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On Möbius Transformations as Polya Vector Fields from Lie Theory Viewpoint

Lucian M. Ionescu[1], Gabriel-Teodor Pripoae[2], Cristina-Liliana Pripoae[3]

[1]Department of Mathematics, Illinois State University, Normal, USA
[2]Faculty of Mathematics and Computer Science, University of Bucharest, Bucharest, Romania
[3]Department of Applied Mathematics, The Bucharest University of Economic Studies, Bucharest, Romania
Email: lmiones@ilstu.edu, gpripoae@yahoo.com, cristinapripoae@csie.ase.ro

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Abstract

This expository article connects the conformal geometry of Möbius transformations with their kinematic interpretation as Pólya vector fields. We review the vector-calculus perspective of holomorphic functions, establishing a rigorous analytic-to-dynamic dictionary that maps complex analytic objects to conservative and solenoidal 2D physical flows. Leveraging the Lie theory of $SL_2(\mathbb{C})$, we explicitly compute the one-parameter subgroups and their associated invariant vector fields, demonstrating how Möbius transformations can be understood dynamically as superpositions of elementary flows via the adjoint representation. Finally, as a motivational outlook for future research, we contextualize these geometric and Lie-theoretic results within a broader mathematical physics framework. We discuss heuristic analogies with 2D-electromagnetism, Hopf fibrations, and the Cayley-Dickson doubling construction as structural stepping stones toward higher-dimensional gauge theories.

Keywords

Mobius Transformations, Polya Vector Fields, Lie Theory, 2D-Electromagnetism, Fluid Dynamics

1. Introduction

This is an expository article relating well-known facts belonging to Complex Analysis and Symplectic Mechanics, well known to be part of a "trinity" [1], aiming to clarify the perspective of (Sophus) Lie Theory from the geometric point of view of Felix Klein [2][1]. While it is part of a larger program relating Math-Physics theories started in [3], it is self-contained when focusing on the understanding of the struc-

[1]It is well known the productive collaboration of these two influential mathematicians, loc. cit.

ture of the 1-parameter groups of Mobius transformations, and the interplay between the conformal geometry of complex transformations and symplectic-Hamiltonian mechanics aspects, at an elementary level. There are numerous "digressions", as "links" to further developments the reader may choose to ignore at this stage [4] [5].

Before proceeding to the technical aspects presented, a few general considerations allow to place the article in a larger context.

1.1. The Conceptual Network Methodology

It is an attempt to build a hierarchic *Conceptual Network* relating not just concepts, but also the *Theories* they belong to, in a mutually beneficial way, hopefully allowing for a better understanding of their roles and fostering new developments and applications.

$$\underline{\text{Complex Geometry}} \quad \overset{\text{Kahler}}{\Leftrightarrow} \quad \underline{\text{Symplectic Geometry}}$$

- $\begin{matrix}\text{Conformal/Projective}\\ \mathrm{U}(1)-\text{Equivariant}\end{matrix} \xrightarrow{\text{Polya's prescription}} \text{Kahler Dynamics}$
- Lie-Cartan Theory $\qquad$ Conformal/Analytic Flows.

1.2. Motivational Outlook: Physics and Advanced Geometries

While the exact derivations in this article are self-contained, they are motivated by a larger program relating these mathematical structures to theoretical physics. Specifically, the Hamiltonian mechanics utilized here rely on what we term an *accidental Lagrangian structure*. When constructing the complex numbers via the Cayley-Dickson doubling procedure (rather than as a simple axiomatic field extension), the resulting space $\mathbb{C}$ canonically inherits the structure of a cotangent bundle phase-space, $\mathbb{C} \cong T^*\mathbb{R}$. From a purely abstract-algebraic perspective, this phase-space geometry appears "accidental", yet it is structurally essential for hosting the Hamiltonian dynamics of the conformal flows.

This profound equivalence between the complex-analytic properties of holomorphic functions and the symplectic-geometric properties of their flows serves as a localized, pointwise analog for *Quantum Mirror Symmetry*—the theoretical framework in modern physics that establishes dualities between complex geometries (A-models) and symplectic geometries (B-models).

We emphasize that references to Quantum Mirror Symmetry, Noether's Theorem, Hopf fibrations, and the Standard Model within this text (particularly in Section 4) are provided as a heuristic, motivational outlook. They illustrate how the exact 2D structures derived here serve as foundational stepping stones toward higher-dimensional gauge theories.

Returning to the main topic of the article, recall that Möbius transformations (MT) play a central role in Complex Analysis and in Physics, via their connection to the group of Lorentz transformations. This lifts to the *Kähler triangle*, the abstract linear version of Kähler structures of manifold theory:

$$\begin{tikzcd}
 & \mathit{Complex\ structure}\ (\mathbb{C}, J) \arrow[dl, "\mathit{Symmetric}"'] \arrow[dr, "\mathit{Antisymmetric}"] \arrow[d, phantom, "\text{Kähler Triangle}"] & \\
\mathit{Metric}\ (R^2, g) \arrow[rr, leftrightarrow] & & \mathit{Symplectic\ form}\ (T^*R, \omega)
\end{tikzcd}$$

where the Galois group $\{Id, \mathbb{C}\}$ as Klein geometry structure plays a crucial role linking *Complex Analysis* and *Symplectic Dynamics*, which briefly will be referred to by $\mathbb{C} = D(\mathbb{R}) = T^*\mathbb{R}$, where the Galois group $\{Id, \mathbb{C}\}$ as Klein geometry structure plays a crucial role linking *Complex Analysis* and *Symplectic Dynamics*, which briefly will be referred to by $\mathbb{C} = D(\mathbb{R}) \cong T^*\mathbb{R}$ [2].

It is a rich structure resulting from the Cayley-Dickson construction, which extends the traditional viewpoints derived from Abstract Algebra and Klein Geometry, which is mentioned in the **Appendix**, to be expanded elsewhere ([4]; see also [5]).

This allows to integrate constructively Lie Theory of $SL(2, \mathbb{C})$, with Cartan Theory of Roots Systems associated to the adjoint representation, with the 2nd Hopf fibration $S^1 \to S^3 \to S^2$, viewed as a Cartan-Klein symmetric space $S^1 \to SU(2) \to S^2$.

Comparing the Polya physics viewpoint, which corresponds to Hamiltonian Dynamics, with Klein-Lie Theory analysis of MT, yields a better understanding of their role as "dynamic transformations", rather then just conformal transformations. By *dynamic transformations* we will refer to mathematical transformation with a Hamiltonian/Symplectic Geometry interpretation due to the *physics related context* of their origin, as a reacher structure due to the respective correspondence.

The main result ties the 1-parameter subgroup $\exp(tX) = \mathfrak{H}^t(z)$ and invariant vector field, of a Möbius transformation $f(z)$ lifted as an $SL_2(\mathbb{C})$ matrix $\mathfrak{H}$ [6]:

$$f(z) = (az + b)/(cz + d), \quad \mathfrak{H} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{1}$$

with the equipotentials and streamlines of the associated *Polya Vector Field* (PVF):

$$V_f(x, y) = \mathrm{Re}(f(z))\frac{\partial}{\partial x} - \mathrm{Im}(f(z))\frac{\partial}{\partial y}. \tag{2}$$

Part of the motivation comes from better documenting the explanations and example from [7], documentation which seems to be missing from the standard literature on the subject. While some elementary, perhaps pedagogically useful points are made on the way, applications to Special Relativity and Quantum Physics are in view, via the connection with Lorentz transformations and Hopf fibration, a "pointwise model" in Gauge Theory.

The article is organized as follows. A background on Möbius transformations is presented in Section §2. Our main result is Theorem 3.1 in Section 3, presenting

[2]The consequence of these isomorphisms and shift of perspective from algebraic to geometric, is a *Lagrangian structure* that we called it "accidental" in §1.2, but essential to the *Kahler triangle* structure.

the Lie Theory viewpoint. It states the relation between the *conformal geometry* of Möbius transformations and their 1-parameter group dynamics.

