ad(\mathcal{D}_1)$ as:

$$\partial_2\lambda=\mu^1,-\partial_1\lambda=\mu^2,-c\lambda=\mu^3$$

We have therefore to consider the same two cases as before:

• $c=0$: We have $V_1\equiv\partial_1\mu^3=-\dfrac{\alpha_1}{\beta}U$, $V_2\equiv\partial_2\mu^3=-\dfrac{\alpha_2}{\beta}U$ and obtain $\partial_1\left(\dfrac{\alpha_2}{\beta}U\right)-\partial_2\left(\dfrac{\alpha_1}{\beta}U\right)=0$, that is U is a torsion element. On the contrary, $im\left(ad(\mathcal{D}_1)\right)$ is defined by $U=\partial_1\mu^1+\partial_2\mu^2=0$ and $\mu^3=0$. It follows that the torsion module $ext^1(M)\neq 0$ is generated by the residue of $\mu^3=\nu'$ because $\alpha\neq 0$ and we may thus suppose that $\alpha_1\neq 0$. As for $ext^2(M)$, this torsion module is just defined by the system $\partial_2\lambda=0$, $\partial_1\lambda=0$ for λ and thus $ext^2(M)\neq 0$. A special situation may be obtained with $\alpha=dx^1$, $\beta=dx^1\wedge dx^2$ giving rise to the Lie pseudogroup $\Gamma=\left\{y^1=x^1+a,y^2=x^2+b\mid a=cst,b=cst\right\}$.

• $c\neq 0$: We must have the new CC:

$$\partial_1\mu^3-c\mu^2=0,\partial_2\mu^3+c\mu^1=0\Rightarrow\partial_1\mu^1+\partial_2\mu^2=0$$

It follows that $ext^1(M)$ is now generated by the residue of $\partial_1\mu^1+\partial_2\mu^2=\nu'$. Finally, $ker\left(ad(\mathcal{D}_1)\right)$ is defined by $\lambda=0$ and thus $ext^2(M)=0$. When $\left(\alpha^1=x^2,\alpha_2=0,\beta=1\right)$, we obtain $c=-1$ and $ker\left(ad(\mathcal{D})\right)$ is defined by $x^2\left(\partial_1\mu^1+\partial_2\mu^2\right)+\mu^2+\partial_1\mu^3=0$, $\partial_2\mu^3-\mu^1=0$ while $im\left(ad(\mathcal{D}_1)\right)$ is defined by adding $\partial_1\mu^1+\partial_2\mu^2=0$ after substituting $\lambda=\mu^3$, obtaining $x^2\partial_2U+2U=0$.

Hence, both $ext^1(M)$ and $ext^2(M)$ highly depend on the Vessiot structure constant c but their explicit computation is not so easy, even on such a simple example.

EXAMPLE 5.3: (*Vessiot II*): This third example also presented by Vessiot in [7] is providing a Lie pseudogroup defined by two first order Lie equations though the corresponding 2-dimensional first order geometric object is not a tensor.

With $m=n=3$ let us consider the Lie pseudogroup $\Gamma=\left\{y^1=f\left(x^1\right),y^2=g\left(x^1,x^2,x^3\right),y^3=h\left(x^1,x^2,x^3\right)\right\}$ defined by the involutive system made by the two PD equations $y_3^1 = 0, y_2^1 = 0$. The corresponding infinitesimal Lie equations are defined by the involutive system:

$$\begin{cases} \xi_3^1 & = & 0 \\ \xi_2^1 & = & 0 \end{cases} \quad \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & \bullet \end{array}}$$

We let the reader prove that a fundamental set of differential invariants providing a Lie form may be $\Phi^2 \equiv \frac{y_3^1}{y_1^1} = \omega^2(x)$, $\Phi^1 \equiv \frac{y_2^1}{y_1^1} = \omega^1(x)$ with $\omega^1 = 0$, $\omega^2 = 0$. The transformation rule of these invariants, namely:

$$\bar{x} = \varphi(x) \Rightarrow u^1 = \frac{\partial_2\varphi^1 + \bar{u}^1\partial_2\varphi^2 + \bar{u}^2\partial_2\varphi^2}{\partial_1\varphi^1 + \bar{u}^1\partial_1\varphi^2 + \bar{u}^2\partial_1\varphi^3},\ u^2 = \frac{\partial_3\varphi^1 + \bar{u}^1\partial_3\varphi^2 + \bar{u}^2\partial_3\varphi^3}{\partial_1\varphi^1 + \bar{u}^1\partial_1\varphi^2 + \bar{u}^2\partial_1\varphi^3}$$

provides a bundle $\mathcal{F}$ of geometric objects over X with local coordinates $(x^1, x^2, x^3, u^1, u^2)$. The system of two infinitesimal Medolaghi equations $\Omega \equiv L(\xi_1)\omega$ obtained by linearization is:

$$\begin{cases} \Omega^2 \equiv \xi_3^1 + \omega^1(x)\xi_3^2 + \omega^2(x)\xi_3^3 - \omega^2(x)\xi_1^1 - \omega^1(x)\omega^2(x)\xi_1^2 - \left(\omega^2(x)\right)^2\xi_1^3 + \xi^r\partial_r\omega^2(x) = 0 \\ \Omega^1 \equiv \xi_2^1 + \omega^1(x)\xi_3^2 + \omega^2(x)\xi_3^3 - \omega^1(x)\xi_1^1 - \left(\omega^1(x)\right)^2\xi_1^2 - \omega^1(x)\omega^2(x)\xi_1^3 + \xi^r\partial_r\omega^1(x) = 0 \end{cases}$$

Now, we notice that:

$$\boxed{d_3\Phi^1 - d_2\Phi^2 + \Phi^1 d_1\Phi^2 - \Phi^2 d_1\Phi^1 = 0} \Rightarrow \boxed{\partial_3\omega^1 - \partial_2\omega^2 + \omega^1\partial_1\omega^2 - \omega^2\partial_1\omega^1 = 0}$$

These conditions are of course satisfied by the special case $\omega^1 = 0$, $\omega^2 = 0$ and $\bar{\omega} \sim \omega \Leftrightarrow \bar{\omega} = \omega$.

In addition, the linearization gives the only CC defining $\mathcal{D}_1$:

$$\boxed{d_3\Omega^1 - d_2\Omega^2 + \omega^1 d_1\Omega^2 - \omega^2 d_1\Omega^1 + \left(\partial_1\omega^2\right)\Omega^1 - \left(\partial_1\omega^1\right)\Omega^2 = 0}$$

Using the 7 parametric jets of strict order 1 to be $\left\{\xi_1^1, \xi_1^2, \xi_2^2, \xi_3^2, \xi_1^3, \xi_2^3, \xi_3^3\right\}$, it is indeed a tedious checking to discover that the condition $R_1^{(1)} = R_1$ providing the only Vessiot structure equation is satisfied without involving any arbitrary structure constant. It is rather extraordinary (and a chance for him) that the system is generically involutive, contrary to the previous one, as no tensorial framework can be used. In the special situation considered, we have $ext^1(M) = 0$ but that $ext^2(M) \neq 0$ because $ad(\mathcal{D}_1)$ is not injective. Indeed, $\mathcal{D}$ is defined by $d_3\xi^1 = \Omega^1$, $d_2\xi^1 = \Omega^2$ and $\mathcal{D}_1$ is defined by $d_3\Omega^2 - d_2\Omega^1 = \zeta$. Multiplying by convenient test functions, we notice that $ad(\mathcal{D})$ generates the CC of $ad(\mathcal{D}_1)$ but that $ad(\mathcal{D}_1)$ is not injective. In the general situation, the study is quite more difficult. Indeed, multiplying on the left the last operator framed by a test function λ and integrating by parts, the key step is to prove that the system $\left(d_3\lambda - \omega^2 d_1\lambda = 0, d_2\lambda - \omega^1 d_1\lambda = 0\right)$ obtained is involutive if and only if the Vessiot structure equation is satisfied, a *very striking* result that can be obtained at once by crossed derivatives and substitution but a concept discovered by Janet… twenty years later (!).

EXAMPLE 5.4: (*Riemann structure*) The constant Riemannian curvature is the *only* example of a Lie group of transformation considered as a Lie pseudogroup such that the *only* Vessiot structure constant c is known and the corresponding geometric object $\omega \in S_2T^*$ is a non-degenerate metric, thus a tensor. The confusion done by Eisenhart in 1918 between IC and CC is explained for the first time through the lemma and it becomes clear that Eisenhart was not aware of the work of Vessiot done in 1903, that is fifteen years before. The other main difficulty is that the first order system of Killing equations is *not* involutive because its second order symbol is vanishing. As such a concept has been introduced for systems of PD equations by Janet in 1920, the use of exterior calculus on jet bundles made by Cartan could not be adapted to such a situation. As a byproduct, the concept of "*curvature alone*" involved in the corresponding Janet sequence with $\mathcal{D}_1$ has *strictly nothing to do* with the "*curvature* + *torsion*" involved in the Spencer sequence with D_2. The situation can be simply explained by the fact that the Janet sequence only depends on the system $R_q \subset J_q(T)$ of order q while the Spencer sequence only depends on the first order system $R_{q+1} \subset J_1(R_q)$. This is the main reason for which the Vessiot structure equations cannot be compared to the Cartan structure equations, a *fact* showing the importance of this specific example. Finally, as it is known that the extension modules do not depend on the differential sequence used in order to define them, they could be studied either by the Janet sequence or by the Spencer sequences, a *fact* that we consider as one of the most difficult results from differential homological algebra that we had to understand during our scientific life.

A Riemann structure on a manifold X of dimension n is a field of symmetric non-degenerate tensors $\omega = (\omega_{ij}(x)) \in S_2T^*$ with $det(\omega) \neq 0$. The finite Lie equations in Lie form are $\Phi_{ij}(y_1) \equiv \omega_{kl}(y) y_i^k y_j^l = \omega_{ij}(x)$ and define a Lie pseudogroup $\Gamma \subset aut(X)$. The corresponding infinitesimal Lie equations in Medolaghi form are $\mathcal{L}(\xi)\omega \equiv \Omega = 0$ with

$\Omega_{ij} \equiv \omega_{rj}(x)\partial_i\xi^r + \omega_{ir}(x)\partial_j\xi^r + \xi^r\partial_r\omega_{ij}(x) = 0$ or simply

$\omega_{rj}\xi_i^r + \omega_{ir}\xi_j^r + \xi^r\partial_r\omega_{ij} = 0$ if we use jet notations for defining the corresponding Killing system. The set of solutions will be denoted by $\Theta \subset T$. We have proved in many books and papers that such a system $R_1 \subset J_1(T)$ is very far from being FI in general, for example in the case of the Schwarzschild (S) or Kerr (K) metrics contrary to the case of the Minkowski (M) metric [27] [28]. Our purpose is rather to work out the corresponding deformation theory by constructing at the same time the Janet sequence and the deformation sequence. Up to our knowledge, there is not a single reference in the whole mathematical literature.