Generalizations and applications are deferred to the concluding Section 4. **Appendix** provides a preview of our interpretation of the role of the Cayley-Dickson construction, resulting into an additional insight regarding the importance of quaternions and octonions, to be dealt with in detail elsewhere [4][3].

2. Möbius Transformations: Fields or Transformations?

Complex numbers have both an algebraic role as a field extension $\mathbb{R}[i]$, with the familiar cartesian coordinates representation of its elements as linear combinations $z = x + iy$, as well as a *geometric role* as the *fundamental representation* of the *linear conformal group* on real vector space $\mathbb{R}^2$:

$$\rho : (\mathbb{C}^*, \cdot) \to Aut_{\mathbb{R}}(\mathbb{R}^2)$$

acts via scalar multiplication and rotations. The polar decomposition reflects (is derived from) the canonical isomorphism $PD : \mathbb{R}_+ \times SO(2) \to (\mathbb{C}^*, \cdot)$. The relation between the two structures is provided by the exponential:

$$\exp : (\mathbb{C}, +) \to (\mathbb{C}^*, \cdot).$$

The 1-point compactification of the complex numbers yields the Riemann sphere, canonically isomorphic with the complex projective line $\mathbb{C}P^1 = \mathbb{C} \times \mathbb{C} /_{\sim}$, and allowing to freely "move" from the complex plane, to the 2D-sphere and spinorial representation, via the stereographic projection and Hermitian model, via $SL_2(\mathbb{C})$, if needed.

2.1. Physics Interpretations and the Analytic-to-Dynamic Dictionary

To streamline the transition from complex analysis to vector calculus and kinematics, we consolidate the classical correspondences into a single, comprehensive framework, based on the framework introduced by the renowned mathematician George Pólya and his colleague Gordon E. Latta in their classic 1974 textbook on complex variables [8].

Theorem 2.1 (Analytic-Kinematic Equivalence). *Let* $f(z) = u(x, y) + iv(x, y)$ *be a complex function defined on a domain in* $\mathbb{C}$. *The analyticity of* f (*satisfaction of the Cauchy-Riemann equations*) *is strictly equivalent to its associated Polya Vector Field* $V_f = u\frac{\partial}{\partial x} - v\frac{\partial}{\partial y}$ *being simultaneously irrotational* ($\mathrm{curl}(V_f) = 0$) *and divergence-free* ($\mathrm{div}(V_f) = 0$).

Under these conditions, V_f *locally admits a complex potential* $F(z) = \phi(x, y) + i\psi(x, y)$ *such that* $F'(z) = f(z)$. *The level curves of the real harmonic potential* ϕ (*equipotentials*) *and the harmonic conjugate* ψ (*streamlines*) *form orthogonal families that govern the geometry of the flow. Furthermore,*

[3]Comments from the reader are welcomed!

the complex line integral over any closed contour C *unifies the classical dynamic quantities*:

$$\int_C f(z)\mathrm{d}z = \mathrm{Work}(V_f;C) - i\,\mathrm{Flux}(V_f;C). \quad (3)$$

To make these correspondences explicit, we define the following Analytic-to-Dynamic Dictionary, mapping complex variables to their vector calculus interpretations via the basis of Wirtinger derivatives ($\partial_z, \partial_{\bar{z}}$) and 2D-Frenet moving frames (See Table 1).

Table 1. Analytic-to-Dynamic dictionary mapping complex analysis structures to vector calculus equivalents.

Analytic Object	Dynamic/Geometric Interpretation
$f(z)$ holomorphic	V_f is an irrotational, solenoidal (conservative) field.
$f(z)$ anti-holomorphic	The position vector field P_f is irrotational and solenoidal.
$V_f = 2\,\mathrm{Re}\left(f(z)\frac{\partial}{\partial \overline{z}} \right)$	$u\partial_x - v\partial_y$: Pólya vector field (Velocity/Force flow).
$P_f = 2\,\mathrm{Re}\left(f(z)\frac{\partial}{\partial z} \right)$	$u\partial_x + v\partial_y$: Position vector field.
W_f, U_f	$v\partial_x + u\partial_y$, $v\partial_x - u\partial_y$: Pauli-matrix transformed basis flows.
$\alpha_f = \mathrm{Re}(f(z)\mathrm{d}z)$	Tangential 1-form evaluated as $V_f \cdot \mathrm{d}r$ (yields Work).
$\beta_f = \mathrm{Im}(f(z)\mathrm{d}z)$	Normal 1-form evaluated as $V_f \times \mathrm{d}r$ (yields -Flux).
$F(z) = \phi(x,y) + i\psi(x,y)$	Complex Potential ($V_f = \nabla\phi$, ψ generates streamlines).
$\int_C \alpha_f = \int_C (V_f)_T\, \mathrm{ds}$	Work (or Circulation) of the vector field along contour C .
$\int_C \beta_f = -\int_C (V_f)_N\, \mathrm{ds}$	Flux of the vector field across contour C .

Remark **2.1.** The four vector fields (V_f, W_f, U_f, P_f) provide a complete local basis representation of the complex function via combinations analogous to Pauli matrices. By utilizing the 2D-Frenet moving frame along a curve C , the work and flux map directly to the restrictions of the dual 1-forms α_f and β_f . If the flow is "straightened" locally via the conformal transformation $F(z)$ (away from critical points), it becomes purely horizontal ($F_*(V)(\phi,\psi) = \psi$).

2.2. Möbius Transformations and Their Associated Polya Vector Fields

In this subsection we exemplify the previous qualitative considerations in canonical Cartesian coordinates.

Fix four complex numbers $a := a^1 + a^2 i$, $b := b^1 + b^2 i$, $c := c^1 + c^2 i$, $d := d^1 + d^2 i$ such that $ac - bd \neq 0$. Denote $\tilde{\mathbb{C}} := \mathbb{C} \setminus \left\{ -\frac{d}{c} \right\}$.

Consider the complex (holomorphic) function $f : \tilde{\mathbb{C}} \to \mathbb{C}$, $f(z) = \frac{az+b}{cz+d}$. Without restraining the generality, we can suppose $ac - bd = 1$, as f is invariant to (non-null) scalar multiplication of the constants.

Notation. We consider the associated matrix $\mathfrak{H} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\sigma := (\text{trace } \mathfrak{H})^2$. Denote, as previously, by $u(x,y) := \operatorname{Re}(f(z))$ and $v(x,y) := \operatorname{Im}(f(z))$. Then

$$u(x,y) = \frac{(b^1 + a^1 x - a^2 y)(d^1 + c^1 x - c^2 y) + (b^2 + a^2 x + a^1 y)(d^2 + c^1 y + c^2 x)}{(d^1 + c^1 x - c^2 y)^2 + (d^2 + c^1 y + c^2 x)^2}$$

$$v(x,y) = \frac{-(b^1 + a^1 x - a^2 y)(d^2 + c^1 y + c^2 x) + (b^2 + a^2 x + a^1 y)(d^1 + c^1 x - c^2 y)}{(d^1 + c^1 x - c^2 y)^2 + (d^2 + c^1 y + c^2 x)^2}$$

are rational functions, as quotients of two polynomials of degree 2 with real coefficients.

***Remark* 2.2.** The potentials ϕ and ψ of the Polya vector field $V_f = u\frac{\partial}{\partial x} - v\frac{\partial}{\partial y}$ can be obtained (through tedious calculations) from the differential equations

$$\frac{\partial \phi}{\partial x} = u, \; \frac{\partial \psi}{\partial x} = -v.$$

By partial fraction decomposition, we get by quadratures the potentials to be "elementary" functions (sums/products/compositions of rational functions, trigonometric functions and logarithmic functions).

In complex notation, the complex potential of a Möbius transformation $f(z) = \frac{az+b}{cz+d}$ is (generically)

$$F(z) = \frac{a}{c}z + \frac{bc-ad}{c^2} \cdot \ln(cz+d) + \alpha,$$

where α is a complex constant.

Instead, the complex potential of the associated Polya vector field V_f is (generically)

$$\Theta(\bar{z}) = \frac{\bar{a}}{\bar{c}}\bar{z} + \frac{\bar{b}\bar{c} - \bar{a}\bar{d}}{\bar{c}^2} \cdot \overline{\ln(cz+d)} + \bar{\alpha}.$$

***Remark* 2.3.** (i) There exists a well-known classification of MTs, into three disjoint families [9] (Corresponds to the type of fixed points §2.5):

- *Parabolic type*, when $\sigma = 4$;
- *Elliptic type*, when $\sigma \in [0,4)$;
- *Loxodromic type*, when σ is (complex but) not in $[0,4]$.

A particular case of the loxodromic type is the *hyperbolic* one, when $\sigma > 4$. A particular case of the elliptic type is the *circular* one, for $\sigma = 0$.

(ii) Of interest are also the MTs with all constants real numbers (this is equivalent with all constants imaginary numbers!—hence some duality).

***Remark* 2.4.** Denote

$$\mathbb{A} = \left\{ (z,a,b,c,d) \in \mathbb{C}^5 \mid z \neq -\frac{d}{c}, ad - bc \neq 0 \right\}$$

and define $F:\mathbb{A}\to\mathbb{C}^6$, $F(z,a,b,c,d)=\left(z,a,b,c,d,\frac{az+b}{cz+d}\right)$. Then F defines a (parameterized) complex hypersurface in $\mathbb{C}^6$, as the union of the graphs of all the Möbius transformations. It is a disjoint union of patches, each one corresponding to a special case of the classification in the previous Remark 2.7. Properties of geometric invariants of F (fundamental forms, curvature(s), geodesics) can establish correspondences between families of such complex hypersurfaces and specific subsets (geometric loci) of $\mathbb{A}$.

The image of F parameterizes the set $MT\times\mathbb{C}^*$, where MT is the set of all Möbius transformations. It corresponds to a restriction of the map
$GL_2(\mathbb{C})\to C^\infty(\mathbb{C})$.

2.3. Towards a Lie Theory Viewpoint of Möbius Transformations

The set of Möbius transformations is a 6-dimensional real Lie group, isomorphic with the Lorentz group $SO(3,1)$ and denoted with $PGL(2,\mathbb{C})=:G$. We denote its Lie algebra with $L(G)$.

2.3.1. Projective Geometry and Group

Note that $PGL(2,\mathbb{C})$ is the same as $PSL_2(\mathbb{C})$, and the quotient map is a 2:1 covering map (see (1)):

$$SL_2(\mathbb{C})\to PSL_2(\mathbb{C}),\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\mapsto f(z)=(az+b)/(cz+d),$$

which provides a convenient "handle" on Möbius transformations. Hence its Lie algebra is canonically isomorphic to $sl_2(\mathbb{C})$, the Lie algebra $sl_2(C)$ of $SL_2(\mathbb{C})$, of complex traceless matrices:

$$sl_2(\mathbb{C}): A=\begin{bmatrix} t & x+iy \\ x-iy & -t \end{bmatrix}, tr(A)=0. \tag{4}$$

In general, the exponential map $\exp: L(G)\to G$ sends left invariant vector fields on G (elements from L(G)) in points of G (Möbius transformations).

Usually, one considers the following *Cartan basis* on $L(G)$:

$$X^+=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ H=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ X^-=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{5}$$

with

$$\left[X^+,X^-\right]=H,\ \left[H,X^+\right]=2X^+,\ \left[H,X^-\right]=-2X^-.$$

The complex 3-dimensional Lie algebra $L(G)$ is simple, hence semi-simple.

Remark **2.5.** This is the same Lie algebra (structure constants) as for its compact form $g=su_2$ with Cartan maximal abelian subalgebra $h=U(1)$; in other words, $sl_2(\mathbb{C})$ is the complexification of su_2.

2.3.2. Connections with Physics

For further applications planned in subsequent work, relating the *Electroweak Theory* (EWT; spinors and $U(1)_{EM}\times SU(2)_L$ gauge group) with *Quantum*

Computing framework (qubits and quantum gates), we provide a brief interpretation here, in the following two Remarks. The role of complexification mentioned here, will be documented elsewhere from the *Hodge structure* point of view [10].

N. B. This subsection can be skipped by the reader interested mainly in the mathematical aspects discussed in this article, as speculative and providing only suggestions for further research in Mathematical-Physics targeting EPP.

***Remark* 2.6.** $SU(2)$, with its fundamental representation on $\mathbb{C}\times\mathbb{C}$ (spinors $Spin(3)\cong SU(2)$), comes with a canonical basis in $\mathbb{C}^2$ of qubits $z=(z_1,z_2)$, $|z_1|^2+|z_2|^2=1$. The isoclinical embedding $\lambda:U(1)\to\mathbb{C}\times\mathbb{C}$, $\lambda(e^{i\theta})=(e^{i\theta},e^{-i\theta})$ of $U(1)$ (EM's gauge group) in $SU(2)$, corresponding to the Hopf fibration $U(1)\to SU(2)\to S^2$, provides a natural Cartan subalgebra $h=\langle H\rangle$, having the above H as a generator. Then the restriction of the adjoint action $ad_h:h\to End(g)$ determines uniquely $X^{\pm}$ as eigenvectors for ad_H, with the above eigenvalues, as in the theory of Cartan of root systems, here for the Lie algebra $sl_2(\mathbb{C})$ of rank 1.

***Remark* 2.7** Note also that the EWT distinguishes a different torus $U(1)_Y\cong U(1)_{EM}$ then our $U(1)_{EM}\cong U(1)$ isoclinic embedding[4]; in EWT the so called Weinberg angle $\theta\approx\pi/6$ is defined, to account for the break of symmetry corresponding to isospin representation [11] of $SU(2)_L$, on u/d quark types of fractional electric charge +2/3 and −1/3[5]. We further suspect it is related to the winding number of $U(1)$ on a Hopf torus with ratio 2:1 as suggested by H. Jehle [12], idea to be investigated elsewhere.

2.4. The 2D-EM Analogy for Möbius Transformations

To clarify the heuristic use of the terms "electric" and "magnetic" in our discussion of conformal flows, we establish an explicit interpretive analogy between (3 + 1)D Electromagnetism (EM) and a simplified 2D-EM framework. This parallel is mathematically rooted in the correspondence between the polar decomposition of the conformal group $(\mathbb{C}^\times,\cdot)$ acting on $(\mathbb{C},+)$, and the Hopf fibration resulting from the adjoint representation applied to the conformal transformations $SL(2,\mathbb{C})$ acting on the spinor space $\mathbb{C}^2$. This is relevant to the emergence of Space-Time Physics from the Quantum Computing spinorial Physics, as documented in [13], beyond the mathematical correspondence between Möbius Transformations and Lorentz Transformations.

Analytically, applying Green's Theorems to the Pölya vector field V_f of a holomorphic function $f(z)$ near a zero or pole allows us to interpret the Cauchy integral period as a dual entity. It behaves simultaneously as an "electric" charge, *i.e.* a 2D analog to Gauss's Theorem for flux in 3D, and a "magnetic" charge, as a 2D analogue of Stokes' Theorem evaluating circulation around a virtual spin z-

[4]All Cartan tori are conjugate.

[5]Here the term "electric" is used to refer to sources of divergence, which for analytic maps localize topologically at zeros and poles. When considering the orthogonal family of equipotential lines, these singularities can be viewed as sources of *magnetic* field lines, with a corresponding associated circulation.

axis.

Within this interpretive analogy, the divergent streamlines emanating from or sinking into a singularity represent a 2D electric field, quantified by its flux. Conversely, the conjugate vector field tangent to the orthogonal family of closed equipotential lines represents the magnetic lines of force. This is quantified by its circulation, acting as an analog to the vector potential in 3D.