In order to explain the type of problems we shall have to overcome, let us consider the specific metric $(\omega_{11} = 1, \omega_{12} = \omega_{21} = 0, \omega_{22} = b(x^1))$. Setting

$B = \frac{1}{2}\left(\frac{\partial_{11}b}{b} - \left(\frac{\partial_1 b}{b}\right)^2\right)$, the first involutive prolongation $R_2 \subset J_2(T)$ and its Janet tabular become:

$$\left\{\begin{array}{lcl} \xi^2_{22} - \frac{1}{2}\partial_1 b\ \xi^2_1 & = & 0 \\ \xi^1_{22} - b\,B\,\xi^1 & = & 0 \\ \xi^2_{12} + B\,\xi^1 & = & 0 \\ \xi^1_{12} & = & 0 \\ \xi^2_{11} + \frac{\partial_1 b}{b}\xi^2_1 & = & 0 \\ \xi^1_{11} & = & 0 \\ \xi^2_2 + \frac{1}{2}\frac{\partial_1 b}{b}\xi^1 & = & 0 \\ \xi^1_2 + b\,\xi^2_1 & = & 0 \\ \xi^1_1 & = & 0 \end{array}\right. \qquad \begin{array}{|cc|} \hline 1 & 2 \\ 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \\ 1 & \bullet \\ 1 & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ \hline \end{array}$$

We have $dim(g_1)=1$ with the only parametric jet ξ^2_1 and $g_2=0$ is trivially involutive. The system $R_2 \subset J_2(T)$ is involutive and thus FI, if and only if no new equation is brought by crossed derivatives with respect to the dots. Indeed, we have identities to zero like $d_2\xi^1_{11} - d_1\xi^1_{12} = \xi^1_{112} - \xi^1_{112} = 0$ or four differential relations like:

$$d_2\left(\xi^2_{11} + \frac{\partial_1 b}{b}\xi^2_1\right) - d_1\left(\xi^2_{12} + B\xi^1\right) = -\left(\partial_1 B + \frac{\partial_1 b}{b}B\right)\xi^1 \Rightarrow \partial_1(bB) = 0$$

This is only possible if a kind of Vessiot structure equation is existing "*at first sight*", namely:

$$\boxed{bB = cst \Rightarrow \partial_{11}b - \frac{(\partial_1 b)^2}{b} = cst}$$

Let us consider two examples that can be found among the exercises of many textbooks on differential geometry, for example in the book published by V. Guedj in 2022 [29]:

- In the plane with coordinates (x,y), let us consider the change of coordinates defined by the formula $(u,v) \to (x = u\cos(v), y = u\sin(v))$ leading to $dx^2 + dy^2 = du^2 + u^2 dv^2$. The initial metric is the standard Euclidean metric of the plane while the second is of the type just considered with $b = (x^1)^2$ and we obtain $bB = 1 - 2 = -1$.
- In $\mathbb{R}^3$, let us consider the unit sphere with standard coordinates $(r = 1, \theta, \phi)$. The arc of length is easily seen to be defined by the metric $ds^2 = d\theta^2 + \sin^2(\theta)d\phi^2$ leading to $b = \sin^2(x^1)$, $\partial_1 b = 2\sin(x^1)\cos(x^1)$, $\partial_{11}b = 2 - 4\sin^2(x^1)$ and thus $bB = (1 - 2\sin^2(x^1)) - 2\cos^2(\theta) = 1 - 2 = -1$, that is *exactly* the same constant but in a completely different situation because the first metric is known to be flat while the second has *surely* a non zero scalar curvature *by assumption*.

A special metric could be the Euclidean metric when $n = 1,2,3$ as in elasticity theory or the Minkowski metric when $n = 4$ as in special relativity. The *first order Medolaghi equations*:

$$\Omega_{ij} \equiv \left(\mathcal{L}(\xi)\omega\right)_{ij} \equiv \omega_{rj}(x)\partial_i\xi^r + \omega_{ir}(x)\partial_j\xi^r + \xi^r\partial_r\omega_{ij}(x) = 0$$

with $\Omega \in S_2T^*$ are also called *Killing equations* for historical reasons [11]. The main problem is that *this system is not involutive* unless we prolong it to order two by differentiating once the equations. For such a purpose, introducing $\omega^{-1} = \left(\omega^{ij}\right)$ as usual, we may define the *Christoffel symbols*:

$$\gamma_{ij}^k(x) = \frac{1}{2}\omega^{kr}(x)\left(\partial_i\omega_{rj}(x) + \partial_j\omega_{ri}(x) - \partial_r\omega_{ij}(x)\right) = \gamma_{ji}^k(x)$$

This is a new geometric object of order 2 providing an isomorphism $j_1(\omega) \simeq (\omega,\gamma)$ and allowing us to obtain the *second order Medolaghi equations* with now $\Gamma \in S_2T^* \times T$:

$$\Gamma_{ij}^k \equiv \left(\mathcal{L}(\xi)\gamma\right)_{ij}^k \equiv \partial_{ij}\xi^k + \gamma_{rj}^k(x)\partial_i\xi^r + \gamma_{ir}^k(x)\partial_j\xi^r - \gamma_{ij}^r(x)\partial_r\xi^k + \xi^r\partial_r\gamma_{ij}^k(x) = 0$$

Surprisingly, using the fact that $\partial_i\gamma_{rj}^r - \partial_j\gamma_{ri}^r = 0$, the following expressions:

$$\rho_{lij}^k(x) \equiv \partial_i\gamma_{lj}^k(x) - \partial_j\gamma_{li}^k(x) + \gamma_{ri}^k(x)\gamma_{lj}^r(x) - \gamma_{rj}^k(x)\gamma_{li}^r(x)$$

$$\rho_{ij}(x) = \rho_{i,rj}^r(x) = \partial_r\gamma_{ij}^r(x) - \partial_j\gamma_{ri}^r(x) + \gamma_{rs}^r(x)\gamma_{ij}^s(x) - \gamma_{ri}^s(x)\gamma_{sj}^r(x)$$

are still first order geometric objects and even tensors with $n^2\left(n^2-1\right)/12$ reduced o $n(n+1)/2$ independent components satisfying the purely algebraic relations:

$$\rho_{lij}^k + \rho_{lji}^k = 0,\ \ \rho_{lij}^k + \rho_{ijl}^k + \rho_{jli}^k = 0,\ \ \omega_{rl}\rho_{kij}^r + \omega_{kr}\rho_{lij}^r = 0,\ \ \rho_{ij} = \rho_{ji}$$

along with the following commutative and exact diagram [11]:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & S_3T^*\otimes T & \to & S_2T^*\otimes S_2T^* & \to & F_1 \to 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & T^*\otimes S_2T^*\otimes T & \to & T^*\otimes T^*\otimes S_2T^* & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \wedge^2T^*\otimes g_1 & \to & \wedge^2T^*\otimes T^*\otimes T & \to & \wedge^2T^*\otimes S_2T^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \wedge^3T^*\otimes T & = & \wedge^3T^*\otimes T & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

in which all the vertical sequences are δ-sequences, exact but the first. We obtain therefore $dim(F_1) = n^2\left(n^2-1\right)/12$ from the top row by using $n^2(n+1)^2/4 - n^2(n+1)(n+2)/6$ or from the first column by using $n^2(n-1)^2/4 - n^2(n-1)(n-2)/6$, changing the + signs into - signs (!!!). We obtain in particular $dim(F_1) = 1$ when $n = 2$.

Therefore, setting $\rho_{ij} = \rho_{i,rj}^r$ and $\rho = \omega^{ij}\rho_{ij}$ as usual, we get $\rho_{r,ij}^r = 0$ and $\rho_{ij} = \rho_{ji}$.

Accordingly, the *Integrability Conditions* (IC) must express that the new first order equations:

$$R_{l,ij}^{k} \equiv \left(\mathcal{L}(\xi)\rho\right)_{lij}^{k} = 0$$

are only linear combinations of the previous ones, that is to say we must have:

$$\rho_{r,ij}^{k}\xi_{l}^{r} + \rho_{l,rj}^{k}\xi_{i}^{r} + \rho_{l,ir}^{k}\xi_{j}^{r} - \rho_{l,ij}^{r}\xi_{r}^{k} + \xi^{r}\partial_{r}\rho_{l,ij}^{k} = 0$$

whenever $\omega_{rj}\xi_{i}^{r} + \omega_{ir}\xi_{j}^{r} + \xi^{r}\partial_{r}\omega_{ij} = 0$.

In particular, we must therefore have (care):

$$\rho_{r,ij}^{k}\xi_{l}^{r} + \rho_{l,rj}^{k}\xi_{i}^{r} + \rho_{l,ir}^{k}\xi_{j}^{r} - \rho_{l,ij}^{r}\xi_{r}^{k} = \left(\rho_{r,ij}^{k}\delta_{l}^{s} + \rho_{l,rj}^{k}\delta_{i}^{s} + \rho_{l,ir}^{k}\delta_{j}^{s} - \rho_{l,ij}^{s}\delta_{r}^{k}\right)\xi_{s}^{r} = 0$$

whenever $\omega_{rj}\xi_{i}^{r} + \omega_{ir}\xi_{j}^{r} = 0$ or $\xi_{i,j} + \xi_{j,i} = 0$, raising or lowering the indices by means of ω, that is to say, using the Kronecker symbol $\delta_{i}^{k} = 0$ if $k \neq i$ or 1 if $k = i$:

$$\left(\rho_{r,ij}^{k}\delta_{l}^{s} + \rho_{l,rj}^{k}\delta_{i}^{s} + \rho_{l,ir}^{k}\delta_{j}^{s} - \rho_{l,ij}^{s}\delta_{r}^{k}\right)\omega^{rt} - \left(\rho_{r,ij}^{k}\delta_{l}^{t} + \rho_{l,rj}^{k}\delta_{i}^{t} + \rho_{l,ir}^{k}\delta_{j}^{t} - \rho_{l,ij}^{t}\delta_{r}^{k}\right)\omega^{rs} = 0$$

Contracting in s and l, we get:

$$\left(n\rho_{r,ij}^{k} + \rho_{i,rj}^{k} + \rho_{j,ir}^{k} - \rho_{s,ij}^{s}\right)\omega^{rt} - \rho_{r,ij}^{k}\omega^{rt} - \rho_{s,rj}^{k}\delta_{i}^{t}\omega^{rs} - \rho_{s,ir}^{k}\delta_{j}^{t}\omega^{rs} + \rho_{s,ij}^{t}\omega^{ks} = 0$$

and thus, using the previous algebraic relations:

$$(n-1)\rho_{r,ij}^{k}\omega^{rt} = \left(\rho_{r,sj}^{k}\delta_{i}^{t} - \rho_{r,si}^{k}\delta_{j}^{t}\right)\omega^{rs} \Rightarrow (n-1)\rho_{l,ij}^{k} = \left(\rho_{r,sj}^{k}\omega_{li} - \rho_{r,si}^{k}\omega_{lj}\right)\omega^{rs}$$

with $\rho_{r,sj}^{k}\omega^{rs} = \omega^{kt}\rho_{tr,sj}\omega^{rs} = -\omega^{kt}\rho_{rt,sj}\omega^{rs} = -\omega^{kt}\rho_{tj}$. Finally, like above, we must also have $R_{ij} \equiv \left(\mathcal{L}(\xi)\rho\right)_{ij} = 0$ whenever $\Omega_{ij} = 0$ and similarly:

$$\left(\rho_{rj}\delta_{i}^{s} + \rho_{ir}\delta_{j}^{s}\right)\omega^{rt} - \left(\rho_{rj}\delta_{i}^{t} + \rho_{ir}\delta_{j}^{t}\right)\omega^{rs} = 0$$

Contracting in s and i, we get $n\rho_{ij} = \rho\omega_{ij}$ and thus $n\rho_{rsj}^{k}\omega^{rs} = \rho\delta_{j}^{k}$. As ρ is a scalar and we must not have any zero order equation, we may set $\rho = n(n-1)c$ where c is a constant and obtain the $n^{2}\left(n^{2}-1\right)/12$ *Vessiot structure equations* with the only *structure constant* c:

$$\boxed{\rho_{l,ij}^{k}(x) = c\left(\delta_{i}^{k}\omega_{lj}(x) - \delta_{j}^{k}\omega_{li}(x)\right) \Rightarrow \rho_{ij} = (n-1)c\omega_{ij}}$$

describing the constant Riemannian curvature condition of Eisenhart (Compare [10] and [30]!). If $\bar{\omega}$ is another nondegenerate metric with structure constant $\bar{c}$, the *equivalence problem* $j_{1}(f)^{-1}(\omega) = \bar{\omega}$ cannot be solved even locally if $\bar{c} \neq c$.