***Remark* 2.8.** This is reminiscent of a fluxon associated to the so-called Abrikosov vortices. Note that the so-called skyrmion textures [14] do not have singularities, but would be more general 3D-manifestations of the Hopf-like fibrations with different indexes, worth studying as a stand-alone topic, with perhaps some insight from our physics considerations.

To cleanly separate the rigorous mathematical statements from these heuristic physical analogies, we provide the mapping dictionary in **Table 2**.

Table 2. Dictionary mapping rigorous complex analysis structures to heuristic 2D-electromagnetism analogies.

Rigorous Mathematical Statement	**Heuristic 2D-EM Physics Analogy**
Singularities (zeros and poles) of $f(z)$	Point sources: "Electric" and "Magnetic" charges (e.g., monopoles/dyons).
Divergent streamlines of V_f	"Electric" field lines governing radial attraction/repulsion.
Flux integral $\int_C (V_f)_N \,\mathrm{d}s$	Gauss's Law measuring enclosed "electric" charge.
Orthogonal closed equipotential lines	"Magnetic" lines of force/Vector potential level curves.
Work/Circulation integral $\int_C (V_f)_T \,\mathrm{d}s$	Stokes' Theorem measuring enclosed "magnetic" charge or spin.
Complex integral period $\int_C f(z)\mathrm{d}z$	Total unified electromagnetic charge (complex sum of flux and circulation).
Helmholtz decomposition (Divergence and Curl components)	Fundamental phenomenological coupling of electric and magnetic fields.

An important pedagogical lesson from this analogy is that these "electric" and "magnetic" counterparts naturally emerge together. Just as the divergence and curl are inextricably linked as complementary components in the Helmholtz decomposition of a vector field, the real and imaginary parts of the complex potential inevitably couple the flux and the circulation. This mathematical necessity provides an elegant structural parallel to observations in Elementary Particle Physics, where any fundamental fermion inherently possesses both an electric charge and a magnetic moment correlated with its quantum spin.

2.5. Normal Form of a Mobius Transformation

In our case, denote $\mathfrak{H} = \exp(X)$, where $X \in sl_2(\mathbb{C})$ and $\mathfrak{H} \in Sl_2(\mathbb{C})$. To make this correspondence explicit, it is convenient to consider the normal form

$\mathfrak{H}(k;z_1,z_2)$ of a Möbius transformation, as recalled below (see also [6]):

(i) In the non-parabolic case, there exist two distinct fixed points z_1 and z_2. Denote $k = re^{i\theta}$ the *characteristic constant*. If the two fixed points are at finite range, then

$$\mathfrak{H}(k;z_1,z_2) = \begin{bmatrix} z_1 - kz_2 & (k-1)z_1z_2 \\ 1-k & kz_1 - z_2 \end{bmatrix}.$$

If z_1 is at finite range and z_2 at infinity, then

$$\mathfrak{H}(k;z_1,\infty) = \begin{bmatrix} k & (1-k)z_1 \\ 0 & 1 \end{bmatrix}.$$

(ii) In the parabolic case, denote z_1 the unique fixed point, with multiplicity 2, and by β the translation length. The characteristic constant is 1. If z_1 is at finite range, then

$$\mathfrak{H}(\beta;z_1) = \begin{bmatrix} 1+\beta z_1 & -\beta(z_1)^2 \\ \beta & 1-\beta z_1 \end{bmatrix}.$$

If z_1 is at infinity, then

$$\mathfrak{H}(\beta;\infty) = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}.$$

2.6. Conjugacy Classes of Möbius Transformations

It is instrumental in "reducing" a non-parabolic MT, having distinct fixed points, to an isometry relative to the canonical metric on the Riemann sphere, via the stereographic projection realizing the completed complex plane in 3D.

Definition 2.1. Given two distinct complex numbers (z_1,z_2), thought-off as fixed points, the MT $g(z) = (z-z_1)/(z-z_2)$ sends them onto the canonical antipodal pair $(0,\infty)$ is called the associated *isometric reduction*.

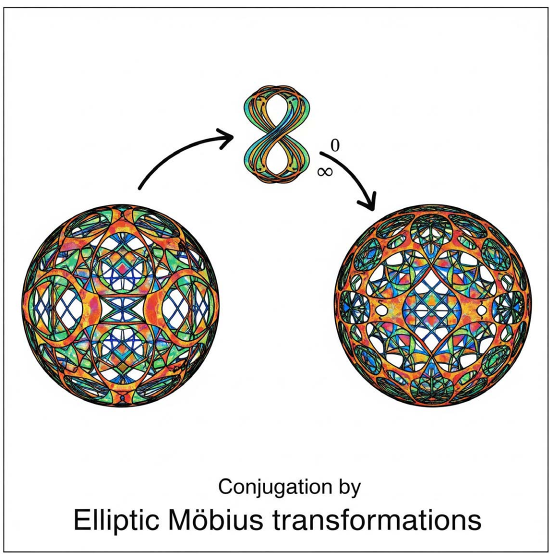
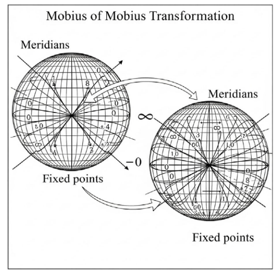


Figure 1. Conjugacy classes of Möbius transformations.

$$\begin{tikzcd} S^2 \arrow[r, "g"] \arrow[d, "f"] & S^2 \arrow[d, "f^g = M_k"] \\ S^2 \arrow[r, "g"] & S^2 \end{tikzcd}$$



Fixed points

It also provides the conjugation relating the (non-parabolic) Möbius transformation $f(z)$ to a multiplier:

$$f^g(z) = g\left(f\left(g^{-1}(z)\right)\right) = kz = M_k(z),\ g(z) = (z - z_1)/(z - z_2).$$

As mentioned above, $g(z)$ "moves"/"straightens" the fixed points z_i of f to the canonical antipodal pair $(0, \infty)$ (non-parabolic case, with distinct fixed points: "non-ramified").

For additional pictures of MT see [15].

Note that reversing the order associated to the fixed points amounts to swapping 0 and ∞, and trading k by its inverse $1/k$.

Then, after conjugation, $H(\lambda) = \mathrm{diag}\left[\lambda, \bar{\lambda}\right]$, with $k = \lambda^2$; the ± ambiguity when defining H is removed under projection to the corresponding MT $f(z)$.

This "reduction" by conjugation allows to investigate the Lie Theory aspects at the level of the relevant subgroups involved:

$$\left(\mathbb{C}^*, \cdot\right) \to G = Aut\left(S^2\right),$$

The multiplicative group of (non-zero) multipliers $M_k(z)$ is maximal abelian in the group G of MT $f(z)$.

2.7. Recall on the Adjoint Action

Conjugation induces the adjoint action of G on its Lie algebra $\mathfrak{g} = sl_2(\mathbb{C})$, which consists of traceless hermitean matrices (see Equation (4)):

$$\begin{array}{ccc} SL_2(\mathbb{C}) & \overset{Ad}{\rightarrow} & Inn\left(SL_2\mathbb{C}\right) \\ {}^{exp}\uparrow & & {}^{exp}\uparrow \\ sl_2(\mathbb{C}) & \overset{ad}{\rightarrow} & End\left(sl_2(\mathbb{C})\right) \end{array} \tag{6}$$

Recall that the inner automorphisms of $SL(2, C)$ form a subgroup of index 2.

3. Lie Theory Viewpoint

In what follows we apply Lie Theory to study MT and the associated vector fields in relation with 1-parameter groups of transformations defined by Lie algebra generators.

In this way we develop the relation between MT and Lorents transformations, with hindsight to applications in Special Relativity.

3.1. The 1-Parameter Subgroup of a MT

The case of multipliers $M_k(z)$, with diagonal matrix $H(\lambda)$, is straight forward.

For this purpose, it is convenient to also consider the *isoclinic embedding* of $\mathbb{C}^*$ into $SL_2\mathbb{C}$:

$$j:(\mathbb{C}^*,\cdot)\to SL_2\mathbb{C},\ \lambda\mapsto \mathrm{diag}\left[\lambda,\lambda^{-1}\right]. \tag{7}$$

This is typical of aiming to define a Hopf algebra structure via a coproduct, and also part of a Hodge structure in Algebraic-Geometry.

3.1.1. The Standard Flows and Its Superpositions

The Lie generator of a multiplier $M_k(z)$ is obtained by composing the Lie exponential of $\mathbb{C}^*$ with the isoclinic embedding:

$$X(H)=j(\log\lambda),\ \log\lambda = 1/2(r+i\theta).$$

The 1-parameter group is $H(\lambda;t)=H^t(\lambda)=H(\lambda^t)$. This is due to the fact that $\mathbb{C}^*$ is commutative.

The corresponding flow is obtained by applying the multiplier kz as a conformal transformation to the *standard flow* by *parallels as streamlines*, and *meridians as equipotentials*.

The *sources of rotation* ("magnetic", positive and negative, relative to the canonical orientation due to the complex structure), and respectively the *sources of divergence* ("electric", negative for a sink), are the two standard fixed points the North and South poles, forming the standard antipodal pair $(0,\infty)$, in this particular order (see Figure 1).

Due to commutativity of $\mathbb{C}^*$ as the subgroup of multipliers, this particular type of MT is a product of an elliptic MT (rotation/"Magnetic" Flow), and a hyperbolic MT (dilation/"Electric Flow")[6]:

$$\begin{tikzcd}
\textit{Cartan subgroup} & \mathbb{C}^* \arrow[r, "j"] \arrow[dr, "j"'] & SL_2\mathbb{C} \arrow[d] & & \lambda \mapsto H(\lambda) \arrow[d] \\
MT: & & G & \supset & \textit{Multipliers}:\ M_k(z)
\end{tikzcd}$$

The combined (superposed) flow is *loxodromic*, *i.e.* that of a spiraling motion from one fixed point (pole) to the other, with an orientation depending on whether $|k|>1$ (South to North) or $|k|<1$ (N to S).

Note that the sources $z_1=0$ and $z_2=\infty$ have both "electric" and "magnetic" charge $q=\log r+i\theta$.

This decomposition of the MT that are multipliers, reflects directly the direct product polar decomposition of $\mathbb{C}^*$.

More precisely, when considering the Polya vector field interpretation, a rotation by i, trades streamlines and equipotentials, yielding the other "harmonic conjugate" potential flow. Together they form a real basis for the 2D-linear flows (vector fields).

[6]The 2D-EM analogy is further detailed in §2.4.

3.1.2. From Group Action to Adjoint Representation

Now, let's conjugate and use the main property of the adjoint action for classical groups $SL_2(\mathbb{C})$ (§2.7 Equation (6)):

$$f \in SL_2(\mathbb{C}), X \in g, Ad(\exp(X)) = \mathrm{Exp}(ad(X)) \Rightarrow Ad(f)X = fXf^{-1}.$$

In order to better understand the main result, we need to recall the meaning of the Cartan basis, in the context of Root Systems for semi-simple Lie algebras like $sl_2(\mathbb{C})$: H is the eigenvector corresponding to the $\lambda = 0$ eigenvalue, while $X^{\pm}$ correspond to ± 1 eigenvalues. Similarly, a general element $X \in g$ satisfies the Cayley-Hamilton equation:

$$\det(X - \lambda Id) = 0 \Leftrightarrow X^2 - Tr(X)X + \det(X) = 0 \Leftrightarrow X^2 + \det(X) = 0,$$

with its two eigenvalues $\omega_i = \pm\sqrt{-\det(X)}$.

Using the result from [16], we obtain the following:

Theorem 3.1. *Let* $G = PSL_2(\mathbb{C})$ *the group of Möbius transformations, with* $g = sl_2(\mathbb{C})$ *its Lie algebra and* $X \in g$. *Then*:

Case 1: If $\det(X) = 0$, we have $\exp(X) = I + X$.

Case 2: If $\det(X) \neq 0$, we have

$$\exp(X) = \frac{1}{2}\sum_{i=1}^{2} e^{\omega_i}\left(I + \frac{1}{\omega_i}X\right).$$

Remark **3.1.** Let $X = \begin{bmatrix} p & q \\ r & -p \end{bmatrix} \in sl_2(\mathbb{C})$. If $p^2 + qr = 0$, then the one-parameter subgroup of X is $\rho_X(t) = \exp(tX): \mathbb{R} \to G$,

$$\rho_X(t) = \begin{bmatrix} 1+tp & tq \\ tr & 1-tp \end{bmatrix} = I + tX,$$

which can be identified with the curve of Möbius transformations

$$f_t(z) = \frac{(1+tp)z + tq}{trz + (1-tp)}. \tag{8}$$

If $p^2 + qr \neq 0$, then the one-parameter subgroup of X is

$$\rho_X(t) = \cosh(t\omega)I + \frac{\sinh(t\omega)}{\omega}X$$
$$= \begin{bmatrix} \cosh(t\omega) + p\cdot\frac{\sinh(t\omega)}{\omega} & q\cdot\frac{\sinh(t\omega)}{\omega} \\ r\cdot\frac{\sinh(t\omega)}{\omega} & \cosh(t\omega) - p\cdot\frac{\sinh(t\omega)}{\omega} \end{bmatrix},$$

which can be identified with the curve of Möbius transformations

$$\mathbf{R} \ni t \mapsto f_t(z) = \frac{\left[\cosh(t\omega) + p\cdot\frac{\sinh(t\omega)}{\omega}\right]z + q\cdot\frac{\sinh(t\omega)}{\omega}}{r\cdot\frac{\sinh(t\omega)}{\omega}\cdot z + \left[\cosh(t\omega) - p\cdot\frac{\sinh(t\omega)}{\omega}\right]} \in MT. \tag{9}$$

Before addressing the general case of how the 1-parameter group of transformations determines the Polya vector field of a corresponding Mobius transformation (see subsection 3.3), we provide a few examples.

3.2. Examples

Consider the case of the natural basis of $sl_2(\mathbb{C})$, *i.e.* the matrices X^+, X^- and H. The first two have null determinant. We apply the formulas from Remark 2.11, (ii) and we get the one-parameter subgroups:

$$\rho_{X^+}(t) = I + tX^+ = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

$$\rho_{X^-}(t) = I + tX^- = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix},$$

$$\rho_H(t) = \cosh(t) I + \sinh(t) H = \begin{bmatrix} \cosh(t) + \sinh(t) & 0 \\ 0 & \cosh(t) - \sinh(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{bmatrix}.$$

The corresponding flows of Möbius transformations are:

$$(f_{X^+})_t(z) = t + z = Id(t + Id(z)) = (Id \circ T_t \circ Id)(z) \ \ (\text{Parabolic case : translations}),$$

$$(f_{X^-})_t(z) = \frac{z}{tz+1} = (S \circ T_t \circ S^{-1})(z) \ \ (\text{Inversion/Conjugation}),$$

where $S(w) = 1/w$ is the inversion map and $T_t(z) = z + t$. The identity map $Id(z) = z$ was introduced for reasons of symmetry. We remark that T_1 is the generator of (integer) translations, which, together with S, generates the modular group $SL(2,\mathbb{Z})$. Moreover, S and all maps T_t generate the whole $SL(2,\mathbb{R})$.