However, there is an open problem concerning the classical or conformal Killing equations not solved by E. Beltrami, L. Bianchi, L.P. Eisenhart, W. Killing, E. Vessiot and followers [31], namely the relation existing between *integrability conditions* (IC), *formal integrability* (FI), *compatibility conditions* (CC) and *constant Riemannian curvature* (CRC). By chance, in the present situation with $n = 2$, the problem is clear because on one side we saw the condition $\partial_{11}b - \frac{(\partial_{1}b)^{2}}{b} = cst$ clearly linking IC with FI on one side and the Vessiot condition $\rho_{ij} = c\omega_{ij}$ in which c is an arbitrary constant on the other side. Indeed, the Christoffel symbols become:

$$\boxed{\gamma_{11}^1 = 0, \gamma_{11}^2 = 0, \gamma_{12}^1 = 0, \gamma_{12}^2 = \frac{1}{2}\frac{\partial_1 b}{b}, \gamma_{22}^1 = -\frac{1}{2}\partial_1 b, \gamma_{22}^2 = 0}$$

After substitution, one obtains $\rho_{11} = -\frac{\partial_{11}b}{2b} + \frac{1}{4}\left(\frac{\partial_1 b}{b}\right)^2 = c = cst$, $\rho_{12} = 0$, $\rho_{22} = cb$. The solution is given by the following striking technical lemma which is valid for any n:

LEMMA: The two conditions are equivalent, even if the constants may be different.

Proof: Differentiating the previous condition, we obtain:

$$-\frac{\partial_{111}b}{2b} + \frac{(\partial_1 b)(\partial_{11}b)}{b^2} - \frac{1}{2}\frac{(\partial_1 b)^3}{b^3} = 0$$

$$\Leftrightarrow \partial_{111}b - 2\frac{(\partial_1 b)(\partial_{11}b)}{b} + \frac{(\partial_1 b)^3}{b^2} = \partial_1\left(\partial_{11}b - \frac{(\partial_1 b)^2}{b}\right) = 0$$

because $det(\omega) = b \neq 0$. In the particular cases considered, we obtain respectively:

- $b = (x^1)^2 \Rightarrow \partial_1 b = 2x^1 \Rightarrow \partial_{11}b = 2 \Rightarrow c = -\frac{1}{(x^1)^2} + \frac{1}{(x^1)^2} = 0$ though the IC constant is equal to $2 - 4 = -2$.
- $b = \sin^2(x^1) \Rightarrow \partial_1 b = 2\sin(x^1)\cos(x^1) \Rightarrow \partial_{11}b = 2\cos^2(x^1) - 2\sin^2(x^1)$ and we obtain $c = -\frac{\cos^2(x^1)}{\sin^2(x^1)} + 1 + \frac{\cos^2(x^1)}{\sin^2(x^1)} = 1$ which is the curvature of the sphere of radius 1 though the IC constant is equal to $2\cos^2(x^1) - 2\sin^2(x^1) - 4\cos^2(x^1) = -2$.

□

Integrating the former involutive system when $bB = 1$, we obtain in both cases the second order equation $\xi_{22}^1 + \xi^1 = 0$ with solutions $(\cos(x^2), \sin(x^2))$ over the constants because $\xi_1^1 = 0$. We finally obtain the 3 infinitesimal generators:

$$\left\{\theta_1 = \partial_2, \theta_2 = \cos(x^2)\partial_1 - \frac{\cos(x^1)}{\sin(x^1)}\sin(x^2)\partial_2, \sin(x^2)\partial_1 + \frac{\cos(x^1)}{\sin(x^1)}\cos(x^2)\partial_2\right\}$$

$$\Rightarrow [\theta_1, \theta_2] = -\theta_3, [\theta_1, \theta_3] = \theta_2, [\theta_2, \theta_3] = \theta_1$$

In both cases, as ω and $\bar{\omega}$ provide the same system if and only if $\bar{\omega} = a\omega$, we obtain $\bar{\gamma} = \gamma$ and thus $\bar{c} = c/a$. Accordingly $N(\Theta)/\Theta$ is of dimension 1 and generated by $x\partial_x + y\partial_y = u\partial_u$ when $c = 0$ while $N(\Theta) = \Theta$ when $c \neq 0$. A direct computation cannot be done without computer algebra!.

The linearization provides:

$$R_{l,ij}^k \equiv -\rho_{l,ij}^r \xi_r^k + \rho_{r,ij}^k \xi_l^r + \rho_{l,rj}^k \xi_i^r + \rho_{l,ir}^k \xi_j^r + \xi^r \partial_r \rho_{l,ij}^k = 0$$

$$\Rightarrow R_{ij} \equiv \rho_{rj}\xi_i^r + \rho_{ir}\xi_j^r + \xi^r \partial_r \rho_{ij} = 0$$

$$F_{ij} \equiv R_{r,ij}^r \equiv \varphi_{rj}\xi_i^r + \varphi_{ir}\xi_j^r + \xi^r \partial_r \varphi_{ij} = 0$$

In the specific dimension $n = 2$ considered, we have by chance the simplified formulas:

$$\rho_{11} = \rho^r_{1,r1} = \rho^2_{1,21},\ \rho_{12} = \rho^r_{1,r2} = \rho^1_{1,12},\ \rho_{21} = \rho^r_{2,r1} = \rho^2_{2,21},\ \rho_{22} = \rho^r_{2,r2} = \rho^1_{2,12}$$

and thus the *specific isomorphism* $\wedge^2 T^* \otimes T^* \otimes T \simeq \wedge^2 T^* \oplus S_2 T^*$ *only for* $n = 2$, defined by:

$$\left(\rho^k_{l,ij}\right) \to \left(\rho_{ij}\right) \to \left(\frac{1}{2}\left(\rho_{ij} - \rho_{ji}\right), \frac{1}{2}\left(\rho_{ij} + \rho_{ji}\right)\right)$$

by counting the dimensions with $1 \times 2 \times 2 = 4 = 1 + 3$ while using the canonical splitting of the short exact δ-sequence:

$$0 \to S_2 T^* \xrightarrow{\delta} T^* \otimes T^* \xrightarrow{\delta} \wedge^2 T^* \to 0$$

Now, we have proved in many books [10]-[12] [18] or papers [19] that, for any dimension n, two sections ω and $\bar{\omega}$ provides the same system of infinitesimal Lie equations if and only if $\bar{\omega} = a\omega$ for $a \neq 0$ the parameter of the multiplicative group of the real line. It follows that we have necessarily a first Vessiot constant c_1 in such a way that:

$$\frac{1}{2}\left(\rho_{ij} + \rho_{ji}\right) = c_1 \omega_{ij}$$

However, after linearization, we also obtain:

$$2det(\omega)\left(\xi^1_1 + \xi^2_2\right) + \xi^r \partial_r det(\omega) = 0$$

and obtain therefore a second Vessiot structure constant c_2 such that we have:

$$\frac{1}{2}\left(\rho_{12} - \rho_{21}\right) = \frac{1}{2}\varphi_{12} = c_2 \left(det(\omega)\right)^{\frac{1}{2}}$$

Finally, we have to take into account that we do want second order *integrability conditions* for the metric ω, that is we must eliminate γ by using the Levi-Civita isomorphism $(\omega, \gamma) \simeq j_1(\omega)$. Then, it is well known that $\varphi_{ij} = 0$ because $\gamma^r_{ri} = \frac{1}{2} det(\omega) \partial_i det(\omega) \Rightarrow \partial_i \gamma^r_{rj} - \partial_j \gamma^r_{ri} = 0$ and we must thus have $c_2 = 0$. Also, using the same diagram as in the previous section, we must have only one second order integrability condition which is indeed the constant curvature condition expressed by means of the Ricci tensor which is now symmetric. In order to describe the structure of electromagnetism (EM), H. Weyl had the idea in 1918 to consider the Lie pseudogroup of conformal transformations defined by the Lie equations $\hat{R}_1 \subset J_1(T)$ [32]-[34]

$$\begin{aligned} &\Phi_{ij}(y_1) \equiv \omega_{kl}(y) y^k_i y^l_j = e^{2a(x)} \omega_{ij}(x) \\ &\Rightarrow \Omega_{ij} \equiv \omega_{rj}(x)\xi^r_i + \omega_{ir}(x)\xi^r_j(x) + \xi^r \partial_r \omega_{ij}(x) = 2A(x)\omega_{ij}(x) \end{aligned}$$

and their prolongations $\hat{R}_2 \subset J_2(T)$ obtained by using the Kronecker symbol δ^k_i:

$$\begin{aligned} \Gamma^k_{ij} &\equiv \xi^k_{ij} + \gamma^k_{rj}(x)\xi^r_i + \gamma^k_{ir}(x)\xi^r_j(x) - \gamma^r_{ij}(x)\xi^k_r + \xi^r \partial_r \gamma^k_{ij}(x) \\ &= \delta^k_i A_j(x) + \delta^k_j A_i(x) - \omega^{kr}(x) A_r(x) \end{aligned}$$

in which $A \in \wedge^0 T^*, A_i dx^i \in \wedge^1 T^*$, a result not depending on the conformal factor

$e^{a(x)}$.

Now, contracting all the indices but one, we obtain successively:

$$\xi_r^r + \xi^r \gamma_{sr}^s = nA \Rightarrow \xi_{ri}^r + \gamma_{sr}^s \xi_i^r + \xi^r \partial_r \gamma_{si}^s = nA_i$$
$$\Rightarrow \left(\partial_i \xi_r^r - \xi_{ri}^r\right) + \gamma_{sr}^s \left(\partial_i \xi^r - \xi_i^r\right) = n\left(\partial_i A - A_i\right)$$

We obtain thus an induced Spencer operator $d : (A, A_i) \to (\partial_i A - A_i)$, a reason for introducing the factor 2 and the EM field $F_{ij} = \partial_i A_j - \partial_j A_i \in \wedge^2 T^*$, contrary to the way that has been used around 1956 at the birth of Gauge theory (GT) with the unitary abelian group $U(1)$ which is not acting on space-time [35]. This result also explains why the most important apparatus in a rocket an accelerometer is associated to an inertial platform, that only allows to measure the difference $\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} - \vec{g} = \partial_4 \xi_4^k - \xi_{44}^k$ between acceleration and gravity and also why the conformal elations are sometimes called "*acceleration*".