On the Cartan (real) torus, we have:

$$(f_H)_t(z) = \mathrm{e}^{2t} z, \ t \in \mathbb{R}.$$

Starting with the flows of functions f_{X^+}, f_{X^-} and f_H, we can associate the families (“flows”) of Polya vector fields, parameterized by $t \in \mathbb{R}$:

$$V^t_{f_{X^+}} = (t+x)\partial_x - y\partial_y,$$

$$V^t_{f_{X^-}} = \frac{t(x^2+y^2)+x}{(tx+1)^2 + t^2y^2} \cdot \partial_x - \frac{y}{(tx+1)^2 + t^2y^2} \cdot \partial_y,$$

$$V^t_{f_H} = \mathrm{e}^{2t} x\partial_x - \mathrm{e}^{2t} y\partial_y.$$

The complex potentials for these Polya vector fields write, respectively, as:

$$\left(\frac{x^2-y^2}{2} + tx\right) + i(xy + ty),$$

$$\left[\frac{x}{t} - \frac{1}{2t^2}\ln\left((tx+1)^2 + t^2y^2\right)\right] + i\left[\frac{y}{t} - \frac{1}{t^2}\arctan\left(\frac{ty}{tx+1}\right)\right]$$

and

$$\frac{1}{2}\mathrm{e}^{2t}\left(x^2 - y^2 + 2ixy\right).$$

***Remark* 3.2.** (i) We can reinterpret these formulas in terms of the isoclinic embedding of $\mathbb{C}^*$, Equation (7), corresponding to the Cartan torus, and in terms of the Cartan Root System basis $\{H, X^{\pm}\}$.

These basis elements (eigenvectors) are canonical representatives of the conjugacy classes of MT which are dilations along the meridians, from "South Pole" 0 towards the "North Pole" ∞, or rotations about the axis through the canonical two fixed antipodal points, along the parallels §2.6.

Exponentiation of their superposition $X = \alpha H + \beta_+ X^+ + \beta_- X^-$ yields loxodromic transformations, and providing a visual interpretation of the classification of MT in terms of multiplier, conform §2.5. This general case is treated in § 3.3.

(ii) The action of the Möbius group on $\mathbb{P}^1\mathbb{C}$, maps the basis of $sl(2,\mathbb{C})$ given by X^+, X^- and H into the basis of the 3D Witt subalgebra $\tilde{X}^+$, $\tilde{X}^-$ and $\tilde{H}$, where

$$\tilde{X}^+ = \frac{\mathrm{d}}{\mathrm{dz}}, \quad \tilde{X}^- = -z^2\frac{\mathrm{d}}{\mathrm{dz}}, \quad \tilde{H} = 2z\frac{\mathrm{d}}{\mathrm{dz}}.$$

Since we do not consider the full *Dolbeault complex*, with $\partial/\partial z, \partial/\partial\overline{z}$ the two corresponding *Wirtinger derivatives* [17], we will use the simpler notation $\mathrm{d}/\mathrm{dz} = \partial_z$ instead. The trajectories of $\tilde{X}^+$ are horizontal lines pointing left/right relative to the axis Oy, as in **Figure 3**. The trajectories of $\tilde{H}$ suggest a monopole singular at the origin, with radial outward directed jet, as in **Figure 4**. The trajectories of $\tilde{X}^-$ suggest instead a dipole centered in the origin, with both inward and outward directed jets, as in **Figure 2**.

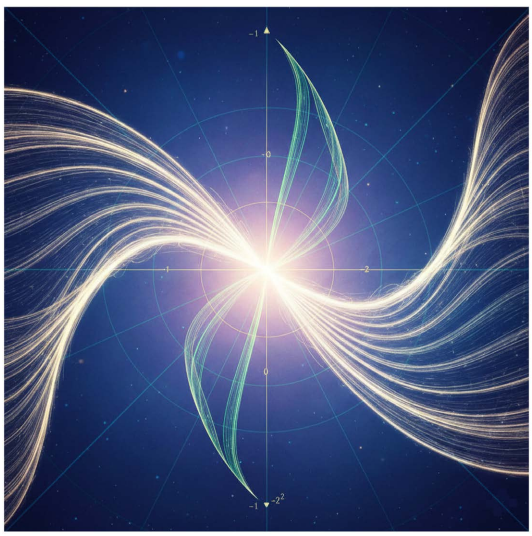


Figure 2. Trajectories of $\tilde{X}^-$.

The vector field $\alpha(z)\frac{\mathrm{d}}{\mathrm{dz}}$, via the Remark 2.2., writes

$$\alpha(z)\frac{\mathrm{d}}{\mathrm{dz}} = \frac{1}{2}P_\alpha + \frac{i}{2}U_\alpha.$$

In particular, we obtain:

$$\tilde{X}^{+}=\frac{1}{2}\partial_{x}-\frac{i}{2}\partial_{y},\ \tilde{X}^{-}=\left[\frac{1}{2}\left(y^{2}-x^{2}\right)\partial_{x}-xy\partial_{y}\right]-\left[xy\partial_{x}+\frac{1}{2}\left(y^{2}-x^{2}\right)\partial_{y}\right]i,$$

$$\tilde{H}=\left(x\partial_{x}+y\partial_{y}\right)+\left(y\partial_{x}-x\partial_{y}\right)i.$$

The double of their real parts provides the Polya vector fields of the following functions, respectively:

$$\tilde{f}^{+}(x,y)=(1,0),\ \tilde{f}^{-}(x,y)=\left(y^{2}-x^{2},2xy\right),\ \tilde{f}^{H}(x,y)=(2x,-2y).$$

We got three interesting anti-holomorphic functions:

- The image of the first one is the fixed point $(1,0)$; we may write it as $\tilde{f}^{+}(z)=\overline{z}^{0}$.
- The second function re-writes as $\tilde{f}^{-}(z)=-\overline{z}^{2}$; it is a special function, with important geometric, analytic and physical roles.
- The third function is the double of the complex conjugation $\tilde{f}^{H}(z)=2\overline{z}$.

(We have here another reason why it is better to extend Polya vector fields theory from the holomorphic to the complex framework.)

It is also noteworthy that the road from the complex functions associated to $\tilde{X}^{+}$, $\tilde{X}^{-}$, $\tilde{H}$, toward the complex functions $\tilde{f}^{+}$, $\tilde{f}^{-}$ and $\tilde{f}^{H}$ was made by complex conjugation.

(iii) The action of $SL(2,\mathbb{C})$ on $\mathbb{C}\times\mathbb{C}$ maps the basis of $sl(2,\mathbb{C})$ given by X^{+}, X^{-} and H into $\hat{X}^{+}$, $\hat{X}^{-}$ and $\hat{H}$, where

$$\hat{X}^{+}=z_{1}\cdot\frac{\partial}{\partial z_{2}},\ \hat{X}^{-}=z_{2}\cdot\frac{\partial}{\partial z_{1}},\ \hat{H}=z_{1}\cdot\frac{\partial}{\partial z_{1}}-z_{2}\cdot\frac{\partial}{\partial z_{2}}.$$

The trajectories of $\hat{X}^{+}$ are complex lines in $\mathbb{C}^{2}$ given by the flow $\dot{z}_{1}=0$ and $\dot{z}_{2}=z_{1}$, resulting in a shear transformation parallel to the z_{2} axis:

$$z_{1}(t)=z_{1}(0),\ z_{2}(t)=z_{2}(0)+tz_{1}(0).$$

The trajectories of $\hat{H}$ are the complex hyperbolas given by:

$$z_{1}(t)=z_{1}(0)\mathrm{e}^{t},\ z_{2}(t)=z_{2}(0)\mathrm{e}^{-t}.$$

This flow preserves the product $z_{1}z_{2}$, representing a hyperbolic scaling that leaves the origin invariant.

The trajectories of $\hat{X}^{-}$ are the complex lines corresponding to the dual shear transformation, parameterized as:

$$z_{1}(t)=z_{1}(0)+tz_{2}(0),\ z_{2}(t)=z_{2}(0).$$

Projective Action on the Riemann Sphere:

When projecting these flows from the fundamental representation in $\mathbb{C}^{2}$ down to the Riemann sphere $\mathbb{P}^{1}\mathbb{C}$ (via the affine coordinate $z=z_{1}/z_{2}$), we recover the *Witt algebra generators* $\tilde{X}^{+}$, $\tilde{X}^{-}$, and $\tilde{H}$ from part (ii). Their geometric trajectories describe the 1-parameter groups of Möbius transformations:

- **Translations ($\tilde{X}^{+}$):** The trajectories in the complex plane form a family of parallel horizontal lines. On the Riemann sphere, these lines compactify into mutually tangent circles that intersect exclusively at a single degenerate node:

the North Pole $z=\infty$ (See **Figure 3**).

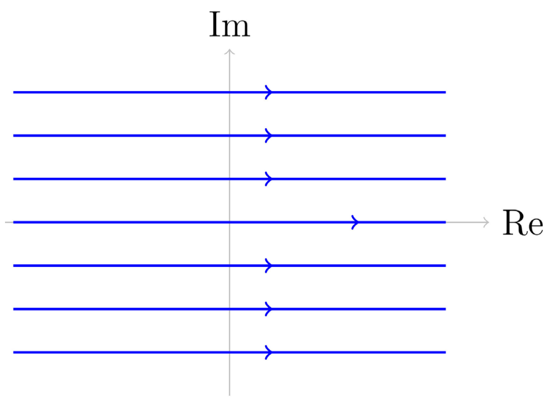


Figure 3. Trajectories of $\tilde{X}^+$ (translations): A uniform horizontal flow in the complex plane.

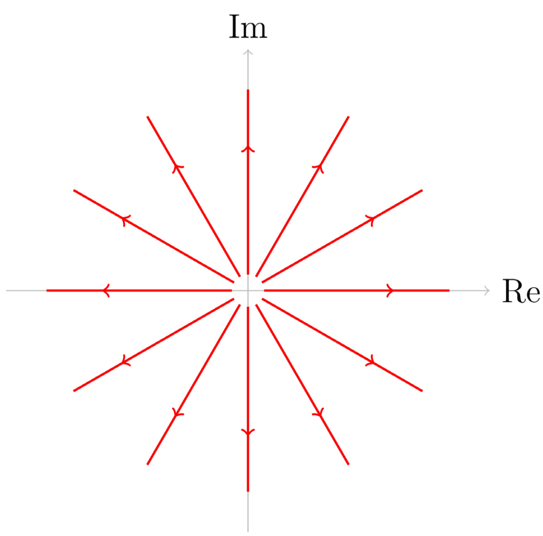


Figure 4. Trajectories of $\tilde{H}$ (dilations): A radial flow representing a hyperbolic scaling from the origin.

- **Dilations ($\tilde{H}$):** The trajectories in the complex plane form rays extending radially from the origin (**Figure 4**). On the Riemann sphere, these correspond to meridians acting as loxodromes flowing from the South Pole (the source at $z=0$) directly to the North Pole (the sink at $z=\infty$).
- **Special Conformal Transformations ($\tilde{X}^-$):** The trajectories in the complex plane form a family of circles that all pass through the origin and are tangent to the real axis (**Figure 5**). On the Riemann sphere, this defines a dipole field whose inward and outward trajectories all meet at the South Pole ($z=0$). This is the exact dual to $\tilde{X}^+$ via the inversion map $S(z)=1/z$.

Note the notation distinguishes between spinorial action of $SL(2,C)$ on the spinor space $\mathbb{C}\times\mathbb{C}$, e.g. $\hat{H}$, versus the corresponding extension of the action on the Riemann sphere, e.g. $\tilde{H}$.

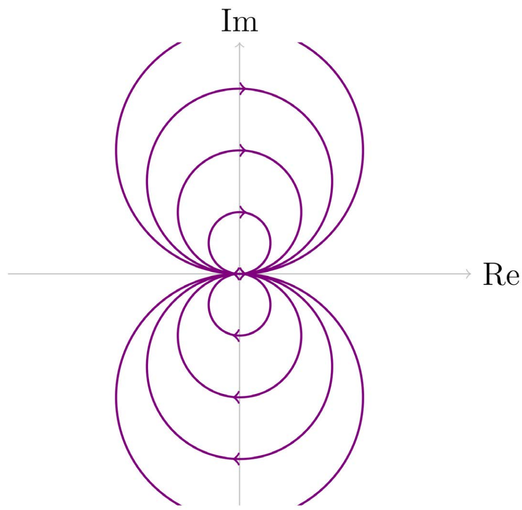


Figure 5. Trajectories of $\tilde{X}^-$ (special conformal): A dipole field with streamlines forming circles tangent to the real axis at the origin.

***Remark* 3.3.** Needham's book [7] is famous for these exact visualizations. Chapter 11 contains beautifully illustrated Polya vector fields on the Riemann Sphere, explicitly showing the dipole flow, source/sinks, and translations mapped to the sphere.

The reader is also invited to consult [18]; this is a widely cited paper and has an accompanying YouTube video. It explicitly visualizes how the 2D plane flows of translations, dilations and rotations, map onto the Riemann Sphere via the stereographic projection.

3.3. From $\mathfrak{sl}(2,\mathbb{C})$ to Pólya Vector Fields: The General Case

We return to the general setting from Theorem 3.1 and Remark 3.1. Let $X = \begin{bmatrix} p & q \\ r & -p \end{bmatrix} \in \mathfrak{sl}_2(\mathbb{C})$ and let ω be one of the (complex) roots of the characteristic equation $z^2 + \det X = 0$.

To explicitly link the Lie-algebraic computation to the geometric flow picture, we provide the following derivation map. This conceptual pipeline demonstrates how the exponential formulas rigorously generate the associated Pólya vector fields:

1) **Lie Algebra Generator (Infinitesimal):** We begin with a specific conformal generator $X \in \mathfrak{sl}_2(\mathbb{C})$.

2) **Exponentiation (Matrix Dynamics):** We compute the 1-parameter subgroup $\rho_X(t) = \exp(tX) \in SL_2(\mathbb{C})$. Theorem 3.1 yields the exact, closed-form algebraic expressions for these matrices depending on whether $\det X$ is zero or non-zero.

3) **Projective Action (Kinematic Flow):** We project the matrix subgroup $\rho_X(t)$ onto the Riemann sphere via the standard Möbius group quotient action. This yields the time-dependent macroscopic conformal flows $f_t(z)$ explicitly

parameterized in equations (8) and (9).

4) **Pólya Vector Field (Geometric Flow):** Finally, we apply our Analytic-to-Dynamic Dictionary (Table 1) to $f_t(z)$. By extracting the real and imaginary parts of the flow $f_t(z)$, we analytically construct the precise Pólya vector field $V_{f_t} = \mathrm{Re}(f_t)\partial_x - \mathrm{Im}(f_t)\partial_y$, defining the physical trajectories (streamlines and equipotentials) on the plane.

Executing this pipeline via direct calculation proves the following three results.

Proposition 3.1. *With the previous notations, consider* V_{f_t} *the family of integral lines (the "flow") of Pólya vector fields associated with* f_t[7]. *Then*:

(i) *if* $\det X = 0$, *we have*

$$\begin{aligned} V_{f_t} = &\left(\frac{tr(1+tp)(x^2+y^2)+\left[1-t^2\left(p^2-qr\right)\right]x+tq(1-tp)}{(trx+1-tp)^2+t^2r^2y^2}\right)\partial_x \\ &-\left(\frac{\left[1-t^2\left(p^2+qr\right)\right]y}{(trx+1-tp)^2+t^2r^2y^2}\right)\partial_y \end{aligned} \tag{10}$$

(ii) *if* $\det X \neq 0$, *we have*

$$\begin{aligned} V_{f_t} = &\frac{\left(rSC+prS^2\right)(x^2+y^2)+\left[C^2-S^2\left(p^2-qr\right)\right]x+\left(qSC-pqS^2\right)}{(rSx+C-pS)^2+r^2S^2y^2}\partial_x \\ &-\frac{\left[C^2-S^2\left(p^2+qr\right)\right]y}{(rSx+C-pS)^2+r^2S^2y^2}\partial_y \end{aligned} \tag{11}$$

where $C := \cosh(t\omega)$ *and* $S := \dfrac{\sinh(t\omega)}{\omega}$.