EXAMPLE 5.5: (*Lie group*) This example is provided in such a way that the reader can understand why the well-known Maurer-Cartan equations are only examples of the Vessiot structure equations obtained when a Lie group G is acting on a manyfold X while $dim(G) = dim(X)$. At the same time, it is explaining why the Chevalley-Eilenberg cohomology is just a particular example of the deformation cohomology we have presented in this paper, a deep reason in order to understand why a Lie pseudogroup must not be considered as an infinite Lie group.

Let us consider the Lie group of transformations of the manifold $X = \mathbb{R}^3$ with local coordinates $x = (x^1, x^2, x^3)$ and the three infinitesimal generators as in ([10], p 346):

$$\left\{\theta_1 = \partial_1, \theta_2 = \partial_2 + x^1 \partial_3, \theta_3 = \partial_3\right\} \Rightarrow [\theta_1, \theta_2] = \theta_3, [\theta_1, \theta_3] = 0, [\theta_2, \theta_3] = 0$$

It follows that all the Lie structure constants vanish but $c_{12}^3 = 1$. In the present situation, the general infinitesimal transformation is known to be of the form $\xi = a\partial_1 + b\partial_2 + (bx^1 + c)\partial_3 \in \Theta \subset T$ when (a, b, c) are arbitrary constants. It is evident that the corresponding trivially involutive first order system $R_1 \subset J_1(T)$ of infinitesimal Lie equations with Janet tabular is made by the following 9 PD equations:

$$\left\{\begin{array}{lcl|ccc|}
d_3\xi^3 & = & 0 & 1 & 2 & 3 \\
d_3\xi^2 & = & 0 & 1 & 2 & 3 \\
d_3\xi^1 & = & 0 & 1 & 2 & 3 \\
d_2\xi^3 & = & 0 & 1 & 2 & \bullet \\
d_2\xi^2 & = & 0 & 1 & 2 & \bullet \\
d_2\xi^1 & = & 0 & 1 & 2 & \bullet \\
d_1\xi^3 - \xi^2 & = & 0 & 1 & \bullet & \bullet \\
d_1\xi^2 & = & 0 & 1 & \bullet & \bullet \\
d_1\xi^1 & = & 0 & 1 & \bullet & \bullet
\end{array}\right.$$

Counting the numbers of single dots (that is 9), pairs of dots (that is 3), we obtain at once the locally and formally exact Janet sequence:

$$\boxed{0 \to \Theta \to T \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} F_2 \to 0}$$

with only first order operators and Euler-Poincaré characteristic $3-9+9-3=0$. As it is a first order system, introducing the corresponding three dimensional Lie algebra $\mathcal{G}$, the Spencer sequence is isomorphic to the tensor product of the Poincaré sequence for the exterior derivative d. The fundamental diagram I linking the Janet and Spencer sequences becomes:

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \Theta & \xrightarrow{j_1} & 3 & \xrightarrow{D_1} & 9 & \xrightarrow{D_2} & 9 & \xrightarrow{D_3} & 3 & \to & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
 & & 0 & \to & 3 & \xrightarrow{j_1} & 12 & \xrightarrow{D_1} & 18 & \xrightarrow{D_2} & 12 & \xrightarrow{D_3} & 3 & \to & 0 \\
 & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Theta & \to & 3 & \xrightarrow{\mathcal{D}} & 9 & \xrightarrow{\mathcal{D}_1} & 9 & \xrightarrow{\mathcal{D}_2} & 3 & \to & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & & & & & 0 & & 0 & & 0 & & & &
\end{array}$$

and only contains first order operators. The most interesting feature of this example is that the operators in the Spencer sequence are completely different from the operators appearing in the Janet sequence though the dimensions $(3,9,9,3)$ are the same ... with a shift by one step!

However, even if we are sure to have a first order system of infinitesimal Lie equations and thus a transitive Lie algebroid $R_1 \subset J_1(T)$ with $[R_1,R_1] \subset R_1$ and $R_1^0 = g_1 = 0$ because $dim(R_1) = dim(T) = 3$, we still do not know what is the underlying geometric object allowing to obtain a finite Lie form or an infinitesimal Medolaghi form that *must* be used for the deformation theory and the deformation sequence.

The general transformations of Γ are $\{y^1 = x^1 + a, y^2 = x^2 + b, y^3 = x^3 + bx^1 + c\}$ with three constants (a,b,c). As we have $dy^3 = dx^3 + (y^2 - x^2)dx^1$ and thus $dy^3 - y^2dy^1 = dx^3 - x^2dx^1$, the invariant geometric object is thus made by three 1-forms $\omega^\tau = \omega_i^\tau(x)dx^i$ and thus $\omega = (\omega^1,\omega^2,\omega^3)$ as a section of the vector bundle of geometric objects $\mathcal{F} = T^* \times_X T^* \times_X T^*$, that is $\omega = (dx^1, dx^2, dx^3 + x^2dx^1)$ in the present situation. The 9 general finite equations in Lie form are thus $\Phi_i^\tau(y_1) = \omega_i^\tau(x)$. The 9 general Medolaghi infinitesimal Lie equations are thus $\omega_r^\tau(x)\xi_i^r + \xi^r\partial_r\omega_i^\tau(x) = 0$ with the algebraic condition $det(\omega) = det(\omega_i^\tau(x)) \neq 0$. Closing the system we obtain the 9 well known Maurer-Cartan equations $d\omega^\tau - c_{\rho\sigma}^\tau\omega^\rho \wedge \omega^\tau = 0$ that are just introducing the structure constants c as particular examples of the Vessiot structure constants, a *fact* still totally ignored today after more than one century! Closing again this system, we get the quadratic *Jacobi identities* $\sum_{cycle} c^{\alpha}_{\rho\sigma} c^{\beta}_{\tau\alpha} = 0$ over the three cyclic permutations of (ρ,σ,τ).

We are now ready to look for the normalizer $N(\Theta)$ of Θ, a tricky technical task. For this, we have to study the condition $[\eta,\xi] = \bar{a}\partial_1 + \bar{b}\partial_2 + (\bar{b}x^1 + \bar{c})\partial_3$, whenever $\xi = a\partial_1 + b\partial_2 + (bx^1 + c)\partial_3$. As the only non-zero derivative of the components of ξ is $\partial_1 \xi^3 = \xi^2$, we are led to the equations:

$$-\xi^r \partial_r \eta^1 = \bar{a},\ \xi^r \partial_r \eta^2 = \bar{b},\ \xi^2 \eta^1 - \xi^r \partial_r \eta^3 + x^1 \xi^r \partial_r \eta^2 = \bar{c}$$

Differentiating the first with respect to x^i, we get the only first order equation $\eta^1_3 = 0$.

Differentiating the second with respect to x^i, we get the only first order equation $\eta^2_3 = 0$.

Differentiating the third with respect to x^i, we get the only first order equation $\eta^1_1 + \eta^2_2 - \eta^3_3 = 0$ and all the second order jets of η^3 vanish but we have $\eta^3_{11} - \eta^2_1 = 0$.

Collecting all these results and using the fact that the normalizer is defined by an involutive second order system $\tilde{R}_2$ with symbol $\tilde{g}_2 = 0$, we finally obtain the 9 infinitesimal generators of $N(\Theta)$:

$$\boxed{\partial_1, \partial_2, \partial_3, x^1\partial_3, x^2\partial_3, x^1\partial_1 - x^2\partial_2, x^2\partial_2 + x^3\partial_3, x^2\partial_1 + \frac{(x^2)^2}{2}\partial_3, x^1\partial_2 + \frac{(x^1)^2}{2}\partial_3}$$

In order to look for the centralizer $C(\Theta)$, we have to solve the equations:

$$-\xi^r \partial_r \eta^1 = 0,\ \xi^r \partial_r \eta^2 = 0,\ \xi^2 \eta^1 - \xi^r \partial_r \eta^3 = 0 \Rightarrow \eta = \alpha\partial_1 + \beta\partial_2 + (\alpha x^2 + \gamma)\partial_3$$

We finally obtain $C(\Theta) = \{\partial_1 + x^2\partial_3, \partial_2, \partial_3\} \Rightarrow Z(\Theta) = \Theta \cap C(\Theta) = \{\partial_3\}$.

After another even more tedious computation, we let the reader prove that $N(N(\Theta))$ is self-normalizing and given by the action of 10 infinitesimal generators made by the 9 former ones and by $(x^1x^2 - 2x^3)\partial_3$.

The following Lemma will allow to study the deformation theory of Lie algebras as in [10] [18]:

LEMMA: If M and $\bar{M}$ are two invertible square matrices defined over X, then:

$$\boxed{\bar{M}^{-1}d\bar{M} = M^{-1}dM \Leftrightarrow \bar{M} = AM, dA = 0}$$

Proof. We have successively:

$$MM^{-1} = id \Rightarrow (dM)M^{-1} + M(dM^{-1}) = 0 \Rightarrow dM^{-1} = -M^{-1}(dM)M^{-1}$$

$$d(\bar{M}M^{-1}) = (d\bar{M})M^{-1} + \bar{M}dM^{-1} = (d\bar{M})M^{-1} - \bar{M}M^{-1}dMM^{-1}$$
$$= \bar{M}(\bar{M}^{-1}d\bar{M} - M^{-1}dM)M^{-1} = 0$$

□

Indeed, considering the following two Medolaghi equations:

$$\boxed{\omega^{\tau}_r(x)\xi^r_i + \xi^r\partial_r\omega^{\tau}_i(x) = 0 \qquad \bar{\omega}^{\tau}_r(x)\xi^r_i + \xi^r\partial_r\bar{\omega}^{\tau}_i(x) = 0}$$

Using the preceding Lemma with the inverse matrix $\alpha = \omega^{-1}$, the two Lie algebroids R_1 and $\overline{R}_1$ will be identical if and only if there exists a constant 3×3 matrix A such that $\overline{\omega}(x) = A\omega(x)$ describing $\Upsilon_0 = \Upsilon(F_0)$ which is isomorphic to the Lie algebra of the label group $\overline{u}^\tau = a^\tau_\rho u^\rho$ with dimension $3\times 3 = 9$. In the present situation, the infinitesimal action is described by $a^\tau_\rho = \delta^\tau_\rho + t\lambda^\tau_\rho + \cdots$ when $t \to 0$. It follows that $\overline{c} = c + tC + \cdots$ with $C = \sum c\lambda$ because c trransforms like a 1-contravariant and 2 covariant tensor. As only $c^3_{12} = 1$ does not vanish, we get $C^\tau_{12} = c^3_{12}\lambda^\tau_3 = \lambda^\tau_3$ must vanish for the cocycles $Z_0(R_1) = N(\Theta)/\Theta \in \Upsilon_0$ while we have $B_0(R_1) = C(\theta)/Z(\Theta) \in \Upsilon_0$. The cohomology $H_0(R_1)$ is a vector space of dimension 4 because $dim(N(\Theta)/\Theta) = 9 - 3 = 6$ and $dim(C(\Theta)/Z(\Theta)) = 3 - 1 = 2$. This result is showing the crucial use of the Janet sequence, both with the link with the Chevalley-Eilenberg cohomology [36] and the deformation theory of Lie algebras [37] but the link with the Vessiot structure equations is still not known.

EXAMPLE 5.6: This example, which is clearly a Lie group of transformations, is thus a Lie pseudogroup defined by a first order system of Lie equations which is not involutive because its second order symbol is vanishing. In this case, the general theory is proving that the Jacobi conditions may *not* be quadratic and could thus be linear, a result showing that the number of Vessiot structure constants involved is not evident at first sight. It has been provided for the first time in a computational fashion in [10] but never acknowledged and in [19] with a printing mistake that the editor of the journal did not accept to correct. It is presented anew in the simplest and shortest way in order to prove that the foundations of Riemannian geometry must be almost entirely revisited.

With $m = n = 2$, let us now consider the Lie group of transformations $\{y^1 = ax^1 + b, y^2 = cx^2 + d \mid a,b,c,d = cst, ac = 1\}$ as an algebraic Lie pseudogroup Γ. It is easy to exhibit the corresponding first order system $\mathcal{R}_1$ of finite Lie equations in Lie form by introducing the three generating differential invariants and the corresponding Lie form:

$$\Phi^1 \equiv \frac{y^1_2}{y^1_1} = 0, \Phi^2 \equiv \frac{y^2_1}{y^2_2} = 0, \Phi^3 \equiv y^1_1 y^2_2 = 1 \Rightarrow \Delta \equiv \Phi^3\left(1 - \Phi^1\Phi^2\right) \equiv y^1_1 y^2_2 - y^1_2 y^2_1 \neq 0$$

The details of the corresponding tricky computations, first done in 1978 [10], have been improved in 2016 [18] and are again revisited in this paper. As we shall see, its major interest is to work out the *two* Vessiot structure constants existing like for the Riemannian structure but without having *any* tensorial framework.

First of all, we notice that this system is of finite type with a vanishing second order symbol and is thus formally integrable but not involutive. For this, we may introduce the six generating differential invariants obtained after one prolongation, exactly like the six Christoffel symbols in 2-dimensional Riemannian geometry:

$$\Phi^4 \equiv \frac{y^1_{11}}{y^1_1} = 0, \Phi^5 \equiv \frac{y^1_{12}}{y^1_1} = 0, \Phi^6 \equiv \frac{y^1_{22}}{y^1_1} = 0, \Phi^7 \equiv \frac{y^2_{22}}{y^2_2} = 0, \Phi^8 \equiv \frac{y^2_{12}}{y^2_2} = 0, \Phi^9 \equiv \frac{y^2_{11}}{y^2_2} = 0$$

We notice Δ is changing like the Jacobian of the change of coordinates $\bar{x}=\varphi(x)$ but we obtain for example $u^1=\left(\partial_2\varphi^1+\bar{u}^1\partial_2\varphi^2\right)/\left(\partial_1\varphi^1+\bar{u}^1\partial_1\varphi^2\right)$ and so on for describing the natural fiber bundle $\mathcal{F}$ with local coordinates $\left(x^1,x^2;u^1,u^2,u^3\right)$ and section $\omega=\left(\omega^1,\omega^2,\omega^3\right)$ becoming $(0,0,1)$ with our choice of the above Lie form for $\mathcal{R}_1$ and $\omega^3\left(1-\omega^1\omega^2\right)=1\neq 0$. Passing to the infinitesimal point of view, we obtain the first order system $R_1\subset J_1(T)$ in the Medolaghi form with jet notation $\Omega\equiv L(\xi_1)\omega=0$ as for the three equations of the Killing system in dimension $n=2$:

$$\begin{cases}\Omega^1\equiv\xi_2^1+\omega^1\xi_2^2-\omega^1\xi_1^1-\left(\omega^1\right)^2\xi_1^2+\xi^r\partial_r\omega^1=0\\ \Omega^2\equiv\xi_1^2+\omega^2\xi_1^1-\omega^2\xi_2^2-\left(\omega^2\right)^2\xi_2^1+\xi^r\partial_r\omega^2=0\\ \Omega^3\equiv\omega^3\left(\xi_1^1+\xi_2^2\right)+\omega^1\omega^3\xi_1^2+\omega^2\omega^3\xi_2^1+\xi^r\partial_r\omega^3=0\end{cases}$$

We can extend them to six *intermediate* new equations equations of order 2, in particular:

$$\Omega^4\equiv\xi_{11}^1+\omega^1\xi_{11}^2+\omega^4\xi_1^1+\left(2\omega^5-\omega^1\omega^4\right)\xi_1^2+\xi^r\partial_r\omega^4=0$$
$$\Omega^5\equiv\xi_{12}^1+\omega^1\xi_{12}^2+\omega^4\xi_2^1+\left(\omega^6-\omega^1\omega^5\right)\xi_1^2+\omega^5\xi_2^2+\xi^r\partial_r\omega^5=0$$

Taking into account these new invariants bringing for example six relations like:

$$\boxed{\begin{array}{ll}\partial_1\omega^1-\omega^5+\omega^1\omega^4=0 & \partial_2\omega^1-\omega^6+\omega^1\omega^5=0\\ \partial_1\omega^2-\omega^9+\omega^2\omega^8=0 & \partial_2\omega^2-\omega^8+\omega^2\omega^7=0\\ \partial_1\omega^3-\omega^3\left(\omega^4+\omega^8\right)=0 & \partial_2\omega^3-\omega^3\left(\omega^5+\omega^7\right)=0\end{array}}$$

The determinant of the 6×6 matrix with respect to $\left(\omega^4,\cdots,\omega^9\right)$ is $\omega^3\left(1-\omega^1\omega^2\right)\neq 0$.

After tedious but elementary substitutions, we obtain for example:

$$d_2\Omega^4-d_1\Omega^5\equiv\left(\partial_2\omega^4-\partial_1\omega^5\right)\left(\xi_1^1+\xi_2^2\right)+\xi^r\partial_r\left(\partial_2\omega^4-\partial_1\omega^5\right)=0$$

However, we have also:

$$\omega^3\left(1-\omega^1\omega^2\right)\left(\xi_1^1+\xi_2^2\right)+\xi^r\partial_r\left(\omega^3\left(1-\omega^1\omega^2\right)\right)=0$$

The quotient of $\partial_2\omega^4-\partial_1\omega^5$ by $\omega^3\left(1-\omega^1\omega^2\right)$ is thus well defined, say equal to $c(x)$, and we get the new zero order equation $\xi^r\partial_r c(x)=0$ contradicting the formal integrability of the given system. We obtain two Vessiot structure equations with two Vessiot structure constants c',c'':

$$\partial_2\omega^4-\partial_1\omega^5=c'\omega^3\left(1-\omega^1\omega^2\right),\ \partial_1\omega^7-\partial_2\omega^8=c''\omega^3\left(1-\omega^1\omega^2\right)$$

There is one Vessiot structure equation of order two with a single structure constant because:

$$\partial_2\left(\omega^4+\omega^8\right)-\partial_1\left(\omega^5+\omega^7\right)=0\ \Rightarrow\ \boxed{c'-c''=0}\ \Rightarrow\ c'=c''=c$$

In the present situation, we may construct the same diagram as the one used in the study of the Killing system because we have also $g_2 = 0 \Rightarrow g_3 = 0$. In particular, applying the Spencer δ-map to the symbol sequence, we obtain the commutative and exact diagram with $dim(F_0) = 3$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & S_3T^* \otimes T & \to & S_2T^* \otimes F_0 & \to & F_1 \to 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & T^* \otimes S_2T^* \otimes T & \to & T^* \otimes T^* \otimes F_0 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 \to & & \wedge^2 T^* \otimes g_1 & \to & \wedge^2 T^* \otimes T^* \otimes T & \to & \wedge^2 T^* \otimes F_0 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

We obtain the isomorphism $F_1 \simeq \wedge^2 T^* \otimes g_1$ by a snake chase with $dim(F_1) = 1$ because $n = 2$. We also obtain $dim(F_1) = dim(S_2T^* \otimes F_0) - dim(S_3T^* \otimes T) = 9 - 8 = 1$ in a coherent way.

For the sake of completeness, we provide the only component of the second order CC, namely:

$$\Omega^1 \equiv \xi^1_2 = 0, \Omega^2 \equiv \xi^2_1 = 0, \Omega^3 \equiv \xi^1_1 + \xi^2_2 = 0 \Rightarrow d_{11}\Omega^1 + d_{22}\Omega^2 - d_{12}\Omega^3 = 0$$

Accordingly, if we do want to solve the equivalence problem $\Phi^1 = \bar{\omega}^1$, $\Phi^2 = \bar{\omega}^2$, $\Phi^3 = \bar{\omega}^3$, *we must know the Vessiot structure equations*. As for the Vessiot structure constant c, two sections ω and $\bar{\omega}$ of $\mathcal{F}$ give the same infinitesimal Lie equations if and only if:

$$\bar{\omega}^1 = \omega^1, \ \bar{\omega}^2 = \omega^2, \ \bar{\omega}^3 = a\omega^3 \ \Rightarrow \ \bar{c} = c/a$$

where $a \neq 0$ is the parameter of the multiplicative group of the real line. It follows that we have $c = 0 \Rightarrow \bar{c} = 0$ both with $c \neq 0 \Rightarrow \bar{c} \neq 0$ and it just remains to exhibit such situations.

In the present situation, we have $\omega^1 = 0, \omega^2 = 0, \omega^3 = 1 \Rightarrow c = 0$. However, the new pseudogroup:

$$\bar{\Gamma} = \left\{ y^1 = \frac{ax^1 + b}{cx^1 + d}, y^2 = \frac{ax^2 + b}{cx^2 + d} \,|\, a, b, c, d = cst \right\}$$

is easily seen to be provided by the new specialization:

$$\bar{\omega}^1 = 0, \ \bar{\omega}^2 = 0, \ \bar{\omega}^3 = 1\Big/\left(x^2 - x^1\right)^2 \Rightarrow \bar{c} = -2$$

leading to the new Lie form:

$$y^1_2 = 0, \ y^2_1 = 0, \ \frac{1}{\left(y^2 - y^1\right)^2} y^1_1 y^2_2 = \frac{1}{\left(x^2 - x^1\right)^2}$$

It follows that the equivalence problem $y^1_2 / y^1_1 = 0, y^2_1 / y^2_2 = 0, y^1_1 y^2_2 = 1\Big/\left(x^2 - x^1\right)^2$ cannot be solved.

Indeed, we notice that $0 = ac = \bar{c} = -2$ for a certain $a \neq 0$ and a contradiction.

Needless to say that no classical tool can produce these formal results.