Proposition 3.2. *With the previous notations, consider* $\tilde{X}$ *the image of* X *via the action of the Möbius group on* $\mathbb{C}$. *Then*:

$$\tilde{X} = \left(-rz^2+2pz+q\right)\frac{\partial}{\partial z}, \tag{12}$$

or, equivalently, expanded into real and imaginary components (where $p = p_1 + ip_2$, $q = q_1 + iq_2$, $r = r_1 + ir_2$ *):*

$$\tilde{X} = \mathrm{Re}\left(\tilde{X}\right) + \mathrm{Im}\left(\tilde{X}\right)i, \tag{13}$$

where

$$\begin{aligned} \mathrm{Re}\left(\tilde{X}\right) = &\frac{1}{2}\left[q_1+2p_1x-2p_2y-r_1\left(x^2-y^2\right)+2r_2xy\right]\frac{\partial}{\partial x} \\ &+\frac{1}{2}\left[q_2+2p_2x+2p_1y-r_2\left(x^2-y^2\right)-2r_1xy\right]\frac{\partial}{\partial y}, \\ \mathrm{Im}\left(\tilde{X}\right) = &\frac{1}{2}\left[q_2+2p_2x+2p_1y-r_2\left(x^2-y^2\right)-2r_1xy\right]\frac{\partial}{\partial x} - \\ &-\frac{1}{2}\left[q_1+2p_1x-2p_2y-r_1\left(x^2-y^2\right)+2r_2xy\right]\frac{\partial}{\partial y}. \end{aligned}$$

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[7]We use V_{f_t} to avoid the more cumbersome notation $V_{f_t}^t$.

Proposition 3.3. *With the previous notations, consider* $\tilde{f}$ *the Polya vector field associated to the double of the real part of* $\tilde{X}$. *Then*:

$$\tilde{f}(z) = -\bar{r}\bar{z}^2 + 2\bar{p}\bar{z} + \bar{q}, \tag{14}$$

or, equivalently,

$$\begin{aligned}\tilde{f}(x,y) = &\left[q_1 + 2p_1x - 2p_2y - r_1\left(x^2 - y^2\right) + 2r_2xy\right] \\ &-i\left[q_2 + 2p_2x + 2p_1y - r_2\left(x^2 - y^2\right) - 2r_1xy\right].\end{aligned} \tag{15}$$

In particular, assuming $r, p, q \in \mathbb{R}$ *for this specific real reduction, we have*:

$$\tilde{f}(x,y) = \left(-rx^2 + ry^2 + 2px + q\right) + i\left(2rxy - 2py\right). \tag{16}$$

Remark **3.4.** One may consider the 1-parameter family f_t as a canonical deformation typical of Lie Theory, and focus on the function $f(z) = f_1(z)$ and its corresponding static formulas to evaluate the flow at a specific evolutionary state.

4. Conclusions and Future Outlook

In this expository article, we have analyzed Möbius transformations by unifying the Klein geometric perspective with the Pólya vector field interpretation. By employing Hamiltonian dynamics and Lie Theory, these viewpoints complement the purely abstract algebraic classification of their structures.

4.1. Rigorous Mathematical Synthesis

Mathematically, our approach explicitly links conformal geometry to symplectic geometry. The normal forms, conjugacy classes, and one-parameter subgroups of $SL(2,\mathbb{C})$ map directly onto the irrotational and solenoidal vector fields in the complex plane. Furthermore, interpreting the complex numbers via the Cayley-Dickson doubling construction—rather than exclusively as a simple field extension $\mathbb{R}[i]$—provides a natural algebraic scaffolding that extends beyond complex numbers to higher-dimensional division algebras, such as quaternions, octonions, and sedonions [3].

4.2. Heuristic Physics Applications and Further Developments

While the mathematical correspondences established here are rigorous, they serve as a structural foundation for several heuristic physical theories and analogies. For instance, the exact correspondence between complex and symplectic structures exemplifies one of "Arnold's trinities", providing a local kinematic analog for broader theoretical frameworks like the AdS/CFT correspondence.

To clearly demarcate our established mathematical hierarchy from its speculative applications in physics, specifically within the *Standard Model* and *Gauge Theories* [19]-[22], we summarize the conceptual territory in **Table 3**. This map highlights both the formal mathematical structures and their anticipated physical correlates, separating established algebraic and geometric facts from their corresponding physical kinematics.

Table 3. Conceptual map distinguishing the rigorous algebraic and geometric structures from their corresponding heuristic physical applications.

Rigorous Mathematical Frameworks				Heuristic Physics Analogies
Abstract Algebra (Galois)	**Abstract Geometry (Klein)**	**Lie Theory ($\exp \mapsto \mathrm{ad}$)**	**Symplectic Dynamics (Pólya V.F.)**	**Gauge Theory (Kähler/Hopf Fibrations)**
$\mathbb{C} = \mathbb{R}[i]$	$\mathbb{C}^*$	$U(1) \times \mathbb{R}_+^*$	$\mathbb{C} \cong T^*\mathbb{R}$	$(J, \mathbb{C}) \to (\mathfrak{g}, \omega)$; 1st Hopf Fibr.
$Sp = \mathcal{D}(\mathbb{C})$	$SL_2(\mathbb{C})$	$Sp \mapsto \mathbb{H}$	MT/Lorentz Transf.	2nd Hopf Fibration
$\mathbb{O} = \mathcal{D}(Sp)$	…?	…?	…?	Standard Model (SM)?
$\mathbb{S} = \mathcal{D}(\mathbb{O})$ (Sedonions)	…?	…?	…?	3rd Hopf Fibration

The pedagogical value of this exposition lies in establishing precise connections between mathematical disciplines that are traditionally isolated in specialized courses. This fits within a larger picture relating Classical Physics and Quantum Physics [4]. The natural next step in this research program is to apply these exact Lie-theoretic and dynamical ideas to the double of the quaternions—the sedonions—and investigate their rigorous mathematical relationship to the 3rd Hopf fibration, thereby laying a formal groundwork for subsequent physics applications [3] [4]. Note also that this direction is being actively considered by other researchers, including [23].

Author Contributions

The research reported in this article is the result of an extensive collaboration of the first two authors, with contributions from the third author, which were essential for the computational aspects contained in the present article.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix. On Cayley-Dickson Doubling Construction

It is well known that the *Cayley-Dickson* (*doubling*) *Construction* (CDC) $\mathcal{D}(A)$ of an algebra with involution A yields also the complex numbers, as an alternative viewpoint to the traditional field extension with Galois Theory within Abstract Algebra, besides its role of defining the quaternions, octonions, sedonions etc.

It is important to distinguish between the double of $\mathbb{C}$, denoted here $Sp = \mathcal{D}(\mathbb{C})$, as spinors related, from Hamilton's quaternions $\mathbb{H} = \{t + xi + yj + zk\}$, since the later is in fact obtained via the adjoint representation of the former, associated to the fundamental representation of $SU(2)$:

$$SU(2) = Aut_R(\mathcal{D}(\mathbb{C})) \overset{ad}{\mapsto} 0 \to (\mathbb{R}, \cdot) \to (\mathbb{H}, \cdot) \to (\mathbb{R}^3, \times) \to 0.$$

The right hand side is a *Lie algebra central extension*, as the "*infinitesimal*" (tangent space) version of conjugation Ad, hence related to the *infinitesimal deformation* of the algebraic structure at hand, which is achieved by the Cayley-Dickson construction.

Reinterpreted this way yields additional insight into the rich structures of quaternions, octonions and sedonions, with deep implications to their use in Physics.

Mirror Symmetry is another direction of research, where the role of the *Cayley-Dickson doubling Construction*, yielding the "accidental" Lagrangian structure mentioned in the introduction, is important, as explained in a recent presentation by the first author [24].

The reader is invited to explore this research direction [4], seemingly not present in the current literature.