We finally prove that there is *almost* no difference with the Killing example though the background group is quite different. Indeed, we may introduce the new generating differential invariants and Lie forms:

$$\Phi_{11} = 2\Phi^2\Phi^3 \equiv 2y_1^1 y_1^2 = 0,$$
$$\Phi_{22} = 2\Phi^1\Phi^3 \equiv 2y_2^1 y_2^2 = 0,$$
$$\Phi_{12} = \Phi^3\left(1+\Phi^1\Phi^2\right) \equiv y_1^1 y_2^2 + y_2^1 y_1^2 = 1$$

in such a way that:

$$\omega_{11} = 2\omega_2\omega_3,\ \omega_{22} = 2\omega_1\omega_3,\ \omega_{12} = \omega_3\left(1+\omega_1\omega_2\right) \Rightarrow d_{11}\Omega_{22} + d_{22}\Omega_{11} - 2d_{12}\Omega_{12} = 0$$

We have $\Phi_{11}\Phi_{22} - \left(\Phi_{12}\right)^2 = -\Delta^2$ and obtain again $\Phi_{ij} \equiv \omega_{kl} y_i^k y_j^l$ as before but now with the strange metric $\omega_{11} = 0$, $\omega_{22} = 0$, $\omega_{12} = 1$ in such a way that $det(\omega) = -1 < 0$ contrary to the previous example where $\bar{\omega}_{11} = 1$, $\bar{\omega}_{22} = 1$, $\bar{\omega}_{12} = 0$ leading to $det(\bar{\omega}) = 1 > 0$. The equivalence problem between these two structures cannot be solved. Indeed, taking the determinants of the equations $\omega_{kl}(x)\partial_i f^k(x)\partial_j f^l(x) = \bar{\omega}_{ij}(x)$, we should obtain $det(\omega)\Delta^2 = det(\bar{\omega}) = 1$ and thus the contradiction $\Delta^2 = -1$.

EXAMPLE 5.7:(*Lie pseudogroup*) Contrary to the preceding example clearly exhibiting a Lie group of transformations and thus a Lie pseudogoup by eliminating the constant parameters, the present example *cannot* be a Lie group of transformations because it clearly depends on one arbitrary function of one variable. However, it is not easy to discover that it is a Lie pseudogroup and to exhibit the corresponding finite and infinitesimal Lie equations while providing the corresponding bundle of geometric objects involved. Also, if the computations can be done by hand while using elementary exterior calculus, the exhibition of the corresponding deformation sequence becomes quite a hard task that cannot be even imagined by using only standard argument of differential geometry. It can be a test example while using computer algebra.

With $n = 3$, let us consider the Lie pseudogroup Γ defined by the following finite transformations that depend on one arbitrary invertible function of x^1:

$$\boxed{y^1 = f\left(x^1\right),\ y^2 = x^2 f'\left(x^1\right),\ y^3 = x^3 + x^2\frac{f''\left(x^1\right)}{f'\left(x^1\right)}}$$

It is not evident at all that we have indeed a Lie pseudogroup. First of all, if we have also:

$$z^1 = g\left(y^1\right),\ z^2 = y^2 g'\left(y^1\right),\ z^3 = y^3 + y^2\frac{g''\left(y^1\right)}{g'\left(y^1\right)}$$

we obtain through the chain rule for derivatives:

$$\begin{aligned} z^1 &= g\left(f\left(x^1\right)\right) = g \circ f\left(x^1\right) = h\left(x^1\right), \\ z^2 &= g'\left(y^1\right) f'\left(x^1\right) = h'\left(x^1\right), \\ z^3 &= x^3 + x^2 \frac{f''\left(x^1\right)}{f'\left(x^1\right)} + x^2 f'\left(x^1\right) \frac{g''\left(y^1\right)}{g'\left(y^1\right)} \end{aligned}$$

and thus $z^3 = x^3 + x^2 \dfrac{h''\left(x^1\right)}{h'\left(x^1\right)}$ in a coherent way because

$h''\left(x^1\right) = g''\left(y^1\right)\left(f'\left(x^1\right)\right)^2 + g'\left(y^1\right) f''\left(x^1\right)$.

Then, though the transformations depend on $j_2(f) = \left(f\left(x^1\right), f'\left(x^1\right), f''\left(x^1\right)\right)$, we discover that the finite transformations are solutions of the following non-linear involutive system of 8 PD equations:

$$\left\{ \begin{array}{lcl} y_3^3 - 1 & = & 0 \\ y_3^2 & = & 0 \\ y_3^1 & = & 0 \\ y_2^3 - \left(y^3 - x^3\right)\dfrac{1}{x^2} & = & 0 \\ y_2^2 - \dfrac{y^2}{x^2} & = & 0 \\ y_2^1 & = & 0 \\ y_1^2 - \left(y^3 - x^3\right)\dfrac{y^2}{x^2} & = & 0 \\ y_1^1 - \dfrac{y^2}{x^2} & = & 0 \end{array} \right. \qquad \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \\ 1 & \bullet & \bullet \end{array}}$$

The infinitesimal Lie equations are solutions of the linear involutive system $R_1 \subset J_1(T)$ with Janet tabular where the 7 dots denote the non-multiplicative variables:

$$\left\{ \begin{array}{lcl} \xi_3^3 & = & 0 \\ \xi_3^2 & = & 0 \\ \xi_3^1 & = & 0 \\ \xi_2^3 - \dfrac{1}{x^2}\xi^3 & = & 0 \\ \xi_2^2 - \dfrac{1}{x^2}\xi^2 & = & 0 \\ \xi_2^1 & = & 0 \\ \xi_1^2 - \xi^3 & = & 0 \\ \xi_1^1 - \dfrac{1}{x^2}\xi^2 & = & 0 \end{array} \right. \qquad \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \\ 1 & \bullet & \bullet \end{array}}$$

The general solution is $\xi^1 = h\left(x^1\right), \xi^2 = x^2 h'\left(x^1\right), \xi^3 = x^2 h''\left(x^1\right)$ with $h\left(x^1\right)$ an arbitrary function of x^1

We let the reader check that each dot allows to exhibit one CC for the corresponding first order involutive Lie operator $\mathcal{D}: T \to F_0$ and we thus obtain the following Janet sequence made only by first order operators:

$$0 \to \Theta \to T \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} F_2 \to 0 \qquad 0 \to \Theta \to 3 \xrightarrow{\mathcal{D}} 8 \xrightarrow{\mathcal{D}_1} 7 \xrightarrow{\mathcal{D}_2} 2 \to 0$$

with Euler-Poincaré characteristic $3-8+7-2=0$.

As we have 4 parametric jets $\left(\xi^1,\xi^2,\xi^3,\xi^3_1\right)$, we could now pass from source to target and look for $(3+9)-4=8$ generating differential invariants. However, we may (*by chance*!) use a particular feature of this example in order to separate the equations into three blocks, namely $\left(\xi^1_1,\xi^1_2,\xi^1_3\right)+\left(\xi^2_1,\xi^2_2,\xi^2_3\right)+\left(\xi^3_2,\xi^3_3\right)$ and discover that the two first blocks provide the Lie derivative of two 1-forms (α,α') while the third block provide the Lie derivative of a 2-form β, namely:

$$\boxed{\alpha=\frac{1}{x^2}dx^1,\ \alpha'=-\frac{x^3}{x^2}dx^1+\frac{1}{x^2}dx^2,\ \beta=\frac{1}{x^2}dx^1\wedge dx^3-\frac{x^3}{\left(x^2\right)^2}dx^1\wedge dx^2 \Rightarrow \alpha\wedge\beta=0}$$

Accordingly, the corresponding geometric object is a section $(\alpha,\alpha',\beta)\in\mathcal{F}=T^*\times_X T^*\times_X \wedge^2 T^*$ provided we add the purely algebraic invariant constraint $\alpha\wedge\beta=0$ in $\wedge^3 T^*$ which is reducing the number of differential invariant from 9 down to 8 while simplifying all the computations.

The general infinitesimal Lie equations are thus:

$$\boxed{A\equiv\mathcal{L}(\xi)\alpha=0,\ \ A'\equiv \text{Ł}(\xi)\alpha'=0,\ \ B\equiv\mathcal{L}(\xi)\beta}$$

We may thus finally set $\omega=\left(\alpha\in T^*,\alpha'\in T^*,\beta\in\wedge^2T^*\right)\Rightarrow\Omega=\left(A\in T^*,A'\in T^*,B\in\wedge^2T^*\right)$ as in the general framework but now with the constraint $\alpha\wedge\beta=0\Rightarrow\alpha\wedge B+\beta\wedge A=0$ in $\wedge^3T^*$ because $\alpha\wedge\beta=\beta\wedge\alpha$. For a later use, we notice that $\alpha'\wedge\beta=-\frac{1}{\left(x^2\right)^2}dx^1\wedge dx^2\wedge dx^3\neq 0$.

The hard step is now to look for the 7 Vessiot structure equations as not a single reference may be found in the whole mathematical literature because of Cartan, Spencer and Kolchin as we already said.

Lemma: When $\alpha'\wedge\beta\neq 0$, there are only two Vessiot structure equations of first kind, namely:

$$\boxed{\alpha\wedge d\alpha=0,\ \ \alpha\wedge d\alpha'=0}$$

Proof. Let $\alpha=\left(\omega^1,\omega^2,\omega^3\right)$, $\alpha'=\left(\omega^4,\omega^5,\omega^6\right)$, $\beta=\left(\omega^7,\omega^8,\omega^9\right)$ with $\alpha\wedge\beta=\omega^1\omega^7+\omega^2\omega^8+\omega^3\omega^9=0$ and $\alpha'\wedge\beta=\omega^4\omega^7+\omega^5\omega^8+\omega^6\omega^9=\sigma\neq 0$. In $\mathbb{R}^3$ let us consider the 2×3 matrix:

$$\begin{pmatrix}\omega^2\omega^6-\omega^3\omega^5 & \omega^3\omega^4-\omega^1\omega^6 & \omega^1\omega^5-\omega^2\omega^4\\ \omega^7 & \omega^8 & \omega^9\end{pmatrix}$$

The right determinant is equal to $\omega^1\left(\omega^5\omega^8-\omega^6\omega^9\right)+\omega^4\left(\omega^3\omega^9+\omega^2\omega^8\right)=\omega^1\left(\omega^4\omega^7+\omega^5\omega^8+\omega^6\omega^9\right)$. Accordingly, the two lines are linearly independent when $\alpha'\wedge\beta\neq 0$. Also, we have

$\mathcal{L}(\xi)\alpha = 0 \Rightarrow \mathcal{L}(\xi)d\alpha = d\mathcal{L}(\xi)\alpha = 0$. It follows that a necessary (but not sufficient!) condition of formal integrability must be, at least, $\alpha \wedge d\alpha = 0$ with a similar result for α', namely $\alpha \wedge d\alpha' = 0$. We point out the fact that, when $n = 3$, the condition $\alpha \wedge \alpha' \neq 0$ is a byproduct because otherwise it amounts to $\alpha' = a(x)\alpha$ and thus $\alpha' \wedge \beta = 0$.

□

More generally, we also obtain $d\alpha = c^1(x)\alpha \wedge \alpha' + c^2(x)\beta$ with $(c^1, c^2) \in \wedge^0 T^*$ and we thus obtain two zero order Medolaghi equations $\mathcal{L}(\xi)c^1 = 0$, $\mathcal{L}(\xi)c^2 = 0$ a result showing that a necessary condition for R_1 to be FI is that $c^1(x) = c^1 = cst$, $c^2(x) = c^2 = cst$. Similarly, we also obtain $d\alpha' = c'^1\alpha \wedge \alpha' + c'^2\beta$. We have thus obtained 2 Vessiot structure equations of first kind (no constant) and 4 Vessiot structure equations of second kind with 4 Vessiot structure constants (c^1, c^2, c'^1, c'^2).

It remains to find out one more CC to reach the 7 CC existing in the Janet sequence already obtained. For this, we notice that the quotient of two 3-forms is a scalar and we must thus have $d\beta = c^3(x)\alpha' \wedge \beta$. Taking the Lie derivative like, we already did and using the fact that $\alpha' \wedge \beta \neq 0$ by assumption, we obtain the new Vessiot structure equation of second kind $d\beta = c^3\alpha' \wedge \beta$.

We have thus obtained $7 - 2 = 5$ Vessiot structure equations of the second class, namely:

$$\boxed{d\alpha = c^1\alpha \wedge \alpha' + c^2\beta,\ d\alpha' = c'^1\alpha \wedge \alpha' + c'^2\beta,\ d\beta = c^3\alpha' \wedge \beta}$$

In the present situation, we have thus obtained the 5 Vessiot structure constants

$$\boxed{c^1 = 1,\ c^2 = 0,\ c'^1 = 0,\ c'^2 = 1,\ c^3 = 0}$$

Finally, closing the exterior systems, we get the two striking quadratic Jacobi identities in a coherent way with the underlying Janet differential sequence:

$$\boxed{c^2\left(c^3 - c^1\right) = 0,\ c^2c'^1 - c^3c'^2 = 0}$$

The next hard step will be to look for the normalizer $N(\Theta)$ without using computer algebra!.

The first task is to look for the Lie group of *label* transformations of the fiber of $\mathcal{F}$ of the form $\bar{u} = g(u, a)$ such that $\bar{R}_1 = R_1$ when $u = \omega(x) \to \bar{u} = \bar{\omega}(x)$. For this, we first notice the relations:

$$\xi_1^1 - \frac{\omega^2\omega^6 - \omega^3\omega^5}{\omega^1\omega^5 - \omega^2\omega^4}\xi_1^3 + \cdots = 0,\ \xi_1^2 - \frac{\omega^1\omega^6 - \omega^3\omega^4}{\omega^1\omega^5 - \omega^2\omega^4}\xi_1^3 + \cdots = 0$$

It follows that we must have $\bar{\alpha} \wedge \bar{\alpha}' = a(x)\alpha \wedge \alpha'$ and thus *necessarily*:

$$\bar{\alpha} = a^1(x)\alpha + a'^1(x)\alpha',\ \bar{\alpha}' = a^2(x)\alpha + a'^2(x)\alpha'$$

However, using the section $\sigma \in \wedge^3 T^*$ of the previous Lemma we have $\sigma\left(\xi_1^1 + \xi_2^2 + \xi_3^3\right) + \xi^r\partial_r\sigma = 0$ and we must thus have $\bar{\sigma} = a^3\sigma$ with $a^3 = cst$, a parameter of the multiplicative group of the real line. This result shows that we must have *necessarily* the relation $\bar{\beta} = a^3\beta + a'^3\alpha \wedge \alpha' \in \wedge^2 T^*$.

Finally, we have

$\bar{\alpha}'\wedge\bar{\beta}=\left(a^1(x)\alpha+a'^1(x)\alpha'\right)\wedge\left(a^3\beta+a'^3\alpha\wedge\alpha'\right)=a'^1a^3\alpha'\wedge\beta=0$ and we must have *necessarily* $a'^1=0$, that is $\bar{\alpha}=a^1\alpha$ as a multiplicative group. We obtain therefore a Lie group of label transformations with 5 parameters $\left(a^1,a^2,a'^2,a^3,a'^3\right)$ with identity $(1,0,0,1,1,0)$:

$$\boxed{\bar{\alpha}=a^1\alpha,\ \bar{\alpha}'=a^2\alpha+a'^2\alpha',\ \bar{\beta}=a^3\beta+a'^3\alpha\wedge\alpha'}$$

In the deformation theory, we have now to look for the effect of a label transformation a on the Vessiot structure constants c. After easy but tedious substitutions we obtain:

$$\boxed{\begin{array}{lcl}
\bar{c}^1 & = & \dfrac{1}{a'^2}c^1-\dfrac{a'^3}{a^2a'^2}c^2\\
\bar{c}^2 & = & \dfrac{a^1}{a^3}c^2\\
\bar{c}'^1 & = & \dfrac{a^2}{a^1a'^2}c^1+\dfrac{1}{a^1}c'^1-\dfrac{a^2a'^3}{a^1a^3a'^2}c^2-\dfrac{a'^3}{a^2a^2}c'^2\\
\bar{c}'^2 & = & \dfrac{a^2}{a^1}c^2+\dfrac{a'^2}{a^3}c'^2\\
\bar{c}^3 & = & \dfrac{1}{a'^2}c^3-\dfrac{a'^3}{a'^2a^3}c^2
\end{array}}$$

Linearizing this framework, we may set $\bar{c}=c+tC+\cdots$ with a small parameter t and:

$$\left\{a^1=1+tA^1+\cdots,a^2=tA^2+\cdots,a'^2=1+tA'^2+\cdots,a^3=1+tA^3,q'^3=tA'^3+\cdots\right\}$$

We obtain thus successively the coboundary space B_1 and cocycle space Z_1 through the formulas:

$$\boxed{B_1\left\{\begin{array}{lcl}
C^1 & = & -A'^2c^1-A'^3c^3\\
C^2 & = & \left(A^1-A^3\right)c^2\\
C'^1 & = & A^2c^1-A^1c'^1-A'^3c'^3\\
C'^2 & = & A^2c^2+\left(A'^2-A^3\right)c'^2\\
C^3 & = & -A'^2c^3-A'^3c^2
\end{array}\right.\subset Z_1\left\{\begin{array}{lcl}
\left(c^3-c^1\right)C^2+c^2\left(C^3-C^1\right) & = & 0\\
c^2C'^1+c'^1C^2-c^3C'^2-c'^2C^3 & = & 0
\end{array}\right.}$$

We check easily that $B_1\subseteq Z_1\subseteq\Upsilon_1\subset F_1$ and may define the cohomology space $H_1=Z_1/B_1$.

In the present example, we obtain by using the structure constants already exhibited:

$$B_1=\left\{C^1=A'^2,C'^1=A'^2-A^2,C^2=0,C'^2=A^3-A'^2,C^3=0\right\}\Rightarrow dim\left(B_1\right)=3$$

$$Z_1=\left\{C^2=0,C^3=0\right\}\Rightarrow dim\left(Z_1\right)=5-2=3\Rightarrow dim\left(H_1\right)=0$$

It follows that $N(\Theta)/\Theta$ is defined by $\left\{A'^2=0,A^3=0,A^2=A'^3\right\}$ or, equivalently by setting $\left\{a'^2=1,a^3=1,a^2=a'^3\right\}$ and has dimension equal to 2, being generated by two vertical vector fields on $\mathcal{F}$, namely:

$$u^1\frac{\partial}{\partial u^1}+u^2\frac{\partial}{\partial u^2},u^3\frac{\partial}{\partial u^3},u^4\frac{\partial}{\partial u^4}+u^5\frac{\partial}{\partial u^5}+u^6\frac{\partial}{\partial u^5}+u^7\frac{\partial}{\partial u^7}+u^8\frac{\partial}{\partial u^8}+u^9\frac{\partial}{\partial u^9}$$

We finally prove that, with the same general structure but with a different Lie pseudogroup and different Vessiot structure constants, the previous results may drastically change.

For this, let us consider the new Lie pseudogroup $\bar{\Gamma}$ with finite transformations:

$$\boxed{y^1=x^1+a,\ \ y^2=x^2+f\left(x^1\right),\ \ y^3=x^3+f'\left(x^1\right)}$$

which seems completely different from the previous one, in particular because, though it also depends on one arbitrary function $f\left(x^1\right)$, now it only depends on $j_1(f)=\left(f\left(x^1\right),f'\left(x^1\right)\right)$.

The infinitesimal Lie equations are solutions of the linear involutive system $\bar{R}_1\subset J_1(T)$ with Janet tabular where the 7 dots denote the non-multiplicative variables:

$$\left\{\begin{array}{lcl|ccc|} \xi^3_3 & = & 0 & 1 & 2 & 3\\ \xi^2_3 & = & 0 & 1 & 2 & 3\\ \xi^1_3 & = & 0 & 1 & 2 & 3\\ \xi^3_2 & = & 0 & 1 & 2 & \bullet\\ \xi^2_2 & = & 0 & 1 & 2 & \bullet\\ \xi^1_2 & = & 0 & 1 & 2 & \bullet\\ \xi^2_1-\xi^3 & = & 0 & 1 & \bullet & \bullet\\ \xi^1_1 & = & 0 & 1 & \bullet & \bullet \end{array}\right.$$

The general solution is the vector field $b\partial_1+f\left(x^1\right)\partial_2+f'\left(x^1\right)\partial_3$ with b an arbitrary constant and $f\left(x^1\right)$ an arbitrary function of x^1.

We may introduce as before the section $\omega=(\alpha,\alpha',\beta)\in\mathcal{F}=T^*\times_X T^*\times_X\wedge^2T^*$ with now:

$$\alpha=dx^1,\ \alpha'=dx^2-x^3dx^1,\ \beta=dx^1\wedge dx^3\Rightarrow d\alpha=0,\ d\alpha'=\beta,\ d\beta=0,\ \alpha\wedge\beta=0$$

but also with $\alpha'\wedge\beta=-dx^1\wedge dx^2\wedge dx^3\neq 0$. The Vessiot tructure constants are now:

$$\boxed{c^1=0,\ \ c^2=0,\ \ c'^1=0,\ \ c'^2=1,\ \ c^3=0}$$

and we have:

$$\boxed{B_1\left\{\begin{array}{lcl} C^1 & = & 0\\ C^2 & = & 0\\ C'^1 & = & -A'^3\\ C'^2 & = & A'^2-A^3\\ C^3 & = & 0\end{array}\right. \quad \subset Z_1\left\{C^3 = 0\right.}$$

We have thus $B_1\subset Z_1$ with

$dim\left(B_1\right)=2, dim\left(Z_1\right)=4\Rightarrow dim\left(H_1\right)=4-2=2$.

We finally obtain $dim\left(N(\Theta)/\Theta\right)=dim\left(\Upsilon_0\right)-dim\left(B_1\right)=5-2=3$, a result that we shall obtain now *directly* through the algebroid point of view, showing thus its importance!.

For any vector field $\eta\in N(\Theta)$ and any vector field $\xi=b\partial_1+f\left(x^1\right)\partial_2+f'\left(x^1\right)\partial_3\in\Theta$, we must have:

$$[\xi,\eta]=c\partial_1+g\left(x^1\right)\partial_2+g'\left(x^1\right)\partial_3$$

Expanding the bracket, we obtain therefore *separately*:

$$\begin{array}{rcl} b\partial_1\eta^1+f\left(x^1\right)\partial_2\eta^1+f'\left(x^1\right)\partial_3\eta^1 & = & c \\ -f'\left(x^1\right)\eta^1+b\partial^1\eta^2+f\left(x^1\right)\partial_2\eta^2=f'\left(x^1\right)\partial_3\eta^2 & = & g\left(x^1\right) \\ -f''\left(x^1\right)\eta^1+b\partial_1\eta^3+f\left(x^1\right)\partial_2\eta^3+f'\left(x^1\right)\partial_3\eta^3 & = & g'\left(x^1\right) \end{array}$$

Using the fact that c is a constant and thus $\partial_i c=0$ for $i=1,2,3$, that $\partial_1 g\left(x^1\right)-g'\left(x^1\right)=0$ and that $\partial_2 g\left(x^1\right)=0$, $\partial_3 g\left(x^1\right)=0$, we obtain after elementary but very tedious substitutions, the 5 first order PD Lie equations for η with jet notations:

$$\eta_1^1-\eta_2^2+\eta_3^3=0,\ \eta_2^1=0,\ \eta_3^1=0,\ \eta_3^2=0,\ \eta_2^3=0$$

As the symbol g_1 of R_1 has dimension $dim\left(g_1\right)=dim\left(g_2\right)=\cdots=1$ we have $\tilde{g}_2=g_2$ which is defined by the following $18-1=17$ second order equations leading to $dim\left(\tilde{g}_2\right)=1$ with the only parametric jet $\left(\xi_{11}^3\right)$:

$$\begin{array}{l} \eta_{12}^3=0,\eta_{13}^3=0,\eta_{22}^3=0,\eta_{23}^3=0,\eta_{33}^3=0 \\ \eta_{11}^2-\eta_1^3=0,\eta_{12}^2=0,\eta_{13}^2=0,\eta_{22}^2=0,\eta_{23}^2=0,\eta_{33}^2=0 \\ \eta_{11}^1=0,\eta_{12}^1=0,\eta_{13}^1=0,\eta_{22}^1=0,\eta_{23}^1=0,\eta_{33}^1=0 \end{array}$$

A tedious integration finally provides the general solution:

$$\boxed{\eta=\left(ax^1+b\right)\partial_1+\left(g\left(x^1\right)+(a+c)x^2\right)\partial_2+\left(g'\left(x^1\right)+cx^3+d\right)\partial_3}$$

with the arbitrary function $g\left(x^1\right)$ and the 4 additional constant parameters (a,b,c,d) instead of the only d as claimed. It follows that $Z_0=N(\Theta)/\Theta$ is generated by the three infinitesimal generators:

$$\left\{x^1\partial_1+x^2\partial_2,?x^2\partial_2+x^3\partial_3,?\partial_3\right\}$$

A similar but quite easy procedure may be use in order to determined $C(\Theta)$ by writing out $[\xi,\eta]=0$ for any $\xi\in\Theta$ in order to obtain $\eta=c\partial_2+d\partial_3$ for two constants (c,d). It follows that $Z(\Theta)=C(\Theta)\cap\Theta$ is generated by ∂_2 and $B_0=C(\Theta)/Z(\Theta)$ is thus finally generated by ∂_3. We obtain therefore $dim\left(H_0\right)=3-1=2\neq 0$.

EXAMPLE 5.8: (*Contact transformations*) This Lie pseudogroup is known to play an important role in analytical mechanics thanks to the work of Vessiot on the Hamilton-Jacobi equation that we have explained in [3]. The main difficulty involved in this elementary situation is that the so-called given *contact 1-form* α cannot be used as an underlying geometric object because it is only invariant up to a factor $\rho(x)$. We have thus indeed a Lie pseudogroup defined by only two first order PD equations after eliminating the factor but the system obtained is not involutive at all. The main difficulty is thus to use the "*up and down*" procedure in order to obtain therefore a new first order system of involutive Lie equations while proving that the corresponding geometric object is no longer a 1-form but becomes a 1-form density ω. Such a situation is not so well known though it also exists in general relativity and electromagnetism (See [35] for details).

With $m=n=3, q=1, K=\mathbb{Q}\left(x^1,x^2,x^3\right)$ or simply $\mathbb{Q}(x)$, we may introduce the 1-form $\alpha = dx^1 - x^3 dx^2 \in T^*$ and consider the Lie pseudogroup of transformations preserving α up to a function factor defined by $j_1(f)^{-1}(\alpha)=\rho(x)\alpha$. Eliminating the factor ρ and linearizing at the identity, we obtain a first order system $R_1 \subset J_1(T)$ which is not even formally integrable and must use the PP procedure to get the new involutive system $R_1^{(1)} \subset R_1 \subset J_1(T)$:

$$\begin{cases} \partial_3\xi^3 + \partial_2\xi^2 - \partial_1\xi^1 + 2x^3\partial_1\xi^2 & = \eta^3 \\ \partial_3\xi^1 - x^3\partial_3\xi^2 & = \eta^2 \\ \partial_2\xi^1 - x^3\partial_2\xi^2 + x^3\partial_1\xi^1 - (x^3)^2\partial_1\xi^2 - \xi^3 & = \eta^1 \end{cases} \quad \begin{array}{|ccc|} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ \hline \end{array}$$

There is thus one CC of order 1 described by the formally surjective first order operator $\mathcal{D}_1$:

$$d_3\eta^1 - d_2\eta^2 - x^3 d_1\eta^2 + \eta^3 = \zeta$$

Multiplying $\left(\eta^1,\eta^2,\eta^3\right)$ by the test functions $\left(\mu^1,\mu^2,\mu''\right)$ and integrating by parts, we obtain (*by chance*!) the involutive operator $ad(\mathcal{D})$:

$$\begin{cases} \partial_3\mu^3 + \mu^1 & = -\nu^3 \\ \partial_3\mu^2 + x^3\partial_1\mu^1 + \partial_2\mu^1 - \partial_1\mu^3 & = -\nu^1 \\ \partial_2\mu^3 + x^3\partial_1\mu^3 - \mu^2 & = -(\nu^2 + x^3\nu^1) \end{cases} \quad \begin{array}{|ccc|} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ \hline \end{array}$$

providing the only first order CC:

$$\boxed{ad\left(\mathcal{D}_{-1}\right):\left(\nu^1,\nu^2,\nu^3\right) \to x^3\partial_3\nu^1 + 2\nu^1 + \partial_3\nu^2 - \partial_2\nu^3 - x^3\partial_1\nu^3 = 0}$$

and the classical injective parametrization operator $\mathcal{D}_{-1}$:

$$\boxed{-x^3\partial_3\phi + \phi = \xi^1, -\partial_3\phi = \xi^2, \partial_2\phi + x^3\partial_1\phi = \xi^3 \Rightarrow \xi^1 - x^3\xi^2 = \phi}$$

Having in mind the Vessiot structure equations, we notice that α is not invariant by the contact Lie pseudogroup. The associated invariant geometric object is a 1-form density ω leading to the system of infinitesimal Lie equations in Medolaghi form:

$$\boxed{\Omega_i \equiv \left(\mathcal{L}(\xi)\omega\right)_i \equiv \omega_r\partial_i\xi^r - \frac{1}{2}\omega_i\partial_r\xi^r + \xi^r\partial_r\omega_i = 0}$$

and to the *only* Vessiot structure equations, *still not known today*:

$$\boxed{\omega_1\left(\partial_2\omega_3 - \partial_3\omega_2\right) + \omega_2\left(\partial_3\omega_1 - \partial_1\omega_3\right) + \omega_3\left(\partial_1\omega_2 - \partial_2\omega_3\right) = c}$$

with the *only* structure constant c. In the present contact situation, we may choose $\omega = \left(1, -x^3, 0\right)$ as we did and get $c=1$ but we may also choose

$\omega=(1,0,0)$ and get $c=0$. This new choice is also bringing an involutive system:

$$\boxed{-2\Omega_1 \equiv \partial_3\xi^3+\partial_2\xi^2-\partial_1\xi^1=0,\ \Omega_2 \equiv \partial_2\xi^1=0,\ \Omega_3 \equiv \partial_3\xi^1=0}$$

having the only CC $d_2\Omega_3-d_3\Omega_2=0$. However, ξ^1 is indeed a torsion element and one cannot find a parametrization. Such an example is thus proving that the existence of a parametrization for systems of Lie equations highly depends on the Vessiot structure constants.

6. Conclusions

E. Vessiot discovered the so-called *Vessiot structure equations* as early as 1903 and, only one year later, E. Cartan discovered the so-called *Cartan structure equations*. Both structure equations depend on a certain number of constants like the single *geometric structure constant* of the constant Riemannian curvature for the first and the many *algebraic structure constants* of Lie algebra for the second.

However, Cartan and followers never acknowledged the existence of another approach which is therefore still totally ignored today, in particular by physicists. Now, it is well known that the structure constants of Lie algebra play a fundamental part in the Chevalley-Eilenberg cohomology of Lie algebras and their deformation theory. Certain counterexamples only existing in high dimensions with about 500 structure constants provided one of the first applications of computer algebra in mathematics around 1970.

It was thus a challenge to associate the strange Vessiot structure constants with other homological properties related to systems of Lie equations, namely the extension modules determined by Lie operators [38]. As a striking consequence, such a possibility opens a new way to understand and revisit the various contradictory works done during the last fifty years or so by different groups of researchers, using Cartan, Gröbner or Janet bases respectively while looking for a modern interpretation of the work done by C. Lanczos from 1938 to 1962. However, the reader must not forget that the Weyl tensor was not known by Lanczos, even as late as 1967 because Lanczos was invited to lecture in France by my PhD supervisor A. Lichnerowicz in 1962. Also, it was not possible to discover any solution of the parametrization problem by potentials through double duality before 1990/1995, that is too late for the many people already engaged in this type of research [39].

As a byproduct, the Bianchi operator generates the CC of the Riemann operator and the Beltrami operator, which is the adjoint of the Riemann operator, generates the CC of the Lanczos operator, which is the adjoint of the Bianchi operator [35]. Meanwhile, the Cauchy operator, adjoint of the Killing operator is parametrized by the Beltrami operator in the following sequence and its adjoint sequence in arbitrary dimension n along the differential *double duality* technique of differential homological algebra [38] [39]:

$$\begin{array}{ccccccccc} & & n & \underset{1}{\overset{Killing}{\rightarrow}} & \frac{n(n+1)}{2} & \underset{2}{\overset{Riemann}{\rightarrow}} & \frac{n^2(n^2-1)}{12} & \underset{1}{\overset{Bianchi}{\rightarrow}} & \frac{n^2(n^2-1)(n-2)}{24} \\ 0 & \leftarrow & n & \underset{1}{\overset{Cauchy}{\leftarrow}} & \frac{n(n+1)}{2} & \underset{2}{\overset{Beltrami}{\leftarrow}} & \frac{n^2(n^2-1)}{12} & \underset{1}{\overset{Lanczos}{\leftarrow}} & \frac{n^2(n^2-1)(n-2)}{24} \end{array}$$

It follows that the equations used for defining gravitational waves in any textbook like [40] are nothing else than the adjoint of the Ricci operator as explained in [24]-[26] [35] [41] [42] and Einstein's general relativity is thus confusing the div operator induced by the Bianchi operator with the Cauchy operator (!).

We finally hope that this paper will open new domains for applying computer algebra while offering a collection of useful test examples.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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